Problems and Solutions
for
Ordinary Diffferential Equations

by
Willi-Hans Steeb
International School for Scientific Computing
at
University of Johannesburg, South Africa

and

by
Yorick Hardy
Department of Mathematical Sciences
at
University of South Africa, South Africa

updated: February 8, 2017

Preface

The purpose of this book is to supply a collection of problems for ordinary differential equations.

Prescribed books for problems.

1) Continous Symmetries, Lie Algebras, Differential Equations and Computer Algebra, second edition

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2007
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6515.html`

2) Problems and Solutions in Theoretical and Mathematical Physics, third edition, Volume I: Introductory Level

by Willi-Hans Steeb
World Scientific, Singapore, 2009
ISBN-13 978-981-4282-14-7
`http://www.worldscibooks.com/physics/7416.html`

3) Problems and Solutions in Theoretical and Mathematical Physics, third edition, Volume II: Advanced Level

by Willi-Hans Steeb
World Scientific, Singapore, 2009
ISBN-13 978-981-4282-16-1
`http://www.worldscibooks.com/physics/7416.html`

The International School for Scientific Computing (ISSC) provides certificate courses for this subject. Please contact the author if you want to do this course or other courses of the ISSC.

e-mail addresses of the author:

steebwilli@gmail.com
steeb_wh@yahoo.com

Home page of the author:

http://issc.uj.ac.za

Contents

Chapter 1

First Order Differential Equations

We consider differential equations of the form

$$\frac{du}{dt} = f(u). \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous differentiable function.

A point u^* is called a *fixed point* of the differential equation if $f(u^*) = 0$.

The *variational equation* of $du/dt = f(u)$ is given by

$$\frac{dy}{dt} = \left(\frac{df}{du}(u(t))\right) y \tag{2}$$

where it is assumed that f is continuous differentiable.

Problem 1. (i) Solve the initial value problem $u(t = 0) = 0$ for the first order ordinary differential equation

$$\frac{du}{dt} = k(a - u)(b - u)$$

where $k > 0$, $a > 0$ and $b > 0$.
(ii) Find the fixed points.
(iii) What happens for $t \to \infty$?

Problem 2. Discuss the qualitative behaviour of the one-dimensional nonlinear differential equation

$$\frac{du}{dt} = r - u^2$$
$$\frac{du}{dt} = ru - u^2$$
$$\frac{du}{dt} = -(1 + r^2)u^2$$

where r is a bifurcation parameter. Study the behaviour of the fixed points.

Problem 3. Find the solution of the initial value problem the linear differential equation

$$\frac{du}{dx} = x + u, \quad u(0) = 0.$$

Problem 4. Consider the nonlinear differential equation

$$\frac{du}{dt} = \sin(u)$$

with the initial value $u(t = 0) = u_0 = \pi/2$.
(i) Find the fixed points.
(ii) Solve the differential equation by direct integration. Hint.

$$\int \frac{du}{\sin(cu)} = \frac{1}{c} \ln\left(\tan\left(\frac{cu}{2}\right)\right).$$

What happens if $t \to \infty$?
(iii) Find the solution of the initial value problem using the *Lie series expansion*

$$u(t) = \exp\left(t \sin(u) \frac{d}{du}\right) u\Big|_{u=u_0}.$$

Problem 5. Consider the initial value problem of the nonlinear differential equation

$$\frac{du}{dt} = u - u^2, \qquad u(t = 0) = u_0 > 0.$$

(i) Solve the differential equation by direct integration. Find $u(t)$ for $t \to \infty$.
(ii) Solve the differential equation using the Lie series

$$u(t) = e^{tV} u\big|_{u \to u_0}$$

where V is the *vector field* V associated with the differential equation

$$V = (u - u^2)\frac{d}{du}.$$

Problem 6. Consider the *logistic equation*

$$\frac{du}{dt} = ru(1-u)$$

with $u(t=0) = u_0$. Find the solution of the initial value problem.

Problem 7. Solve the *Bernoulli equation*

$$\frac{du}{dx} + P(x)u = Q(x)u^n, \qquad n \neq 0, 1. \tag{1}$$

Problem 8. An ordinary differential equation

$$H\left(t, u(t), \frac{du}{dt}\right) = 0 \tag{1}$$

may often be simplified or reduced to a standard form by introducing new variables, T, U by means of the equations

$$T(t) = G(t, u(t)), \qquad U(T(t)) = F(t, u(t)). \tag{2}$$

We assume that $H(t, u, \dot{u})$, $G(t, u)$ and $F(t, u)$ are smooth functions. All considerations are local. Geometrically, we regard (2) as a *point transformation* , for it transforms points (t, u) of the tu-plane to points (T, U) of the TU-plane. We assume that the *Jacobian determinant*

$$J := \frac{\partial(T, U)}{\partial(t, u)} = \begin{vmatrix} \dfrac{\partial G}{\partial t} & \dfrac{\partial G}{\partial u} \\ \dfrac{\partial F}{\partial t} & \dfrac{\partial F}{\partial u} \end{vmatrix} \neq 0 \tag{3}$$

over a region R of the tu-plane. There is then no functional relation between u and U; for this would imply $J = 0$. Moreover, if the point (T_1, U_1) corresponds to (t_1, u_1) we can solve equations (2) uniquely for t, u in the neighbourhood of t_1, u_1. We thus obtain the inverse transformation

$$t(T) = Q(T, U(T)), \qquad u(t(T)) = P(T, U(T)). \tag{4}$$

Thus we can transform (1) into another

$$\tilde{H}\left(T, U(T), \frac{dU}{dT}\right) = 0. \tag{5}$$

If (5) can be integrated to give

$$\varphi(T, U(T), C) = 0 \tag{6}$$

we obtain a solution of (1) on replacing u and v by their values (2) in terms of t and u. Show that

$$\frac{dU}{dT} = \frac{\dfrac{\partial F}{\partial u}\dfrac{du}{dt} + \dfrac{\partial F}{\partial t}}{\dfrac{\partial G}{\partial u}\dfrac{du}{dt} + \dfrac{\partial G}{\partial t}}. \tag{7}$$

Problem 9. Solve the *Bernoulli equation*

$$\frac{du}{dx} + u = u^3. \tag{1}$$

Problem 10. Solve the *Bernoulli equation*

$$\frac{du}{dx} + u \tan x = u^3 \sec^4 x \tag{1}$$

where $\sec x := 1/\cos x$.

Problem 11. Solve the *Riccati equation*

$$\frac{du}{dx} - e^{-x}u^2 - u - e^x = 0. \tag{1}$$

Problem 12. Find the solution of the initial value problem of the special *Riccati equation*

$$\frac{du}{dt} = u^2 + t, \qquad u(0) = 1. \tag{1}$$

Problem 13. Prove the following. If in the *generalized Riccati equation*

$$\frac{du}{dx} = f(x)u^2 + g(x)u + h(x) \tag{1}$$

the coefficients f, g and h, defined and continuous in some open interval $(a, b) \subset \mathbb{R}$, are related as

$$f + g + h = \frac{d}{dx} \ln \frac{\alpha}{\beta} - \frac{\alpha - \beta}{\alpha\beta}(\alpha f - \beta h) \tag{2}$$

with $\alpha(x)$ and $\beta(x)$ properly chosen functions differentiable in (a, b) such that $\alpha\beta > 0$, then (1) is integrable by quadratures.

Problem 14. Consider the initial value problem for the *Riccati equation*

$$\frac{du}{dx} = x^2 + u + 0.1u^2, \qquad u(0) = 0. \tag{1}$$

There is no elementary solution. Therefore one neglects the quadratic term and solves the linear equation

$$\frac{du_1}{dx} = x^2 + u_1, \qquad u_1(0) = 0. \tag{2}$$

(i) Show that this gives a first approximate solution

$$u_1(x) = 2e^x - (x^2 + 2x + 2). \tag{3}$$

(ii) Reintroduce this solution into (1) and now solve the differential equation

$$\frac{du_2}{dx} = x^2 + u_2 + 0.1(u_1(x))^2, \qquad u_2(0) = 0. \tag{4}$$

(iii) Show that

$$u_2(x) = u_1(x) + \frac{2}{5}e^{2x} - \frac{2}{15}e^x(x^3+3x^2+6x-54) - \frac{1}{10}(x^4+8x^3+32x^2+72x+76). \tag{5}$$

Problem 15. *Abel's differential equation* of the first kind is written in the form

$$\frac{du}{dt} = a_0(t) + a_1(t)u + a_2(t)u^2 + a_3(t)u^3 \tag{1}$$

where a_j $(j = 0, 1, 2, 3)$ are known smooth functions of t. (i) Show that (1) can be put into the standard form as

$$\frac{dz}{dx} = z^3 + p(t) \tag{2}$$

by introducing the following transformations

$$u(t) = a(t)z(x(t)) + b(t), \qquad x(t) = \int^t a^2(s)a_3(s)ds \tag{3}$$

with

$$a(t) := \exp\left(\int^t \left(a_1(s) - \frac{a_2^2(s)}{3a_3(s)}\right) ds\right), \qquad b(t) = \frac{a_2(t)}{3a_3(t)}. \tag{4}$$

Here $p(t)$ in (2) has the form

$$p(t) = \left(a_0 - \frac{1}{3}\frac{a_1a_2}{a_3} + \frac{2}{27}\frac{a_2^3}{a_3^2} + \frac{1}{3}\frac{d}{dt}\frac{a_2}{a_3}\right)(a^3a_3)^{-1}. \tag{5}$$

(ii) Show that if $a_3 = 0$, (1) reduces to the Ricatti equation.
(iii) Show that for $a_0 = 0$, $a_1 \neq 0$ and either $a_2 = 0$, $a_3 \neq 0$ or $a_2 \neq 0$, $a_3 = 0$, it becomes the nonlinear differential equation of Bernoulli type, which has an explicit general solution.

Problem 16. Show that the homogeneous equation

$$\frac{du}{dx} + f\left(\frac{u}{x}\right) = 0 \tag{1}$$

is transformed by

$$y = x, \qquad v(y(x)) = \frac{u(x)}{x} \tag{2}$$

into the separable equation

$$v + y\frac{dv}{dy} + f(v) = 0. \tag{3}$$

Problem 17. Show that the differential equation

$$\frac{du}{dx} = f\left(\frac{ax + bu + c}{\alpha x + \beta u + \gamma}\right) \tag{1}$$

can be integrated by means of a *point transformation*.

Problem 18. Solve

$$\frac{du}{dx} = \frac{2x + 3u - 4}{4x + u - 3}. \tag{1}$$

Problem 19. Solve the initial value problem of the differential equation

$$\frac{du}{dt} = k(a - u)(b - u), \qquad a > b > 0 \tag{1}$$

by direct integration if $u(t = 0) = 0$.

Problem 20. (i) Find the fixed points of the differential equation

$$\frac{du}{dt} = u - u^2. \tag{1}$$

(ii) Find the solution of (1) with the initial condition $u(t = 0) = u_0$.
(iii) Find the variational equation.
(iv) Study the stability of the fixed points.

Problem 21. Consider the nonlinear ordinary differential equation

$$\frac{du}{dt} = -qu^{1+(1/q)}, \qquad t > 0 \tag{1}$$

with $u(0) = 1$, where q is any positive integer.
(i) Show that using (1) recursively, we obtain

$$\frac{d^n u}{dt^n} = (-1)^n q(q+1)\cdots(q+n-1)u^{1+n/q}, \quad n = 1, 2, \cdots. \tag{3}$$

(ii) Show that the *Taylor series expansion* of u is then given by

$$u(t) = 1 - qt + \cdots + \frac{(-1)^n}{n!}q(q+1)\cdots(q+n-1)t^n + \cdots. \tag{4}$$

(iii) Show that from the Cauchy-Hadamard theorem it follows that the radius of convergence of series (4) is 1. Show that the solution of the initial value problem is given by

$$u(t) = (1+t)^{-q}. \tag{5}$$

Thus, $t = 1$ is not a singular point of u and the solution of the problem is defined for any $t > 0$. But Taylor series (4) in its original form does not give any informations about u for $t > 1$. (iv) Apply *Padé approximation* to (4), and show that the $[N/N+L]$ $(L \geq 0)$ approximant to it is given by

$$[N/N+L] = \frac{1 + a_1 t + \cdots + a_N t^N}{1 + b_1 t + \cdots + b_{N+L} t^{N+L}} \tag{6}$$

where

$$a_m = \frac{N(N-1)\cdots(N-m+1)(N+L-q)(N+L-1-q)\cdots(N+L-m+1-q)}{m!(2N+L)(2N+L-1)\cdots(2N+L-m+1)} \tag{7}$$

with $m = 1, 2, \cdots, N$ and

$$b_m = \frac{(N+L)(N+L-1)\cdots(N+L-m+1)(N+q)(N+q-1)\cdots(N+q-m+1)}{m!(2N+L)(2N+L-1)\cdots(2N+L-m+1)} \tag{8}$$

with $m = 1, 2, \cdots, N+L$. Show that the $[N/N+L]$ approximants for $N + L \geq q$ in a form of irreducible rational fractions reduce to the exact solution (5).

Problem 22. Try to solve the classical *brachystochrone problem*

$$\left(\frac{du}{dt}\right)^2 = \frac{L}{u} - 1 \qquad \text{or} \qquad u\left(\frac{du}{dt}\right)^2 + u = L \tag{1}$$

by a series solution. We suppose $h = 0$ and the initial value $u(0) = 0$.

Problem 23. Consider the nonlinear differential equation

$$\frac{du}{dt} = -u^2 \tag{1}$$

with the initial condition $u(t = 0) \equiv u_0 = 1$ and $t \in [0, \infty)$.
(i) Show that the exact solution of the initial value problem is given by

$$u(t) = \frac{1}{1+t}. \tag{2}$$

(ii) Solve initial value problem of the differential equation (1) with the help of *Lie series*.

Problem 24. Find the solution of the differential equation

$$\frac{du}{dt} = r - u^2, \qquad u(0) = 0 \tag{1}$$

as a function of the bifurcation parameter r, where $r > 0$, $r = 0$ and $r < 0$.

Problem 25. Discuss the behaviour of the differential equation

$$\frac{du}{dt} = ru + u^3 \tag{1}$$

as a function of the bifurcation parameter r.

Problem 26. Discuss the behaviour of the differential equation

$$\frac{du}{dt} = \mu - u^2, \qquad u(0) = 0 \tag{1}$$

as a function of the bifurcation parameter μ, where $\mu > 0$, $\mu = 0$ and $\mu < 0$.

Problem 27. Define a function $u(x)$ (the *inverse error function*)

$$x = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-t^2) dt \tag{1}$$

and show that it satisfies the differential equation

$$\frac{du}{dx} = \frac{\sqrt{\pi}}{2} \exp(u^2), \qquad u(0) = 0. \tag{2}$$

Obtain recursion formulas for its Taylor coefficients.

Problem 28. Show that the differential equation

$$\frac{du}{dx} = (-u\sin x + \tan x)u, \qquad u(\pi/6) = \frac{2}{\sqrt{3}} \tag{1}$$

admits the solution

$$u(x) = \frac{1}{\cos(x)}. \tag{2}$$

Problem 29. (i) Show that the general solution of

$$\left|\sin\theta\frac{d\phi}{d\theta}\right| = |\sin\phi| \tag{1}$$

is given by

$$\cos\phi(\theta) = \frac{\sinh c + \cosh c\cos\theta}{\cosh c + \sinh c\cos\theta} \tag{2}$$

where c is the constant of integration with $-\infty < c < \infty$. Show that when

$$c = 0 \to \phi(\theta) = \theta$$

when

$$c > 0 \to \phi(\theta) \le \theta$$

and when

$$c < 0 \to \phi(\theta) \ge \theta$$

Hint. Notice that

$$\frac{d}{d\theta}\cos\phi(\theta) = -\sin\phi(\theta)\frac{d\phi}{d\theta}. \tag{3}$$

Problem 30. The majorizing differential equation is given by

$$\frac{du}{dx} = \frac{M}{(1-x/r)(1-u/r)}, \qquad u(0) = 0 \tag{1}$$

where M and r are positive constants. Find the solution.

Problem 31. Consider the initial problem

$$\frac{du}{dt} = t + u^2, \qquad u(0) = 1 \tag{1}$$

Show that the solution explodes somewhere in the interval $[\pi/4, 1]$.

Problem 32. Discuss the fixed points (equilibrium points) of the nonlinear ordinary differential equation

$$\frac{du}{dt} = \gamma\sin(\omega t)\sin^{1/3}\left(\frac{\omega u}{\alpha}\right)$$

where γ, ω, and α are positive constants.

Problem 33. Solve the initial problem for the differential equation

$$\frac{du}{dt} = -u^{1/3}, \qquad u(0) \equiv u_0 > 0. \tag{1}$$

Study the stability of the fixed point $u^* = 0$.

Problem 34. A first order differential equation modelling the concentration $u(t)$ of an allosteric enzyme is given by

$$\frac{du}{dt} = -u\frac{a+bu}{c+u+du^2}$$

where $u(t=0) > 0$ and a, b, c, d are positive constants. Find the fixed points and study their stability.

Problem 35. Let f be an analytic function of x and u. In *Picard's method* one approximates a solution of a first order differential equation

$$\frac{du}{dx} = f(x,u)$$

with initial conditions $u(x_0) = u_0$ as follows. Integrating both sides yields

$$u(x) = u_0 + \int_{x_0}^{x} f(s, u(s))ds.$$

Now starting with u_0 this formula can be used to approach the exact solution iteratively if the series converges. The next approximation is given by

$$u_{k+1}(x) = u_0 + \int_{x_0}^{x} f(s, u_k(s))ds, \qquad k = 0, 1, 2, \ldots.$$

Apply this approach to the linear differential equation

$$\frac{du}{dx} = x + u$$

where $x_0 = 0$, $u(x_0) = 1$.

Problem 36. Consider the nonlinear differential equation

$$\frac{du}{dt} = \frac{1}{u}, \qquad u(t=0) = u_0 > 0.$$

Solve the initial value problem using

$$u(t) = e^{tV}u\big|_{u \to u_0}$$

where the vector field V is given by

$$V = \frac{1}{u}\frac{d}{du}.$$

solve the initial value problem by direct integration of the differential equation. Compare the two solutions. Discuss.

Problem 37. (i) Consider

$$\frac{du}{dt} = -2u + 3u^2 - u^3$$

where $u(t=0) = u_0 > 0$. Find the fixed points of the differential equation.
(ii) Calculate the Gateaux derivative of

$$\frac{du}{dt} + 2u - 3u^2 + u^3$$

and thus find the variational equation.
(iii) Study the stability of the fixed points.

Problem 38. (i) Consider

$$\frac{du}{dt} = \cos(u)$$

where $u(t=0) = u_0 > 0$. Find the fixed points of the differential equation.
(ii) Calculate the Gateaux derivative of

$$\frac{du}{dt} - \cos(u).$$

and thus find the variational equation. Study the stability of the fixed points.

Problem 39. (i) Consider

$$\frac{du}{dt} = \cos(u)\sin(u)$$

where

$$u(t=0) = u_0 > 0.$$

Find the fixed points of the differential equation.
(ii) Calculate the Gateaux derivative of

$$\frac{du}{dt} - \cos(u)\sin(u)$$

and thus find the variational equation. Study the stability of the fixed points.

Problem 40. Consider the intial value problem for the nonlinear differential equation

$$\frac{du}{dt} = e^u - 1.$$

Find the fixed points and study their stability.

Problem 41. Find the solutions of the initial value problem of the nonlinear differential equation

$$\frac{du}{dt} = 3u^{2/3}, \qquad u(0) = 0.$$

Problem 42. Consider the first order linear scalar differential equation

$$\frac{du}{dt} = r(t) + k(t)u$$

where $r(t)$ and $k(t)$ are continuous functions on a finite closed interval $[a, b]$ of the real line. Let τ be an arbitrary point of $[a, b]$. Show that the general solution of this differential equation is given by

$$u(t) = \exp\left(\int_\tau^t k(s)ds\right) u(\tau) + \int_\tau^t \exp\left(\int_s^t k(s_1)ds_1\right) r(s)ds.$$

Problem 43. Find the solution of the initial value problem

$$\frac{du}{dt} = u^2, \qquad u(0) = 1.$$

Discuss.

Problem 44. Consider the nonlinear differential equation

$$\frac{du}{dt} = \cos(u)\sin(u).$$

(i) Find all fixed points.
(ii) Find the variational equation and study the stability of one of the fixed point.
(iii) Find the solution of the initial value problem $u(t = 0) = u_0$.

Problem 45. Calculate

$$\exp\left(tx^3 \frac{d}{dx}\right) x.$$

All terms must be summed up. Describe the connection with the solution of the differential equation

$$\frac{dx}{dt} = x^3, \qquad x(t=0) = x_0.$$

Problem 46. Consider the first order differential equation

$$\frac{du}{dt} = u - u^3.$$

Find the fixed points. Study the stability of the fixed points. Show that the differential equation can be written as

$$\frac{du}{dt} = -\frac{\partial V}{\partial u}.$$

Find the potential V. Give the solution for the initial value problem.

Problem 47. Consider the initial value problem

$$\frac{du}{dt} + 2tu = 0, \qquad u(0) = 1.$$

Let $Lu := -2tu$ and

$$u(t) = u(0) - 2L^{-1}(tu(t))$$

with

$$L^{-1}f(t) := \int_0^t f(s)ds.$$

Use the recursion

$$u_{j+1}(t) = -2L^{-1}(tu_j(t)), \qquad j = 0, 1, 2, \ldots$$

with $u_0 = u(0) = 1$ to find the solution of the intial value problem.

Problem 48. Consider the nonlinear ordinary differential equation

$$\frac{du}{dt} = \cos^2(u)$$

with the initial condition $u(t=0) = u_0 = 0$.

(i) Find the (local) solution of the initial value problem using the exponential map

$$u(t) = \exp(tV)u|_{u \to u(0)=0}$$

where V is the vector field

$$V(u) = \cos^2(u)\frac{d}{du}.$$

Discuss the radius of convergence.
(ii) Find the solution of the initial value problem by direct integration of the differential equation. Use

$$\int \frac{du}{\cos^2(u)} = \tan(u).$$

Compare the result from (i) and (ii). Discuss.
(iii) Find $u(t \to \infty)$ and compare with the solutions of the equation $\cos^2(u) = 0$ (fixed points).

Problem 49. Solve the initial value problem of the nonlinear differential equation

$$\frac{du}{dt} = u - u^3$$

where $u(t = 0) = u_0 > 0$ and $u_0 < 1$. What happens for $t \to \infty$?

Problem 50. By an ϵ-neighbourhood of a point $\mathbf{x}_0 \in \mathbb{R}^n$, we define an open ball of positive radius ϵ, i.e.

$$N_\epsilon(\mathbf{x}_0) := \{\, \mathbf{x} \in \mathbb{R}^n \, : \, |\mathbf{x} - \mathbf{x}_0| < \epsilon \,\}.$$

A function $\mathbf{f}$ is said to satisfy a Lipschitz condition on E if there is a positive constant K such that for all $\mathbf{x}, \mathbf{y} \in E$

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le K|\mathbf{x} - \mathbf{y}|.$$

A function $\mathbf{f}$ is said to be *locally Lipschitz* on E if for each point $\mathbf{x}_0 \in E$ there is an ϵ-neighbourhood of $\mathbf{x}_0$, $N_\epsilon(\mathbf{x}_0) \subset E$ and a constant $K_0 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in N_\epsilon(\mathbf{x}_0)$

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le K_0|\mathbf{x} - \mathbf{y}|.$$

Let E be an open subset of $\mathbb{R}^n$. Let $\mathbf{f} : E \to \mathbb{R}^n$. Show that if $\mathbf{f} \in C^1(E)$, then the function $\mathbf{f}$ is locally Lipschitz on E.

Problem 51. Calculate

$$\exp\left(tx^3\frac{d}{dx}\right)x.$$

All terms must be summed up. Describe the connection with the solution of the initial value problem of the differential equation

$$\frac{dx}{dt} = x^3, \qquad x(t=0) = x_0.$$

Problem 52. Consider the linear first order delay-differential equation

$$\frac{du}{dt} = -u(t-1).$$

Find the solution with the ansatz

$$u(t) = Ce^{\lambda t}.$$

Problem 53. Consider the first order differential equation

$$\left(\frac{du}{dx}\right)^3 - 3u^2 = 0.$$

Find the singular solution. Find the general solution.

Problem 54. Solve the Pfaffian differential equation

$$(x_2 + x_3)dx_1 + (x_3 + x_1)dx_2 + (x_1 + x_2)dx_3 = 0.$$

Problem 55. Describe the behaviour of the differential equation

$$\frac{du}{dt} = u - u^3$$

at the fixed points.

Problem 56. Consider the function $u : \mathbb{R} \to \mathbb{R}$

$$u(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

The function is continuously differentiable. Show that the function satisfies the first order differential equation

$$x\frac{du}{dx} - 2u = 0.$$

Problem 57. Solve the first order ordinary differential equation

$$\frac{du}{dx} = -2\lambda x u.$$

Problem 58. Consider the differential equation

$$\frac{du}{dt} = 1 - u^2.$$

(i) Show that the fixed point $u^* = 1$ is an asymptotically stable solution.
(ii) Show that the fixed point $u^* = -1$ is not stable.

Problem 59. When two hermitian operators A, B do not commute, i.e. $[A, B] = iC$, the product of their uncertainties satisfies the relation

$$(\Delta A)(\Delta B) \geq \frac{1}{2}|\langle C \rangle|.$$

Equality only holds when the hermitian operators A and B are proportional. This means the states with minimal uncertainty satisfy the equation

$$(B - \langle B \rangle I)u = \frac{i}{2}\frac{\langle C \rangle}{(\Delta A)^2}(A - \langle A \rangle I)u$$

where I is the identity operator. Assume that A is the one-dimensional position operator x and B is the one dimensional momentum operator p_x, i.e.

$$\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p_x \rangle\right) u(x) = \frac{i\hbar}{2(\Delta x)^2}(x - \langle x \rangle)u(x).$$

Solve this linear first order differential equation.

Problem 60. Find the general solution of the first order linear ordinary differential equation

$$\frac{du}{dt} + f(t)u = f(t)$$

where f is an analytic function.

Problem 61. Solve the ordinary differential equation

$$u\frac{dv}{dt} = v\frac{du}{dt}.$$

Problem 62. Consider the one-dimensional stochastic differential equation

$$\frac{dx}{dt} = f(x)\xi(t)$$

where the Gaussian random force $\xi(t)$ is defined by the correlators

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\xi(t') \rangle = D\delta(t - t')$$

and f is a given smooth function of x. Show that under the transformation

$$y(x) = \int_{x_0}^{x} \frac{dx'}{f(x')}$$

we obtain

$$\frac{dy}{dt} = \xi(t).$$

Problem 63. Given the first order differential equation

$$\frac{du}{dt} = f(u) = u^2.$$

(i) Apply the *hodographic transformation*

$$\bar{t}(t) = u(t), \qquad \bar{u}(\bar{t}(t)) = t.$$

(ii) Solve both differential equations.

Problem 64. Show that ($u_0 > 0$)

$$u(t) = u_0 \cosh(t/u_0)$$

satisfies the differential equation

$$\left(\frac{du}{dt}\right) = \frac{1}{u_0^2} u^2 - 1.$$

This differential equation plays a role in general relativity.

Problem 65. Consider the initial value problem for the autonomous system of first order ordinary differential equation

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0$$

where it is assumed that the vector field $\mathbf{f}$ is defined in the whole of $\mathbb{R}^n$ and is analytic. Runge's second order method is given by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + h\mathbf{f}\left(\mathbf{u}_n + \frac{h}{2}\mathbf{f}(\mathbf{u}_n)\right), \quad n = 0, 1, 2, \ldots$$

where h denotes the time step and $\mathbf{u}_n$ is the solution at time $t_n := nh$. Apply Runge's second order method to the differential equation ($n = 1$)

$$\frac{du}{dt} = u(1-u).$$

Problem 66. Consider the general Riccati equation

$$\frac{du}{dx} = f(x)u^2 + g(x)u + h(x)$$

where the continuously differentiable functions f, g and h are defined on an interval (a, b). Show that if

$$f(x) + g(x) + h(x) = \frac{d}{dx} \ln\left(\frac{\alpha(x)}{\beta(x)}\right) - \frac{\alpha(x) - \beta(x)}{\alpha(x)\beta(x)}(\alpha(x)f(x) - \beta(x)h(x))$$

with the differentiable function α, β properly chosen such that $\alpha(x)\beta(x) > 0$, then the general Riccati equation is integrable by quadrature.
Hint. The general Riccati equation is invariant with respect to a linear fractional transformation given by

$$u(x) = \frac{\alpha(x)v(x) + \gamma(x)}{\beta(x)v(x) + \delta(x)}$$

with $\Delta := \alpha\delta - \beta\gamma \neq 0$.

Problem 67. (i) Solve the initial value problem ($u(t = 0) = u_0 > 0$ of the differential equation

$$\frac{du}{dt} = u^3$$

using the Lie series technique.
(ii) Solve the initial value problem by direct integration of the differential equation.

Problem 68. Find the solution of the ordinary differential equation

$$u_1 \frac{du_2}{dx} = u_2 \frac{du_1}{dx}.$$

Problem 69. Let $c_1, c_2, c_3 \in \mathbb{R}$ with $c_3 \neq 0$. Consider the vector field

$$V = (c_1 + c_2 x + c_3 x^2)\frac{d}{dx}.$$

Calculate $x(t) = \exp(tV)x$. Compare with the solution of the initial value problem of the nonlinear differential equation

$$\frac{dx}{dt} = c_1 + c_2 x + c_3 x^2.$$

Problem 70. (i) Consider the initial value problem of the nonlinear ordinary differential equation

$$\frac{dx}{dt} = x^3, \qquad x(t=0) = x_0 > 0.$$

Find a solution using the Lie series technique The Lie series expansion is given by

$$x(t) = \exp(tV)x|_{x=x_0} = (1 + tV + \frac{1}{2!}V^2 + \frac{1}{3!}V^3 + \cdots)x\bigg|_{x=x_0}$$

with the vector field

$$V = x^3 \frac{d}{dx}.$$

(ii) Solve the initial value problem by direct integration of the differential equation. Compare the two solutions.

Problem 71. Consider the initial value problem of the differential equation

$$\frac{dx}{dt} = \frac{1}{2x}, \qquad x(t=0) = x_0 = 1.$$

Use the Lie series technique to solve this differential equation.

Problem 72. Consider the Riccati equation

$$\frac{du}{dt} = c_2 u^2 + c_1 u + c_0$$

where c_0, c_1, c_2 are constants. Show that $v(t) = 1/u(t)$ also satisfies a Riccati equation.

Problem 73. Find the solutions to the first order differential equation

$$\left(\frac{du}{dx}\right)^2 - 4u = 0.$$

Problem 74. Consider the first order differential equation

$$\frac{du}{dt} = \sin(t)u$$

with $u(0) > 0$.
(i) Solve the initial value problem by direct integration.
(ii) Let $v_1(t) = \sin(t)$ and therefore $dv_1(t)/dt = \cos(t) = v_2$. Thus we can consider the autonomous system of first order ordinary differential equations

$$\frac{du}{dt} = v_1 u, \quad \frac{dv_1}{dt} = v_2, \quad \frac{dv_2}{dt} = -v_1.$$

The corresponding vector field is

$$V = v_1 u \frac{\partial}{\partial u} + v_2 \frac{\partial}{\partial v_1} - v_1 \frac{\partial}{\partial v_2}.$$

Solve the autonomous system with the Lie series technique.

Problem 75. Solve

$$\int_0^x e^{-x} f(s) ds = e^{-x} + x - 1$$

applying differentiation.

Problem 76. Solve the initial value problem for the differential equation

$$\frac{du}{dt} = \frac{1}{1+u^2}$$

with $u(0) > 0$. First find the fixed points of the differential equation and study their stability. Can the vector field

$$V = \frac{1}{1+u^2} \frac{d}{du}$$

be used to solve the initial value problem via the Lie series?

Problem 77. Let ω be a fixed frequency and t the time. Solve the initial value problem of

$$\frac{du}{dt} = \omega \sin(\omega t) u$$

with $u(t=0) \equiv u(0) > 0$. What happens for $t \to \infty$?

Problem 78. Let $r > 0$. Solve the ordinary differential equation

$$u^2 \left(\frac{du}{dx} \right)^2 + u^2 = r^2.$$

Problem 79. Let $c > 0$. Find trivial and nontrivial solutions of

$$\frac{du(x)}{dx} = c \left(e^{u(x)} - e^{u(x+1)} \right).$$

Problem 80. Consider the first order differential equation

$$\frac{du}{dx} = xu^{1/2}.$$

Show that

$$u(x) = \left(\frac{1}{4}x^2 + C\right)^2$$

is a solution. Show that $u(x) = 0$ is a singular solution.

Problem 81. Consider the initial value problem

$$\frac{du}{d\tau} = 2\tau u + 2, \quad u(\tau = 0) = 1.$$

Show that the solution is given by

$$u(\tau) = 2e^{\tau^2} \int_0^\tau e^{-s^2} ds + e^{\tau^2}.$$

Problem 82. Consider the equation of motion for the average velocity of a Brownian particle in one dimension

$$m\langle du(t)/dt\rangle + m \int_{-\infty}^{t} \gamma(t-t')\langle u(t')\rangle dt' = F(t)$$

where m is the mass of the particle, $u(t)$ is the velocity, $\gamma(t-t')$ represents the retarded effect of the frictional force and $K(t)$ is the external force. Show that under the Fourier transform

$$u(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} u(t)e^{i\omega t}dt, \quad F(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} F(t)e^{i\omega t}dt$$

the differential equation takes the form

$$\langle u(\omega)\rangle = \frac{1}{m}\frac{1}{\hat{\gamma}(\omega) - i\omega}F(\omega)$$

where

$$\hat{\gamma}(\omega) = \int_0^\infty e^{i\omega t}dt.$$

Chapter 2

Second Order Differential Equations

Problem 1. (i) Find the general solution to the linear differential equation

$$\frac{d^2u}{dx^2} + u = 0$$

where $u(x)$ is a real valued function.
(ii) Solve the initial value problem

$$u(x=0) = 0, \qquad \frac{du(x=0)}{dx} = 1.$$

(iii) Solve the following three boundary value problems

$$u(0) = 1 \qquad u(1) = 1$$

$$u(0) = 1 \qquad u(\pi) = -1$$

$$u(0) = 1 \qquad u(\pi) = -2$$

Problem 2. Integrate the second order nonlinear differential equation

$$\frac{d^2a}{dt^2} = -\frac{4}{3}\pi G \frac{1}{a^2} \tag{1}$$

once, where G is a constant.

Problem 3. Consider the one-dimensional *Schrödinger equation*

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi(x) = 0$$

where the potential V is given by

$$V(x) = \sum_{j=1}^{k} v_j x^{2j}, \qquad v_k > 0.$$

(i) Find the differential equation for the function

$$f(x) = -\frac{1}{\psi}\frac{d\psi}{dx} + \frac{p}{x}$$

where $p = 0$ for the ground state and $p = 1$ for the first excited state.
(ii) Assume that the function f is regular and can be expanded in a Taylor series around the origin

$$f(x) = \sum_{j=0}^{\infty} f_j x^{2j+1}.$$

Find the recursion relation for the coefficients f_j.

Problem 4. Consider the power series

$$u(x) = \sum_{j=0}^{\infty} c_j x^j.$$

Then

$$\frac{d}{dx}\ln(u(x)) = \frac{1}{u(x)}\frac{du}{dx} = \frac{\sum_{j=0}^{\infty} j c_j x^{j-1}}{\sum_{j=0}^{\infty} c_j x^j}$$

which is the ratio of two power series. Thus formally

$$u(x) = \exp\left(\int^x \frac{1}{u(s)}\frac{du(s)}{ds}ds\right).$$

We can truncate the power series for u and expand $(1/u(x))du/dx$ as a continued fraction which when summed and integrated yields an approximation for u that is more accurate then the original truncate power series. Consider the second order nonlinear differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u(1+u^2) = 0.$$

Let

$$u(x) = \sum_{\substack{n=0 \\ n\ even}}^{\infty} b_n x^{n+1}.$$

Apply the method described above to find a solution of this differential equation.

Problem 5. Consider the motions of non-sero-rest-mass particles in a gravitional field created by a mass M and characterized by the *Schwarzschild metric*

$$ds^2 = -(1-2M/r)dt^2 + \frac{1}{1-2M/r}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

where r, θ, and ϕ are spherical co-ordinates and units are chosen such that $G = c = 1$. One can take advantage of the spherical symmetric by choosing the equatorial plane $\theta = \pi/2$.
(i) Show that the (bound) orbits satisfy the second order differential equation

$$\frac{d^2u}{d\phi^2} - 3u^2 + u - \frac{1}{L^{+2}} = 0 \qquad (2)$$

where $u := M/r$, μ is the particle rest mass and $L^+ := L/(M\mu)$.
(ii) Find that the solution of this differential equation using Jacobi elliptic functions.

Problem 6. Solve

$$\frac{d^2u}{dx^2} = f(u). \qquad (1)$$

Problem 7. Solve

$$\frac{d^2u}{dx^2} = u^2 \qquad (1)$$

given that $du/dx = 0$ when $u = 0$ and $x = 0$ when $u = \infty$.

Problem 8. Solve

$$\frac{d^2u}{dx^2} = f\left(u, \frac{du}{dx}\right). \qquad (1)$$

Problem 9. Solve

$$u\frac{d^2u}{dx^2} = \left(\frac{du}{dx}\right)^2 - \left(\frac{du}{dx}\right)^3. \qquad (1)$$

Problem 10. When a solid sphere of radius a and density σ falls vertically in a viscous liquid of density ρ ($< \sigma$) and coefficient of viscosity μ, the viscous resistance according to *Stokes' law* is $6\pi a\mu v$, where v is the downward velocity at any stage. Find the velocity when the sphere has fallen a depth y from rest. Let v be the velocity, dv/dt the acceleration at y below the initial position.

Problem 11. Consider the falling body of mass m where *air resistance* varies as v^2. Choose the origin and x-axis. The body falls from rest, i.e.

$$x(t=0)=0, \qquad v(t=0)=0. \tag{1}$$

Find v and x.

Problem 12. Consider the *free-particle equation*

$$\frac{d^2U}{dT^2}=0. \tag{1}$$

An *invertible point transformation* is given by

$$U=F(u,t), \qquad u=P(U,T) \tag{2a}$$

$$T=G(u,t), \qquad t=Q(U,T) \tag{2b}$$

with

$$\Delta \equiv \frac{\partial G}{\partial t}\frac{\partial F}{\partial u}-\frac{\partial G}{\partial u}\frac{\partial F}{\partial t}\neq 0. \tag{3}$$

(i) Show that we find the following equation

$$\frac{d^2u}{dt^2}+\Lambda_3\left(\frac{du}{dt}\right)^3+\Lambda_2\left(\frac{du}{dt}\right)^2+\Lambda_1\frac{du}{dt}+\Lambda_0=0 \tag{4}$$

where

$$\Lambda_3=\frac{1}{\Delta}\left(\frac{\partial G}{\partial u}\frac{\partial^2 F}{\partial u^2}-\frac{\partial^2 G}{\partial u^2}\frac{\partial F}{\partial u}\right)$$
$$\Lambda_2=\frac{1}{\Delta}\left(\frac{\partial G}{\partial t}\frac{\partial^2 F}{\partial u^2}+2\frac{\partial G}{\partial u}\frac{\partial^2 F}{\partial t\partial u}-2\frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial t\partial u}-\frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial u^2}\right)$$
$$\Lambda_1=\frac{1}{\Delta}\left(\frac{\partial G}{\partial u}\frac{\partial^2 F}{\partial t^2}+2\frac{\partial G}{\partial t}\frac{\partial^2 F}{\partial t\partial u}-2\frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial t\partial u}-\frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial t^2}\right)$$
$$\Lambda_0=\frac{1}{\Delta}\left(\frac{\partial G}{\partial t}\frac{\partial^2 F}{\partial t^2}-\frac{\partial^2 G}{\partial t^2}\frac{\partial F}{\partial t}\right)$$

Show that the Λ_i satisfy

$$\frac{\partial^2\Lambda_1}{\partial u^2}-2\frac{\partial^2\Lambda_2}{\partial u\partial t}+3\frac{\partial^2\Lambda_3}{\partial t^2}+6\Lambda_3\frac{\partial\Lambda_0}{\partial u}+3\Lambda_0\frac{\partial\Lambda_3}{\partial u}-3\Lambda_3\frac{\partial\Lambda_1}{\partial t}-3\Lambda_1\frac{\partial\Lambda_3}{\partial t}-\Lambda_2\frac{\partial\Lambda_1}{\partial u}+2\Lambda_2\frac{\partial\Lambda_2}{\partial t}=0$$

$$-\frac{\partial^2\Lambda_2}{\partial t^2}+2\frac{\partial^2\Lambda_1}{\partial u\partial t}-3\frac{\partial^2\Lambda_0}{\partial u^2}+6\Lambda_0\frac{\partial\Lambda_3}{\partial t}+3\Lambda_3\frac{\partial\Lambda_0}{\partial t}-3\Lambda_0\frac{\partial\Lambda_2}{\partial u}-3\Lambda_2\frac{\partial\Lambda_0}{\partial u}-\Lambda_1\frac{\partial\Lambda_2}{\partial t}+2\Lambda_1\frac{\partial\Lambda_1}{\partial u}=0.$$

Problem 13. The *Pinney equation* is given by

$$\frac{d^2u}{dt^2}+u=\frac{\delta}{u^3}. \tag{1}$$

Show that the general solution (which is expressible in terms of the general solution of the linear version ($\delta = 0$)) is given by

$$u(t) = B(1 + a\cos(2t + c))^{1/2} \tag{2}$$

provided B and a satisfy the relation

$$B^4(1 - a^2) = \delta. \tag{3}$$

The constants of integration are a and c.

Problem 14. Let C be a real number.
(i) Show that the differential equation $d^2u/dx^2 + u = C$ admits the particular solution

$$u(x) = C.$$

(ii) Show that the differential equation $d^2u/dx^2 + u = C\cos(x)$ admits the particular solution

$$u(x) = \frac{1}{2}Cx\sin(x).$$

(i) Show that the differential equation $d^2u/dx^2 + u = C\cos^2(x)$ admits the particular solution

$$u(x) = \frac{1}{2}C - \frac{1}{6}C\cos(2x).$$

Problem 15. Consider the quantum mechanical *eigenvalue problem*

$$\left(\frac{d^2}{dx^2} - V(x)\right)u_m(x) = E_m u_m(x). \tag{1}$$

Consider the *logarithmic derivative* of the mth excited-state wave function,

$$g_m(x) = \frac{d}{dx}\ln u_m(x) \tag{2}$$

which has proven to be useful in classifying most of the exactly solvable Hamiltonians. Show that g_m satisfies a *Ricatti equation.*

Problem 16. Consider

$$\frac{d^2u}{dx^2} - u + u^3 = 0. \qquad (1)$$

Let

$$v := \frac{1}{u}\frac{du}{dx}. \qquad (2)$$

Show that

$$u(x) = \exp\left(\int^x v(s)ds\right) \qquad (3)$$

and

$$u\left(\frac{dv}{dx} + v^2\right) - u + u^3 = 0 \qquad (4)$$

or

$$\frac{dv}{dx} + v^2 - 1 + \exp\left(2\int^x v(s)ds\right) = 0. \qquad (5)$$

Problem 17. Consider the weakly nonlinear *van der Pol equation*

$$\frac{d^2u}{dt^2} + u = \epsilon\left(\frac{du}{dt} - \frac{1}{3}\left(\frac{du}{dt}\right)^3\right). \qquad (1)$$

Insert the "naive" expansion

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots \qquad (2)$$

and find the functions u_0, u_1, u_2, ... by comparing coefficients for ϵ^n. Show that *secular terms* arise.

Problem 18. According to the *Thomas-Fermi model*, the number of electrons per unit volume in an isolated neutral atom is given by

$$\rho(r) = \frac{8\pi}{3h^3}(2me)^{3/2}(V(r) - V_0)^{3/2} \qquad (1)$$

in which r is the distance from the nucleus, $V(r)$ is the electrostatic potential, and V_0 is a reference value of the potential. The electrostatic potential can be expressed in SI units as

$$V - V_0 = -\frac{Ze\Phi(x)}{4\pi\epsilon_0} \qquad (2)$$

in which $x := \gamma r$,

$$\gamma := \frac{(32\pi^2/3)^{2/3}me^2Z^{1/3}}{2\pi\epsilon_0 h^2} \qquad (3)$$

and Φ is a solution to the dimensionless equation

$$\frac{d^2\Phi}{dx^2} = x^{-1/2}\Phi^{3/2} \tag{4}$$

in which $\Phi(0) = 1$. The initial slope $d\Phi(0)/dx$ is to be determined so that $\Phi(\infty) = 0$. (i) Show that the transformation

$$x(t) = t^2, \qquad \Phi(x(t)) = u(t) \tag{5}$$

yields

$$t\frac{d^2u}{dt^2} - \frac{du}{dt} = 4t^2u^{3/2} \tag{6}$$

(ii) Show that expanding (6) in a Taylor series

$$u(t) = \sum_{j=0}^{\infty} a_j t^j \tag{7}$$

around $t = 0$ yields

$$a_0 = 1, \quad a_1 = 0, \quad a_3 = \frac{4}{3}, \quad a_4 = 0, \quad a_5 = \frac{2a_2}{5} \tag{8}$$

and

$$a_{n+4} = (n+1)^{-1}((n+3)^2 - 1)^{-1}$$
$$\times \left(\frac{3}{2}\sum_{j=1}^{n}(j+1)((n+2-j)^2-1)a_{j+1}a_{n+3-j} - \sum_{j=0}^{n-2}(j+1)((j+3)^2-1)a_{j+4}a_{n-j} \right) \tag{9}$$

where $n = 2, 3, \ldots$. Thus all coefficients a_j can be calculated in terms of the initial slope a_2 according to this recurrence relation.
(iii) Consider the transformation

$$s(x) = x^r, \qquad \Phi(x(s))\frac{144}{x^3}v(s)$$

where $r = (7 - 73^{1/2})/2$. Show that (4) takes the form

$$r^2s^2\frac{d^2v}{ds^2} + 6s\frac{dv}{ds} + 12v = 12v^{3/2}$$

Remark. This transformation is useful for the studying of the asymptotic behaviour (i.e. large values of x of Φ. Expand v into a Taylor series around $s = 0$.

Problem 19. Consider the first *Painlevé transcendent*

$$\frac{d^2u}{dx^2} = 6u^2 + \lambda x \tag{1}$$

where λ is an arbitrary parameter.
(i) Show that the parameter can be set equal to 1 by using the transformation

$$x(t) = \lambda^{-1/5}t, \qquad u(x(t)) = \lambda^{2/5}w(t). \tag{2}$$

(ii) Find the general solution of (1) of the form

$$u(x) = \frac{a_{-2}}{(x-x_0)^2} + \frac{a_{-1}}{x-x_0} + a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{4}$$

Problem 20. Consider the equation

$$\frac{d^2u}{dt^2} + 3u\frac{du}{dt} + u^3 = 0 \tag{1}$$

which arises in the investigation of univalued functions defined by second-order differential equations and in the study of the *modified Emden equation*. Perform a *Painlevé analysis* of this equation.

Problem 21. Consider

$$\frac{d^2w}{dz^2} = \frac{1}{w^2}\left(\frac{dw}{dz}\right)^2. \tag{1}$$

(i) Let $dw/dz = u$. Show that (1) takes the form

$$\frac{dw}{dz} = u, \qquad \frac{du}{dz} = \frac{u^2}{w^2} \tag{2}$$

(ii) Let $w = \lambda W$ and $u = \lambda^2 U$. Show that

$$\frac{dW}{dz} = \lambda U, \qquad \frac{dU}{dz} = \frac{U^2}{W^2}. \tag{3}$$

(iii) The solutions of (3) are analytic functions of the parameter λ. Thus they can be expanded with respect to λ. For $\lambda = 0$ we write

$$\frac{dW_0}{dz} = 0, \qquad \frac{dU_0}{dz} = \frac{U_0^2}{W_0^2}. \tag{4}$$

The solutions of (3) are denoted by $W(z,\lambda)$ and $U(z,\lambda)$ and

$$W(z,\lambda) = W(z,0) + \lambda w_1 + \cdots, \qquad U(z,\lambda) = U(z,0) + \lambda u_1 + \cdots. \tag{5}$$

Consider the λ independent initial conditions

$$W(z_0,\lambda) = w_0 \neq 0, \qquad U(z_0,\lambda) = u_0 \neq 0. \tag{6}$$

Show that

$$W_0 = W(z,0) = w_0, \qquad U_0 = U(z,0) = \frac{w_0^2 u_0}{w_0^2 - u_0(z-z_0)}. \tag{7}$$

(iv) Show that

$$\frac{dw_1}{dz} = U(z,0) \tag{8}$$

and

$$w_1(z) = w_0^2 \ln\left(\frac{w_0^2}{w_0^2 - u_0(z-z_0)}\right) \tag{9}$$

and therefore w_1 has a logarithmic singularitry at $z_* = z_0 + \frac{w_0^2}{u_0}$.

Problem 22. Consider the first Painlevé transcendent

$$\frac{d^2u}{dz^2} = 6u^2 + z. \tag{1}$$

Let

$$u(z) = \frac{1}{(z-z_0)^2} + f(z). \tag{2}$$

Find the differential equation for f.

Problem 23. Consider the nonlinear differential equation

$$\frac{d^2u}{dx^2} + \ell u + mu^3 + nu^5 = 0. \tag{1}$$

Let

$$u(x) = \sqrt{\phi(x)} \tag{2}$$

(i) Show that ϕ satisfies the differential equation

$$2\phi\frac{d^2\phi}{dx^2} - \left(\frac{d\phi}{dx}\right)^2 + 4\ell\phi^2 + 4m\phi^3 + 4n\phi^4 = 0. \tag{3}$$

(ii) Insert the ansatz

$$u(x) = \frac{A\exp(\alpha(x+x_0))}{(1+\exp(\alpha(x+x_0)))^2 + B\exp(\alpha(x+x_0))} \tag{4}$$

where A, B, α are undetermined and x_0 is a fixed real number. Show that A, B and α satisfy the condition

$$\alpha^2 + 4\ell = 0, \qquad 2mA + 4\ell(2+B) - \alpha^2(2+B) = 0$$

$$-5\alpha^2 + 2\ell(2+(2+B)^2) + 2mA(2+B) + 2nA^2 = 0. \tag{5}$$

Problem 24. Consider the quantum mechanical eigenvalue problem

$$\frac{d^2u}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))u = 0. \tag{1}$$

Let

$$u(x) = \exp(iw(x)/\hbar), \qquad w(x) := S(x) + \frac{\hbar}{i}\ln(A(x)). \tag{2}$$

(i) Show that inserting (2) into (1) leads to

$$\left(\frac{dS}{dx}\right)^2 - 2m(E - V(x)) = \hbar^2 \frac{1}{A}\frac{d^2A}{dx^2}, \qquad 2\frac{dA}{dx}\frac{dS}{dx} + A\frac{d^2S}{dx^2} = 0. \tag{3}$$

(ii) Show that the second equation of (3) can be integrated to yield

$$A(x) = C\left(\frac{dS}{dx}\right)^{-1/2} \tag{4}$$

where C is a constant.
(iii) Show that substituting this expression for A into the first equation of (3) results in

$$\left(\frac{dS}{dx}\right)^2 = 2m(E - V(x)) + \hbar^2\left(\frac{3}{4}\left(\frac{\frac{d^2S}{dx^2}}{\frac{dS}{dx}}\right)^2 - \frac{1}{2}\frac{\frac{d^3S}{dx^3}}{\frac{dS}{dx}}\right). \tag{5}$$

Remark. In the *WKB approximation* one expands S in a power series in $\hbar^2$

$$S = S_0 + \hbar^2 S_1 + \cdots \tag{6}$$

then one substitutes this expansion into (5) and keeps only zero-order terms.

Problem 25. The equation

$$\frac{d^2u}{dt^2} + f(u)\frac{du}{dt} + g(u) = 0 \tag{1}$$

has a unique stable *limit cycle* under the following conditions:

(a) f is even, g odd, both continuous for all u, and $f(0) < 0$;

(b) $ug(u) > 0$ for $u \neq 0$;

(c) for every interval $|u| < K$ there is an L such that

$$|g(u_1) - g(u_2)| < L|u_1 - u_2|, \qquad \text{(Lipschitz condition)};$$

(d) $F(u) := \int_0^u f(s)ds \uparrow \infty$ as $u \uparrow \infty$;

(e) F has a single positive zero at $u = a$ and is monotone increasing for $u \geq a$.

Give a physical interpretation of the criterion for the existence of a limit cyle.

Problem 26. Consider the eigenvalue problem

$$\hat{H}u = Eu \tag{1}$$

with the non-dilation Hamilton operator of a one dimensional anharmonic oscillator

$$\hat{H} = -\frac{d^2}{dx^2} + x^2 - \lambda x^4, \qquad \lambda > 0. \tag{2}$$

(i) Show that a function

$$\varphi(x) := -\frac{d}{dx}\ln(u(x)) \tag{3}$$

where u is a solution of the eigenvalue equation (1), obeys the equation

$$\frac{d\varphi(x)}{dx} - \varphi^2 = E - x^2 + \lambda x^4 \tag{4}$$

where E is an eigenvalue for $\hat{H}$ in (1). (ii) Show that in the common case the function φ has pole singularities which correspond to nodes of the function u.

Problem 27. Consider the Schrödinger equation (in units of $\hbar = 2m = 1$)

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x) \tag{1}$$

for a potential V. Consider the transformation

$$x(u) = f(u), \quad u(x) = f^{-1}(x), \qquad \psi(x) = \sqrt{f'(u(x))}\xi(u(x)) \tag{2}$$

where the prime denotes differentiation with respect to the variable u. (i) Show that the Schrödinger equation (1) takes the form

$$\left(-\frac{d^2}{du^2} + V_T(u)\right)\xi(u) = E_T\xi(u) \tag{3}$$

where

$$V_T(u) - E_T = (f'(u))^2(V(f(u)) - E) + \Delta V(u) \tag{4}$$

and

$$\Delta V(u) = \left[-\frac{1}{2}\frac{f'''(u)}{f'(u)} + \frac{3}{4}\left(\frac{f''(u)}{f'(u)}\right)^2\right]. \tag{5}$$

(ii) Choose any exactly solvable potential as V_T and find the transformation functions $f(u)$ such that one would have new analytically solvable potentials $V(x)$. The non-trivial part is the proper choice of $f(u)$ so that $V(x)$ as well as energy eigenvalues and eigenfunctions can be expressed in a closed form.
(iii) Consider the mapping function in (2) as

$$x = f(u) = \log(\sinh u) \qquad \text{or} \qquad \sinh u = e^x. \tag{6}$$

Obviously, the domain of the variable u is $0 \leq u \leq \infty$ corresponding to $-\infty \leq x \leq \infty$. Find $\Delta V(u)$.

Problem 28. The second order differential equation

$$\frac{d^2u}{dt^2} + b\left(\frac{du}{dt}\right)^2 + \omega^2 u = 0$$

represents a classical one-dimensional damped harmonic oscillator, where the force of friction is proportional to the square of the velocity. Find the first integral, the Lagrangian function and the Hamilton function.

Problem 29. Consider the motion of a free particle in a medium with quadratic damping. The equation of motion takes the form

$$\frac{d^2u}{dt^2} = -k\left(\frac{du}{dt}\right), \qquad k > 0 \tag{1}$$

Let $u = u_1$ and $du_1/dt = u_2$. Then

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = -ku_2^2. \tag{2}$$

Let

$$U = u_2\frac{\partial}{\partial u_1} - ku_2^2\frac{\partial}{\partial u_2} \tag{3}$$

the vectoir field associated with the first order system (2). Let

$$V = -ku_2^2 e^{ku_1}\frac{\partial}{\partial u_2}. \tag{4}$$

(i) Show that

$$[U, V] = fY, \qquad f(u_1, u_2) = -ku_2 \exp(ku_1). \tag{5}$$

(ii) Show that

$$-f\text{div}V + L_U(\text{div}V = 0 \tag{6}$$

where $L_U(.)$ denotes the Lie derivative. (iii) Show that

$$f(u_1, u_2) + \text{div}U = -3ku_2 \exp(ku_1) \tag{7}$$

is a constant of motion.

Problem 30. Consider the nonlinear second order differential equation

$$\frac{d^2u}{dt^2} = ce^u.$$

Show that there is a first integral.

Problem 31. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Consider the first order differential equation

$$\frac{du}{dt} = f(u).$$

Differentiating the differential equation and inserting $du/dt = f(u)$ yields the second order differential equation

$$\frac{d^2u}{dt^2} = \frac{df}{du}f.$$

Show that the second order differential equation

$$\frac{d^2u}{dt^2} = u - 3u^2 + 2u^3$$

can be derived from a first order differential equation using the approach given above. Show that the solution of this first order differential equation is also a (particular) solution of the second order differential equation.

Problem 32. Show that the general solution of the linear second order differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{3}xu = 0$$

is given by

$$u(x) = \sqrt{x}(AJ_{1/3}(\xi) + BJ_{-1/3}(\xi))$$

where $\xi^2 = 4x^3/27$, A, B are the constants of integration and $J_{1/3}$, $J_{-1/3}$ are Bessel functions.

Problem 33. Solve the initial value problem of

$$\frac{d^2x}{dt^2} + \alpha\left(\frac{dx}{dt}\right)^2 = b$$

or

$$\frac{dv}{dt} + \alpha v^2 = b$$

where $v = dx/dt$ and $v(t = 0) = dx(t = 0)/dt = 0$.

Problem 34. Solve the initial value problem

$$\frac{d^2x}{dt^2} + \beta \left(\frac{dx}{dt} \right)^n = 0, \qquad n = 0, 1, 2, \ldots$$

with $\beta > 0$ and $dx(t = 0)/dt > 0$. Write $dx/dt = v$ and use the first order differential equation

$$\frac{dv}{v^n} = -\beta dt.$$

Problem 35. Solve the initial value problem

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} = b$$

with $a > 0$ and $dx(t = 0)/dt > 0$. Write $dx/dt = v$ and solve first

$$\frac{dv}{dt} + av = b \quad v(0) > 0.$$

Problem 36. Find the solution of the second order differential equation

$$\frac{d^2z}{dt^2} + 2i\frac{dz}{dt} - z = 0$$

where z is complex valued.

Problem 37. Integrate the ordinary differential equation

$$\frac{d^2f}{dx^2} - \frac{3}{x}\frac{df}{dx} + 2\frac{d^2f(0}{dx^2} = 0.$$

Problem 38. Two solid iron spheres, each 1 m in diameter, are in a region of interstellar space where the gravitational field of the rest of the universe is negligible. Intially, they are at rest with respect to each other and the distance between teir centres is 10 m.
(i) Calculate the absolute speed at which the two sphere collide.
(ii) Find the time required for the contact to be effected.

Problem 39. Consider the eigenvalue problem

$$-\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

with the boundary condition

$$\frac{d}{dx}\psi(x=0) + c\psi(x=0) = 0$$

where $c < 0$. Let H be the Heaviside step function. Show this problem admits the solution

$$\psi(x) = H(-x)\sqrt{-2c}\exp(-cx).$$

Problem 40. Let $c > 0$. Show that the linear differential equation

$$\left(\frac{d}{dx} + \frac{2}{x}\right)\frac{du}{dx} = 2cu$$

can be solved exactly to give the asymptotic solution for u

$$u(x) \sim \frac{1}{x}e^{-\sqrt{2c}x}.$$

Problem 41. Consider the differential equations

$$\frac{d^2q}{dt^2} + \omega^2(t)q = 0 \qquad (1)$$

and

$$\frac{d^2\rho}{dt^2} + \omega^2(t)\rho = \frac{1}{\rho^3}. \qquad (2)$$

Show that under the invertible point transformation

$$Q(T(t)) = \frac{q(t)}{\rho(t)}, \qquad T(t) = \int^t \frac{1}{\rho^2(s)}ds \qquad (3)$$

(1) takes the form

$$\frac{d^2Q}{dT^2} + Q = 0 \qquad (4)$$

where ρ satisfies (2). We have

$$\frac{dQ}{dt} = \frac{dQ}{dT}\frac{dT}{dt} = \frac{1}{\rho}\frac{dq}{dt} - q\frac{1}{\rho^2}\frac{d\rho}{dt} \qquad (5a)$$

and

$$\frac{dT}{dt} = \frac{1}{\rho^2}. \tag{5b}$$

Thus

$$\frac{dQ}{dT}\frac{1}{\rho^2} = \frac{1}{\rho}\frac{dq}{dt} - \frac{q}{\rho^2}\frac{d\rho}{dt}. \tag{6}$$

Problem 42. At time $t = 0$ a dog is at the x-axis at point $x_0 > 0$ and runs with constant speed w in the direction of his master, who walks with constant speed v along the y-axis. Show that this leads to the differential equation

$$x\frac{d^2y}{dx^2} = \frac{v}{w}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} \tag{1}$$

(ii) Solve the differential equation.

Problem 43. Consider the nonlinear differential equation

$$\frac{d^2u}{dx^2} + \ell u + mu^3 + nu^5 = 0. \tag{1}$$

Let

$$u(x) = \sqrt{\phi(x)}. \tag{2}$$

(i) Show that ϕ satisfies the differential equation

$$2\phi\frac{d^2\phi}{dx^2} - \left(\frac{d\phi}{dx}\right)^2 + 4\ell\phi^2 + 4m\phi^3 + 4n\phi^4 = 0. \tag{3}$$

(ii) Insert the ansatz

$$u(x) = \frac{A\exp(\alpha(x+x_0)}{(1+\exp(\alpha(x+x_0)))^2 + B\exp(\alpha(x+x_0))} \tag{4}$$

where A, B, α are undetermined and x_0 is a fixed real number. Show that A, B and α satisfy the condition

$$\alpha^2 + 4\ell = 0, \qquad 2mA + 4\ell(2+B) - \alpha^2(2+B) = 0$$

$$-5\alpha^2 + 2\ell(2 + (2+B)^2) + 2mA(2+B) + 2nA^2 = 0. \tag{5}$$

Problem 44. Consider the pair of coupled nonlinear differential equations relevant to the quantum field theory of charged solitons

$$\frac{d^2\sigma}{dx^2} = -\sigma + \sigma^3 + d\rho^2\sigma \tag{1a}$$

$$\frac{d^2\rho}{dx^2} = f\rho + \lambda\rho^3 + d\rho(\sigma^2 - 1) \tag{1b}$$

where σ and ρ are real scalar fields and d, f, λ are constants. Try to find an excat solution with the ansatz

$$\rho(x) = b_1 \tanh(\lambda_0(x + c_0)), \qquad \sigma(x) = \sum_{n=1} a_n \tanh^n(\lambda_0(x + c_0)). \tag{2}$$

Problem 45. Show that

$$u(x) = \frac{m}{\sqrt{c}} \tanh(mx/\sqrt{2})$$

satisfies the nonlinear second order differential equation

$$\frac{d^2u}{dx^2} - m^2 u + cu^3 = 0.$$

Problem 46. Solve the inital value problem of the second order differential equation

$$\frac{d^2u}{dt^2} + \left(\frac{1}{u} - 1\right)\frac{du}{dt} + u = 0$$

with $u(t = 0) = 1$ and $du(t = 0)/dt = 0$.

Problem 47. The Schrödinger equation for the radial function χ has the form

$$\frac{d^2\chi(r)}{dr^2} + \frac{2m}{\hbar^2}(E - U(r))\chi(r) = 0.$$

Let

$$k^2 := \frac{2mE}{\hbar^2}, \qquad V(r) := \frac{2mU(r)}{\hbar^2}.$$

(i) Find the Schrödinger equation in this form.
(ii) Assume that $\chi(r) = \chi_0(r)f(r)$ with

$$\chi_0(r) = \frac{1}{k}\sin(kr).$$

Find $V(r)$ as function of f.
(iii) We define

$$C(r) := \frac{1}{f}\frac{df}{dr}.$$

Show that

$$V(r) = C^2(r) + \frac{dC(r)}{dr} + 2k\mathrm{ctg}(kr)C(r).$$

Problem 48. Let $a, b > 0$. Find the differential equation the function

$$u(x) = a\tanh(x) - \frac{b}{\cosh(x)}$$

obeys.

Problem 49. The *Airy function* $Ai(z)$ is the solution of the second order differential equation

$$\frac{d^2 Ai}{dz^2} - zAi(z) = 0.$$

Show that its asymptotoc behaviour at large $|z|$ is given by

$$Ai(z) \approx \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right)$$

where $|\arg(z)| < \pi$ and $|z| \gg 1$.

Problem 50. Consider the second order differential equation

$$\frac{d^2 w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

Show that the transformation

$$w(z) = y(z)\exp\left(-\frac{1}{2}\int^z p(z')dz'\right)$$

casts the differential equation into its normal form

$$\frac{d^2 y}{dz^2} + I(z)y = 0$$

where I is given by

$$I(z) = q(z) - \frac{1}{2}\frac{dp(z)}{dz} - \frac{1}{4}p^2(z).$$

Problem 51. Consider the second order linear ordinary differential equation and boundary condition for the wave function u of a one-dimensional particle confined between a hard wall and a gravitional potential

$$\frac{d^2 u}{dx^2} + \frac{2m^2 g}{\hbar^2}\left(\frac{E}{mg} - x\right)u = 0 \quad \text{for} \quad x > 0$$

and $u(x) = 0$ for $x \le 0$.

(i) Show that with the variable transform

$$y(x) = (2m^2g/\hbar^2)^{1/3}(x - E/(mg)), \quad \widetilde{u}(y(x) = u(x)$$

the ordinary differential equation takes the form

$$\frac{d^2\widetilde{u}}{dy^2} - y\widetilde{u} = 0.$$

(ii) Show that the solution of this differential equation is given as a linear combination of the Airy functions, $Ai(y)$ and $Bi(y)$. Show that by including the boundary conditions the solution $Bi(y)$ has to be excluded. Find the energy eigenvalues by imposing the boundary condition at $x = 0$.

Problem 52. Let σ_1 and σ_3 be the Pauli spin matrices and

$$W(\zeta) = \begin{pmatrix} w_{11}(\zeta) & w_{12}(\zeta) \\ w_{21}(\zeta) & w_{22}(\zeta) \end{pmatrix}.$$

Consider the matrix differential equation

$$\frac{dW}{d\zeta} = \left(2\zeta^2\sigma_3 - \frac{1}{2\zeta}\sigma_1\right) W.$$

This differential equation has an irregular singularity of order 3 at infinity, and $T = (2/3)\zeta^3\sigma_3$. Consequently there are six Stokes sectors defined by the rays $\theta = \pm\pi/6$, $\theta = \pm\pi/2$, $\theta = \pm 5\pi/6$. Stokes sectors are the angular regions inside an angle drwan by the two Stokes half lines. Let Ai and Bi be the Airy functions. Show that defining $Ai_1(z) := Ai(ze^{-2\pi i/3})$ the fundamental solution valid in the sector $\{\zeta : -\pi/6 < \arg\zeta < \pi/2\}$ is

$$W_1(\zeta) = \sqrt{\frac{\pi}{\zeta}} \begin{pmatrix} Ai_1'(\zeta^2) + \zeta Ai_1(\zeta^2) & Ai'(\zeta^2) + \zeta Ai(\zeta^2) \\ Ai_1'(\zeta^2) - \zeta Ai_1(\zeta^2) & Ai'(\zeta^2) - \zeta Ai(\zeta^2) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & -1 \end{pmatrix}$$

and in the sector $\{\zeta : \pi/6 < \arg\zeta < 5\pi/6\}$ one has

$$\begin{aligned} W_2(\zeta) = & \sqrt{\frac{\pi}{4\zeta}} \begin{pmatrix} -i(Ai'(\zeta^2) + \zeta Ai(\zeta^2)) & -(Ai'(\zeta^2) + \zeta Ai(\zeta^2)) \\ -i(Ai'(\zeta^2) - \zeta Ai(\zeta^2)) & -(Ai'(\zeta^2) - \zeta Ai(\zeta^2)) \end{pmatrix} \\ & + \sqrt{\frac{\pi}{4\zeta}} \begin{pmatrix} Bi'(\zeta^2) + \zeta Bi(\zeta^2) & i(Bi'(\zeta^2) + \zeta Bi(\zeta^2)) \\ Bi'(\zeta^2) - \zeta Bi(\zeta^2) & i(Bi'(\zeta^2) - \zeta Bi(\zeta^2)) \end{pmatrix}. \end{aligned}$$

These two solutions are single-valued and holomorphic in the region $\arg\zeta \neq \pi$.

Problem 53. Consider the nonlinear second order differential equation

$$\frac{d^2u}{dx^2} = \sinh(u).$$

(i) Show that

$$\left(\frac{du}{dx}\right)^2 = 2\cosh(u) + C$$

is a first integral and C denotes a constant of integration.
(ii) Show that for $C = -2$ one has the solution

$$u(x) = \ln(\coth^2(x/2)).$$

Problem 54. Consider the Hilbert space $L_2(\mathbb{R})$ and the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2} + V(x)\right) u(x) = Eu(x)$$

where the potential V is given by

$$V(x) = x^2 + \frac{ax^2}{1+bx^2}$$

where $b > 0$. Insert the ansatz

$$u(x) = e^{-x^2/2}v(x)$$

and find the differential equation for v. Discuss. Make a polynomial ansatz for v.

Problem 55. Consider the second order differential equation

$$\frac{d^2\phi}{dx^2} = (u(x) + \lambda)\phi \tag{1}$$

where u is a smooth function. This equation can be viewed as an eigenvalue equation. The transformation

$$\psi(x) = f(x)\phi(x) + g(x)\frac{d\phi(x)}{dx} \tag{2}$$

is called a *Darboux transformation* (with f and g are smooth functions) if ψ satisfies the second order differential equation

$$\frac{d^2\psi}{dx^2} = (v(x) + \lambda)\psi. \tag{3}$$

Consider the case $g(x) = 1$. Find the condition on the smooth function f such that we have a Darboux transformation.

Problem 56. Consider the Hilbert space $L_2(\mathbb{R})$. Let $g > 0$. Consider the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2} + x^2 + \frac{\lambda x^2}{1+gx^2}\right) u(x) = Eu(x).$$

Find a solution of the second order differential equation by making the ansatz

$$u(x) = A(1+gx^2)\exp(-x^2/2).$$

Problem 57. Let $g, m, \epsilon > 0$. Solve the second order nonlinear ordinary differential equation

$$\frac{d^2u}{dx^2} - 2m\epsilon u + 6mgu^5 = 0$$

with the boundary conditions

$$u(x) \to 0, \quad \frac{du(x)}{dx} \to 0 \quad \text{for} \quad x \to \pm\infty.$$

Problem 58. Consider the the second order ordinary differential equation

$$f(x, u(x), du/dx, d^2u/dx^2) = 0$$

where $f(x, u, u_x, u_{xx})$ is an analytic function. Find the f such that the second order ordinary differential equation admits the Lie symmetry vector fields (projective group)

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x}, \quad u\frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial u}$$

$$u\frac{\partial}{\partial x}, \quad xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}, \quad x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}.$$

Problem 59. Let J be a positive integer. Consider the one-dimensional Schrödinger equation

$$-\frac{d^2u}{dx^2} + (V(x) - E)u(x) = 0$$

where the potential V is given by

$$V(x) = x^6 - (4J-1)x^2.$$

Consider the ansatz

$$u(x) = e^{-x^4/4} \sum_{k=0}^{J-1} c_k x^{2k}.$$

Find the recursion relation for the coefficients c_k. Then consider the special case $J = 2$.

Problem 60. Consider the second order nonlinear differential equation

$$\frac{d^2u}{dx^2} + u\frac{du}{dx} - \frac{1}{2}u + \frac{1}{9}u^3 = 0.$$

Show that the equation can be solved with the ansatz

$$u(x) = \frac{3}{v(x)}\frac{dv}{dx}.$$

Problem 61. Consider the system of first order differential equations

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = -(a + bu_1^2)u_2 + cu_1 - u_1^3.$$

Show that

$$I(u_1, u_2) = \exp((3/b)t)\left(u_2 + \frac{b}{3}u_1^3 + \frac{1}{b}u_1\right).$$

is a first integral of the system if $a = (4/b)$ and $c = -(3/b^2)$

Problem 62. Consider the second order ordinary differential equation

$$\frac{d^2u}{dx^2} + (\lambda - V(x))u = 0$$

with the boundary conditions $u(+\infty) = 0$ and $u(-\infty) = 0$. Let

$$V(x) = -2\mathrm{sech}^2(x).$$

Show that

$$u(x) = \mathrm{sech}(x)$$

with $\lambda = -1$ is a solution of the differential equation.

Problem 63. (i) Show that the second order linear ordinary differential equation

$$x^2\frac{d^2u}{dx^2} - x(2x-1)\frac{du}{dx} - (4+2x)u = -\frac{\sqrt{2}}{3}x^3 + \frac{2\sqrt{2}}{3}x^4$$

admits the particular solution

$$u(x) = -\frac{\sqrt{2}}{6}x^3 - \frac{\sqrt{2}}{4}x^2.$$

(ii) Show that the second order linear ordinary differential equation

$$x^2\frac{d^2u}{dx^2} - x(2x-1)\frac{du}{dx} - 4u = -\frac{\sqrt{2}}{3}x^3$$

admits the particular solution

$$u(x) = \frac{1}{\sqrt{3}}x^2.$$

Problem 64. Solve the nonlinear differential equation

$$u\frac{d^2u}{dx^2} = \left(\frac{du}{dx}\right)^2$$

by inspection.

Problem 65. Let $\epsilon > 0$. Consider the second order linear differential equation

$$\frac{d^2u}{dt^2} + e^{-\epsilon t}u = 0.$$

Show that under the change of the independent variable

$$s(t) = \frac{2}{\epsilon}\exp(-\epsilon t/2), \quad \widetilde{u}(s(t)) = u(t)$$

the differential equation reduces to a zeroth order Bessel differential equation

$$\frac{d^2\widetilde{u}}{ds^2} + \frac{1}{s}\frac{d\widetilde{u}}{ds} + \widetilde{u} = 0$$

with the general solution

$$\widetilde{u}(s) = c_1J_0(s) + c_2Y_0(s).$$

Problem 66. Consider the second order ordinary differential equation

$$\frac{d^2}{dx^2}u(x) + k^2(1+V(x))u(x) = 0$$

where $V : \mathbb{R} \to \mathbb{R}$ is an analytic function. Set

$$u(x) = A(x)e^{ikx} + B(x)e^{-ikx}$$

with $A(x)$ and $B(x)$ satisfying the condition

$$\frac{dA}{dx}e^{ikx} + \frac{dB}{dx}e^{-ikx} = 0.$$

Show that

$$\frac{d}{dx}\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = i\frac{k}{2}V(x)\begin{pmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{pmatrix}\begin{pmatrix} A(x) \\ B(x) \end{pmatrix}.$$

Problem 67. Show that the solution of of the initial value problem $u(0) = 1$ of the differential equation

$$\left(\frac{du}{dx}\right)^2 = \frac{2}{9}\frac{x^2}{u^3}$$

is given by

$$u(x) = \left(\frac{5}{6\sqrt{2}}(x^2 - 1) + 1\right)^{2/5}.$$

Problem 68. Let $r \geq 0$, $a > 0$ and $c > 0$. Consider the potential

$$V(u) = C(u^2 - a^2)^2.$$

Study the second order differential equation

$$\frac{d^2u}{dr^2} + \frac{3}{r}\frac{du}{dr} = \frac{d^2V}{du^2}$$

with $u(\infty) = c$ and $du(r = 0)/dr = 0$.

Problem 69. Solve the initial value problem of the second order differential equation

$$\frac{d^2u}{dx^2} + u\frac{du}{dx} = 0$$

with $u(0) > 0$ and $du(0)/dx > 0$.

Problem 70. (i) Solve the initial value problem of

$$\frac{d^2u_1}{dx^2} = u_1^2.$$

(ii) Solve the initial value problem of

$$\frac{d^2u_1}{dx^2} = u_1^2, \quad \frac{d^2u_2}{dx^2} = u_1u_2.$$

Problem 71. Consider the second order ordinary differential equation

$$-\frac{d^2u}{dx^2} + u^3 = 0.$$

Apply the transformation

$$\widetilde{x}(x) = x, \qquad \widetilde{u}(\widetilde{x}(x)) = \sinh(u(x)).$$

Problem 72. Solve the integral equation

$$\int_0^x ((x-y)^2 - 2)f(y)dy = -4x$$

applying differentiation and the solving the resulting differential equation.

Problem 73. Let $\alpha > 0$ and $\beta > 0$. Consider the *van der Pol equation*

$$\frac{d^2u}{dt^2} - (\alpha - \beta u^2)\frac{du}{dt} + u = 0.$$

We set $f(u, du/dt) = u - (\alpha - \beta u^2)du/dt$. Insert the ansatz

$$u(t) = A(t)\cos(\omega t)$$

into the differential equation with $A(t)$ a slowly changing function of t, i.e. $dA/dt \ll A\omega$, $d^2A/dt^2 \ll A\omega^2$ to find a approximate solution for the van der Pol equation.

Problem 74. Study the driven one-dimensional harmonic oscillator

$$\frac{d^2u}{dt^2} + \omega^2(1 + k\cos(\omega t))u = 0.$$

Problem 75. *Bessel's equation* of order zero is given by

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0$$

which is a second order linear differential equation. Apply the transformation

$$u = x, \qquad v = \frac{1}{y}\frac{dy}{dx}$$

and find dv/du.

Problem 76. Solve the second order ordinary differential equation

$$\frac{d^2u}{dx^2} - \frac{1}{2}\left(\frac{du}{dx}\right)^2 + 2 = 0.$$

Problem 77. Show that the second order ordinary linear differential equation

$$\frac{d^2u}{dt^2} = 0$$

admits the eight Lie symmetries

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial u}, \quad t\frac{\partial}{\partial t}, \quad t\frac{\partial}{\partial u}$$

$$u\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial t}, \quad ut\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u}, \quad ut\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t}.$$

Find the commutators. Classify the Lie algebra.

Problem 78. Let C be a real constant.
(i) Show that the differential equation $d^2u/dx^2 + u = C$ admits the particular solution $u(x) = C$.
(ii) Show that the differential equation $d^2u/dx^2 + u = C\cos(x)$ admits the particular solution

$$u(x) = \frac{1}{2}Cx\sin(x).$$

(iii) Show that the differential equation $d^2u/dx^2 + u = A\cos^2(x)$ admits the particular solution

$$u(x) = \frac{1}{2}C - \frac{1}{6}C\cos(2x).$$

Chapter 3

First Order Autonomous Systems in the Plane

We consider exercises for first order autonomous systems in the plane

$$\frac{du_1}{dt} = f_1(u_1, u_2), \qquad \frac{du_2}{dt} = f_2(u_1, u_2)$$

where f_1, f_2 are continuous differentiable functions. The *fixed points* are the solutions of the equations

$$f_1(u_1^*, u_2^*) = 0, \qquad f_2(u_1^*, u_2^*) = 0.$$

A differentiable function $I(u_1(t), u_2(t))$ is called a *first integral* if

$$\frac{d}{dt}I(u_1(t), u_2(t)) = \frac{\partial I}{\partial u_1}\frac{du_1}{dt} + \frac{\partial I}{\partial u_2}\frac{du_2}{dt} = \frac{\partial I}{\partial u_1}f_1 + \frac{\partial I}{\partial u_2}f_2 = 0.$$

The second order differential equation $d^2u_1/dt^2 = f(u_1, du_1/dt)$ can be cast into an first order autonomous system

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = f(u_1, u_2).$$

Problem 1. Show that the polynomial system

$$\frac{dx}{dt} = x(ax + c), \qquad \frac{dy}{dt} = y(2ax + by + c)$$

admits the first integral

$$I(x, y) = \frac{(ax + c)(ax + by)}{x(ax + by + c)}$$

where a, b, c are nonzero real numbers.

Problem 2. Show that the polynomial system

$$\frac{dx}{dt} = -y - b(x^2 + y^2), \qquad \frac{dy}{dt} = x$$

admits the first integral

$$I(x, y) = e^{2by}(x^2 + y^2)$$

where b is a real number.

Problem 3. Show that the polynomial system

$$\frac{dx}{dt} = 2(1 + 2x - 2ax^2 + 6xy), \qquad \frac{dy}{dt} = 8 - 3a - 14ax - 2axy - 8y^2$$

with $0 < a < 1/4$ possesses the (irreducible) invariant algebraic curve

$$H(x, y) \equiv \frac{1}{4} + x - x^2 + ax^3 + xy + x^2y^2 = 0.$$

Problem 4. Consider the $x - y$ plane. At time $t = 0$, there is a man at origin $(0, 0)$ and a dog on the y-axis at $(0, a)$, where $a > 0$. At $t = 0$ the man starts moving along the x-axis at a constant velocity v. At the same time, the dog starts moving towards the man at a constant velocity kv, where $k > 1$. The dog moves towards the man at all times.
(i) Find the differential equation for the motion of the dog.
(ii) How long does it take for the dog to catch up with the man?

Problem 5. Consider the system of differential equations

$$\frac{du_1}{dt} = (u_1 - u_2)(1 - u_1^2 - u_2^2), \qquad \frac{du_2}{dt} = (u_1 + u_2)(1 - u_1^2 - u_2^2). \qquad (1)$$

(i) Show that every point on the circle $C : u_1^2 + u_2^2 = 1$, is a fixed point.

(ii) Show that there is also an isolated fixed point $(0,0)$.
(iii) Study the stability of the fixed point $(0,0)$ and show that the origin $(0,0)$ is an unstable focus.

Hint. Let $U(u_1,u_2) := u_1^2 + u_2^2$. Then

$$\frac{d}{dt}U(u_1,u_2) = 2(1-u_1^2-u_2^2)(u_1^2+u_2^2) \qquad (2)$$

is positive definite within C.

(iv) Show that every trajectory that starts from a point (u_1,u_2) inside the circle C will end on C.
(v) Show that outside of C, $dU(u_1,u_2)/dt < 0$ and every trajectory that starts from a point (u_1,u_2) outside of C will also on C.
(vi) Show that a change to *polar coordinates*

$$u_1(r,\theta) = r\cos(\theta), \qquad u_2(r,\theta) = r\sin(\theta) \qquad (3)$$

reduces (1) to

$$\frac{dr}{dt} = r(1-r^2), \qquad \frac{d\theta}{dt} = 1-r^2. \qquad (4)$$

Problem 6. (i) Show that the equations

$$\frac{du_1}{dt} = u_1 - u_2 - u_1(u_1^2+u_2^2), \qquad \frac{du_2}{dt} = u_1 + u_2 - u_2(u_1^2+u_2^2) \qquad (1)$$

have their only fixed point at $(0,0)$.
(ii) Show that the origin is an unstable focus for the linear approximation and also for the nonlinear system. (iii) Express (1) in polar coordinates and solve the system of differential equations.

Problem 7. Consider the system of differential equations

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = 4u_1 - u_1^3. \qquad (1)$$

(i) Show that it admits three fixed points $(0,0)$, $(\pm 2,0)$ and study the stability of the fixed points.
(ii) Show that (1) admits the first integral

$$I(u_1,u_2) = 4u_1^2 - \frac{1}{2}u_1^4 - u_2^2. \qquad (2)$$

(iii) Discuss the phase portrait.

Problem 8. Consider the system of differential equations

$$\frac{du_1}{dt} = P(u_1, u_2), \qquad \frac{du_2}{dt} = Q(u_1, u_2). \tag{1}$$

Let P, Q, $\partial P/\partial u_1$, $\partial Q/\partial u_2$ be continuous in the open region $\mathcal{U}$ bounded by a simple closed curve. Show that if $\partial P/\partial u_1 + \partial Q/\partial u_2$ has a fixed sign in $\mathcal{U}$, the equations can have no limit cycle C in $\mathcal{U}$.

Hint. Apply *Green's theorem* to the cycle C

$$\oint_C (P du_2 - Q du_1) = \int_{\mathcal{U}} \int \left(\frac{\partial P}{\partial u_1} + \frac{\partial Q}{\partial u_2} \right) du_1 du_2.$$

Note that if $\mathbf{f} = (P, Q)$, then

$$\text{div}\mathbf{f} := \frac{\partial P}{\partial u_1} + \frac{\partial Q}{\partial u_2}$$

and the circuit integral is the normal flux of $\mathbf{f}$ through C.

Problem 9. Apply *Bendixon's theorem* to show that the *Van der Pol equation*

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = a(1 - u_1^2)u_2 - u_1 \qquad a > 0$$

can have no limit cycle within the circle $u_1^2 + u_2^2 = 1$ in the phase plane.

Problem 10. We consider the *Van der Pol oscillator*

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = a(1 - u_1^2)u_2 - u_1 \tag{1}$$

where $a > 0$. Study the stability of the fixed points.

Problem 11. The system of differential equations

$$\frac{du_1}{dt} = u_1(2 - u_1 - u_2), \qquad \frac{du_2}{dt} = u_2(3 - 2u_1 - u_2)$$

describes competing species $u_1 \geq 0$, $u_2 \geq 0$. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

Problem 12. Two species u_1, u_2 ($u_1 \geq 0$, $u_2 \geq 0$ are in symbiosis if an increase of either population leads to an increase in the growth rate of the other. Thus we assume

$$\frac{du_1}{dt} = M(u_1, u_2)u_1, \qquad \frac{du_2}{dt} = N(u_1, u_2)u_2 \tag{1}$$

with

$$\frac{\partial M}{\partial u_2} > 0, \qquad \text{and} \qquad \frac{\partial N}{\partial u_1} > 0. \tag{2}$$

We also suppose that the total food supply is limited. Hence for some $A > 0$, $B > 0$ we have

$$M(u_1, u_2) < 0 \text{ if } u_1 > A, \qquad N(u_1, u_2) < 0 \text{ if } u_2 > B. \tag{3}$$

If both populations are very small, they both increase; hence

$$M(0,0) > 0 \quad \text{and} \quad N(0,0) > 0.$$

Assuming that the intersections of the curves $M^{-1}(0)$, $N^{-1}(0)$ are finite, and all are transverse, show that:

(a) every trajectory tends to an equilibrium in the region $0 < u_1 < A$, $0 < u_2 < B$;

(b) there are no sources;

(c) there is at least one sink;

(d) if $\partial M/\partial u_1 < 0$ and $\partial N/\partial u_2 < 0$, there is a unique sink.

Problem 13. A system describing the time evolution (*Goodwin model*) of a metabolic feedback control cycle of protein synthesis are given by

$$\frac{dX}{dt} = \frac{a}{A + kY} - b, \qquad \frac{dY}{dt} = \alpha X - \beta \tag{1}$$

where $a, b, A, k, \alpha, \beta$ are positive constants. The X and Y variables measure m-RNA and protein concentrations, respectively. (i) Find the fixed points. (ii) Show that there is a constant of motion.

Problem 14. Consider the *Lotka Volterra model*

$$\frac{du_1}{dt} = c_1 u_1 - c_2 u_1 u_2, \qquad \frac{du_2}{dt} = -c_3 u_2 + c_4 u_1 u_2 \tag{1}$$

where $u_1 > 0$, $u_2 > 0$ and c_1, c_2, c_3 and c_4 are positive constants. Show that

$$I(u_1, u_2) = u_1^{c_3} u_2^{c_1} \exp(-c_4 u_1) \exp(-c_2 u_2) \tag{2}$$

is a first integral of the Lotka Volterra model.

Problem 15. (i) Show that the system of first order ordinary differential equations

$$\frac{du_1}{dt} = -\frac{1}{2} u_1 + u_2^2, \qquad \frac{du_2}{dt} = -\frac{1}{2} u_2 - u_1 u_2 \tag{1}$$

can be cast into the form

$$\frac{du_1}{dt} + \frac{1}{2}u_1 + u_1^2 = c\exp(t). \qquad (2)$$

(ii) Show that this equation can be linearized by using the transformation

$$u_1 = \frac{1}{v}\frac{dv}{dt}. \qquad (3)$$

Problem 16. The *Selkov model* is given by

$$\frac{du_1}{dt} = a - cu_1 - u_1u_2^2 + du_2^2, \qquad \frac{du_2}{dt} = b - u_2 + u_1u_2^2 - du_2^3 \qquad (1)$$

where a, b, c and d are constants. Show that the system admits the time dependent first integral

$$I(u_1, u_2, t) = (u_1 + u_2 - a - b)e^t \qquad (2)$$

if $c = 1$.

Problem 17. The Selkov model can also be written as

$$\frac{du_1}{dt} = 1 - bu_1 + u_1u_2^2$$
$$\frac{du_2}{dt} = a(u_1u_2^2 - u_2)$$

where a, b be positive bifurcation parameters. Find the fixed points and study the stability of the fixed points.

Problem 18. Consider the motion of a free particle in a medium with quadratic damping. The equation of motion takes the form

$$\frac{d^2u}{dt^2} = -k\left(\frac{du}{dt}\right)^2, \qquad k > 0. \qquad (1)$$

Let $u = u_1$ and $du_1/dt = u_2$. Then

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = -ku_2^2. \qquad (2)$$

Let

$$U = u_2\frac{\partial}{\partial u_1} - ku_2^2\frac{\partial}{\partial u_2} \qquad (3)$$

the vector field associated with the first order system (2). Let

$$V = -ku_2^2\exp(ku_1)\frac{\partial}{\partial u_2}. \qquad (4)$$

(i) Show that the commutator $[V, U]$ can be written as

$$[U, V] = fU. \tag{5}$$

Find f.
(ii) Show that

$$-f\,\mathrm{div}U + L_V(\mathrm{div}U) = 0 \tag{6}$$

where $L_V(.)$ denotes the *Lie derivative*.
(iii) Show that $u_2 \exp(ku_1)$ is a first integral.

Problem 19. Let f be a C^1 vector field on a neighbourhood of the annulus

$$A = \{\, \mathbf{x} \in \mathbb{R}^2 \ : \ 1 \le |x| \le 2 \,\}.$$

Suppose that f has no zeros and that f is transverse to the boundary, pointing inward.

(a) Prove there is a closed orbit.

(b) If there are exactly seven closed orbits, show that one of them has orbits spiraling toward it from both sided.

Problem 20. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 vector field with no zeros. Suppose that flow Φ, generated by $\mathbf{f}$ preserves area (that is, if S is any open set, the area of $\Phi_t(S)$ is independent of t). Show that every trajectory is a closed set.

Problem 21. Let $\mathbf{f}$ and $\mathbf{g}$ be C^1 vector fields on $\mathbb{R}^2$ such that $\langle \mathbf{f}(u), \mathbf{g}(u)\rangle = 0$ for all u. If $\mathbf{f}$ has a closed orbit, prove that $\mathbf{g}$ has a zero.

Problem 22. Let $\mathbf{f}$ be a C^1 vector field on an open set $W \subset \mathbb{R}^2$ and $H : W \to \mathbb{R}$ a C^1 function such that

$$DH(\mathbf{u})\mathbf{f}(\mathbf{u}) = 0$$

for all u. Prove that:

(a) H is constant on solution curves of $d\mathbf{u}/dt = f(\mathbf{u})$;

(b) $DH(\mathbf{u}) = 0$ if $\mathbf{u}$ belongs to a limit cycle;

(c) If $\mathbf{u}$ belongs to a compact invariant set on which DH is never 0, then $\mathbf{u}$ lies on a closed orbit.

Problem 23. Consider the two-dimensional autonomous system in the (x, y) plane described by

$$\frac{dx}{dt} = y^3(x^2 - 1)(2 + xy), \qquad \frac{dy}{dt} = x^3(y^2 - 1)(2 - xy). \tag{1}$$

(i) Show that the fixed points are given by $A(-1, -1)$, $B(1, 1)$, $C(1, -1)$, $D(-1, 1)$, $E(-2, 1)$, $F(1, 2)$, $G(2, -1)$, $H(-1, -2)$ and $I(0, 0)$ in the $x - y)$ plane.
(ii) Show that the fixed points A, B, C, D are saddles with eigenvalues -6 and $+2$. The fixed points E, F, G, H are stable attractors whilst I is neutral.
(iii) The phase space is clearly not compact. Show but it can be made so by a coordinate change to a four dimensional system (u, v, w, z) where

$$z = x^{-1}, \ u = yx^{-1}, \ we = y^{-1}, \ v = xy^{-1}.$$

A new time coordinate τ can be introduced to simplify the system if we define it by $d\tau/dt = c^6$. The system can be restored to two dimensions by examining its behaviour on the slice where $z = 0$ and $w = 0$. On this plane we have

$$\frac{dz}{d\tau} = zu^3(l - z^2)(u + 2z^2)$$

$$\frac{du}{d\tau} = (u^2 - z^2)(-u + 2z^2) + u^4(l - z^2)(u + 2z^2)$$

and the new system has the fixed points $A - I$ within a bounded circular region. These fixed points are augmented by J–Q on the circular boundary. The separatrix diagram indicates the generic fate of any trajectory. For example, a trajectory lying in the cell CADB will wind around in a spiral clockwise, indicative of quasi-periodic, oscillatory behaviour. The motion of a generic trajectory through the cell complex can be determined and a discrete mapping set up to desribe the sequence of separatrix changes.

Problem 24. *Hopf bifurcation theorem* is as follows. Let G be an open connected domain in $\mathbb{R}^n$, $c > 0$, and let $\mathbf{F}$ be a real analytic function defined on $G \times [-c, c]$. Consider the differential system

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}, \mu), \quad \text{where} \quad \mathbf{u} \in G, \quad |\mu| < c. \tag{1}$$

Suppose there is an analytic, real, vector function $\mathbf{g}$ defined on $[-c, c]$ such that

$$\mathbf{F}(\mathbf{g}(\mu), \mu) = \mathbf{0}. \tag{2}$$

This one can expand $\mathbf{F}(\mathbf{u}, \mu)$ about $\mathbf{g}(\mu)$ in the form

$$\mathbf{F}(\mathbf{u}, \mu) = L_\mu \bar{\mathbf{u}} + \mathbf{F}^*(\bar{\mathbf{u}}, \mu), \qquad \bar{\mathbf{u}} := \mathbf{u} - \mathbf{g}(\mu) \tag{3}$$

where L_μ is an $n \times n$ real matrix which depends only on μ, and $\mathbf{F}^*(\bar{\mathbf{u}}, \mu)$ is the nonlinear part of $\mathbf{F}$. Suppose there exist exactly two complex conjugate eigenvalues $\alpha(\mu)$, $\bar{\alpha}(\mu)$ of L_μ with the properties

$$\Re(\alpha(0)) = 0 \qquad \text{and} \qquad \Re(\alpha'(0)) \neq 0 \qquad ('{:=}\, d/d\mu). \tag{3}$$

Then there exists a periodic solution $\mathbf{P}(t, \epsilon)$ with period $T(\epsilon)$ of (2) with $\mu = \mu(\epsilon)$, such that $\mu(0) = 0$, $\mathbf{P}(t, 0) = \mathbf{g}(0)$ and $\mathbf{P}(t, \epsilon) \neq \mathbf{g}(\mu(\epsilon))$ for all sufficiently small $\epsilon \neq 0$. Morover $\mu(\epsilon)$, $\mathbf{P}(t, \epsilon)$, and $T(\epsilon)$ are analytic at $\epsilon = 0$, and

$$T(0) = \frac{2\pi}{|\Im\alpha(0)|}. \tag{4}$$

These "small" periodic solutions exist for exactly one of three cases: either only for $\mu > 0$, or only for $\mu < 0$, or only for $\mu = 0$.

Consider the system of differential equation

$$\frac{du_1}{dt} = u_1^2 u_2 - Bu_1 - u_1 + A, \qquad \frac{du_2}{dt} = -u_1^2 u_2 + Bu_2 \tag{5}$$

where A and B are positive constants. Apply Hopf bifurcation theorem to (5).

Problem 25. (i) Show that the system of first order ordinary differential equations

$$\frac{du_1}{dt} = -\frac{1}{2}u_1 + u_2^2, \qquad \frac{du_2}{dt} = -\frac{1}{2}u_2 - u_1 u_2 \tag{1}$$

can be cast into the form

$$\frac{du_1}{dt} + \frac{1}{2}u_1 + u_1^2 = c\exp(t). \tag{2}$$

(ii) Show that this equation can be linearized by using the transformation

$$u_1 = \frac{1}{v}\frac{dv}{dt}. \tag{3}$$

Problem 26. Let $\mathbf{f}$ and $\mathbf{g}$ be C^1 vector fields on $\mathbb{R}^2$ such that $\langle \mathbf{f}(u), \mathbf{g}(u) \rangle = 0$ for all u. If $\mathbf{f}$ has a closed orbit, prove that $\mathbf{g}$ has a zero.

Problem 27. Consider the two-dimensional autonomous system in the (x, y) plane described by

$$\frac{dx}{dt} = y^3(x^2 - 1)(2 + xy), \quad \frac{dy}{dt} = x^3(y^2 - 1)(2 - xy). \tag{1}$$

(i) Show that the fixed points are given by $A(-1,-1)$, $B(1,1)$, $C(1,-1)$, $D(-1,1)$, $E(-2,1)$, $F(1,2)$, $G(2,-1)$, $H(-1,-2)$ and $I(0,0)$ in the $x-y)$ plane.
(ii) Show that the fixed points A, B, C, D are saddles with eigenvalues -6 and $+2$. The fixed points E, F, G, H are stable attractors whilst I is neutral.
(iii) The phase space is clearly not compact. Show but it can be made so by a coordinate change to a four dimensional system (u,v,w,z) where

$$z = x^{-1}, \quad u = yx^{-1}, \quad we = y^{-1}, \quad v = xy^{-1},$$

A new time coordinate τ can be introduced to simplify the system if we define it by $d\tau/dt = c^6$. The system can be restored to two dimensions by examining its behaviour on the slice where $z = 0$ and $w = 0$. On this plane we have

$$\frac{dz}{d\tau} = zu^3(l - z^2)(u + 2z^2)$$

$$\frac{du}{d\tau} = (u^2 - z^2)(-u + 2z^2) + u^4(l - z^2)(u + 2z^2)$$

and the new system has the fixed points $A - I$ within a bounded circular region. These fixed points are augmented by $J-Q$ on the circular boundary. The separatrix diagram indicates the generic fate of any trajectory. For example, a trajectory lying in the cell CADB will wind around in a spiral clockwise, indicative of quasi-periodic, oscillatory behaviour. The motion of a generic trajectory through the cell complex can be determined and a discrete mapping set up to desribe the sequence of separatrix changes.

Problem 28. Show that a special solutions of the equations

$$\frac{dZ_n}{dt} = i \sum_{m \neq n}^{N} \frac{\Gamma_m}{Z_n^* - Z_m^*} \tag{1}$$

describing the motion of *point vortices* in an ideal two-dimensional fluid is given by

$$Z_n(t) = \rho \exp(i\omega t + i\varphi_n) \tag{2}$$

where

$$\omega = \Gamma(N-1)/(2\rho^2), \qquad \varphi_n = 2\pi n/N, \qquad 0 \leq n \leq N-1. \tag{3}$$

Consider first the case $N = 2$, i.e.

$$\frac{dZ_0}{dt} = i\frac{\Gamma_1}{Z_0^* - Z_1^*}, \qquad \frac{dZ_1}{dt} = i\frac{\Gamma_0}{Z_1^* - Z_0^*}.$$

Problem 29. We consider the *Van der Pol oscillator*

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = a(1-u_1^2)u_2 - u_1 \tag{1}$$

where $a > 0$. Study the stability of the fixed points.

Problem 30. The system of differential equations

$$\frac{du_1}{dt} = u_1(2 - u_1 - u_2)$$

$$\frac{du_2}{dt} = u_2(3 - 2u_1 - u_2)$$

describes competing species $u_1 \geq 0$, $u_2 \geq 0$. Why do these equations make it mathematically possible, but extremely unlikely, for both species to survive?

Problem 31. (i) Show that the system of first order ordinary differential equations

$$\frac{du_1}{dt} = -\frac{1}{2}u_1 + u_2^2, \qquad \frac{du_2}{dt} = -\frac{1}{2}u_2 - u_1u_2 \tag{1}$$

can be cast into the form

$$\frac{du_1}{dt} + \frac{1}{2}u_1 + u_1^2 = c\exp(t). \tag{2}$$

(ii) Show that this equation can be linearized by using the transformation

$$u_1 = \frac{1}{v}\frac{dv}{dt}. \tag{3}$$

Problem 32. The system of differential equations

$$\frac{du_1}{dt} = u_1(2 - u_1 - u_2), \qquad \frac{du_2}{dt} = u_2(3 - 2u_1 - u_2)$$

describes competing species $u_1 \geq 0$, $u_2 \geq 0$. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

Problem 33. Consider the initial value problem for the two linear systems of differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \qquad \frac{d\mathbf{v}}{dt} = B\mathbf{v}$$

where

$$A = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}.$$

Let

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad R^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then $B = RAR^{-1}$. Given the solution of the initial value problem of the first system $\mathbf{u}(t) = e^{tA}\mathbf{u}_0$. Show that $\mathbf{v}(t) = Re^{tA}\mathbf{v}_0$ is the solution of the second system through $R\mathbf{u}_0$.

Problem 34. The *cycloide* is given by

$$x(t) = a(t - \sin t), \qquad y(t) = a(1 + \cos t)$$

where $a > 0$. Find the corresponding system of differential equations with the initial values.

Problem 35. Consider the *Lotka-Volterra system*

$$\frac{du_1}{dt} = u_1 - u_1 u_2$$
$$\frac{du_2}{dt} = -u_2 + u_1 u_2.$$

(i) Find the variational equation.
(ii) Assume that $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2)$ satisfy the variational equaton. Show that

$$\mathbf{v} \wedge \mathbf{w} = (v_1 w_2 - v_2 w_1) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

where $\wedge$ denotes the exterior product.
(iii) Calculate the time-evolution of

$$a(t) := v_1(t) w_2(t) - v_2(t) w_1(t).$$

Problem 36. Consider the *van der Pol oscillator*

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = r(1 - u_1^2)u_2 - u_1$$

where $r > 0$. Study the stability of the fixed point

$$(u_1^*, u_2^*) = (0, 0)$$

and apply the *Hopf bifurcation theorem.*

Problem 37. The *Brusselator model* is given by

$$\frac{du_1}{dt} = a - (1+b)u_1 + u_1^2 u_2$$
$$\frac{du_2}{dt} = bu_1 - u_1^2 u_2.$$

where $u_1 > 0$, $u_2 > 0$ denote concentrations and a, b are positive constants. Find the fixed points and study their stability.

Problem 38. Given one of the 16 binary matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

As underlying field we consider $\mathbb{R}$. The solution of the initial value problem

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = A\mathbf{u} = A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix}.$$

is given by

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = e^{tA}\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}$$

where A is one of the binary matrices given above. Obviously, we have

$$\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = e^{-tA}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

Given $t^* > 0$ and

$$\begin{pmatrix} u_1(t^*) \\ u_2(t^*) \end{pmatrix}, \qquad \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}$$

for one of these binary matrices. Can we reconstruct this matrix from this data? Obviously the solution depends on t^* and the chosen initial conditions. Discuss.

Problem 39. A predator-prey model with Michaelis-Menton-type functional response is given by

$$\frac{dU}{d\tau} = RU\left(1 - \frac{U}{K}\right) - \frac{AUV}{V + AHU}$$
$$\frac{dV}{d\tau} = \frac{BAUV}{V + AHU} - DV$$

where U and V stand for prey and predator density, respectively. The parameters are positive constants. R stands for maximal growth rate of the prey, K for carrying capacity, A for capture rate, H for handling time, B for conversion efficiency, and D for predator death rate. Introducing the scaling

$$u = \frac{AHU}{BK}, \quad v = \frac{AHV}{B^2K}, \quad t = \frac{B\tau}{H}$$

and

$$r = \frac{RH}{B}, \quad d = \frac{HD}{B}, \quad s = \frac{AH}{B}$$

we obtain

$$\frac{du}{dt} = ru\left(1 - \frac{u}{s}\right) - \frac{suv}{v+su}$$
$$\frac{dv}{dt} = \frac{suv}{v+su} - dv$$

where $u(0) > 0$ and $v(0) > 0$ for the initial populations. Find the fixed points and study their stability.

Problem 40. Let $a, b, \alpha, \beta > 0$. Consider the autonomous system in the plane

$$\frac{du_1}{dt} = u_2$$
$$\frac{du_2}{dt} = -au_1 - bu_2 + \alpha u_1^2 + \beta u_2^2.$$

Use the *Dulac function* $B : \mathbb{R}^2 \to \mathbb{R}^+$

$$B(u_1, u_2) = b\exp(-2\beta u_1)$$

to show that this autonomous system has no limit cycle.

Problem 41. Consider the *Emden-Fowler differential equation*

$$\frac{d}{d\zeta}\left(\zeta^2 \frac{du}{d\zeta}\right) + \zeta^\alpha u^n = 0 \tag{1}$$

or

$$\frac{d^2u}{d\zeta^2} = -\frac{2}{\zeta}\frac{du}{d\zeta} - \zeta^{\alpha-2}u^n. \tag{2}$$

Find the system of differential equations under the transformation

$$x(t(\zeta)) = \frac{\zeta}{u(\zeta)}\frac{du}{d\zeta}, \quad y(t(\zeta)) = \frac{\zeta^{\alpha-1}u^n(\zeta)}{\frac{du}{d\zeta}}, \quad t(\zeta) = \ln(|\zeta|). \tag{3}$$

Problem 42. Solve the system

$$\frac{dM_x}{dt} = \gamma H_0 M_y - \frac{M_x}{T_2}$$
$$\frac{dM_y}{dt} = -\gamma H_0 M_x - \frac{M_y}{T_2}.$$

with the ansatz

$$M_x(t) = m\cos(\omega t)\exp(-t/T), \qquad M_y(t) = -m\sin(\omega t)\exp(-t/T).$$

Problem 43. Find the solution of initial value problem of the non-autonomous system of differential equation

$$\frac{du_1}{dt} = -ku_1$$
$$\frac{du_2}{dt} = he^{-kt}u_1 + (2t\sin(2t) - \cos(2t) - 2k)u_2.$$

Problem 44. Let f, g be analytic functions. Consider the autonomous system of first order differential equations

$$\frac{du_1}{dt} = f(\epsilon, u_1), \qquad \frac{du_2}{dt} = 2\epsilon u_1 f(\epsilon, u_1) - g(\epsilon, u_1)(u_2 - \epsilon u_1^2)$$

where $f(\epsilon, 0) = g(\epsilon, 0) = 0$ and $\epsilon \in \mathbb{R}$ is the bifurcation parameter.
(i) Show that

$$V = (u_2 - \epsilon u_1^2)\frac{\partial}{\partial u_2}$$

is Lie symmetry.
(ii) Show that for each fixed ϵ there is a flow-invariant manifold

$$M = \{\, (u_1, u_2) \, : \, u_2 = \epsilon u_1^2 \,\}.$$

Problem 45. Consider the autonomous system

$$\frac{du_1}{dt} = -u_1^3 + u_1u_2^2 + u_2^3, \qquad \frac{du_2}{dt} = -u_1^2 u_2 + u_2^3.$$

Show that $(0,0)$ is a fixed point. Is the fixed point stable? Show that the divergence of the corresponding vector changes sign in any neighbourhood of the fixed point $(0,0)$.

Problem 46. Study the coupled oscillator

$$\frac{d\theta_1}{dt} = \omega_1 + k\sin(\theta_2 - \theta_1)$$
$$\frac{d\theta_2}{dt} = \omega_2 + k\sin(\theta_1 - \theta_2).$$

Hint: Set $\phi(t) = \theta_1(t) - \theta_2(t)$.

Problem 47. Consider the autonomous system of differential equations

$$\frac{du_1}{dt} = e^{u_3}, \quad \frac{du_2}{dt} = e^{u_1} + e^{u_3}, \quad \frac{du_3}{dt} = ce^{u_1} + e^{u_2}.$$

Show that

$$I(u_1, u_2, u_3) = e^{u_2 - u_1} + c(u_2 - u_1) - u_3$$

is a first integral.

Problem 48. (i) Find the solution of the initial value problems of the system of differential equations

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = -u_1.$$

(ii) Find the solution of the initial value problems of the system of differential equations

$$\frac{du_1}{dt} = \sin(u_2), \qquad \frac{du_2}{dt} = -\sin(u_1).$$

(iii) Find the solution of the initial value problems of the system of differential equations

$$\frac{du_1}{dt} = \sinh(u_2), \qquad \frac{du_2}{dt} = -\sinh(u_1).$$

Problem 49. Consider the autonomous system of first order ordinary differential equations

$$\frac{du_1}{dt} = f_1(u_1, u_2), \qquad \frac{du_2}{dt} = f_2(u_1, u_2).$$

Find the conditions (and solve it) on the smooth functions f_1, f_2 such that

$$S_1 = u_1^2 \frac{\partial}{\partial u_1} + u_1 u_2 \frac{\partial}{\partial u_2}, \qquad S_2 = u_1 u_2 \frac{\partial}{\partial u_1} + u_2^2 \frac{\partial}{\partial u_2}$$

are Lie symmetry vector fields of the differential equations.

Problem 50. Find the solution of the initial value of the system of nonlinear differential equations equation

$$\frac{du_1}{dt} = u_2^2, \qquad \frac{du_2}{dt} = u_1 u_2$$

by solving first

$$\frac{du_1}{u_2^2} = \frac{du_2}{u_1 u_2} = dt$$

to find a constant of motion. Find the commutator of the two vector fields

$$V_1 = u_2^2 \frac{\partial}{\partial u_1}, \qquad V_2 = u_1 u_2 \frac{\partial}{\partial u_2}.$$

Discuss.

Problem 51. (i) Let $k > 0$. Consider the autonomous system of differential equations

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = k u_1 (u_1 - 1).$$

Find the fixed points and study their stability.
(ii) Motivated by the Lie series expansion for the solution of the system of differential equations and truncation we replace the system of differential equations by the two-dimensional map

$$f_1(x_1, x_2) = x_1 + x_2 + k x_1 (x_1 - 1), \qquad f_2(x_1, x_2) = x_2 + k x_1 (x_1 - 1).$$

Study this map.

Problem 52. Consider the non-autonomous linear system of differential equations

$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = \begin{pmatrix} -\sin(2t) & cos(2t) - 1 \\ \cos(2t) + 1 & \sin(2t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Show that the system admits the solutions

$$\mathbf{u}_1(t) = \begin{pmatrix} e^t (\cos(t) - \sin(t)) \\ e^t (\cos(t) + \sin(t)) \end{pmatrix}, \qquad \mathbf{u}_2(t) = \begin{pmatrix} e^{-t} (\cos(t) + \sin(t)) \\ e^{-t} (-\cos(t) + \sin(t)) \end{pmatrix}.$$

Problem 53. Consider a two-dimensional phase space with motion described by the differential equations

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = \alpha \tag{1}$$

where α is a real parameter. We take as a subset of phase space of finite measure the unit square

$$0 \leq x \leq 1, \qquad 0 \leq y \leq 1 \tag{2}$$

and impose periodic boundary conditions. This, in fact, is equivalent to choosing the phase space to be a torus. Discuss the solution of (1) as a function of α and show that the system is ergodic if α is irrational.

Problem 54. Solve the initial value problem for the autonomous system of first order differential equations

$$\frac{du_1}{dt} = -u_1^2 + u_2^2, \quad \frac{du_2}{dt} = -u_2^2 + u_1^2.$$

Hint: Set

$$n(t) := u_1(t) + u_2(t), \quad j(t) = u_1(t) - u_2(t).$$

Problem 55. Solve the initial value problem for the autonomous system of differential equations

$$\frac{du_1}{dt} = -\frac{2u_1}{\sqrt{u_1^2 + u_2^2}}, \qquad \frac{du_2}{dt} = 1 - \frac{2u_2}{\sqrt{u_1^2 + u_2^2}}$$

with the initial conditions $u_1(0) = 1$, $u_2(0) = 0$.

Problem 56. Find solutions of the autonomous first order system of differential equations

$$\frac{du_1}{dt} = \cos^2(u_1), \qquad \frac{du_2}{dt} = sin(u_1).$$

Show that

$$I(u_1, u_2) = sec(u_1) - u_2 \equiv \frac{1}{\cos(u_1)} - u_2$$

is a first integral.

Problem 57. Let x_0 be a fixed point of $dx/dt = f(x)$ in $M \subset \mathbb{R}^n$. A central manifold is an invariant manifold that touches in the fixed point a eigenspace E_c belonging to the eigenvalues with vanishing real part. Using the two-dimensional example

$$\frac{d}{dt}u_1 = u_1^2, \quad \frac{d}{dt}u_2 = -u_2$$

show that the central manifold does not need to be unique.

Problem 58. Evaluate the behavior of the autonomous system

$$du_1/dt = -u_1 u_2, \quad du_2/dt = -\beta u_2 + u_1^2$$

with $\beta > 0$, near the fixed point $(0,0)$.

Problem 59. Study the fixed points of the dynamical system

$$\frac{du_1}{dt} = u_2 + u_1^2, \qquad \frac{du_2}{dt} = -u_1^3.$$

Show that the dynamical system has a nilpotent center at $(0,0)$. Show that the dynamical system admits the symmetry

$$(u_1, u_2, t) \mapsto (-u_1, u_2, -t).$$

Problem 60. Let $\epsilon \in (0,1)$. Consider the three linear systems of differential equations

$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = \begin{pmatrix} 0 & -\epsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = \begin{pmatrix} 0 & 1-\epsilon \\ 1-\epsilon & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Find the fixed points for the three systems and find the eigenvalues of the three matrices.

Chapter 4

Higher Order Differential Equations

We consider differential equations of the form

$$\frac{d^n u}{dt^n} = f(t, u, \ldots, d^{n-1}u/dt^{n-1})$$

for $n \geq 3$ and autonomous systems of first order differential equations of the form

$$\frac{du_j}{dt} = f_j(u_1, u_2, \ldots, u_n), \qquad j = 1, 2, \ldots, n$$

where $n \geq 3$.

Problem 1. Consider the initial value problem for the system of linear first order ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

with $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$, A is an $n \times n$ matrix over $\mathbb{R}$ and $\mathbf{u}_0 \equiv \mathbf{u}(t = 0)$. Then the solution of the initial value problem is given by

$$\mathbf{u}(t) = e^{tA}\mathbf{u}_0.$$

Find the solution of the initial value problem

$$\frac{du_1}{dt} = u_3, \quad \frac{du_2}{dt} = u_2, \quad \frac{du_3}{dt} = u_1$$

i.e. we have

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

and $\mathbf{u}_0 = (1, 0, 1)^T$.

Problem 2. Consider the initial value problem for the system of linear first order ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{b}(t)$$

with $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, A is an $n \times n$ matrix over $\mathbb{R}$ and $\mathbf{u}_0 \equiv \mathbf{u}(t = 0)$. Then the solution of the initial value problem is given by

$$\mathbf{u}(t) = e^{tA}\mathbf{u}_0 + e^{tA} \int_0^t e^{-\tau A}\mathbf{b}(\tau) d\tau.$$

Find the solution of the initial value problem for

$$\frac{du_1}{dt} = u_3, \quad \frac{du_2}{dt} = u_2, \quad \frac{du_3}{dt} = u_1$$

i.e. we have

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{b}(t) = (\cos(t), \sin(t), 0)^T$$

and $\mathbf{u}_0 = (1, 0, 1)$.

Problem 3. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Consider the systems of linear equations with constant coefficients

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \qquad \frac{d\mathbf{y}}{dt} = B\mathbf{y}$$

with the initial conditions $\mathbf{x}(t = 0) = \mathbf{x}_0$ and $\mathbf{y}(t = 0) = \mathbf{y}_0$, respectively. Derive the differential equation for

$$\mathbf{z}(t) = \begin{pmatrix} z_{11}(t) \\ z_{12}(t) \\ \vdots \\ z_{1n}(t) \\ z_{21}(t) \\ \vdots \\ z_{nn}(t) \end{pmatrix} = \begin{pmatrix} x_1(t)y_1(t) \\ x_1(t)y_2(t) \\ \vdots \\ x_1(t)y_n(t) \\ x_2(t)y_1(t) \\ \vdots \\ x_n(t)y_n(t) \end{pmatrix} \equiv \mathbf{x}(t) \otimes \mathbf{y}(t)$$

where $\otimes$ denotes the *Kronecker product* and find the solution.

Problem 4. The *geodesic flow* on a sphere S^n

$$\langle \mathbf{x}, \mathbf{x} \rangle \equiv \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|^2 = 1 \tag{1}$$

where $\mathbf{x}(t) = (x_0(t), x_1(t), \ldots, x_n(t))^T$ (^{T} denotes the transpose, $\langle \, , \, \rangle$ the scalar product and $\|.\|$ the norm is described by the autonomous system of differential equations

$$\frac{d^2\mathbf{x}}{dt^2} = \lambda \mathbf{x} \tag{2}$$

where the Lagrange parameter λ is determined such that (1) is compatible with the differential equation (2). Find this compatibility and show that

$$\frac{d^2\mathbf{x}}{dt^2} = -\|d\mathbf{x}/dt\|^2 \mathbf{x}. \tag{3}$$

Problem 5. Consider the initial value problem of the autonomous system

$$\frac{du_1}{dt} = -u_2 + u_1 u_3^2$$
$$\frac{du_2}{dt} = u_1 + u_2 u_3^2$$
$$\frac{du_3}{dt} = -u_3(u_1^2 + u_2^2).$$

Show that

$$I(u_1, u_2, u_3) = u_1^2 + u_2^2 + u_3^2$$

is a first integral. Discuss.

Problem 6. In the study of the Lagrangian structure of the *ABC-flow* we consider the dynamical system

$$\frac{dx}{dt} = A \sin z + C \cos y$$
$$\frac{dy}{dt} = B \sin x + A \cos z$$
$$\frac{dz}{dt} = C \sin y + B \cos x$$

where A, B, C are real parameters. Since the right-hand side is 2π-periodic in x, y, and z we have a dynamical system defined on the three-dimensional torus T^3. Find a first integral when $C = 0$.

Problem 7. The autonomous system of ordinary differential equations

$$\frac{dx_1}{dt} = x_2x_3 - x_1(x_2 + x_3)$$
$$\frac{dx_2}{dt} = x_3x_1 - x_2(x_3 + x_1)$$
$$\frac{dx_3}{dt} = x_1x_2 - x_3(x_1 + x_2)$$

is called the *Darboux-Halphen system*.
(i) Show that the three hyperplanes

$$H_1(\mathbf{x}) := x_1 - x_2 = 0, \quad H_2(\mathbf{x}) := x_1 - x_3 = 0, \quad H_3(\mathbf{x}) := x_2 - x_3 = 0$$

are invariant by the flow of the Darboux-Halphen system.
(ii) Let $y = -2(x_1+x_2+x_3)$. Find the differential equation for y. Calculate up to d^3y/dt^3.
(iii) Consider the matrix-valued differential equation

$$\frac{dM}{dt} = (\det M)(M^{-1})^T + M^T M - (\mathrm{tr}M)M$$

where M is a 3×3 matrix-valued function of t. Show that the Darboux-Halphen system can be obtained by setting M to the diagonal matrix

$$M(t) = \begin{pmatrix} x_1(t) & 0 & 0 \\ 0 & x_2(t) & 0 \\ 0 & 0 & x_3(t) \end{pmatrix}.$$

Problem 8. (i) Find the first integrals of the system

$$\frac{dx_1}{dt} = x_1x_2 - x_1x_3 \equiv x_1(x_2 - x_3)$$
$$\frac{dx_2}{dt} = x_2x_3 - x_1x_2 \equiv x_2(x_3 - x_1)$$
$$\frac{dx_3}{dt} = x_3x_1 - x_2x_3 \equiv x_3(x_1 - x_2).$$

(ii) Find the first integrals of the system

$$\frac{dx_1}{dt} = x_1(c - x_2 + x_3)$$
$$\frac{dx_2}{dt} = x_2(c - x_3 + x_1)$$
$$\frac{dx_3}{dt} = x_3(c - x_1 + x_2).$$

Problem 9. The *Halphen-Darboux system* of ordinary differential equations is given by

$$\frac{du_1}{dt} = u_2u_3 - u_1(u_2 + u_3)$$
$$\frac{du_2}{dt} = u_1u_3 - u_2(u_1 + u_3)$$
$$\frac{du_3}{dt} = u_1u_2 - u_3(u_1 + u_2).$$

The system is invariant under the six permutations of the u_j's. One generating set of invariants of the permutation group is

$$v_1 = u_1 + u_2 + u_3$$
$$v_2 = u_1u_2 + u_2u_3 + u_1u_3$$
$$v_3 = u_1u_2u_3.$$

Express the system of differential equations in terms of the variables v_1, v_2, v_3.

Problem 10. Consider the autonomous system of first order differential equations

$$\frac{d\omega_1}{dt} = \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \qquad \frac{d\tau_1}{dt} = -\tau_1(\omega_2 + \omega_3)$$
$$\frac{d\omega_2}{dt} = \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2, \qquad \frac{d\tau_2}{dt} = -\tau_2(\omega_3 + \omega_1)$$
$$\frac{d\omega_3}{dt} = \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2) + \tau^2, \qquad \frac{d\tau_3}{dt} = -\tau_3(\omega_1 + \omega_2)$$

where $\tau^2 := \tau_1^2 + \tau_2^2 + \tau_3^2$. Using the ansatz

$$\omega_1(t) = -\frac{1}{2}\frac{d}{dt}\ln\left(\frac{\dot{s}}{s(s-1)}\right), \qquad \tau_1(t) = \frac{\kappa_1\dot{s}}{\sqrt{s(s-1)}}$$
$$\omega_2(t) = -\frac{1}{2}\frac{d}{dt}\ln\left(\frac{\dot{s}}{s-1}\right), \qquad \tau_2(t) = \frac{\kappa_2\dot{s}}{s\sqrt{s-1}}$$
$$\omega_3(t) = -\frac{1}{2}\frac{d}{dt}\ln\left(\frac{\dot{s}}{s}\right), \qquad \tau_3(t) = \frac{\kappa_3\dot{s}}{\sqrt{s}(s-1)}$$

find the differential equation for $s(t)$, where $\dot{s} \equiv ds/dt$. Here κ_j are constants.

Problem 11. Find all time dependent 2×2 matrices A_1, A_2, A_3 such that

$$\frac{dA_i}{dt} = \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}[A_j(t), A_k(t)], \qquad i = 1, 2, 3$$

where $\varepsilon_{123} = \varepsilon_{321} = \varepsilon_{132} = 1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$. All other ε_{ijk} are equal to 0.

Problem 12. Consider the nonlinear ordinary differential equation of third order

$$F \equiv \frac{d^3u}{dx^3} + 4u\frac{d^2u}{dx^2} + 6u^2\frac{du}{dx} + 3\left(\frac{du}{dx}\right)^2 + u^4 = 0. \tag{1}$$

Show that this equation can be linearized with the ansatz

$$\frac{dv}{dx} = u(x)v(x). \tag{2}$$

Find the general solution to (1).

Problem 13. Show that

$$\frac{d^3u}{dx^3} - u\frac{du}{dx} = 0 \tag{1}$$

admits the first integrals

$$F_1(u) = \frac{d^2u}{dx^2} - \frac{1}{2}u^2, \qquad F_2(u) = u\frac{d^2u}{dx^2} - \frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{1}{3}u^3. \tag{2}$$

Problem 14. The *Lorenz model* is given by

$$\frac{dx}{dt} = \sigma(y - x), \qquad \frac{dy}{dt} = -xz + rx - y, \qquad \frac{dz}{dt} = xy - bz \tag{1}$$

where σ, r and b are three real positive parameters. Show that by eliminating y and z from the above system, we obtain the third-order equation

$$x\frac{d^3x}{dt^3} - \left(\frac{dx}{dt} - (\sigma + b + 1)x\right)\frac{d^2x}{dt^2} - (\sigma + 1)\left(\frac{dx}{dt}\right)^2 +$$
$$\left(x^3 + b(\sigma + 1)x\right)\frac{dx}{dt} + \sigma(x^4 + b(1 - r)x^2) = 0. \tag{2}$$

Problem 15. Consider the diffential equation

$$x\frac{d^3x}{dt^3} - \left(\frac{dx}{dt} - (\sigma + b + 1)x\right)\frac{d^2x}{dt^2} - (\sigma + 1)\left(\frac{dx}{dt}\right)^2 +$$
$$\left(x^3 + b(\sigma + 1)x\right)\frac{dx}{dt} + \sigma\left(x^4 + b(1 - r)x^2\right) = 0. \tag{1}$$

Find the Lie symmetry vector fields of (1). Assume that the Lie symmetry vector field is of the form

$$V = (V_0(t, x, \dot{x}) + V_1(t, x, \dot{x})\ddot{x})\frac{\partial}{\partial x}. \tag{2}$$

Hint. The invariance condition is

$$pr^{(3)}V(\Delta) \doteq 0 \tag{3}$$

where Δ is given by

$$x\,\dddot{x} - (\dot{x} - (\sigma + b + 1)x)\,\ddot{x} - (\sigma + 1)\,(\dot{x})^2 +$$

$$\left(x^3 + b(\sigma + 1)x\right)\dot{x} + \sigma\left(x^4 + b(1 - r)x^2\right) = 0. \tag{4}$$

and $pr^{(3)}V$ denotes the third prolongation of V. Let

$$\chi := V_0(t, x, \dot{x}) + V_1(t, x, \dot{x})\ddot{x}.$$

Then

$$pr^{(3)}V := \chi\frac{\partial}{\partial x} + \left(\frac{d}{dt}\chi\right)\frac{\partial}{\partial \dot{x}} + \left(\frac{d^2}{dt^2}\chi\right)\frac{\partial}{\partial\,\ddot{x}} \tag{5}$$

Hint. The total derivative operator d/dt is given by

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \cdots \tag{6}$$

Problem 16. The *Lorenz model* is given by

$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = -y - xz + rx, \quad \frac{dz}{dt} = xy - bz. \tag{1}$$

The system (1) can be represented as a third order differential equation

$$u\frac{d^3u}{dt^3} - \frac{du}{dt}\frac{d^2u}{dt^2} + u^3\frac{du}{dt} + (b + \sigma + 1)u\frac{d^2u}{dt^2} +$$

$$(\sigma + 1)\left(bu\frac{du}{dt} - \left(\frac{du}{dt}\right)^2\right) + \sigma u^4 + b(1 - r)\sigma u^2 = 0 \tag{2}$$

where $u \equiv x$. Show that (2) admits the following first integrals:

1) $b = 2\sigma$

$$I_1 = \left(-2\sigma(r - 1) + u^2 + 2(\sigma + 1)\frac{1}{u}\frac{du}{dt} + 2\frac{1}{u}\frac{d^2u}{dt^2}\right)e^{2\sigma t}$$

2) $b=0$, $\sigma=1/3$

$$I_2 = \left(-\frac{3}{4}u^4 + 3\left(\frac{du}{dt}\right)^2 - 3u\frac{d^2u}{dt^2}\right)e^{4t/3}$$

3) $b=1$, $r=0$

$$I_3 = \Bigg(1 + u^2 + 2(\sigma+1)\frac{1}{\sigma u}\frac{du}{dt} + \frac{2}{\sigma}u\frac{du}{dt} + \frac{1}{\sigma^2}\left(\frac{du}{dt}\right)^2\left(1+\frac{(\sigma+1)^2}{u^2}\right)$$

$$+2\frac{1}{\sigma u}\frac{d^2u}{dt^2} + \frac{1}{\sigma^2u^2}\left(\frac{d^2u}{dt^2}\right)^2 + 2(\sigma+1)\frac{1}{\sigma^2u^2}\frac{du}{dt}\frac{d^2u}{dt^2})e^{2t}$$

4) $b=4$, $\sigma=1$

$$I_4 = (-4(r-1)^2 + 2(r-1)u^2 - \frac{1}{4}u^4 + 8(r-1)\frac{1}{u}\frac{du}{dt}$$

$$-2u\frac{du}{dt} + \left(\frac{du}{dt}\right)^2 + 4(r-1)\frac{1}{u}\frac{d^2u}{dt^2} - u\frac{d^2u}{dt^2})e^{4t}$$

5) $b=1$, $\sigma=1$

$$I_5 = ((r-1)^2 - (r-1)u^2 - 4(r-1)\frac{1}{u}\frac{du}{dt} + 2u\frac{du}{dt} + \left(\frac{du}{dt}\right)^2$$

$$+4\frac{1}{u^2}\left(\frac{du}{dt}\right)^2 - 2(r-1)\frac{1}{u}\frac{d^2u}{dt^2} + \frac{1}{u^2}\left(\frac{d^2u}{dt^2}\right)^2 + 4\frac{1}{u^2}\frac{du}{dt}\frac{d^2u}{dt^2})e^{2t}$$

6) $b=6\sigma-2$, $r=2\sigma-1$

$$I_6 = \Bigg(\sigma^{-1}(\sigma-1)(3\sigma-1)u^2 - \frac{1}{4}\sigma^{-1}u^4 - \sigma^{-1}(3\sigma-1)u\frac{du}{dt}$$

$$+\sigma^{-1}\left(\frac{du}{dt}\right)^2 - \sigma^{-1}u\frac{d^2u}{dt^2}\Bigg)e^{4\sigma t}.$$

Problem 17. Consider the *Kuramoto differential equation* in the complex domain

$$\frac{1}{2}u^2 + \frac{du}{dz} + \frac{d^3u}{dz^3} = 0. \tag{1}$$

Show that the equation admits the general solution (psi series)

$$u(z) = \frac{1}{(z-z_0)^3}\left(120 + P(A(z-z_0)^r) + Q(B(z-z_0)^{r^*})\right) \tag{2}$$

where P and Q are two power series without constant term

$$P(y) := \sum_{m=1}^{\infty} a_m y^m, \qquad Q(y) := \sum_{m=1}^{\infty} b_m y^m. \tag{3}$$

Problem 18. The *Schwarzian derivative* plays an important role in several branches of complex analysis. Let w be a holomorphic function. The Schwarzian derivative $\{w; z\}$ of w is defined by

$$\{w; z\} := \frac{\dfrac{d^3w}{dz^3}}{\dfrac{dw}{dz}} - \frac{3}{2}\left(\frac{\dfrac{d^2w}{dz^2}}{\dfrac{dw}{dz}}\right)^2 . \tag{1}$$

(i) Show that if $w(z) = z$ we find $\{w; z\} = 0$.
(ii) Prove the following *Theorem.* Let y_1 and y_2 be two linearly independent solutions of the equation

$$\frac{d^2y}{dz^2} + Q(z)y = 0 \tag{2}$$

which are defined and holomorphic in some simply connected domain D in the complex plane $\mathbb{C}$. Then

$$w(z) = \frac{y_1(z)}{y_2(z)} \tag{3}$$

satisfies the differential equation

$$\{w; z\} = 2Q(z) \tag{4}$$

at all points of D where $y_2(z) \neq 0$. Conversely, if w is a solution of (4), holomorphic in some neighbourhood of a point $z_0 \in D$, then one can find two linearly independent solutions, $u(z)$ and $v(z)$, of (2) defined in D so that

$$w(z) = \frac{u(z)}{v(z)}, \tag{5}$$

and, if $v(z_0) = 1$, the solutions u and v are uniquely defined.

Problem 19. The *Lorenz equations* are

$$\frac{du_1}{dt} = \sigma(u_2 - u_1), \qquad \frac{du_2}{dt} = u_1u_3 + ru_1 - u_2, \qquad \frac{du_3}{dt} = u_1u_2 - bu_3$$

where σ, r and b are three real positive parameters.

(i) Show that these equations are invariant under the discrete transformation

$$(u_1, u_2, u_3) \to (-u_1, -u_2, u_3).$$

Repeating this parity transformation gives back the identity mapping.
(ii) Show that there can exist orbits which are invariant under this reflection or pairs of orbits which are mapped into each other.

Problem 20. The *Lorenz model* is given by

$$\frac{du_1}{dt} = \sigma(u_2 - u_1), \qquad \frac{du_2}{dt} = u_1 u_3 + r u_1 - u_2, \qquad \frac{du_3}{dt} = u_1 u_2 - b u_3 \quad (1)$$

where σ, r and b are three real positive parameters. Find the fixed points and study their stability.

Problem 21. A *Volterra's dynamics* for the populations N_i of m interacting species is

$$\frac{dN_i}{dt} = \epsilon_i N_i + \frac{1}{\beta_i} N_i \sum_{k=1}^{m} \alpha_{ki} N_k, \qquad i = 1, 2, \ldots, m \tag{1}$$

where $N_i > 0$. (i) Show that the stationary population levels $N_i = q_i$ (fixed points) occur for

$$\epsilon_i = \frac{1}{\beta_i} \sum_{k=1}^{m} \alpha_{ki} q_k \tag{2}$$

(all $dN_i/dt = 0$) and will be unique when α is non-singular (this requires the number of species m to be even, since otherwise odd-order skew α is necessarily singular). (ii) Show that introducing the new dependent varibles

$$v_i(t) := \ln\left(\frac{N_i(t)}{q_i(t)}\right) \tag{3}$$

brings the Volterra dynamics to

$$\frac{dv_i}{dt} = \sum_{k=1}^{m} \gamma_{ki} \frac{\partial G}{\partial v_k}, \qquad G := \sum_{\alpha=1}^{m} \tau_\alpha (\exp(v_\alpha) - v_\alpha) \tag{2}$$

where

$$\gamma_{ij} := \frac{\alpha_{ij}}{\beta_i \beta_j} = -\gamma_{ji}, \qquad \tau_i \equiv q_i \beta_i. \tag{3}$$

(iii) Show that in the one-predator/one-prey case of Lotka-Volterra, these equations are in Hamiltonian form, with

$$v_1 \equiv Q, \qquad v_2 \equiv P, \qquad \gamma_{12} = -\gamma_{21} = -\gamma \tag{4}$$

and

$$H(Q,P) = \gamma\tau_1(e^Q - Q) + \gamma\tau_2(e^P - P) \tag{5}$$

where the Hamilton equations of motion are

$$\frac{dQ}{dt} = \frac{\partial H}{\partial P}, \qquad \frac{dP}{dt} = -\frac{\partial H}{\partial Q}. \tag{6}$$

Problem 22. In the *Jaynes-Cummings model* a single-mode field and a two-state atom couple to each other via the undamped Bloch-Maxwell equations

$$\frac{ds_1}{dt} = -s_2, \qquad \frac{ds_2}{dt} = s_1 + s_3 E, \qquad \frac{ds_3}{dt} = s_2 E \tag{1a}$$

$$\frac{d^2E}{dt^2} + \mu^2 E = as_1 \tag{1b}$$

where the dimensionless parameter $\mu = \omega/\omega_0$, the coupling constant $a = 8\pi N d^2 \omega_0 \hbar^{-1}$ and N is the number of two-level atoms. In (1) s_1, s_2, s_3 are components of Bloch's vector describing polarization and inversion. The electric field $E = 2d\tilde{E}/\hbar\omega_0$, d being the electric dipole moment of the atom, is dimensionless and equals the ratio of the Rabi frequency and the atomic transition frequency ω_0. The dimensionless time t is scaled with the atomic transition frequency ω_0. The model of a two-level atom described by (1) is valid under the assumption that $E \ll 1$.
(i) Show that the system (1) possesses conservation laws for length of the Bloch vector and energy

$$s_1^2 + s_2^2 + s_3^2 = 1 \tag{2}$$

$$W = as_3 - as_1 E + \frac{1}{2}\mu^2 E^2 + \frac{1}{2}\left(\frac{dE}{dt}\right)^2. \tag{3}$$

(ii) Show that the system (1) admits the following particular solution

$$E(t) = E_0 \mathrm{cn}(\Omega t, k) \tag{4}$$

$$\Omega^4 = \frac{1}{3}\left(\mu^2 - \frac{2}{27}\right) + \frac{1}{3}\left(a^2 - 4\left(\mu^2 - \frac{1}{9}\right)^3\right)^{1/2} \tag{5}$$

$$k^2 = \frac{1}{4\Omega^2}\left(\mu^2 - \frac{1}{3}\right) + \frac{1}{2}, \qquad E_0^2 = 4\left(\mu^2 - \frac{1}{3}\right) + 8\Omega^2. \tag{6}$$

(iii) Show that the inversion s_3 and the components of the dipole moment expressed in terms of E takes the form

$$s_3 = \frac{1}{a}\left(-\frac{3}{2}\left(\mu^2 - \frac{1}{9}\right)^2 + \left(a^2 - 4\left(\mu^2 - \frac{1}{9}\right)^3\right)^{1/2} + \frac{3}{4}\left(\mu^2 - \frac{1}{9}\right)E^2 - \frac{3}{32}E^4\right) \tag{7}$$

$$s_1 = \frac{1}{a}\left(\frac{3}{2}\left(\mu^2 - \frac{1}{9}\right)E - \frac{1}{8}E^3\right), \qquad s_2 = -\frac{ds_1}{dt}. \tag{8}$$

(iv) Show that the solution is valid only for

$$W = -2\left(\mu^2 - \frac{1}{9}\right)\left(\mu^2 - \frac{5}{9}\right) + \frac{5}{3}\left(a^2 - 4\left(\mu^2 - \frac{1}{9}\right)^3\right)^{1/2}. \tag{9}$$

Problem 23. Consider the coupled *Riccati equation*

$$\frac{du_i}{dt} = d_i + \sum_{j=1}^{N} e_{ij}u_j + \sum_{j,k=1}^{N} f_{ijk}u_j u_k, \qquad i = 1, 2, \ldots, N. \tag{1}$$

(i) Find analytical solutions for the special cases, e.g. if $f_{ijk} = f_j\delta_{ik}$ (projective Riccatis).
(ii) Show that the $N \times N$ matrix Riccati equation

$$\frac{d\mathbf{u}}{dt} = \mathbf{a} + \mathbf{b}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{b}' + \mathbf{u}\cdot c\cdot\mathbf{u} \tag{2}$$

is linearizable due to the non-commutative property of matrices, where $\mathbf{b}$ and $\mathbf{c}$ are $N \times N$ matrices.
(iii) Insert

$$\mathbf{u}(t) = \sum_{i=1}^{N} \mathbf{A}_i(t)u_i(t) + \mathbf{A}_0(t) \tag{3}$$

into (2) and use (1). Show that one gets the compatibility relations

$$\frac{d\mathbf{A}_0}{dt} + \sum_{i=1}^{N}\mathbf{A}_i d_i = \mathbf{a} + \mathbf{b}\cdot\mathbf{A}_0 + \mathbf{A}_0\mathbf{b}' + \mathbf{A}_0\mathbf{c}\cdot\mathbf{A}_0 \tag{4a}$$

$$\frac{d\mathbf{A}_i}{dt} + \sum_{j=1}^{N}\mathbf{A}_j e_{ji} = (\mathbf{b} + \mathbf{A}_0\mathbf{c})\mathbf{A}_i + \mathbf{A}_i(\mathbf{b}' + \mathbf{c}\cdot\mathbf{A}_0, \qquad i = 1, \ldots, N \tag{4b}$$

$$2\sum_{i=1}^{N}\mathbf{A}_i f_{ijk} = \mathbf{A}_j\mathbf{c}\cdot\mathbf{A}_k + \mathbf{A}_k\mathbf{c}\cdot\mathbf{A}_j, \qquad j, k = 1, \ldots, N. \tag{4c}$$

Problem 24. Consider the differential difference isotropic *Heisenberg spin chain*

$$\frac{d}{dt}\mathbf{S}_n = \frac{\mathbf{S}_n \times \mathbf{S}_{n+1}}{1 + \mathbf{S}_n\cdot\mathbf{S}_{n+1}} - \frac{\mathbf{S}_{n-1}\times\mathbf{S}_n}{1 + \mathbf{S}_{n-1}\cdot\mathbf{S}_n}. \tag{1}$$

(i) Show that this differential-difference solition equation can be reduced to a difference equation assuming a simple time-dependence

$$\mathbf{S}_n(t) = (\cos\phi_n \cos\omega t, \cos\phi_n \sin\omega t, \sin\phi_n) \tag{2}$$

with

$$x_n := \tan\frac{1}{2}\phi_n \tag{3}$$

where

$$x_{n+1} = \frac{(2x_n^3 + \omega x_n^2 + 2x_n - \omega - x_{n-1}(-x_n^4 - \omega x_n^3 + \omega x_n + 1))}{(-x_n^4 - \omega x_n^3 + \omega x_n + 1 - x_{n-1}(\omega x_n^4 - 2x_n^3 - \omega x_n^2 - 2x_n))}. \tag{4}$$

(ii) Show that its one-parameter family of invariant curves is given by the symmetric biquadratic relation

$$(1+2K)x_n^2 x_{n+1}^2 + \omega(1+K)(x_n^2 x_{n+1} + x_n x_{n+1}^2) + (1+K)(x_n^2 + x_{n+1}^2) +$$

$$2Kx_n x_{n+1} + \omega(1+K)(x_n + x_{n+1}) + 1 + 2K = 0 \tag{4}$$

where K is the invariant parametrizing the family curves.

Problem 25. Consider a discrete modified Korteweg-de Vries equation

$$\frac{d}{dt}x_n = (1+x_n^2)(x_{n-1} - x_{n+1} + \frac{1}{2}(x_{n+2}+x_n)(1+x_{n+1}^2) - \frac{1}{2}(x_n + x_{n-2})(1+x_{n-1}^2)). \tag{1}$$

(i) Show that stationary solutions of this equation are given by

$$x_{n+1} - \frac{1}{2}(x_n + x_{n+2})(1 + x_{n+1}^2) = x_{n-1} - \frac{1}{2}(x_{n-2} + x_n)(1 + x_{n-1}^2). \tag{2}$$

(ii) Equation (2) defines a 4 dimensional mapping.
(iii) Show that this mapping can be integrated to a two dimensional mapping

$$x_{2n} = \frac{2x_{2n-1} + 2K_2 - x_{2n-2}(1 + x_{2n-1}^2)}{1 + x_{2n-1}^2} \tag{3}$$

$$x_{2n+1} = \frac{2x_{2n} + 2K_1 - x_{2n-1}(1 + x_{2n}^2)}{1 + x_{2n}^2}. \tag{4}$$

where K_1, K_2 are integration constants. This mapping is integrable. Its one-parameter family of invariant curves is given by the asymmetric biquadratic relation

$$x_{2n}^2 x_{2n+1}^2 + x_{2n}^2 + x_{2n+1}^2 - 2x_{2n}x_{2n+1} - 2K_1 x_{2n+1} - 2K_2 x_{2n} + K_3 = 0 \tag{5}$$

where K_3 is the third integration constant.

Problem 26. Consider the autonomous system

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) = \text{grad}V(\mathbf{u}) \tag{1}$$

which is called a *gradient system* of ordinary differential equations. Show that a gradient system cannot have any limit cycle solutions.

Problem 27. Within the *rotating wave approximation* the dynamical system of perturbed *Maxwell-Bloch equations* is

$$\frac{d\mathcal{E}}{dt} = \mathcal{P}, \qquad \frac{d\mathcal{P}}{dt} = (\mathcal{E} + \epsilon e^{i\omega t})\mathcal{D}$$

$$\frac{d\mathcal{D}}{dt} = -\frac{1}{2}((\mathcal{E} + \epsilon e^{i\omega t})\mathcal{P}^* + (\mathcal{E}^* + \epsilon e^{-i\omega t})\mathcal{P}). \tag{1}$$

The variables in this set of equations are dimensionless. $\mathcal{E}$ denotes the self-consistent electric field, $\mathcal{P}$ is the polarizability of the matter, and $\mathcal{D}$ is the difference of its occupation numbers, assuming the material response may be modeled by two levels - a ground state and an excited state. Here, $\mathcal{E}$ and $\mathcal{P}$ are complex scalar functions of time, $\mathcal{D}$ is real, ϵ is the (constant) amplitude of the external driving field, and ω is the detuning of the laser probe frequency from resonance with the two-level atoms.
(i) Show that for non-zero ϵ and ω, these equations possess two first integrals

$$H = \frac{1}{2}|\mathcal{P}|^2 + \frac{1}{2}\mathcal{D}^2 \tag{2}$$

$$L = \frac{1}{2}\omega|\mathcal{E}|^2 + \omega\mathcal{D} + \frac{1}{2i}((\mathcal{E} + \epsilon e^{i\omega t})\mathcal{P}^* - (\mathcal{E}^* + \epsilon e^{-i\omega t})\mathcal{P}). \tag{3}$$

These two first integrals result from unitarity (H) and energy conservation (L). The three summands in L involve the self-consistent electric field energy, $\frac{1}{2}|\mathcal{E}|^2$, the excitation energy of the atoms, $\mathcal{D}$, and the interaction energy of the polarizable medium with the total electric field, $\mathcal{E} + \epsilon e^{i\omega t}$. Notice that

$$\frac{d}{dt}|\mathcal{P}|^2 = \frac{d}{dt}\mathcal{P}\mathcal{P}^* = \frac{d\mathcal{P}}{dt}\mathcal{P}^* + \mathcal{P}\frac{d\mathcal{P}^*}{dt}. \tag{4}$$

Problem 28. The unperturbed *Maxwell-Bloch equations* are

$$\frac{d\mathcal{E}}{dt} = \mathcal{P}, \qquad \frac{d\mathcal{P}}{dt} = \mathcal{E}\mathcal{D}, \qquad \frac{d\mathcal{D}}{dt} = -\frac{1}{2}(\mathcal{E}\mathcal{P}^* + \mathcal{E}^*\mathcal{P}). \tag{1}$$

Show that these unperturbed equations posses three integrals of motion

$$H = \frac{1}{2}|\mathcal{P}|^2 + \frac{1}{2}\mathcal{D}^2, \tag{2a}$$

$$J = \frac{1}{2i}(\mathcal{E}\mathcal{P}^* - \mathcal{E}^*\mathcal{P}), \tag{2b}$$

$$K = \frac{1}{2}|\mathcal{E}|^2 + \mathcal{D}. \tag{2c}$$

Problem 29. The surface of a two-dimensional *ellipsoid* can be imbedded in three-dimensional Euclidean space by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

where the axes satisfy the inequality $a > b > c > 0$. When a particle with mass m moves freely on this surface, it is subject to a force that is always perpendicular to the tangent plane, and whose direction is, therefore, given by the vector $(x/a^2, y/b^2, z/c^2)$ times some factor λ to be determined. (i) Show that the equations of motion then become

$$\frac{du}{dt} = -\lambda\frac{x}{a^2}, \qquad \frac{dv}{dt} = -\lambda\frac{y}{b^2}, \qquad \frac{dw}{dt} = -\lambda\frac{z}{c^2} \tag{2a}$$

$$\frac{dx}{dt} = \frac{u}{m}, \qquad \frac{dy}{dt} = \frac{v}{m}, \qquad \frac{dz}{dt} = \frac{w}{m} \tag{2b}$$

where (u, v, w) is the momentum. (ii) Show that by taking two derivatives with respect to time in (1) and replacing the second derivatives of (x, y, z) according to (2), one gets the condition,

$$m\lambda = \frac{u^2/a^2 + v^2/b^2 + w^2/c^2}{x^2/a^4 + y^2/b^4 + z^2/c^4}. \tag{3}$$

(iii) Show that if the initial position of the particle satisfies (1), and the initial momentum is tangential to (1), then the whole trajectory stays on the ellipsoid. (iv) Show that the quantity

$$A = u^2 + \frac{(xv - yu)^2}{a^2 - b^2} + \frac{(xw - zu)^2}{a^2 - c^2} \tag{4}$$

and similar ones, B and C, which are obtained by the cyclic permutation of the triples (x, y, z), (u, v, w), and (a, b, c) are first integrals. These three quantities are not independent since one has the relation $A + B + C = u^2 + v^2 + w^2$, where the right-hand side is the kinetic energy which is in fact the Hamiltonian of the system. A, B, and C are in involution.

Problem 30. Show that a special solutions of the equations

$$\frac{dZ_n}{dt} = i\sum_{m \neq n}^{N} \frac{\Gamma_m}{Z_n^* - Z_m^*} \tag{1}$$

describing the motion of *point vortices* in an ideal two-dimensional fluid is given by

$$Z_n(t) = \rho \exp(i\omega t + i\varphi_n) \tag{2}$$

where

$$\omega = \Gamma(N-1)/(2\rho^2), \qquad \varphi_n = 2\pi n/N, \qquad 0 \le n \le N-1. \tag{3}$$

Consider first the case $N = 2$, i.e.

$$\frac{dZ_0}{dt} = i\frac{\Gamma_1}{Z_0^* - Z_1^*}, \qquad \frac{dZ_1}{dt} = i\frac{\Gamma_0}{Z_1^* - Z_0^*}.$$

Problem 31. Consider the system of differential equations

$$\frac{dP_k}{d\tau} = -(k-1)P_k - (zk - 2k + 2)P_{k+1} \tag{1}$$

where $k \ge 1$. Here the first term is self-explanatory. In the second term the probability P_{k+1} is used because a larger cluster must be actually occupied in order for a reaction event involving a site outside the original k-site cluster to proceed. The most interesting quantity is $P_1(\tau) = c(\tau)$. It is expected to decrease in time but remain finite as $\tau \to \infty$. All other probabilities $P_{k>1}$ are expected to vanish for large times. Show that the solution of the differential equation (1) can be obtained by the ansatz

$$P_k(\tau) = c(\tau)[\sigma(\tau)]^{k-1} \tag{2}$$

where $\sigma(0) = \rho$, eliminates the k dependence.

Problem 32. A model for *epidemics* is given by

$$\frac{dS}{dt} = -rSI, \qquad \frac{dI}{dt} = rSI - aI, \qquad \frac{dR}{dt} = aI \tag{1}$$

where S stands for the number of susceptibles, I for those infected and R denotes the removals. The constants a, r determine the infection and removal rates of infectives, respectively. Show that the system admits the two first integrals

$$H_1 = S + I + R, \qquad H_2 = R + \frac{a}{r}\ln(S). \tag{2}$$

Problem 33. Find the Lie symmetries and first integrals of the system

$$\frac{dx}{dt} = ax + by + z - 2y^2$$
$$\frac{dy}{dt} = ay - bx + 2xy$$
$$\frac{dz}{dt} = -2z - 2zx.$$

This system of differential equations corresponds to the interaction of three quasi-synchronous waves in a plasma with quadratic nonlinearities.
(i) Let

$$y \Rightarrow y + b/2$$

Show that the system then takes the form

$$\frac{dx}{dt} = ax - by + z - 2y^2 \tag{1a}$$

$$\frac{dy}{dt} = ay + ab/2 + 2xy \tag{1b}$$

$$\frac{dz}{dt} = -2z - 2xz. \tag{1c}$$

Problem 34. Consider the system of ordinary differential equations

$$\frac{dP_i}{dt} = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}(A_{ijkl}P_kP_l - A_{klij}P_iP_j) \tag{1}$$

where $i = 1, 2, \ldots, n$ and $A_{ijkl} \geq 0$. Assume that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ijkl} = 1 \tag{2}$$

for all pairs (k, l). The quantities denotes transition probabilities. Show that

$$\sum_{i=1}^{n} P_i = \text{const} \tag{3}$$

The model given above is a caricature of *Boltzmann's equation.* . The j-th component P_j is the probability to find a particle in the j-th phase space (or configuration) cell. Obviously, $P_j \geq 0$ for $j = 1, 2, \ldots, n$ and

$$\sum_{j=1}^{n} P_j(t = 0) = 1. \tag{4}$$

Owing to (3) we find that

$$\sum_{j=1}^{n} P_j(t) = 1. \tag{5}$$

Problem 35. Consider the equations of motion

$$m_i\frac{d^2x_i}{dt^2} = -\frac{\partial U}{\partial x_i}, \quad m_i\frac{d^2y_i}{dt^2} = -\frac{\partial U}{\partial y_i}, \quad m_i\frac{d^2z_i}{dt^2} = -\frac{\partial U}{\partial z_i}, \quad i = 1, 2, 3 \tag{1}$$

where

$$U = -m_1 m_2 F(r_{12}^2) - m_2 m_3 F(r_{23}^2) - m_3 m_1 F(r_{31}^2), \qquad m_i = 1, \quad i = 1,2,3 \tag{2}$$

(x_k, y_k, z_k) are the coordinates of the k-th body, $k = 1,2,3$, $F(r^2)$ is an arbitrary, sufficiently smooth function, and

$$r_{ij} := ((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{1/2}. \tag{3}$$

(i) Show that (1) is invariant under the 10-parameter Galilean group $G(1,3)$.
(ii) Show that the Lie algebra of this group has a basis consisting of the following infinitesimal generators

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3},$$

$$X_3 = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}, \quad X_4 = t\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right),$$

$$X_5 = t\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right), \quad X_6 = t\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}\right),$$

$$X_7 = y_k \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial y_k}, \quad X_8 = z_k \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial z_k}, \quad X_9 = x_k \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial x_k}. \tag{4}$$

Remark. Ten integrals of motion of the spatial three-body problem were known already to Lagrange

Problem 36. Euler's equations for rotation of a rigid body about a fixed point are given by

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3 \tag{1a}$$

$$I_2 \frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1 \tag{1b}$$

$$I_3 \frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2 \tag{1c}$$

A plane lamina has principal moments of inertia I, $2I$ and $3I$. It is rotating freely with angular velocity n about an axis through its centre of mass perpendicular to its plane when it is given an additional angular velocity $\sqrt{3}\, n$ about its principal axis with moment of inertia I.
(i) Prove that, at time t later, the components of the angular velocity of the lamina along its principal axes are

$$(\sqrt{3}n \operatorname{sech}(nt), \sqrt{3}n \tanh(nt), \operatorname{sech}(nt)). \tag{2}$$

(ii) What is the eventual motion as $t \to \infty$?

Problem 37. Find the solution of the initial value problem

$$\frac{d^3u}{dt^3} + u = 0$$

with $u(t=0) = u_0$, $du(t=0)/dt = u_{t0}$, $d^2u(t=0)/dt^2 = u_{tt0}$ and u is a real-valued function.

Problem 38. Consider the *Lorenz model* of the form

$$\begin{aligned}
\frac{dw}{dt} &= R - zy - w \\
\frac{dz}{dt} &= wy - z \\
\frac{dy}{dt} &= \sigma(z - y).
\end{aligned}$$

Consider the case that $R \gg \sigma$ and $R\sigma \gg 1$. Find the Lorenz model under the scaling

$$t \to \epsilon t, \quad w \to \frac{w}{\epsilon^2 \sigma}, \quad z \to \frac{z}{\epsilon^2 \sigma}, \quad y \to \frac{y}{\epsilon}, \quad \epsilon = \frac{1}{\sqrt{R\sigma}}.$$

Show that for $\epsilon = 0$ we find two first integrals (constants of motion). Thus for this case the system is completely integrable.

Problem 39. Consider the system of differential equations

$$\begin{aligned}
\frac{du_1}{dt} &= u_1 u_2 - u_1 u_3 \equiv u_1(u_2 - u_3) \\
\frac{du_2}{dt} &= u_2 u_3 - u_1 u_2 \equiv u_2(u_3 - u_1) \\
\frac{du_3}{dt} &= u_3 u_1 - u_2 u_3 \equiv u_3(u_1 - u_2).
\end{aligned}$$

(i) Show that $u_1^* = u_2^* = u_3^* = c$ ($c \in \mathbb{R}$) is a fixed point.
(ii) Find the variational equation.
(iii) Study the stability of the fixed point.

Problem 40. Consider the dynamical system

$$\frac{du_1}{dt} = -u_2 u_3, \qquad \frac{du_2}{dt} = u_1 u_3, \qquad \frac{du_3}{dt} = -u_3^2. \tag{1}$$

This system has a simple system-theoretic interpretation: u_1 and u_2 are the states of an oscillator, frequency modulated by u_3.

(i) Show that for $u_3(0) \geq 0$ the solution of the initial value problem is given by

$$u_1(t) = R\cos(\ln(1+tu_3(0))+\delta), \qquad u_2(t) = R\sin(\ln(1+tu_3(0))+\delta)$$

$$u_3(t) = \frac{u_3(0)}{(1+tu_3(0))} \tag{2}$$

where

$$R := \sqrt{u_1^2(0)+u_2^2(0))} \tag{3}$$

and

$$\delta := \arctan\left(\frac{u_1(0)}{u_2(0)}\right). \tag{4}$$

(ii) Show that the trajectories are bounded when $u_3(0) \geq 0$. (iii) Show that if $u_3(0) > 0$, then the trajectory u has no autocorrelation, that is, the limit in

$$R_u(\tau) := \lim_{T\to\infty} u(t)u(t+\tau)dt \tag{5}$$

does not exist, and hence u has no spectrum.

Problem 41. Suppose that an elementary magnet is situated at the origin and that its axis corresponds to the z-axis. The trajectory ($x(t)$, $y(t)$, $z(t)$) of an electrical particle in this magnetic field is then given as the autonomous system of second order ordinary differential equations

$$\frac{d^2x}{dt^2} = \frac{1}{r^5}\left(3yz\frac{dz}{dt} - (3z^2-r^2)\frac{dy}{dt}\right)$$
$$\frac{d^2y}{dt^2} = \frac{1}{r^5}\left((3z^2-r^2)\frac{dx}{dt} - 3xz\frac{dz}{dt}\right)$$
$$\frac{d^2z}{dt^2} = \frac{1}{r^5}\left(3xz\frac{dy}{dt} - 3yz\frac{dx}{dt}\right)$$

where $r^2 := x^2+y^2+z^2$. Show that the system can be simplified by introducing *polar coordinates*

$$x(t) = R(t)\cos(\phi(t)), \qquad y(t) = R(t)\sin(\phi(t)).$$

Problem 42. Consider the pair of coupled nonlinear differential equations relevant to the quantum field theory of charged solitons

$$\frac{d^2\sigma}{dx^2} = -\sigma+\sigma^3+d\rho^2\sigma, \quad \frac{d^2\rho}{dx^2} = f\rho+\lambda\rho^3+d\rho(\sigma^2-1) \tag{1}$$

where σ and ρ are real scalar fields and d, f, λ are constants.

(i) Try to find an excat solution with the ansatz

$$\rho(x) = b_1 \tanh(\lambda_0(x+c_0)), \qquad \sigma(x) = \sum_{n=1} a_n \tanh^n(\lambda_0(x+c_0)). \qquad (2)$$

Problem 43. The coupled system of ordinary differential equations

$$\frac{d^2u}{dt^2} + \omega(t)u = \frac{1}{u^2 v} f_1\left(\frac{v}{u}\right) \qquad (1a)$$

$$\frac{d^2v}{dt^2} + \omega(t)v = \frac{1}{uv^2} f_2\left(\frac{u}{v}\right) \qquad (1b)$$

is a so-called *Ermakov-type system*, where F_1 and f_2 are arbitrary differentiable functions. Show that the system admits the first integral

$$F = \frac{1}{2}\left(u\frac{dv}{dt} - v\frac{du}{dt}\right)^2 + \int^{v/u} f_1(s)ds + \int^{u/v} f_2(s)ds. \qquad (2)$$

Problem 44. Consider the linear system of first order ordinary differential equations

$$\frac{du_1}{dt} = au_2, \quad \frac{du_2}{dt} = -au_1, \quad \frac{du_3}{dt} = bu_3$$

where $a > 0$ and $b > 0$. The corresponding vector field is given by

$$V = au_2\frac{\partial}{\partial u_1} - au_1\frac{\partial}{\partial u_2} + bu_3\frac{\partial}{\partial u_3}$$

with the Lie series solution of the initial value problem

$$\mathbf{u}(t) = e^{tV}\mathbf{u}\big|_{\mathbf{u}\to\mathbf{u}_0}$$

where $\mathbf{u}(t=0) = \mathbf{u}_0$. Find the *curvature* $\kappa(t)$ and *torsion* $\omega(t)$ of this curve, where

$$\kappa^2(t) := \frac{1}{\rho^2(t)} = \frac{\dot{\mathbf{u}}^2\ddot{\mathbf{u}}^2 - (\dot{\mathbf{u}}^T\ddot{\mathbf{u}})^2}{(\dot{\mathbf{u}}^2)^3}$$

$$\omega(t) = \rho^2(t)\frac{\det(\dot{\mathbf{u}}\ddot{\mathbf{u}}\;\dddot{\mathbf{u}})}{(\dot{\mathbf{u}}^2)^3}$$

Problem 45. Consider the *Lorenz model*

$$\frac{du_1}{dt} = -\sigma(u_1 - u_2), \qquad \frac{du_2}{dt} = -u_1u_3 + ru_1 - u_2, \qquad \frac{du_3}{dt} = u_1u_2 - bu_3.$$

Find the system of differential equation under the transformation

$$\tau(t) = t\sigma(r-1)^{1/2}$$
$$u(\tau(t)) = \frac{1}{(2\sigma(r-1))^{1/2}} u_1(t)$$
$$m(\tau(t)) = \frac{1}{(r-1)} u_3(t) - u^2(\tau(t)).$$

Set $\epsilon = 1/(r-1)^{1/2}$ and study the case that ϵ is small which relates to high Rayleigh numbers $r \gg 1$.

Problem 46. Consider the system of first order differential equations

$$\frac{du_1}{dt} = -\nu u_1 + m_1 u_2 u_3$$
$$\frac{du_2}{dt} = -\nu u_2 + m_2 u_3 u_1$$
$$\frac{du_3}{dt} = -\nu u_3 + m_3 u_1 u_2$$

where ν, m_1, m_2, m_3 are nonzero constants. The system plays a role in nonlinearly coupled positive and negative energy waves in plasma physics. Find the system of differential equations under the transformation

$$\bar{u}_j(\bar{t}(t)) = u_j(t)\exp(\nu t), \qquad \bar{t}(t) = \nu^{-1}(1 - \exp(-\nu t))$$

where $j = 1, 2, 3$.

Problem 47. Consider the nonlinear autonomous system

$$\frac{du_1}{dt} = -2u_2 + u_2 u_3, \quad \frac{du_2}{dt} = u_1 - u_1 u_3, \quad \frac{du_3}{dt} = u_1 u_2.$$

Show that there is one fixed point (equilibrium point). Study the stability of this fixed point. Let

$$V(\mathbf{u}) = c_1 u_1^2 + c_2 u_2^2 + c_3 u_3^2.$$

Find the time evolution of V, i.e. dV/dt. Find the condition on the coefficients c_1, c_2, c_3 such that $V(\mathbf{u}) > 0$ for $\mathbf{u} \neq \mathbf{0}$ and $dV/dt = 0$ for all $\mathbf{u} \in \mathbb{R}^3$.

Problem 48. Consider the initial value problem of the autonomous system of differential equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \qquad \mathbf{u}(t=0) = \mathbf{u}_0 \tag{1}$$

where $\mathbf{f} \in C^1(\mathbb{R}^n)$. Show that for each $\mathbf{u}_0 \in \mathbb{R}^n$ the initial value problem

$$\frac{d\mathbf{u}}{dt} = \frac{\mathbf{f}(\mathbf{u})}{1 + |\mathbf{f}(\mathbf{u})|}, \qquad \mathbf{u}(t=0) = \mathbf{u}_0 \tag{2}$$

has unique solution $\mathbf{u}(t)$ for all $t \in \mathbf{R}$, i.e. system (2) defines an autonomous system on $\mathbb{R}^n$ and (2) is topologically equivalent to (1) on $\mathbb{R}^n$.

Problem 49. Consider the *Lorenz model*

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = xy - bz.$$

(i) Show that the Lorenz model is invariant under the involution $(x, y, z) \to (-x, -y, z)$.
(ii) Find the solution for $z(t)$ if $x(t) = 0$ and $y(t) = 0$. Is the solution stable.

Problem 50. Consider an autonomous system of first order differential equation

$$\frac{d\mathbf{u}}{dt} = f(\mathbf{u}), \qquad \mathbf{u} \in \mathbb{R}^n$$

where $f : \mathbb{R} \to \mathbb{R}^n$ is an analytic function. Assume it admits a first integral $I(\mathbf{u})$, i.e.

$$f(\mathbf{u}) \cdot \nabla I(\mathbf{u}) \equiv f_1(\mathbf{u})\frac{\partial I}{\partial u_1} + f_2(\mathbf{u})\frac{\partial I}{\partial u_2} + \cdots + f_n(\mathbf{u})\frac{\partial}{\partial u_n} = 0$$

Then the system can be written as

$$\frac{d\mathbf{u}}{dt} = S(\mathbf{u})\nabla I(\mathbf{u})$$

where $S(\mathbf{u})$ is a skew-symmetric $n \times n$ matrix. Consider the autonomous system

$$\frac{du_1}{dt} = u_1(u_2 - u_3)$$
$$\frac{du_2}{dt} = u_2(u_3 - u_1)$$
$$\frac{du_3}{dt} = u_3(u_1 - u_2).$$

(i) It admits the first integral

$$I(\mathbf{u}) = u_1 + u_2 + u_3.$$

Find the representation described above.
(ii) The autonomous system also admits the first integral

$$I(\mathbf{u}) = u_1 u_2 u_3.$$

Find the representation described above.

Problem 51. Consider the first order autonomous system

$$\frac{dx}{dt} = -ax + y + 10yz$$
$$\frac{dy}{dt} = -x - 0.4y + 5xz$$
$$\frac{dz}{dt} = bz - 5xy$$

where a and b are real and positive bifurcation parameters. Find the fixed points and study their stability. Set $a = 0.4$. Show that there is Hopf bifurcation.

Problem 52. Consider a set n identical elements, where each of them is characterized by a state variable $u_j(t)$ with $-1 \leq u_j \leq +1$ and $j = 1, 2, \ldots, n$. Without coupling the individual dynamics the state variable $u_j(t)$ obeys the nonlinear differential equation

$$\frac{du}{dt} = u - u^3$$

with $-1 \leq u(0)l + 1$. This differential equation describes an overdamped motion in the one-dimensional potential $V(u) = -u^2/2 + u^4/4$. The fixed points are $0, \pm 1$. The solution of the initial value problem is

$$u(t) = \frac{u_0}{\sqrt{e^{-2t}(1 - u_0^2) + u_0^2}}$$

with $u_0 = u(0)$. For $t \to \infty$ $u(t)$ preserves its sign and approaches the fixed points ± 1. The fixed points $u^* = 0$ is unstable. Define

$$\bar{u}(t) := \frac{1}{n} \sum_{j=1}^{n} u_j(t).$$

Study the behaviour of the autonomous system

$$\frac{du_j}{dt} = u_j - u_j^3 + k(\bar{u} - u_j) = (1-k)u_j + k\bar{u} - u_j^3$$

where $k = 1$.

Problem 53. Consider the initial value problem of the system of linear differential equations

$$\frac{dY}{dt} = A(t)Y(t), \qquad Y(t=0) = Y_0$$

where the $n \times n$ matrix is a smooth function of t. We express the solution as

$$Y(t) = \exp(\Omega(t))Y_0$$

and find $\Omega(t)$ as expansion (Magnus expansion)

$$\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t)$$

One finds for the first three terms

$$\Omega_1(t) = \int_0^t A(t_1)dt_1$$

$$\Omega_2(t) = \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [A(t_1, A(t_2)]$$

$$\Omega_3(t) = \frac{1}{6}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_3} dt_3 ([A(t_1), [A(t_2), A(t_3)]] + [A(t_3), [A(t_2), A(t_1)]]).$$

Let

$$A(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Find Ω_1, Ω_2, Ω_3.

Problem 54. Consider a model for the Belousov-Zhabotinskii reaction

$$\frac{dx}{dt} = s(y - xy + x - qx^2)$$

$$\frac{dy}{dt} = s^{-1}(fz - y - xy)$$

$$\frac{dz}{dt} = w(x - z)$$

where f, s, q and w are positive real parameters and (x, y, z) are concentrations and therefore nonnegative. Show the existence of periodic solutions applying the theorem of Hopf.

Problem 55. The time evolution equations of a particle of mass m and electric charge q in an electric potential are given by

$$\frac{d^2y}{dt^2} + \frac{q\Phi_0}{mr_0^2}\cos(\Omega t)y(t) = 0$$
$$\frac{d^2z}{dt^2} + \frac{q\Phi_0}{mr_0^2}\cos(\Omega t)z(t) = 0$$
$$\frac{d^2x}{dt^2} = 0.$$

Let

$$b := \frac{2q\Phi_0}{mr_0^2\Omega^2}, \qquad \zeta := \frac{\Omega t}{2}.$$

Then we can write the first two differential equations as

$$\frac{d^2y(\zeta)}{d\zeta^2} + 2b\cos(2\zeta)y(\zeta) = 0$$
$$\frac{d^2z(\zeta)}{d\zeta^2} - 2b\cos(2\zeta)z(\zeta) = 0.$$

This a special cases of the *Mathieu differential equation*

$$\frac{d^2y(\zeta)}{d\zeta^2} + (a + 2b\cos(2\zeta))y(\zeta) = 0.$$

Study the stability of the general solution

$$y(\zeta) = \sum_{n\in\mathbb{Z}} C_{2n}(\lambda_+ e^{+\mu\zeta}e^{+2in\zeta} + \lambda_- e^{-\mu\eta}e^{-2in\zeta})$$

with general integration constants $\lambda_\pm$ which have to mached with the inital conditions and certain constants C_{2n} and μ depending on a, b.

Problem 56. The autonomous system of ordinary differential equations for energy level motion in quantum mechanics is given by

$$\frac{dE_n}{d\epsilon} = p_n$$
$$\frac{dp_n}{d\epsilon} = 2\sum_{m(\neq n)} \frac{V_{nm}V_{mn}}{E_n - E_m}$$
$$\frac{dV_{mn}}{d\epsilon} = \sum_{\ell(\neq m,n)} V_{m\ell}V_{\ell n}\left(\frac{1}{E_m - E_\ell} + \frac{1}{E_n - E_\ell}\right) - \frac{1}{E_m - E_n}V_{mn}(p_m - p_n)$$

where $m \neq n$ and $V_{mn} = V_{nm}$. Consider the case with three level, i.e. $n = 0, 1, 2$. Write down the differential equation for this case and solve the initial value problem

$$E_0(\epsilon = 0) = -1, \quad E_1(\epsilon = 0) = 0, \quad E_2(\epsilon = 0) = 1$$

$$p_0(\epsilon = 0) = 0, \quad p_1(\epsilon = 0) = 0, \quad p_2(\epsilon = 0) = 0$$

$$V_{01}(\epsilon = 0) = 1, \quad V_{02}(\epsilon = 0) = 0, \quad V_{12}(\epsilon = 0) = 1.$$

Problem 57. Consider the system of first order ordinary differential equations

$$\frac{dz_m}{dt} = (i\omega_m + 1 - |z_m|^2)z_m + \frac{K}{N}\sum_{n=1}^{N}(z_n - z_m)$$

where $z_m(t)$ $(m = 1, \ldots, N)$ is a complex number representing the amplitude $r_m(t)$ and phase $\theta_m(t)$ of the m-th oscillator, i.e.

$$z_m(t) = r_m(t)\exp(i\theta_m(t))$$

and ω_m is the natural frequency. The natural frequency is chosen from a distribution $g(\omega)$. Rewrite the system using $r_m(t)$ and $\theta(t)$. Discuss the behaviour of the dynamical system.

Problem 58. The *Möbuis band* can be represented in parameter representation as

$$x_1(t, \lambda) = (1 + \lambda\cos(t/2))\cos(t)$$
$$x_2(t, \lambda) = (1 + \lambda\cos(t/2))\sin(t)$$
$$x_3(t, \lambda) = \lambda\sin(t/2).$$

We consider λ as a fixed parameter. Find the autonomous system of differential equations for $x_1(t)$, $x_2(t)$, $x_3(t)$.

Problem 59. Consider the autonomous system of differential equations

$$\frac{du_1}{dt} = u_2u_3$$
$$\frac{du_2}{dt} = u_1u_3$$
$$\frac{du_3}{dt} = u_1u_2.$$

(i) Show that

$$I = \frac{1}{2}(u_1^2 - u_2^2)$$

is a first integral.
(ii) Show that the system can be written as

$$\frac{d\mathbf{u}}{dt} = S\nabla I$$

where S is a 3×3 skew-symmetric matrix ($S^T = -S$) and

$$\nabla I = \begin{pmatrix} \partial I/\partial u_1 \\ \vdots \\ \partial I/\partial u_n \end{pmatrix}.$$

Problem 60. Let $\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t) \in \mathbb{R}^3$. Solve the initial value problem of the nonlinear autonomous system of first order differential equations

$$\frac{d\mathbf{u}_1}{dt} = \mathbf{u}_2 \times \mathbf{u}_3, \quad \frac{d\mathbf{u}_2}{dt} = \mathbf{u}_3 \times \mathbf{u}_1, \quad \frac{d\mathbf{u}_3}{dt} = \mathbf{u}_1 \times \mathbf{u}_2$$

where $\times$ denotes the vector product.

Problem 61. Let c be a constant. Solve the differential equations

$$u(x) + \frac{du/dx}{du/dx + d^2u/dx^2} = c$$

$$u(x) + \cfrac{du/dx}{du/dx + \cfrac{d^2u/dx^2}{d^2u/dx^2 + d^3u/dx^3}} = c.$$

Problem 62. Consider the autonomous system

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + (1 - r^2)\mathbf{u}$$

where $\mathbf{u} = (u_1\, u_2\, u_3)^T$, $r^2 = u_1^2 + u_2^2 + u_3^2$ and A is the 3×3 matrix

$$A = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}.$$

Here μ is a bifurcation parameter. Discuss the behaviour of the system.

Problem 63. Study Hopf bifurcation for the coupled system of first order differential equations

$$\frac{du_1}{dt} = -u_2 + \mu u_1 + v_1(v_1^2 + v_2^2)$$

$$\frac{du_2}{dt} = u_1 + \mu u_2 + v_2(v_1^2 + v_2^2)$$
$$\frac{dv_1}{dt} = v_2 + \mu v_1 + u_1(u_1^2 + u_2^2)$$
$$\frac{dv_2}{dt} = -v_1 + \mu v_2 + u_2(u_1^2 + u_2^2)$$

where μ is the bifurcation parameter. Utilize the symmetry of the problem.

Problem 64. Solve the initial value problem of the autonomous system of first order differential equations

$$\frac{du_1}{dt} = u_1$$
$$\frac{du_2}{dt} = (1 - u_1)u_2$$
$$\frac{du_3}{dt} = (1 - u_1)(1 - u_2)u_3$$
$$\vdots$$
$$\frac{du_n}{dt} = (1 - u_1)(1 - u_2)\cdots(1 - u_{n-1})u_n.$$

Problem 65. Consider the autonomous system of differential equations

$$\frac{du_1}{dt} = c_1 u_1$$
$$\frac{du_2}{dt} = c_{13} u_1 u_3$$
$$\frac{du_3}{dt} = c_3 u_3$$

where c_1, c_3, c_{13} are nonzero bifurcation parameters. Find the fixed points. Solve the initial value problem.

Problem 66. Consider the autonomous system of first order differential equations

$$\frac{du_1}{dt} = -8u_7, \quad \frac{du_2}{dt} = 4u_5, \quad \frac{du_3}{dt} = 2(u_4 u_7 - u_5 u_6). \quad \frac{du_4}{dt} = 4u_2 u_5 - u_7,$$

$$\frac{du_5}{dt} = u_6 - 4u_2 u_4, \quad \frac{du_6}{dt} = -u_1 u5 + u_2 u_7, \quad \frac{du_7}{dt} = u_1 u_4 - 2u_2 u_6 - 4u_3.$$

(i) Show that this system admits the first integrals

$$I_1(\mathbf{u}) = u_1 + 4u_2 - 8u_4, \quad I_2(\mathbf{u}) = u_1 u_2 + 4u_6, \quad I_3(\mathbf{u}) = u_3 + u_4^2 + u_5^2,$$

$$I_4(\mathbf{u}) = u_2u_3 + u_4u_6 + u_5u_7, \quad I_5(\mathbf{u}) = -u_1u_3 + u_6^2 + u_7^2.$$

(ii) Consider the autonomous system in the complex domain ($t \to z, u_j(t) \to w_j(z)$) and perform a Painlevé analysis, i.e. insert the ansatz

$$w_j(z) = (z - z_0)^{-n_j} \sum_{k=0}^{\infty} c_{kj}(z - z_0)^k$$

where $j = 1, \ldots, 7$.

Problem 67. Consider the first order autonomous system of ordinary differential equations

$$\frac{du_j}{dt} = u_j(u_{j+1} - u_{j-1}), \quad j = 1, \ldots, N$$

and $u_0 = 0$, $u_{N+1} = 0$. Solve the initial value problem.

Problem 68. Consider the driven *van der Pol equation*

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt}\left(x^2 - 1\right) + x = b\cos(\omega t) \tag{1}$$

where $a \neq 0$. Extend the equation into the complex domain and perfom a singular point analysis. Show that all of its solutions posses only square-root singularities in the complex time plane.

Problem 69. Consider the differential equation

$$\frac{d\mathbf{x}}{dt} = P(t)\mathbf{x}$$

where $P(t)$ is periodic with principal period T and differentiable. Thus T is the smallest positive number for which $P(t+T) = P(t)$ and $-\infty < t < \infty$. Can we conclude that all solutions are periodic? For example, consider

$$\frac{dx}{dt} = (1 + \sin t)x.$$

Problem 70. (i) Find the first integrals of the system

$$\frac{dx_1}{dt} = x_1x_2 - x_1x_3$$
$$\frac{dx_2}{dt} = x_2x_3 - x_1x_2$$
$$\frac{dx_3}{dt} = x_3x_1 - x_2x_3$$

(ii) Find the first integrals of the system

$$\frac{dx_1}{dt} = x_1(c - x_2 + x_3)$$
$$\frac{dx_2}{dt} = x_2(c - x_3 + x_1)$$
$$\frac{dx_3}{dt} = x_3(c - x_1 + x_2).$$

Problem 71. Solve the initial value problem for the system of linear differential equations

$$\frac{dc_0}{dt} = \frac{1}{2} i\Omega e^{-i\phi} e^{i(\omega-\nu)t} c_1$$
$$\frac{dc_1}{dt} = \frac{1}{2} i\Omega e^{i\phi} e^{-i(\omega-\nu)t} c_0$$

where Ω, ω, ν are constant frequencies. Note the the system depends explicitly on the time t. Then study the special case $\omega = \nu$.

Problem 72. Given the surface in $\mathbb{R}^3$

$$f(t,\theta) = \left(\left(1 + t \sin\frac{\theta}{2}\right)\cos\theta, \left(1 + t\cos\frac{\theta}{2}\right)\sin\theta, t\sin\frac{\theta}{2} \right)$$

where

$$t \in \left(-\frac{1}{2}, \frac{1}{2}\right) \qquad \theta \in \mathbb{R}$$

(i) Build three models of this surface using paper, glue and a scissors. Color the first model with the South African flag. For the second model keep t fixed (say $t = 0$) and cut the second model along the θ parameter. For the third model keep θ fixed (say $\theta = 0$) and cut the model along the t parameter. Submit all three models.
(ii) Describe the curves with respect to t for θ fixed. Describe the curve with respect to θ for t fixed.

voluntary (iii) The map given above can also be written in the form

$$x(t,\theta) = \left(1 + t\sin\frac{\theta}{2}\right)\cos\theta$$

$$y(t,\theta) = \left(1 + t\cos\frac{\theta}{2}\right)\sin\theta$$

$$z(t,\theta) = t\sin\frac{\theta}{2}.$$

For fixed t the curve

$$(x(\theta), y(\theta), z(\theta))$$

can be considered as a solution of a differential equation. Find this differential equation. Then t plays the role of a bifurcation parameter.

Problem 73. Consider the dynamical system of two coupled harmonic oscillators

$$\frac{d^2u_1}{dt^2} + \omega_1^2 u_1 = \sin(u_1 - u_2), \qquad \frac{d^2u_2}{dt^2} + \omega_2^2 u_2 = \sin(u_2 - u_1).$$

Solve the initial value problem.

Problem 74. (i) Solve the initial value problem of the linear autonomous system of differential equations

$$\frac{du_1}{dt} = k_1(u_2 - u_3)$$
$$\frac{du_2}{dt} = k_2(u_3 - u_1)$$
$$\frac{du_3}{dt} = k_3(u_1 - u_2)$$

where $k_1 = k_2 = k_3 = k$.
(ii) Solve the initial value problem of the nonlinear autonomous system of differential equations

$$\frac{du_1}{dt} = k_1 \sin(u_2 - u_3)$$
$$\frac{du_2}{dt} = k_2 \sin(u_3 - u_1)$$
$$\frac{du_3}{dt} = k_3 \sin(u_1 - u_2)$$

where $k_1 = k_2 = k_3 = k$. Can the system show chaotic behaviour?
(iii) Solve the initial value problem of the nonlinear autonomous system of differential equations

$$\frac{d^2u_1}{dt^2} = k_1 \sin(u_2 - u_3)$$
$$\frac{d^2u_2}{dt^2} = k_2 \sin(u_3 - u_1)$$
$$\frac{d^2u_3}{dt^2} = k_3 \sin(u_1 - u_2)$$

where $k_1 = k_2 = k_3 = k$. Can the system show chaotic behaviour?

Problem 75. Solve the initial value problem of the first order autonomous system of differential equations

$$\frac{du_1}{dt} = u_1, \quad \frac{du_2}{dt} = u_1 u_2, \quad \frac{du_3}{dt} = u_1 u_2 u_3.$$

Problem 76. Let $\mathbf{u}(t) \in \mathbb{R}^3$. Solve the initial value problem for the differential equation

$$\frac{d^2\mathbf{u}}{dt^2} = \mathbf{u} \times \frac{d\mathbf{u}}{dt}$$

where $\times$ denotes the vector product.

Problem 77. The Chazy class III third order differential equation is given by

$$\frac{d^3u}{dt^3} - 2u\frac{d^2u}{dt^2} + 3\left(\frac{du}{dt}\right)^2 = 0.$$

Show that the differential equation admits the two-parameter particular solution

$$u(t) = -6\frac{t - c_1}{(t - c_2)^2}$$

where c_1, c_2 arbitrary in the complex plane.

Problem 78. Let ω_0 be real and positive. Solve the coupled system of linear equations

$$\frac{d^2u_1}{dt^2} = -\omega_0\frac{du_2}{dt}, \qquad \frac{d^2u_2}{dt^2} = \omega_0\frac{du_1}{dt}.$$

Problem 79. Show that the *Lorenz model*

$$\frac{du_1}{dt} = \sigma(u_2 - u_1)$$
$$\frac{du_2}{dt} = u_1(r - u_3) - u_2$$
$$\frac{du_3}{dt} = u_1 u_2 - b u_3$$

can be written as

$$\frac{d\mathbf{u}}{dt} = \text{grad}H_1 \times \text{grad}H_2 + \text{grad}D$$

where

$$H_1(u_1, u_2, u_3) = \frac{1}{2}(u_2^2 + (u_3 - r)^2), \quad H_2(u_1, u_2, u_3) = \sigma u_3 - \frac{1}{2}u_1^2$$

and

$$D(u_1, u_2, u_3) = -\frac{1}{2}(\sigma u_1^2 + u_2^2 + bu_3^2)$$

Problem 80. Consider the nonlinear differential equation

$$3u\frac{du}{dx} = 2\frac{du}{dx}\frac{d^2u}{dx^2} + u\frac{d^3u}{dx^3}.$$

Show that $u(x) = e^{-|x|}$ is a solution in the sense of generalized function.

Problem 81. Study the system of differential equations

$$\frac{d\theta_1}{dt} = \omega_1 + \sin(\theta_1 - \bar{\theta})$$
$$\frac{d\theta_2}{dt} = \omega_2 + \sin(\theta_2 - \bar{\theta})$$
$$\frac{d\theta_3}{dt} = \omega_3 + \sin(\theta_3 - \bar{\theta})$$

where

$$\bar{\theta}(t) = \frac{1}{3}(\theta_1(t) + \theta_2(t) + \theta_3(t)).$$

Hint. Consider the sum $\theta(t) := \theta_1(t) + \theta_2(t) + \theta_3(t)$.

Problem 82. Study Hopf bifurcation for the coupled oscillator

$$\frac{d^2u_1}{dt^2} = -u_1 + (\mu - u_1^2 - \alpha u_2^2)\frac{du_1}{dt}$$
$$\frac{d^2u_2}{dt^2} = -u_2 + (\mu - u_2^2 - \alpha u_1^2)\frac{du_2}{dt}.$$

Problem 83. Let $z_m(t) = r_m(t)\exp(i\theta_m(t))$, $\omega_m > 0$, $K > 0$ and $m = 1, 2, \ldots, N$. Study the autonomous system of first order differential equations

$$\frac{dz_m}{dt} = (i\omega_m + 1 - |z_m|^2)z_m + \frac{K}{N}\sum_{n=1}^{N}(z_n - z_m).$$

Problem 84. Consider the linear operators

$$L := -\frac{d^2}{dx^2} + u(x), \qquad A := -4\frac{d^3}{dx^3} + 6u(x)\frac{d}{dx} + 3\frac{du}{dx}.$$

Let $[,]$ be the commutator. Show that the condition $[L, A]\psi(x) = \epsilon\psi(x)$ provides the nonlinear third order differential equation

$$\frac{d^3u}{dx^3} = 6u\frac{du}{dx} + \epsilon.$$

Problem 85. Consider the linear fourth order differential equation

$$\frac{d^4u}{dx^4} = Eu.$$

Show that (even solution)

$$u(x) = A\cos(kx) + B\cosh(kx)$$

satisfies the differential equation with $E = k^4$. Show that (odd solution)

$$u(x) = A\sin(kx) + B\sinh(kx)$$

satisfies the differential equation with $E = k^4$.

Problem 86. The governing Einstein equations for the mixmaster metric tensor fields are given by the autonomous system of second order ordinary differential equations

$$\frac{d^2f}{d\tau^2} = \frac{1}{2}((e^{2g} - e^{2h})^2 - 4e^{4f})$$
$$\frac{d^2g}{d\tau^2} = \frac{1}{2}((e^{2h} - e^{2f})^2 - 4e^{4g})$$
$$\frac{d^2h}{d\tau^2} = \frac{1}{2}((e^{2f} - e^{2g})^2 - 4e^{4h})$$

where f, g, h are the scale factors of the metric tensor field and the derivative is with respect to the (logarithmic) time variable τ. Find the discrete symmetries of the system. Show that this coupled system of differential equations admits the first integral

$$I(f,g,h) = 4\left(\frac{df}{d\tau}\frac{dg}{d\tau} + \frac{dg}{d\tau}\frac{dh}{d\tau} + \frac{dh}{d\tau}\frac{df}{d\tau}\right) - e^{4f} - e^{4g} - e^{4h} + 2(e^{2(f+g)} + e^{2(g+h)} + e^{2(h+f)}).$$

Problem 87. (i) Consider the autonomous system of first order differential equations

$$\frac{du_j}{dt} = f_j(\mathbf{u}), \qquad j = 1, \ldots, n$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ are analytic functions. Assume that the system is invariant under the transformation $\mathbf{u} \to -\mathbf{u}$. Show that if $\mathbf{v}(t)$ satisfies the system then $\mathbf{w}(t) = -\mathbf{v}(t)$ also satisfies the system.
(ii) Assume that $\mathbf{f}(\mathbf{u}) = \mathbf{f}(-\mathbf{u})$. Show that if $\mathbf{v}(t)$ is a solution of the system, then $\mathbf{w}(t) = -\mathbf{v}(-t)$ is also a solution.

Problem 88. Consider an autonomous system of first order differential equations

$$\frac{du_j}{dt} = f_j(\mathbf{u}), \qquad j = 1, \ldots, n$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions. The traditional *flow box theorem* asserts that if $\mathbf{f}$ is a C^1 vector field and $u_0 \in \mathbb{R}^n$ is not a fixed point, i.e. $f(u_0) \neq \mathbf{0}$, then there is a diffeomorphism which maps the vector field $\mathbf{f}$ near u_0 to a constant vector field. In other words, the local flow of the vector field $\mathbf{f}$ is conjugate via diffeomorphism to translation. Apply the flow box theorem to the autonomous system in the plane

$$\frac{du_1}{dt} = -\frac{3u_2^2}{1+2u_2}, \qquad \frac{du_2}{dt} = \frac{1}{1+2u_2}$$

(which admits no fixed points) and the transformation

$$v_1(u_1, u_2) = u_1 + u_2^2, \qquad v_2(u_1, u_2) = u_2 + u_2^2.$$

Problem 89. Consider the autonomous system

$$\frac{du_1}{dt} = f(u_2), \quad \frac{du_2}{dt} = f(u_3), \quad \frac{du_3}{dt} = f(u_1)$$

where $f : \mathbb{R} \to \mathbb{R}$ is an analytic function. We also assume that the autonomous system admits a fixed point at $(0, 0, 0)$ and the first term in the Taylor expansion of f around 0 is u. The following five functions with these properties are studied

$$f_1(u) = \sin(u), \quad f_2(u) = \arctan(u), \quad f_3(u) = \sinh(u),$$
$$f_4(u) = \tanh(u), \quad f_5(u) = \sinh^{-1}(u).$$

First study the stability of the fixed pont $(0, 0, 0)$. What are the discrete symmetries of the autonomous system? Note that the divergence of the vector field of the autonomous system is 0. Apply the Lie series technqiue to find solutions of the initial value problem.

Problem 90. (i) Consider the autonomous systems of first order differential equation

$$\frac{du_j}{dt} = f(u_j) + c \sum_{k=1, k \neq j}^{n} u_k, \quad j = 1, \ldots, n \qquad (1)$$

where c is a positive constant and f is an analytic function. Assume we know the solution of $du_j/dt = f(u_j)$ for $j = 1, 2, \ldots, n$. What can be said about the solution of (1)? Let $n \geq 3$. Can we find a function f and a constant c such that equation (1) shows chaotic behaviour?
(ii) Consider the autonomous systems of first order differential equation

$$\frac{du_j}{dt} = f(u_j) + c \prod_{k=1, k\neq j}^{n} u_k, \quad j = 1, \ldots, n \tag{2}$$

where c is a positive constant and f is an analytic function. Assume we know the solution of $du_j/dt = f(u_j)$ for $j = 1, 2, \ldots, n$. What can be said about the solution of (2)? Let $n \geq 3$. Can we find a function f and a constant c such that equation (1) shows chaotic behaviour?

Problem 91. Consider the autonomous system of first order ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}).$$

Assume that I is a first integral and that ∇I is nonzero. Then the system can be written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) = A(\mathbf{x})\nabla I(\mathbf{x})$$

where A is antisymmetric. Show that the matrix A is given by

$$A = \frac{1}{|\nabla I|^2}\left(\mathbf{f}(\nabla I)^T - (\nabla I)\mathbf{f}^T\right).$$

Problem 92. Consider the *Lorenz model*

$$\frac{dx}{dt} = \sigma y - \sigma x$$
$$\frac{dy}{dt} = -xz + rx - y$$
$$\frac{dz}{dt} = -xy - bz.$$

Find the fixed points of the system and study the stability.

Problem 93. Consider the dynamical system

$$\frac{dx}{dt} = \sigma y - \sigma x, \quad \frac{dy}{dt} = -y + rx - xz, \quad \frac{dz}{dt} = -bz + xy \tag{1}$$

where σ, r and b are control parameters. The behaviour of this system was first investigated numerically by Lorenz in 1963 for $r = 28$, $\sigma = 10$ and

$b = \frac{8}{3}$. It was found that the system starts a rotation around one of the (unstable) focuses with amplitude increasing with time, thereby forming a divergent spiral. After a number of such oscillations, the system suddenly leaves this regime and goes monotonically towards the second available (unstable) focus around which it starts again an oscillatory motion along a divergent spiral. Again, after a certain number of oscillations around this focus, the system jumps anew towards the vicinity of the previous focus, from which it starts again a divergent oscillatory trajectory and so on. The interesting thing is that the time intervals the system spends in the vicinity of each focus before jumping into the vicinity of the other focus are stochastically distributed and there is no regularity, whatsoever, in the process, which nevertheless is created by the unfolding of a deterministic (non-linear) dynamics. (i) Show that eliminating y in (1) and solving z in terms of x^2 we obtain the following equation for $x(t)$

$$\frac{d^2x}{dt^2}+(1+\sigma)\frac{dx}{dt}+\sigma\left(1-r+\frac{1}{2\sigma^2}x^2+\left(1-\frac{b}{2\sigma}\right)\int_0^\infty (x(t-\tau))^2\exp(-b\tau)d\tau\right)x=0. \tag{2}$$

(ii) We assume that the motion started at $-\infty$, Show that if $b > 0$ we can split $\exp(-\tau)$ into a δ-function and the deviation from it to obatin

$$\frac{d^2x}{dt^2}+(1+\sigma)\frac{dx}{dt}+\frac{dU}{dx}+\left(\left(\sigma-\frac{b}{2}\right)\int_0^\infty (x^2(t-\tau)-x^2(t))e^{-b\tau}d\tau\right)x=0 \tag{3}$$

where

$$U(x)=\sigma\left(\frac{1-r}{2}x^2+\frac{1}{4b}x^4\right) \tag{4}$$

depends on the history of motion. (iii) Discuss (3).

Problem 94. (i) Consider the autonomous systems of first order differential equation

$$\frac{du_j}{dt}=f(u_j)+c\sum_{k=1,k\neq j}^{n}u_k, \quad j=1,\ldots,n \tag{1}$$

where c is a positive constant and f is an analytic function. Assume we know the solution of $du_j/dt = f(u_j)$ for $j = 1, 2, \ldots, n$. What can be said about the solution of (1)? Let $n \geq 3$. Can we find a function f and a constant c such that equation (1) shows chaotic behaviour?
(ii) Consider the autonomous systems of first order differential equation

$$\frac{du_j}{dt}=f(u_j)+c\prod_{k=1,k\neq j}^{n}u_k, \quad j=1,\ldots,n \tag{2}$$

where c is a positive constant and f is an analytic function. Assume we know the solution of $du_j/dt = f(u_j)$ for $j = 1, 2, \ldots, n$. What can be said about the solution of (2)? Let $n \geq 3$. Can we find a function f and a constant c such that equation (1) shows chaotic behaviour?

Problem 95. (i) Solve the initial value problem of the linear autonomous system of differential equations

$$\begin{aligned}\frac{du_1}{dt} &= k_1(u_2 - u_3)\\ \frac{du_2}{dt} &= k_2(u_3 - u_1)\\ \frac{du_3}{dt} &= k_3(u_1 - u_2)\end{aligned}$$

where $k_1 = k_2 = k_3 = k$.
(ii) Solve the initial value problem of the nonlinear autonomous system of differential equations

$$\begin{aligned}\frac{du_1}{dt} &= k_1 \sin(u_2 - u_3)\\ \frac{du_2}{dt} &= k_2 \sin(u_3 - u_1)\\ \frac{du_3}{dt} &= k_3 \sin(u_1 - u_2)\end{aligned}$$

where $k_1 = k_2 = k_3 = k$. Can the system show chaotic behaviour?
(iii) Solve the initial value problem of the nonlinear autonomous system of differential equations

$$\begin{aligned}\frac{d^2u_1}{dt^2} &= k_1 \sin(u_2 - u_3)\\ \frac{d^2u_2}{dt^2} &= k_2 \sin(u_3 - u_1)\\ \frac{d^2u_3}{dt^2} &= k_3 \sin(u_1 - u_2)\end{aligned}$$

where $k_1 = k_2 = k_3 = k$. Can the system show chaotic behaviour?

Problem 96. Let $N \geq 2$ and $j = 0, 1, \ldots, N-1$. Study the initial value problem of the coupled system of differential equations

$$\frac{dx_j}{dt} = f(x_j(t)) + \sum_{j=0}^{N-1} a_{ij}(h(x_j(t)) - h(x_i(t))), \quad i = 0, 1, \ldots, N-1$$

where $A = (a_{ij})$ $(i, j = 0, 1, \ldots, N-1)$ is the coupling matrix with $a_{ii} = 0$ for $i = 0, 1, \ldots, N-1$. Assume that $N = 3$, $f(x) = 4x(1-x)$ and $h(x) = \tanh(x)$.

Problem 97. Let f_{11}, f_{22}, f_{33}, f_{44} be analytic functions $f_{jj} : \mathbb{R} \to \mathbb{R}$. Consider the 4×4 matrix

$$M = \begin{pmatrix} f_{11} & 0 & 0 & 0 \\ 1 & f_{22} & 0 & 1 \\ 0 & 0 & f_{33} & 1 \\ f'_{11} & f'_{22} & f'_{33} & 0 \end{pmatrix}$$

where $'$ denotes differentiation with respect to x. Find the determinant of the matrix and write down the ordinary differential equation which follows from $\det(M) = 0$. Find solutions of the differential equation.

Problem 98. Let $c > 0$. Consider the autonomous system of first order differential equations

$$\begin{aligned} \frac{du_1}{dt} &= \frac{1}{2}u_2(u_3 - c) \\ \frac{du_2}{dt} &= -\frac{1}{2}u_1(u_3 - c) - \frac{1}{2}v_3 \\ \frac{du_3}{dt} &= v_2 \\ \frac{dv_1}{dt} &= v_2 u_3 - v_3 u_2 \\ \frac{dv_2}{dt} &= v_3 u_1 - v_1 u_3 \\ \frac{dv_3}{dt} &= v_1 u_2 - v_2 u_1. \end{aligned}$$

Show that this system admits the first integrals

$$\begin{aligned} H &= u_1^2 + u_2^2 + \frac{1}{2}u_3^2 - v_1 \\ L &= u_1 v_1 + u_2 v_2 + \frac{1}{2}(u_3 + c)v_3 \\ K &= (u_1^2 - u_2^2 + v_1)^2 + (2u_1 u_2 + v_2)^2 + 2c((u_3 - c)(u_1^2 + u_2^2) + 2u_1 v_3). \end{aligned}$$

Problem 99. Solve the initial value problem of the system of ordinary differential equations

$$\frac{du_1}{dt} = u_2 u_4 - u_1 u_3$$

$$\frac{du_2}{dt} = -\frac{du_1}{dt}$$
$$\frac{du_3}{dt} = \frac{du_1}{dt}$$
$$\frac{du_4}{dt} = -\frac{du_1}{dt}$$

with $0 < u_j(t) < 1$ ($j = 1, 2, 3, 4$) and

$$\sum_{j=1}^{4} u_j(t) = 1.$$

Problem 100. Study the coupled limit cycle system

$$\begin{aligned}
\frac{du_1}{dt} &= u_1(r - u_1^2 - u_2^2) - u_2 + s(u_3 - u_1) \\
\frac{du_2}{dt} &= u_2(r - u_1^2 - u_2^2) + u_1 + s(u_4 - u_2) \\
\frac{du_3}{dt} &= u_3(r - u_3^2 - u_4^2) - u_4 + s(u_5 - u_3) \\
\frac{du_4}{dt} &= u_4(r - u_3^2 - u_4^2) + u_3 + s(u_6 - u_4) \\
\frac{du_5}{dt} &= u_5(r - u_5^2 - u_6^2) - u_6 + s(u_1 - u_5) \\
\frac{du_6}{dt} &= u_6(r - u_5^2 - u_6^2) + u_5 + s(u_2 - u_6).
\end{aligned}$$

with $r = 1$ and $s = 2$. Obviously $(u_1, u_2, u_3, u_4, u_5, u_6) = (0, 0, 0, 0, 0, 0)$ is a fixed point. Study the stability of the fixed point. Does Hopf bifurcation occur? Now consider s as a bifurcation parameter. Does the system admit limit cycles?

Problem 101. Consider the system of second order ordinary differential equations

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{\alpha}{r^2}\frac{d\mathbf{r}}{dt} + \frac{\mu}{r^3}\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(i) Find

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \frac{\alpha}{r^2}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right).$$

(ii) Let $\mathbf{L} := \mathbf{r} \times d\mathbf{r}/dt$ (angular momentum). Find the time evolution of $\mathbf{L}$.
(iii) Find $d\mathbf{L}/dt \times \mathbf{L}$.

Problem 102. The *coagulation equation* for the concentrations of ℓ-mers $u_\ell(t)$, $\ell = 1, 2, 3, \ldots$, $t \geq 0$

$$\frac{du_\ell}{dt} = \frac{1}{2} \sum_{j+k=\ell} jk u_j u_k - \ell u_\ell \sum_{k=1}^{\infty} k u_k$$

together with the initial condition $u_\ell(t=0) = \delta_{1,\ell}$. Show that the solution of the initial value problem is given by

$$u_\ell(t) = \frac{\ell^{\ell-2}}{\ell!} t^{\ell-1} \exp(-\ell t)$$

for $0 \leq t < 1$.
(ii) Show that for $t \in [0,1)$ the quantity $\sum_{\ell=1}^{\infty} \ell u_\ell$ is conserved.
(iii) Show that at $t = 1$ a singularity occurs where the second moment diverge for $t > 1$.

Problem 103. Solve the initial value problem for the system of differential equations

$$\frac{d^2u_1}{dt^2} = -\frac{u_1}{(u_1^2 + u_2^2)^{3/2}}, \qquad \frac{d^2u_2}{dt^2} = -\frac{u_2}{(u_1^2 + u_2^2)^{3/2}}$$

with

$$u_1(0) = 1/2, \;\; u_2(0) = 0, \qquad du_1(0)/dt = 0, \;\; du_2(0)/dt = \sqrt{3}.$$

Problem 104. Let c_1, c_2, c_3 be real constants. Solve the initial value problem of the autonomoua system of first order ordinary differential equations

$$\frac{du_1}{dt} = c_1 + c_2 u_1 - c_3 u_1^2, \quad \frac{du_2}{dt} = c_2 - 2c_3 u_1, \quad \frac{du_3}{dt} = c_3 \exp(u_2)$$

with $u_1(0) = u_2(0) = u_3(0) = 0$.

Problem 105. Study the initial value problem for the two coupled oscillator

$$\frac{d^2u_1}{dt^2} = -k_1 u_1 - k_2(u_1 - u_2), \quad \frac{d^2u_2}{dt^2} = -k_1 u_2 + k_2(u_1 - u_2).$$

Problem 106. Consider the autonomous system of first order differential equations

$$\frac{du_1}{dt} = u_1(u_1 - u_2 - u_3), \quad \frac{du_2}{dt} = u_2(u_2 - u_3 - u_1), \quad \frac{du_3}{dt} = u_3(u_3 - u_1 - u_2).$$

Introduce the new variables

$$v_1 := u_1 + u_2 + u_3, \quad v_2 := u_1u_2 + u_2u_3 + u_3u_1, \quad v_3 := u_1u_2u_3$$

and express the system in these variables.

Problem 107. Consider the Darboux-Halphen system

$$\frac{dx_1}{dt} = x_2x_3 - x_1x_2 - x_3x_1, \quad \frac{dx_2}{dt} = x_3x_1 - x_1x_2 - x_2x_3, \quad \frac{dx_3}{dt} = x_1x_2 - x_3x_1 - x_2x_3$$

with the corresponding vector field

$$V = (x_2x_3 - x_1x_2 - x_3x_1)\frac{\partial}{\partial x_1} + (x_3x_1 - x_1x_2 - x_2x_3)\frac{\partial}{\partial x_2} + (x_1x_2 - x_3x_1 - x_2x_3)\frac{\partial}{\partial x_3}.$$

(i) Is the autonomous system of differential equations invariant under the transformation ($\alpha\delta - \beta\gamma \neq 0$)

$$(t, x_j) \mapsto \left(\frac{\alpha t + \beta}{\gamma t + \delta}, 2\gamma\frac{\gamma t + \delta}{\alpha\delta - \gamma\beta} + \frac{(\gamma t + \delta)^2}{\alpha\delta - \gamma\beta}x_j\right)$$

with $j = 1, 2, 3$.
(ii) Consider the vector fields

$$U = 2(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}), \qquad W = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

Find the commutators $[U, V]$, $U, W]$, $[V, W]$. Do we have basis of a Lie algebra? Discuss.

Problem 108. Consider the system of second order differential equations

$$m_1\frac{d^2u_1}{dt^2} = f(u_2 - u_1), \quad m_2\frac{d^2u_2}{dt^2} = -f(u_2 - u_1)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.
(i) Let $v = u_2 - u_1$. Show that

$$m\frac{d^2v}{dt^2} = f(v)$$

where $1/m = 1/m_1 + 1/m_2$.
(ii) Show that

$$\frac{d}{dt}\left(m_1\frac{du_1}{dt} + m_2\frac{du_2}{dt}\right) = 0.$$

Problem 109. Solve the initial value problem for the infinite system of linear equations

$$\frac{dC_1}{dt} = -C_1 - N, \quad \frac{dN}{dt} = -N$$

$$\frac{dC_k}{dt} = C_{k-1} - C_k, \quad k \geq 2$$

with $C_k(t=0) = \delta_{1k}$.

Problem 110. Let $\mathbf{C}$ be a fixed nonzero vector in $\mathbb{R}^3$. Solve the initial value problem of the first order autonomous system of differential equations in $\mathbb{R}^3$

$$\frac{d\mathbf{u}}{dt} = \mathbf{u} \times \mathbf{C}$$

where $\times$ denotes the vector product, i.e.

$$\begin{pmatrix} du_1/dt \\ du_2/dt \\ du_3/dt \end{pmatrix} = \begin{pmatrix} u_2c_3 - u_3c_2 \\ u_3c_1 - u_1c_3 \\ u_1c_2 - u_2c_1 \end{pmatrix}$$

with the vector field

$$V = (u_2c_3 - u_3c_2)\frac{\partial}{\partial u_1} + (u_3c_1 - u_1c_3)\frac{\partial}{\partial u_2} + (u_1c_2 - u_2c_1)\frac{\partial}{\partial u_3}.$$

First find the fixed points and study their stability. The bifurcation parameters are c_1, c_2, c_3. Find the first integeral of the dynamical system if there are any.

Problem 111. Let

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

where we assume that $\mathbf{C}$ is nonzero vector. We consider the autonomous system of first differential equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{u} \times (\mathbf{u} \times \mathbf{C})$$

or written in components

$$\begin{pmatrix} du_1/dt \\ du_2/dt \\ du_3/dt \end{pmatrix} = \begin{pmatrix} c_1(-u_2^2 - u_3^2) + c_2u_1u_2 + c_3u_1u_3 \\ c_1u_1u_2 + c_2(-u_1^2 - u_3^2) + c_3u_2u_3 \\ c_1u_1u_3 + c_2u_2u_3 + c_3(-u_1^2 - u_2^2) \end{pmatrix}.$$

Thus the corresponding vector field is given by

$$V = (c_1(-u_2^2 - u_3^2) + c_2u_1u_2 + c_3u_1u_3)\frac{\partial}{\partial u_1}$$
$$+(c_1u_1u_2 + c_2(-u_1^2 - u_3^2) + c_3u_2u_3)\frac{\partial}{\partial u_2} + (c_1u_1u_3 + c_2u_2u_3 + c_3(-u_1^2 - u_2^2))\frac{\partial}{\partial u_3}.$$

(i) Apply the Jacobi identity to the right-hand side of the system of differential equations. Discuss.
(ii) The fixed points are the solutions of the equations

$$\mathbf{u} \times (\mathbf{u} \times \mathbf{C}) = \mathbf{0}.$$

Find the fixed points.
(iii) Write down the variational equation and then study the stability of the fixed points.
(iv) Let $L_V(.)$ denote the Lie derivative and $\Omega = du_1 \wedge du_2 \wedge du_3$. Calculate $L_V\Omega$ and thus find the divergence of the vector field.
(v) Let

$$\alpha = u_1du_2 + u_2du_3 + u_3du_1.$$

Find $L_V\alpha$. Discuss.
(vi) Consider the vector fields

$$S = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}.$$

Find the commutator $[V, S]$. Discuss.
(vii) Is $u_1^2 + u_2^2 + u_3^2$ a first integral?

Problem 112. Consider the system of first order ordinary differential equations

$$\frac{d^2u_1}{dx^2} = u_1u_2, \quad \frac{d^2u_2}{dx^2} = u_1u_2.$$

Show that a solution is given by

$$u_1(x) = u_2(x) = \frac{u_0}{(1 + (u_0/6)^{1/2}x)^2}$$

with $u_1(0) = u_2(0) = u_0$.

Problem 113. Solve the initial value problem of

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \mathbf{x}(t) \times \mathbf{y}(t) = \begin{pmatrix} x_2(t)y_3(t) - x_3(t)y_2(t) \\ x_3(t)y_1(t) - x_1(t)y_3(t) \\ x_1(t)y_2(t) - x_2(t)y_1(t) \end{pmatrix}$$

$$\begin{pmatrix} dy_1/dt \\ dy_2/dt \\ dy_3/dt \end{pmatrix} = \mathbf{y}(t) \times \mathbf{x}(t) = \begin{pmatrix} y_2(t)x_3(t) - y_3(t)x_2(t) \\ y_3(t)x_1(t) - y_1(t)x_3(t) \\ y_1(t)x_2(t) - y_2(t)x_1(t) \end{pmatrix}$$

First find the fixed points (if there are any) and study their stability. Find first integrals (if there are any).

Problem 114. Study the system of nonlinear differential equations

$$\frac{du_1}{dt} = (2 - \pi u_2)u_1, \qquad \frac{du_2^{-1}}{dt} = 2\pi^3 u_1^2.$$

This system plays a role for renormalization groups.

Problem 115. Solve the initial value problem of linear system of differential equations

$$\begin{pmatrix} du_1/dt \\ du_2/dt \\ du_3/dt \\ du_4/dt \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix}.$$

Problem 116. Let $k = 1, 2, \ldots$. Show that the function

$$f_k(z) = \sum_{n=1}^{\infty} \frac{z^{kn}}{(n!)^k}$$

satisfies a linear differential equation of order k with coefficients in $\mathbb{C}(z)$.

Problem 117. Show that the third order ordinary linear differential equation

$$\frac{d^3u}{dt^3} = 0$$

admits the seven Lie symmetries

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial u}, \quad t\frac{\partial}{\partial t}, \quad t\frac{\partial}{\partial u}$$
$$t^2\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial u}, \quad ut\frac{\partial}{\partial u} + \frac{1}{2}t^2\frac{\partial}{\partial t}.$$

Find the commutators.

Problem 118. Consider the coupled system of second order ordinary differential equations

$$\frac{d^2x_1}{dt^2} = -x_1 + (\epsilon - x_1^2 - \alpha x_2^2)\frac{dx_1}{dt}$$
$$\frac{d^2x_2}{dt^2} = -x_2 + (\epsilon - x_2^2 - \alpha x_1^2)\frac{dx_2}{dt}.$$

Show that if the bifurcation parameter ϵ varies through 0, there is a *Hopf bifurcation* with D_4 symmetry. The D_4 symmetry consists of four rotations each of $\pi/2$ and four reflection mirrors with each angle betweem them being $\pi/4$.

Chapter 5

Elliptic Functions and Differential Equations

The *Jacobi elliptic functions* can be defined as inverse of the *elliptic integral* of first kind. Thus, if we write

$$x(\phi, k) = \int_0^\infty \frac{dx}{\sqrt{1 - k^2 \sin^2 s}}$$

where $k \in [0, 1]$ we can define the following functions

$$\mathrm{sn}(x, k) := \sin(\phi), \qquad \mathrm{cn}(x, k) := \cos(\phi), \qquad \mathrm{dn}(x, k) := \sqrt{1 - k^2 \sin^2 \phi}.$$

Here k is called the modulus. For $k = 0$ we obtain

$$\mathrm{sn}(x, 0) \equiv \sin(x), \qquad \mathrm{cn}(x, 0) \equiv \cos(x), \qquad \mathrm{dn}(x, k) \equiv 1$$

and for $k = 1$ we have

$$\mathrm{sn}(x, 1) \equiv \tanh(x), \qquad \mathrm{cn}(x, 1) \equiv \mathrm{dn}(x, 1) \equiv \frac{2}{e^x + e^{-x}}.$$

We have the following identities

$$\mathrm{sn}(x, k) \equiv \frac{2\mathrm{sn}(x/2, k)\mathrm{cn}(x/2, k)\mathrm{dn}(x/2, k)}{1 - k^2\mathrm{sn}^4(x/2, k)}$$

$$\mathrm{cn}(x, k) \equiv \frac{1 - 2\mathrm{sn}^2(x/2, k) + k^2\mathrm{sn}^4(x/2, k)}{1 - k^2\mathrm{sn}^4(x/2, k)}$$

$$\mathrm{dn}(x, k) \equiv \frac{1 - 2k^2\mathrm{sn}^2(x/2, k) + k^2\mathrm{sn}^4(x/2, k)}{1 - k^2\mathrm{sn}^4(x/2, k)}.$$

The expansions of the Jacobi elliptic functions in powers of x up to order 4 are given by

$$\begin{aligned}
\mathrm{sn}(x,k) &= x - (1+k^2)\frac{x^3}{3!} + \cdots \\
\mathrm{cn}(x,k) &= 1 - \frac{x^2}{2!} + (4k^2+k^2)\frac{x^4}{4!} - \cdots \\
\mathrm{dn}(x,k) &= 1 - k^2\frac{x^2}{2!} + (4k^2+k^2)\frac{x^4}{4!} - \cdots
\end{aligned}$$

For x sufficiently small this will be a good approximation.

Problem 1. Consider the equations of motion

$$I_1\frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3, \quad I_2\frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1, \quad I_3\frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2. \tag{1}$$

We assume that all the principle moments of inertia I_k have different values.
(i) Show that (1) admits the constant of motion

$$l^2 := I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2, \qquad 2E := I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2. \tag{2}$$

(ii) Show that we can to eliminate two of the variables from (1) to obtain an equation for the third alone.
(iii) Find the solution of this equation.

Problem 2. Consider the equation

$$\left(\frac{dr}{dx}\right)^2 = Ar^4 + Br^2 + C + Dr^{-2} \tag{1}$$

where A, B, C and D are given scalar constants. Show that the equation can be reduced to the standard form

$$\left(\frac{dy}{dt}\right)^2 = \mu^2(1-y^2)(1-k^2y^2) \tag{2}$$

by the change of variables

$$r^2 := ay^2 + b, \qquad y := \mathrm{sn}(\mu x, k) \tag{3}$$

and a, b, μ, are constant.

Problem 3. Establish the derivatives

$$\frac{d}{dx}\mathrm{sn}(x,k) = \mathrm{cn}(x,k)\mathrm{dn}(x,k), \quad \frac{d}{dx}\mathrm{cn}(x,k) = -\mathrm{sn}(x,k)\mathrm{dn}(x,k), \quad \frac{d}{dx}\mathrm{dn}(x,k) = -k^2\mathrm{sn}(x,k)\mathrm{cn}(x,k). \tag{1}$$

Problem 4. Show that $y(t) = \mathrm{cn}(\mu t, k)$ and $r(t) = \mathrm{dn}(\mu t, k)$ are solutions of the differential equations

$$\left(\frac{dy}{dt}\right)^2 = \mu^2(1-y^2)(k'^2 + k^2 y) \tag{1}$$

$$\left(\frac{dr}{dt}\right)^2 = \mu^2(1-r^2)(r^2 - k'^2) \tag{2}$$

where $k'^2 := 1 - k^2$.

Problem 5. Find a change of variables that transforms

$$\left(\frac{dw}{dx}\right)^2 = Aw + Bw^2 + Cw^3 + Dw^4 \tag{1}$$

into an equation of the form

$$\left(\frac{dr}{dx}\right)^2 = A'r^4 + B'r^2 + C' + D'r^{-2}. \tag{2}$$

Problem 6. Show that

$$\mathrm{dn}(z+y)\mathrm{dn}(z-y) = \frac{\mathrm{dn}^2 y - k^3 \mathrm{cn}^2 y \mathrm{sn}^2 z}{1 - k^2 \mathrm{sn}^2 y \mathrm{sn}^2 z}.$$

Problem 7. Prove that

$$1 + \mathrm{dn} 2z = \frac{2\mathrm{dn}^2 z}{1 - k^2 \mathrm{sn}^4 z}. \tag{1}$$

Problem 8. Prove that

$$\mathrm{sn}(z+y) = \frac{\mathrm{sn} z \mathrm{cn} y \mathrm{dn} y + \mathrm{sn} y \mathrm{cn} z \mathrm{dn} z}{1 - k^2 \mathrm{sn}^2 z \mathrm{sn}^2 y}. \tag{1}$$

Problem 9. Show that

$$\mathrm{sn}(z + iK') = \frac{1}{k\mathrm{sn} z} = \frac{1}{kz}\left(1 - \frac{1}{6}(1+k^2)z^2 + \cdots\right)^{-1} = \frac{1}{kz} + \frac{1+k^2}{6k}z + \cdots$$

and similarly

$$\mathrm{cn}(z + iK') = \frac{-i}{kz} + \frac{2k^2-1}{6k} iz + \cdots$$

and

$$\operatorname{dn}(z+iK') = -\frac{i}{z} + \frac{2-k^2}{6} iz + \cdots$$

It follows that at the point $z = iK'$; the functions $\operatorname{sn}z$, $\operatorname{cn}z$, $\operatorname{dn}z$ have simple poles, with the residues

$$\frac{1}{k}, \qquad -\frac{i}{k}, \qquad -i$$

respectively.

Problem 10. Directed by the expansion

$$\frac{1}{\sin^2 z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n\pi)^2} \tag{1}$$

Weistrass defined a new function (*elliptic function of Weierstrass*)

$$\wp(z) = \frac{1}{z^2} + {\sum_{m,n}}' \left(\frac{1}{(z-2m\omega-2n\omega')^2} - \frac{1}{(2m\omega+2n\omega')^2} \right) \tag{2}$$

where the summation is taken over all positive and negative integral values of m and n, including zero, except when m and n are simultaneously zero. ω and ω' are two numbers the ration of which is nor real. Note that $\operatorname{cosec}(\alpha) = 1/\sin(\alpha)$. Prove that

$$\wp(z) = C + \left(\frac{\pi}{2\omega_1}\right)^2 \operatorname{cosec}^2 \left(\frac{z-2n\omega_2}{2\omega_1}\pi\right) \tag{3}$$

where

$$C := -\left(\frac{\pi}{2\omega_1}\right)^2 \left(\frac{1}{3} + \sum_{n=-\infty}^{\infty} \operatorname{cosec}^2 \frac{2n\omega_2}{\omega_1}\pi\right). \tag{4}$$

(i) Show that the function $\wp(z)$ is an even function of z, i.e. it satisfies the equation

$$\wp(z) = \wp(-z). \tag{5}$$

(ii) Show that

$$\wp(z+2\omega_1) = \wp(z). \tag{6}$$

(iii) Show that

$$\wp(z+2\omega_2) = \wp(z) \tag{7}$$

and generally

$$\wp(z+2m\omega_1+2n\omega_2) = \wp(z)$$

where m and n are any integers. Therefore the function $\wp(z)$ admits the two periods $2\omega_1$ and $2\omega_2$.

Problem 11. Show that the function $\wp(z)$ satisfies the differential equation

$$\left(\frac{d\wp}{dz}\right)^2 = 4\wp^3(z) - g_2\wp(z) - g_3 \tag{1}$$

where g_2 and g_3 (called the invariants) are given in terms of the periods of $\wp(z)$ by the equations

$$g_2 = 60\sum(2m\omega_1 + 2n\omega_2)^{-4}, \qquad g_3 = 140\sum(2n\omega_1 + 2n\omega_2)^{-6}. \tag{2}$$

Show that this differential equation can be written in the form

$$\left(\frac{dt}{dz}\right)^2 = 4t^3 - g_2 t - g_3 \tag{3}$$

where

$$t = \wp(z) \tag{4}$$

and therefore (since $\wp(z)$ is infinite when z is zero) we have

$$z = \int_{\wp(z)}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt \tag{5}$$

which is the required expression of $\wp(z)$ in terms of an integral.

Problem 12. If $y = \wp(z)$, show that

$$-\frac{1}{2}\frac{\frac{d^3y}{dz^3}}{\left(\frac{dy}{dz}\right)^3} + \frac{4}{3}\frac{\left(\frac{d^2y}{dz^2}\right)^2}{\left(\frac{dy}{dz}\right)^4} = \frac{3}{16}((y-e_1)^{-2} + (y-e_2)^{-2} + (y-e_3)^{-2})$$

$$-\frac{3}{8}y(y-e_1)^{-1}(y-e_2)^{-1}(y-e_3)^{-1} \tag{1}$$

where e_1, e_2, e_3 are the roots of the equation

$$4y^3 - g_2 y - g_3 = 0. \tag{2}$$

Problem 13. Consider the differential equation

$$\begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(z) & \wp'(z) \\ 1 & \wp(y) & \wp'(y) \end{vmatrix} = 0. \tag{1}$$

Show that

$$\wp(z+y) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right)^2 - \wp(z) - \wp(y). \tag{2}$$

Problem 14. Integrate

$$(ax^4 + 4bx^3 + 6cx^2 + 4dx + e)^{-\frac{1}{2}}. \quad (1)$$

Problem 15. Solve the the integration-problem discussed in the previous problem.

Problem 16. Show that the theorem given above may be stated somewhat differently as follows.

Problem 17. The function $\zeta(z)$ is defined by the differential equation

$$\frac{d\zeta(z)}{dz} = -\wp(z) \quad (1)$$

with the condition that $\zeta(z) - z^{-1}$ be equal to zero when $z = 0$. Since the infinite series which represents $\wp(z)$ is uniformly convergent, it can be integrated term by term. Show that

$$\zeta(z) = -\int (z^{-2} + \sum((z - 2m\omega_1 - 2n\omega_2)^{-2} - (2m\omega_1 + 2n\omega_2)^{-2}))dz. \quad (2)$$

The summation is extended over all positive and negative integer and zero values of m and n, except simultaneous zero values.
(ii) Show that

$$\zeta(z) = z^{-1} + \sum((z - 2m\omega_1 - 2n\omega_2)^{-1} + (2m\omega_1 + 2n\omega_2)^{-1} + z(2m\omega_1 + 2n\omega_2)^{-2}). \quad (3)$$

Hint: Since the condition, which $\zeta(z)$ has to satisfy at $z = 0$, is satisfied by this choice of the constant of integration.

Problem 18. If $z + y + z = 0$, show that

$$(\zeta(z) + \zeta(y) + \zeta(z))^2 + \zeta'(x) + \zeta'(y) + \zeta'(z) = 0. \quad (1)$$

Problem 19. The function

$$\sigma(z; \omega_1, \omega_2) = z \prod_k{}' \left(1 - \frac{z}{\Omega_k}\right) \exp\left[\frac{z}{\Omega_k} + \frac{1}{2}\left(\frac{z}{\Omega_k}\right)^2\right] \quad (1)$$

with

$$\Omega_k = m_k\omega_1 + n_k\omega_2 \quad (2)$$

where m_k, n_k is the sequence of all pairs of integers. The prime after the product sign indicates that the pair $(0,0)$ should be omitted. ω_1 and ω_2 are two complex numbers with

$$\Im\left(\frac{\omega_1}{\omega_2}\right) \neq 0. \tag{3}$$

In the following the dependence on ω_1 and ω_2 will be omitted. The logarithmic derivative $\sigma'(z)/\sigma(z)$ where $\sigma'(z) \equiv d\sigma/dz$ is the meromorphic function $\zeta(z)$ of Weierstrass. The function $\wp(z) = -\zeta'(z)$ is a meromorhic, doubly periodic (or elliptic) function with periods ω_1, ω_2 whose only singularities are double poles $m\omega_1 + n\omega_2$. We find

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + {\sum_k}' \left(\frac{1}{(z-\Omega_k)^2} - \frac{1}{(\Omega_k)^2}\right). \tag{4}$$

Remark: The function $\wp$ is called *Weierstrass' $\wp$ function.*
(i) Show that

$$\sigma(z) = z + c_5 z^5 + c_7 z^7 + \cdots \tag{5}$$

(ii) Show that

$$\frac{\sigma(2u)}{\sigma^4(u)} = -\wp'(u), \qquad 2\zeta(2u) - 4\zeta(u) = \frac{\wp''(u)}{\wp'(u)}. \tag{6}$$

Problem 20. Let $a(q)$ be a meromorphic function. Let $r,\ k,\ l = 1,2,\ldots,N$. Assume that a satifies the following equation

$$\frac{a(q_k-q_r)a'(q_r-q_l) - a'(q_k-q_r)a(q_r-q_l)}{a(q_k-q_l)} = g(q_k-q_r) - g(q_r-q_l) \tag{1}$$

where a' denotes differentiation with respect to the arguments and $k \neq l$. Let

$$U(q) = a^2(q). \tag{2}$$

Show that

$$(U(x)U'(y) - U'(x)U(y)) + (U(y)U'(z) - U'(y)U(z)) + (U(z)U'(x) - U'(z)U(x)) = 0 \tag{4}$$

where

$$x + y + z = (q_k - q_r) + (q_r - q_l) + (q_l - q_k) = 0. \tag{5}$$

Problem 21. A large class of polynomials can be reduced to solving a differential equation of the standard form

$$\left(\frac{dy}{dx}\right)^2 = (1-y^2)(1-k^2y^2) \tag{1}$$

for $0 \le k \le 1$ and $-1 \le y \le 1$. The solution of this equation for the conditions

$$y(x=0)=0, \qquad \frac{dy(x=0)}{dx} > 0 \tag{2}$$

is denoted by

$$y = \mathrm{sn}(x,k). \tag{3}$$

This function depends on the value of the parameter k, which is called the *modulus*. Direct integration of (1) produces the inverse function

$$x = \mathrm{sn}^{-1} y = \int_0^y \frac{dy}{((1-y^2)(1-k^2y^2))^{1/2}}. \tag{4}$$

(i) Show that it is an odd function of y, which increases steadily from 0 to

$$K(k) := \int_0^1 \frac{dy}{((1-y^2)(1-k^2y^2))^{1/2}}. \tag{5}$$

as y increases from 0 to 1.
(ii) Show that $y = \mathrm{sn}(x,k)$ is an odd function of x, and it has period $4K$. This means

$$\mathrm{sn}(x+4K(k),k) = \mathrm{sn}(x,k). \tag{6}$$

The integral (5) is called a *complete elliptic integral of the first kind*. The function $\mathrm{sn}(x,k)$ is called a *Jacobi elliptic function*.
(iii) Show that it can be evaluated by a change of variables and a series expansion

$$K(k) = \int_0^{\pi/2} \frac{du}{(1-k^2\sin^2 u)^{1/2}} = \int_0^{\pi/2} \left(1 + \frac{k^2}{2}\sin^2 u + \cdots\right) du$$

$$= \frac{\pi}{2}\left(1 + \sum_{n=1}^{\infty} \left(\frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}\right)^2 k^{2n}\right). \tag{7}$$

Remark: Two other Jacobi elliptic functions cn x and dn x can be defined by the equations

$$\mathrm{cn}^2(x,k) = 1 - \mathrm{sn}^2(x,k) \quad \mathrm{cn}(0,k)=1, \qquad \mathrm{dn}^2(x,k) = 1-k^2\mathrm{sn}^2(x,k) \quad \mathrm{dn}(0,k)=1. \tag{8}$$

Problem 22. Prove the following identity for the Jacobi elliptic functions

$$k^2\mathrm{sn}(u+a,k)\mathrm{cn}(u-v,k)\mathrm{sn}(v+a,k) = \mathrm{dn}(u-v,k) - \mathrm{dn}(u+a,k)\mathrm{dn}(v+a,k)$$
$$\mathrm{sn}(u+a,k)\mathrm{dn}(u-v,k)\mathrm{sn}(v+a,k) = \mathrm{cn}(u-v,k) - \mathrm{cn}(u+a,k)\mathrm{cn}(v+a,k).$$

Problem 23. Consider (*elliptic cylindrical coordinates*)

$$u_1(\alpha,\beta,\phi,k) = \mathrm{sn}(\alpha,k)\mathrm{dn}(\beta,k')\cos(\phi), \quad u_2(\alpha,\beta,\phi,k) = \mathrm{sn}(\alpha,k)\mathrm{dn}(\beta,k')\sin(\phi),$$

$$u_3(\alpha,\beta,k) = \mathrm{dn}(\alpha,k)\mathrm{sn}(\beta,k'), \qquad u_4(\alpha,\beta,k) = \mathrm{cn}(\alpha,k)\mathrm{cn}(\beta,k')$$

where k and $k' = \sqrt{1-k^2}$ are the modulus and complementary modulus, respectively. Show that

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1.$$

Problem 24. The Jacobi elliptic functions $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ with $k \in [0,1]$ and $k^2 + k'^2 = 1$ have the properties

$$\mathrm{sn}(x,0) = \sin(x), \quad \mathrm{cn}(x,0) = \cos(x), \quad \mathrm{dn}(x,0) = 1$$

and

$$\mathrm{sn}(x,1) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \mathrm{cn}(x,1) = \frac{2}{e^x + e^{-x}} = \mathrm{dn}(x,1).$$

We define

$$\begin{aligned} u_1(x,y,k,k') &= \mathrm{sn}(x,k)\mathrm{dn}(y,k') \\ u_2(x,y,k,k') &= \mathrm{cn}(x,k)\mathrm{cn}(y,k') \\ u_3(x,y,k,k') &= \mathrm{dn}(x,k)\mathrm{sn}(y,k'). \end{aligned}$$

(i) Find $u_1(x,y,0,1)$, $u_2(x,y,0,1)$, $u_3(x,y,0,1)$ and calculate $u_1^2(x,y,0,1) + u_2^2(x,y,0,1) + u_3^2(x,y,0,1)$.
(ii) Find $u_1(x,y,1,0)$, $u_2(x,y,1,0)$, $u_3(x,y,1,0)$ and calculate $u_1^2(x,y,1,0) + u_2^2(x,y,1,0) + u_3^2(x,y,1,0)$.

Chapter 6

Nonautonomous Systems

Problem 1. Consider the driven *van der Pol equation*

$$\frac{d^2u}{dt^2} + a\frac{du}{dt}\left(u^2 - 1\right) + u = b\cos(\omega t) \tag{1}$$

where $a \neq 0$. Extend the equation into the complex domain and perfom a singular point analysis. Show that all of its solutions posses only square-root singularities in the complex time plane.

Problem 2. Consider the differential equation

$$\frac{d\mathbf{u}}{dt} = P(t)\mathbf{u}$$

where $P(t)$ is periodic with principal period T and differentiable. Thus T is the smallest positive number for which $P(t+T) = P(t)$ and $-\infty < t < \infty$. Can we conclude that all solutions are periodic? For example, consider

$$\frac{du}{dt} = (1 + \sin t)u.$$

Problem 3. Solve the initial value problem for the system of linear differential equations

$$\frac{dc_0}{dt} = \frac{1}{2}i\Omega e^{-i\phi}e^{i(\omega-\nu)t}c_1$$
$$\frac{dc_1}{dt} = \frac{1}{2}i\Omega e^{i\phi}e^{-i(\omega-\nu)t}c_0.$$

where Ω, ω, ν are constant frequencies. Note the the system depends explicitly on the time t. Then study the special case $\omega = \nu$.

Problem 4. Consider the differential equations

$$\frac{d^2q}{dt^2} + \omega^2(t)q = 0 \tag{1}$$

and

$$\frac{d^2\rho}{dt^2} + \omega^2(t)\rho = \frac{1}{\rho^3}. \tag{2}$$

Show that under the invertible point transformation

$$Q(T(t)) = \frac{q(t)}{\rho(t)}, \qquad T(t) = \int^t \frac{1}{\rho^2(s)} ds \tag{3}$$

(1) takes the form

$$\frac{d^2Q}{dT^2} + Q = 0 \tag{4}$$

where ρ satisfies (2). We have

$$\frac{dQ}{dt} = \frac{dQ}{dT}\frac{dT}{dt} = \frac{1}{\rho}\frac{dq}{dt} - q\frac{1}{\rho^2}\frac{d\rho}{dt} \tag{5a}$$

and

$$\frac{dT}{dt} = \frac{1}{\rho^2}. \tag{5b}$$

Thus

$$\frac{dQ}{dT}\frac{1}{\rho^2} = \frac{1}{\rho}\frac{dq}{dt} - \frac{q}{\rho^2}\frac{d\rho}{dt}. \tag{6}$$

Problem 5. A system of differential equations describing the forced negative-resistance oscillator is given by

$$L\frac{di(t)}{dt} + Ri(t) + v(t) = E\cos(\omega t)$$

$$i_1(t) = C\frac{dv(t)}{dt}, \qquad i(t) = i_1(t) + i_2(t)$$

where the voltage-current characteristic is given by

$$i_2(t) = f(v(t)) \equiv -Sv(t)\left(1 - \frac{v^2(t)}{V_s^2}\right)$$

with $S = 1/R$ and V_s constants. R is the resistance of the inductor L. Write these equations in dimensionless form using the MKSA-system.

Hint. Note that $1V = 1m^2s^{-3}kgA^{-1}$ and we have the following dimensions

$$[R] = 1VA^{-1} = 1\text{Ohm}, \quad [C] = 1AsV^{-1}, \quad [L] = 1VsA^{-1} = 1\text{Henry}$$

$$[v] = [V_s] = [E] = 1V, \quad [i] = 1A, \quad [t] = 1s, \quad [\omega] = 1s^{-1}.$$

Problem 6. Consider the driven anharmonic system

$$\frac{d^2u}{d\tau^2} + k\frac{du}{d\tau} - au + bu^3 = A\cos(\bar{\omega}\tau)$$

where $k, a, b > 0$. Find the differential equation under the transformation

$$t(\tau) = \tau\sqrt{2a}, \qquad x(t(\tau)) = u(\tau)\sqrt{\frac{b}{a}}.$$

Problem 7. The driven *Morse oscillator* is given by

$$\frac{d^2u}{dt^2} + \alpha\frac{du}{dt} + \beta\exp(-u)(1-\exp(-u)) = k\cos(\omega t).$$

Find the driven Morse oscillator under the transformation

$$t(\tau) = \tau, \qquad v(\tau) = \exp(-u(t(\tau))).$$

Problem 8. Find solutions of the linear system of non-autonomous first order differential equations

$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Problem 9. Study *Hill's equation*

$$\frac{d^2u}{dt^2} = -f(t)u, \quad f(t+T) = f(t)$$

where

$$f(t) := \begin{cases} \omega^2 + \epsilon, & 0 \le t \le \pi \\ \omega^2 - \epsilon, & \pi \le t < 2\pi \end{cases}$$

and $f(t+2\pi) = f(t)$. Derive the shape of the *Arnold tongues* for $0 < \epsilon \ll 1$.

Problem 10. Study the initial value problem of the second order differential equation

$$\frac{d^2u}{dt^2} + k\frac{du}{dt} + \omega^2 u = C_1\cos(\omega t) + C_2\sin(\omega t)$$

with $u(0) > 0$ and $du(0)/dt > 0$.

Chapter 7

Hamilton Systems

Consider the Hamilton function

$$H(\mathbf{p},\mathbf{q}) = \sum_{k=1}^{N} \frac{p_k^2}{2m_k} + V(\mathbf{q}) \tag{1}$$

The first term of the right hand side is the kinetic part of the Hamilton function and the second term is the potential part. Then the *Hamilton equations of motion* are given by

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \qquad \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}. \tag{2}$$

From (1) and (2) we find

$$\frac{d^2\mathbf{q}}{dt^2} = -\frac{\partial V}{\partial \mathbf{q}}. \tag{3}$$

A function $I(\mathbf{p}(t),\mathbf{q}(t),t)$ is called a first integral if

$$\frac{dI}{dt} \equiv \sum_{j=1}^{N} \left(\frac{\partial I}{\partial p_j}\frac{dp_j}{dt} + \frac{\partial I}{\partial q_j}\frac{dq_j}{dt} \right) + \frac{\partial I}{\partial t} = 0. \tag{4}$$

Inserting (2) into (4) yields

$$\sum_{j=1}^{N} \left(\frac{\partial I}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial I}{\partial p_j}\frac{\partial H}{p q_j} \right) + \frac{\partial I}{\partial t} = 0. \tag{5}$$

The *Poisson bracket* is defined as

$$\{A(\mathbf{p},\mathbf{q}),B(\mathbf{p},\mathbf{q})\} := \sum_{k=1}^{N}\left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k}\right). \tag{6}$$

Definition. Two first integrals I_1 and I_2 are called in *involution* if

$$\{I_1, I_2\} = 0.$$

Definition. A Hamilton system is called integrable if there are n first integrals.

Problem 1. Consider the *Toda lattice* with cyclic boundary conditions and two equal particles. The Hamilton function for this system can be written as

$$H(\mathbf{p},\mathbf{q}) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}V_0(e^{a(q_1-q_2)} + e^{a(q_2-q_1)} - 2) \tag{1}$$

where V_0 and a are constants which fix the scale of the potential. Consider a canonical transformation to center-of-mass and relative coordinates

$$P := p_1 + p_2, \qquad p := \frac{1}{2}(p_1 - p_2) \tag{2a}$$

$$Q := \frac{1}{2}(q_1 + q_2), \qquad q := q_1 - q_2. \tag{2b}$$

Assume that $P = 0$. Find the Hamilton function for this coordinate system.

Problem 2. Given the Hamilton function

$$H(p_\rho, p_\zeta, \rho, \zeta) = \frac{1}{2}p_\rho^2 + \frac{1}{2}p_\zeta^2 + V(\rho,\zeta) \tag{1}$$

with

$$V(\rho,\zeta) = \frac{1}{2}\rho^2 + \frac{1}{2}\lambda^2\zeta^2 + \frac{\nu^2}{2\rho^2} + \frac{1}{(\rho^2+\zeta^2)^{1/2}} \tag{2}$$

and

$$p_\rho := \frac{d\rho}{dt}, \qquad \rho_\zeta := \frac{d\zeta}{dt}. \tag{3}$$

This Hamilton function describes the relative motion of two charged particles in a *Paul trap* in the pseudo potential approximation with λ and ν related to the asymmetry of the time average trapping potential and the relative angular momentum, respectively. It can be interpreted as the Hamilton function of a single particle moving in two dimensions, ρ and ζ, respectively.

(i) Show that the equations of motion derived from (1) are

$$\frac{d^2\rho}{dt^2} = \frac{\nu^2}{\rho^3} - \rho + \frac{\rho}{(\rho^2+\zeta^2)^{3/2}}, \tag{4}$$

$$\frac{d^2\zeta}{dt^2} = -\lambda^2\zeta + \frac{\zeta}{(\rho^2+\zeta^2)^{3/2}}. \tag{5}$$

(ii) The Hamilton function H is conservative and autonomous. Therefore, E, the total energy of the system, is a constant of the motion. Show that for arbitrary ν two further integrals of the motion exist for $\lambda = 2$ and $\lambda = \frac{1}{2}$, respectively.
(iii) Show that the first integral F, which applies for the case $\lambda = 2$, is given by

$$F\left(\rho, \frac{d\rho}{dt}, \zeta, \frac{d\zeta}{dt}, \nu\right) = \zeta\left(\frac{d\rho}{dt}\right)^2 - \rho\frac{d\zeta}{dt}\frac{d\rho}{dt} + \frac{\zeta}{(\rho^2+\zeta^2)^{1/2}} - \rho^2\zeta + \frac{\nu^2\zeta}{\rho^2}. \tag{6}$$

(iv) Show that for $\lambda = \frac{1}{2}$ a first integral is given by

$$G\left(\rho, \frac{d\rho}{dt}, \zeta, \frac{d\zeta}{dt}, \nu\right) = I_\rho^2 + I_\phi^2 + \nu^2(\rho^2+\zeta^2) \tag{7}$$

where

$$I_\rho := \frac{\nu^2}{\rho} + \rho\left(\frac{d\zeta}{dt}\right)^2 - \zeta\frac{d\rho}{dt}\frac{d\zeta}{dt} + \frac{\rho}{(\rho^2+\zeta^2)^{1/2}} - \frac{1}{4}\zeta^2\rho \tag{8}$$

and

$$I_\phi := -\frac{\nu}{\rho}\left(\rho\frac{d\rho}{dt} + \zeta\frac{d\zeta}{dt}\right). \tag{9}$$

Problem 3. Let

$$\mathbf{r} := (x, y, z) \tag{1}$$

be a triplet of dynamical variables (canonical triplet) which span a three-dimensional phase space. This is a formal generalization of the conventional phase space spanned by a canonical pair (p, q). Next introduce two functions, H and G, of (x, y, z), which serve as a pair of Hamilton functions to determine the motion of points in phase space. We define the following Hamilton equations

$$\frac{dx}{dt} = \frac{\partial(H,G)}{\partial(y,z)} \qquad \frac{dy}{dt} = \frac{\partial(H,G)}{\partial(z,x)} \qquad \frac{dz}{dt} = \frac{\partial(H,G)}{\partial(x,y)} \tag{2}$$

or in vector notation

$$\frac{d\mathbf{r}}{dt} = \vec{\nabla}H \times \vec{\nabla}G. \tag{3}$$

(i) Show that for any function $F(x,y,z)$

$$\frac{dF}{dt} = \frac{\partial(F,H,G)}{\partial(x,y,z)} = \vec{\nabla}F \cdot (\vec{\nabla}H \times \vec{\nabla}G). \tag{4}$$

We may call the right-hand side of (2) a *generalized Poisson bracket*, to be denoted by $[F,H,G]$.

(ii) Show that the generalized Poisson bracket is antisymmetric under interchange of any pair of its components. (iii) Show that

$$\frac{dH}{dt} = 0, \qquad \frac{dG}{dt} = 0 \tag{5}$$

i.e., both H and G are constants of motion.
(iv) Show that the orbit of a system in phase space is thus determined as the intersection of two surfaces,

$$H(x,y,z) = C_1, \qquad G(x,y,z) = C_2 \tag{6}$$

where C_1 and C_2 are constants.
(v) Show that the velocity field $d\mathbf{r}/dt$ ($\mathbf{r} = (x,y,z)$) is divergenceless,

$$\vec{\nabla} \cdot (\vec{\nabla}H \times \vec{\nabla}G) \equiv 0 \tag{7}$$

and that this amounts to a Liouville theorem in the three dimensional phase space.

Problem 4. Consider the Hamilton function

$$H(p,q) = \frac{1}{2}(p^2 + e^{-2q}).$$

Find the Hamilton equations of motion and solve the initial value problem $p(0) = 0$, $q(0) = 0$.

Problem 5. The construction of integrable Hamilton systems can be extended as follows: In *Nambu mechanics* the phase space is spanned by an n-tuple of dynamical variables u_i ($i = 1, \ldots, n$). The equations of motion of the Nambu mechanics (i.e., the autonomous system of first order ordinary differential equations) is now constructed as follows: Let $I_k : \mathbb{R}^n \to \mathbb{R}$ ($k = 1, \ldots, n-1$) be smooth functions. Then

$$\frac{du_i}{dt} = \frac{\partial(u_i, I_1, \ldots, I_{n-1})}{\partial(u_1, u_2, \ldots, u_n)}, \tag{1}$$

where $\partial(u_1, I_1, \ldots, I_{n-1})/\partial(u_1, u_2, \ldots, u_n)$ denotes the Jacobian. Consequently, the equations of motion can also be written as (summation convention)

$$\frac{du_i}{dt} = \epsilon_{ijk\ldots\ell}\partial_j I_{1\ldots}\partial_\ell I_{n-1} \tag{2}$$

where $\epsilon_{ijk\ldots\ell}$ is the generalized Levi-Cevita symbol and $\partial_j \equiv \partial/\partial u_j$. Show that I_1, I_2, $\ldots$, I_{n-1} are first integrals of (1).

Problem 6. In the discussion of the modulational instability of a Langmuir condensate one finds the following set of equations

$$(i\frac{d}{dt} + \delta_1)A_1 = -\Gamma(|A_0|^2 A_1 + A_0^2 A_2^*) \tag{1a}$$

$$(i\frac{d}{dt} + \delta_2)A_2 = -\Gamma(|A_0|^2 A_2 + A_0^2 A_1^*) \tag{1b}$$

$$i\frac{d}{dt}A_0 = -\Gamma(A_0(|A_1|^2 + |A_2|^2) + 2A_0^* A_1 A_2) \tag{1c}$$

where A_0, A_1, A_2 are complex quantities and δ_1, δ_2, and Γ are real constants.
(i) Show that system (1) can be derived from a Hamilton function H given by

$$H(\mathbf{A}, \mathbf{A}^*) = \delta_1|A_1|^2 + \delta_2|A_2|^2 + \Gamma(|A_0|^2(|A_1|^2 + |A_2|^2) + A_0^2 A_1^* A_2^* + A_0^{*2} A_1 A_2) \tag{2}$$

using the *canonical equations*

$$i\frac{dA_j}{dt} = -\frac{\partial H}{\partial A_j^*}, \qquad j = 1, 2, 3 \tag{3}$$

where the A_j and A_j^* are the canonically conjugate variables.
(ii) Show that system (1) can also be derived from a Langragian L, where

$$L(\mathbf{A}, \dot{\mathbf{A}}) = \frac{1}{2}i\sum_{j=0}^{2}(A_j\dot{A}_j^* - A_j^*\dot{A}_j) - H. \tag{4}$$

(iii) As H does not contain the time explicitly, it is an integral of the motion. Show that L is invariant under the *gauge transformation*

$$A_j \to A_j \exp(i\phi). \tag{5}$$

(iv) We define

$$\{C_1, C_2\} := \sum_{j=0}^{2}\left(\frac{\partial C_1}{\partial A_j}\frac{\partial C_2}{\partial A_j^*} - \frac{\partial C_1}{\partial A_j^*}\frac{\partial C_2}{\partial A_j}\right). \tag{6}$$

Let

$$C_1 = \sum_{j=0}^{2} |A_j|^2, \qquad C_2 = |A_2|^2 - |A_1|^2. \tag{7}$$

Calculate $\{C_1, C_2\}$, $\{C_1, H\}$, $\{C_2, H\}$. Discuss.

Problem 7. We begin with the definition of the $SU(\nu)$ Calogero spin system. Particles have $su(\nu)$ spins as an internal degree of freedom, and move about on a line interacting through spin-dependent inverse square interactions. We denote by N and a the number of particles and a parameter for the interaction strength, respectively. The Hamilton operator is (the units $\hbar = 1$ and $2m = 1$ will be employed)

$$\hat{H} = \sum_{j=1}^{N} \hat{p}_j^2 + 2 \sum_{1 \le j < k \le N} \frac{a^2 - aP_{jk}}{(x_j - x_k)^2} \tag{1}$$

where $\hat{p}_j := -i\partial/\partial x_j$ denotes the momentum operator. Here P_{jk} is a permutation operator in spin space, and exchanges the spin state of the jth and the kth particles. As a basis of the $su(\nu)$ Lie algebra, we use $\nu^2 - 1$ traceless Hermitian matrices t^α which are normalized to be

$$\mathrm{tr}(t^\alpha t^\beta) = \frac{1}{2}\delta_{\alpha\beta}. \tag{2}$$

The commutation relation is

$$[t_j^\alpha, t_k^\beta] = \delta_{jk} \sum_\gamma f^{\alpha\beta\gamma} t_k^\gamma \tag{3}$$

where t_j^α acts on the jth particle and $f^{\alpha\beta\gamma}$ is the structure constant. In terms of these operators t_j^α, the permutation operator P_{jk} is expressed as

$$P_{jk} := \frac{1}{\nu} + 2\sum_\alpha t_j^\alpha t_k^\alpha. \tag{4}$$

(i) Show that the Lax operators L and M_2, which are $N \times N$ operator-valued matrices, are found to be

$$L_{jk} = \delta_{jk}\hat{p}_j + (1 - \delta_{jk}) ia \frac{P_{jk}}{x_j - x_k}, \tag{5}$$

$$(M_2)_{jk} = \delta_{jk} 2a \sum_{l \ne j} \frac{P_{jl}}{(x_j - x_l)^2} - (1 - \delta_{jk}) 2a \frac{P_{jk}}{(x_j - x_k)^2}. \tag{6}$$

(ii) Show that the Lax equation

$$[\hat{H}, L_{jk}] = [L, M_2]_{jk} = \sum_l [L_{jl}(M_2)_{lk} - (M_2)_{jl} L_{lk}], \tag{7}$$

with (2) and (3) yields the Heisenberg equation of motion. The operator M_2 satisfies the sum-to-zero condition

$$\sum_j (M_2)_{jk} = \sum_k (M_2)_{jk} = 0. \tag{8}$$

Problem 8. The Hamilton function for the three-body periodic Toda lattice can be written in a dimensionless form as

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2}(p_1^2+p_2^2+p_3^2)+\exp(q_1-q_2)+\exp(q_2-q_3)+\exp(q_3-q_1). \tag{1}$$

(i) Show that there are three conserved quantities, the total momentum

$$P(\mathbf{p},\mathbf{q}) = p_1+p_2+p_3 \tag{2}$$

the energy E and an additional quantity A. The third quantity A is the third-order polynomial of the momenta,

$$A(\mathbf{p},\mathbf{q}) = p_1p_2p_3 - p_1\exp(q_2-q_3) - p_2\exp(q_3-q_1) - p_3\exp(q_1-q_2). \tag{3}$$

(ii) Show that the first integrals are in involution.

Problem 9. Consider the one-parameter family of Hamilton functions

$$H(p_x,p_y,x,y) = \frac{1}{2}\left(p_x^2+p_y^2+(x^2y^2)^{1/a}\right) \tag{1}$$

where $0 \le \alpha \le 1$. In the limit $a \to 0$ we obtain the hyperbola billiard. Increasing a means a gradual softening of the billiard walls and when $a=1$ we recover the frequently studied x^2y^2 potential. The symmetry group of this family is C_{4v}.
(i) Show that the periodic orbits in these systems may be described by a *symbolic dynamics* using a three letter [2,1,0] alphabet.
(ii) Show that the motion in these horn regions may thus be treated in the *adiabatic approximation*.

Problem 10. Consider the equation of *motion for a charged particle*

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{e}{m_0}\mathbf{E}(\mathbf{r},t) + \frac{e}{m_0c}\frac{d\mathbf{r}}{dt}\times\mathbf{B}_0 \tag{1}$$

where the constant magnetic field $\mathbf{B}_0$ is directed along the z-axis and

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\sin(k_xx+k_zz-v_0t). \tag{2}$$

(i) Show that from (1) and (2) one obtains the following equations of motion

$$\frac{d^2x}{dt^2} + \omega_H^2 x = \frac{e}{m} E_{0x} \sin(k_x x + k_z z - v_0 t) \tag{3a}$$

$$\frac{d^2z}{dt^2} = \frac{e}{m} E_{0z} \sin(k_x x + k_z z - v_0 t) \tag{3b}$$

where

$$\beta = \frac{E_{0z}}{E_{0x}} = \frac{k_z}{k_x} = \text{const.} \tag{4}$$

(ii) Find a Hamilton function for system (3).

Problem 11. Let

$$|\mathbf{x}| := \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{1}$$

Let

$$H(\mathbf{y}, \mathbf{x}) = \frac{1}{2}|\mathbf{y}|^2 + V(|\mathbf{x}|) \tag{2}$$

be a Hamilton function in $\mathbb{R}^6$ invariant under the orthogonal group $SO(3)$

$$\mathbf{x} \to R\mathbf{x}, \quad \mathbf{y} \to R\mathbf{y}, \qquad \text{where} \quad R \in SO(3). \tag{3}$$

(i) Show that

$$F_1(\mathbf{x}, \mathbf{y}) = x_2 y_3 - x_3 y_2, \quad F_2(\mathbf{x}, \mathbf{y}) = x_3 y_1 - x_1 y_3, \quad F_3(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1 \tag{4}$$

are first integrals, defining the angular momentum vector.
(ii) Show that the Hamilton function (2) can be reduced to a one-dimensional Hamilton function.

Problem 12. Consider the non-relativistic motion of a charged particle of mass m and charge q moving in the field of a magnetic dipole of magnetic moment $\mathbf{M}$. It is described by the Hamilton function

$$H(\mathbf{p}, \mathbf{r}) = \frac{1}{2m}(\mathbf{p} - q\mathbf{A}) \tag{1}$$

where the vector potential $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{1}{r^3}(\mathbf{M} \times \mathbf{r}). \tag{2}$$

(i) Show that by choosing the z axis in the direction of $\mathbf{M}$, i.e. $\mathbf{M} = (0, 0, M)$, we have

$$H(p_x, p_y, x, y) = \frac{1}{2}\left(\left(p_x + \frac{ay}{r^3}\right)^2 + \left(p_y - \frac{ax}{r^3}\right)^2 + p_z^2\right). \tag{3}$$

(ii) Show that with $m = 1$, $r := (x^2 + y^2 + z^2)^{1/2}$ and $a = qM$ we obtain

$$H(p_x, p_y, x, y) = \frac{1}{2}(p_x^2 + p_y^2 + p_x^2) + \frac{a}{r^3}(yp_x - xp_y) + \frac{a^2}{2r^6}(x^2 + y^2). \quad (4)$$

(iii) Show that the equations of motion for this system are time-independent, axisymmetric and have also a scale symmetry.
(iv) Show that the following first integrals of motion exist: the Hamilton function (4) and the projection of the angular momentum in the direction of $\mathbf{M}$

$$L_z = xp_y - yp_x. \quad (5)$$

(v) Show that if we choose cylindrical coordinates (ρ, ϕ, z), the Hamilton function (4) becomes

$$H(p_\rho, p_\phi, p_z, \rho, \phi, z) = \frac{1}{2}(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2) + \frac{a^2\rho^2}{2}\frac{1}{(\rho^2 + z^2)^3} + ap_\phi \frac{1}{(\rho^2 + z^2)^{3/2}} \quad (6)$$

and $p_\phi = L_z =$ constant of motion.

Problem 13. Consider the *anisotropic Kepler problem*, whose Hamilton function reads

$$H(p_x, p_y, x, y) = \frac{p_x^2}{2\mu} + \frac{p_y^2}{2\nu} - \frac{1}{\sqrt{x^2 + y^2}}. \quad (1)$$

In the anisotropic Kepler problem the effective masses μ and ν, are different. This system is effectively chaotic when the mass ration μ/ν is sufficiently high (≥ 5). (i) Show that a symbolic coding can be obtained as follows. We take a set of trajectories starting on the x positive axis, with zero initial p_x momentum. We fix a constant energy surface, e.g. $H = -1/2$. Then x labels an unique trajectory, for any $0 \leq x \leq 2$. Following the time evolution, one records a bit sequence b_j : $b_i = 0$ if the i-th intersection with the x-axis occurs for $x \leq 0$, and $b_i = 1$ otherwise. A coding function F is then defined via

$$F(x) = \sum_i b_i 2^{-i}. \quad (2)$$

(ii) Show that this function is non-decreasing, and shows multifractal features.

Problem 14. (i) For a system of N particles with central two-body interactions described by the Hamilton function

$$H = \sum_{k=1}^{N} \frac{\mathbf{p}_k^2}{2m_k} + V \quad (1a)$$

where

$$V = \frac{1}{2} \sum_{\substack{k,l=1 \\ k \neq l}}^{N} V_{kl}(r_{kl}) \tag{1b}$$

and $r_{kl} := |\mathbf{r}_k - \mathbf{r}_l|$. Show that the first integrals are given by

$$H = \sum_{k=1}^{N} \frac{\mathbf{p}_k^2}{2m_k} + V, \qquad \mathbf{p}_k := \frac{\partial L}{\partial \mathbf{v}_k} = m_k \mathbf{v}_k$$

$$\mathbf{P} = \sum_{k=1}^{N} \mathbf{p}_k, \qquad \mathbf{J} = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{p}_k, \qquad \mathbf{G} = \sum_{k=1}^{N} m_k \mathbf{r}_k - \mathbf{P}t.$$

(ii) Show that if the potential is of the form $V_{KL} = C_{KL}/r_{KL}^2$ then there two additional first integrals

$$D = 2Ht - \sum_{k=1}^{N} \mathbf{r}_k \cdot \mathbf{p}_k, \qquad A = Ht^2 - \sum_{k=1}^{N} (\mathbf{r}_k \cdot \mathbf{p}_k t - \frac{1}{2} m_k \mathbf{r}_k^2).$$

Problem 15. The *geodesic flow* on a sphere S^n

$$|\mathbf{x}| = 1, \qquad |\mathbf{x}| := \sqrt{x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2} \tag{1}$$

where $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ is described by the differential equation

$$\frac{d^2\mathbf{x}}{dt^2} = \lambda \mathbf{x} \tag{2}$$

where the Lagrange parameter λ is determined such that

$$|\mathbf{x}| = 1 \tag{3}$$

Show that $d^2\mathbf{x}/dt^2 = -|d\mathbf{x}/dt|^2 \mathbf{x}$ and that this differential equation can be derived from the Hamilton function

$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2} |\mathbf{x}|^2 |\mathbf{y}|^2. \tag{4}$$

Problem 16. Consider the Hamilton function

$$H(p_r, p_\phi, r, \phi) = \frac{p_r^2}{2} + \frac{p_\phi^2}{2r^2} + \frac{\cos\phi + c}{2r^2}. \tag{1}$$

The Hamilton equations of motions are given by

$$\frac{dr}{dt} = \frac{\partial H}{\partial p_r} = p_r, \qquad \frac{d\phi}{dt} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{r^2} \tag{2a}$$

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2 + \cos\phi + c}{r^3}, \qquad \frac{dp_\phi}{dt} = -\frac{\partial H}{\partial \phi} = \frac{\sin\phi}{2r^2}. \tag{2b}$$

Show that the system admits the first integral

$$I(p_r, p_\phi, r, \phi) = p_\phi^2 + \cos\phi + c. \tag{3}$$

Problem 17. Consider the Hamilton function

$$H(p_x, p_y, x, y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y) \tag{1}$$

where

$$V(x, y) = x^4 + ax^2y^2 + by^4. \tag{2}$$

(i) Show that the system is integrable in the following five cases

(a) $b = 1$, $a = 0$, separable in x, y;
(b) $b = 1$, $a = 2$, separable in polar coordinates;
(c) $b = 1$, $a = 6$, separable in $x \pm y$;
(d) $b = 16$, $a = 12$, separable in parabolic coordinates;
(e) $b = 8$, $a = 6$, which possesses an invariant quartic in momenta,

$$C^2 = p_x^4 + 4x^2(x^2+6y^2)p_x^2 - 16x^3yp_xp_y + 4x^4p_y^2 + 4x^4(x^4+4x^2y^2+4y^4). \tag{3}$$

(ii) Find the singular behaviour of the equation of motions for $b = 8$ and $a = 6$.

Problem 18. Consider the *Calogero-Moser system* in the case of two degrees of freedom. The Hamilton function is given by

$$H(p_x, p_y, x, y) = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) \tag{1}$$

where

$$V(x, y) = \frac{k}{(x-y)^2} + \frac{1}{2}(Ax^2 + By^2) \qquad k, A, B > 0 \tag{2}$$

This system is completely integrable only in the symmetric case $A = B$. The equations of motion are

$$\frac{d^2x}{dt^2} = -Ax + \frac{2k}{(x-y)^3}, \qquad \frac{d^2y}{dt^2} = -By - \frac{2k}{(x-y)^3}. \tag{3}$$

(i) Show that for $A = B$ the equations of motion (3) uncouple under the transformation

$$z(x,y) := x - y, \qquad w(x,y) := x + y. \tag{4}$$

Find an additional first integral.
(iii) Show that in the case $A = B$ the solutions possess exactly two Riemann sheets corresponding to the $\pm$ choice in taking the square root of (7).
(iv) The local two-sheetedness of solutions can also be revealed, by singularity analysis. Show that, for general A and B, one can expand $x(t)$ and $y(t)$ near a (movable) singularity $t = t_*$ of (3) and that the only leading behaviour allowed is of the form

$$x(\tau) = \alpha + c_1\tau^{1/2} + \cdots, \quad y(\tau) = \alpha + c_2\tau^{1/2} + \cdots \qquad \tau = t - t_*$$

where α is a free constant and $c_1 = -c_2 = (-k)^{1/4}$. The only type of singularity, therefore, in this problem occurs when the equations of motion themselves are singular, at $x - y = 0$. These singularities are, of course, finite, since the configuration variables $x(t)$, $y(t)$ are finite at $t = t_*$. No logaritmic terms enter the expansion (8).

Problem 19. Consider the *Hénon-Heiles model*

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2}(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2) + q_1^2 q_2 - \frac{1}{3}\epsilon q_2^3. \tag{1}$$

(i) Show that the following three cases are integrable

$$\epsilon = -1, \qquad A = B \tag{2a}$$

$$\epsilon = -6, \qquad \text{for all} \quad A, B \tag{2b}$$

$$\epsilon = -16, \qquad B = 16A \tag{2c}$$

(ii) Find the first integrals.

Problem 20. The Hamilton function of the two-dimensional *hydrogen atom* in a uniform electric field F reads (we set electron mass and charge $m_e = 1$, $|e| = 1$)

$$H(p_x, p_y, x, y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - \frac{1}{(x^2 + y^2)^{1/2}} + Fx. \tag{1}$$

(i) Show that the system has simple discrete symmetry: the Hamilton function is invariant under reflections through the x-axis, $y \to -y$, i.e. under the canonical transformation

$$(x, p_x, y, p_y) \to (x, p_x, -y, -p_y). \tag{2}$$

(ii) Show that the system is separable in *semi-parabolic coordinates*

$$y(u,v) = uv, \qquad x(u,v) = \frac{1}{2}(u^2 - v^2) \tag{3}$$

in which the Hamilton function (1) takes the form

$$H(p_u, p_v, u, v) = \frac{1}{u^2+v^2}\left(\frac{p_u^2}{2} + \frac{p_v^2}{2}\right) - \frac{2}{u^2+v^2} + F\frac{u^2-v^2}{2}. \tag{4}$$

Problem 21. We consider a classical particle with charge e, mass m, and energy E moving in a two-dimensional periodic potential under the influence of a homogeneous magnetic field

$$\mathbf{B} = B\mathbf{z} = (0,0,B) \tag{1}$$

described by the Hamilton function

$$H(p_x, p_y, x, y) = \frac{1}{2m}\left(\left(p_x + \frac{eBy}{2}\right)^2 + \left(p_y - \frac{eBx}{2}\right)^2\right) + V(x,y) \tag{2a}$$

where

$$V(x,y) = V_0(2 + \cos(2\pi x/a) + \cos(2\pi y/a)) \tag{2b}$$

is an isotropic (superlattice) potential. We measure energy in units of V_0, lengths in units of the lattice constant a, and time in units of the inverse harmonic frequency

$$\omega_0 := \left(\frac{4\pi^2 V_0}{a^2 m}\right)^{1/2}. \tag{3}$$

This leads to scaled variables

$$\widetilde{H} := \frac{H}{V_0}, \qquad \widetilde{x} := 2\pi\frac{x}{a}, \qquad \widetilde{y} := 2\pi\frac{y}{a}, \qquad \tau := \omega_0 t. \tag{4}$$

(i) Show that the equations of motion then read (omitting the tildes for convenience)

$$\frac{dx}{d\tau} = v_x, \qquad \frac{dv_x}{d\tau} = \sin x + 2\lambda v_y \tag{5a}$$

$$\frac{dy}{d\tau} = v_y, \qquad \frac{dv_y}{d\tau} = \sin y - 2\lambda v_x \tag{5b}$$

corresponding to the Hamilton function

$$H(p_x, p_y, x, y) = \frac{1}{2}(p_x + \lambda y)^2 + \frac{1}{2}(p_y - \lambda x)^2 + V(x,y) \tag{6}$$

with

$$V(x, y) = 2 + \cos x + \cos y. \tag{7}$$

(ii) Show that there are two integrable limits in this model, that is $\lambda \to 0$ and $\lambda \to \infty$.
(iii) Show that the potential V of (7) has minima at the energy $E = 0$, saddle points at $E = 2$, and maxima at $E = 4$.

Remark 1. Thus in the regime $E \leq 2$ all orbits are restricted to one unit cell for all values of λ. For $E > 2$, localized and delocalized orbits may coexist.

Remark 2. The dimensionless quantity

$$\lambda := \frac{eBa}{(16\pi^2 m V_0)^{1/2}} = \frac{\omega_c}{2\omega_0} \tag{8}$$

proportional to the applied magnetic field $\mathbf{B}$ describes the nonintegrable coupling between the two degrees of freedom and is related to the bare cyclotron frequency ω_c.

Problem 22. Consider the equations of motion

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial U}{\partial x_i}, \quad m_i \frac{d^2 y_i}{dt^2} = -\frac{\partial U}{\partial y_i}, \quad m_i \frac{d^2 z_i}{dt^2} = -\frac{\partial U}{\partial z_i}, \quad i = 1, 2, 3 \tag{1}$$

where

$$U = -m_1 m_2 F(r_{12}^2) - m_2 m_3 F(r_{23}^2) - m_3 m_1 F(r_{31}^2), \qquad m_i = 1, \quad i = 1, 2, 3, \tag{2}$$

(x_k, y_k, z_k) are the coordinates of the k-th body, $k = 1, 2, 3$, $F(r^2)$ is an arbitrary, sufficiently smooth function, and

$$r_{ij} := ((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{1/2}. \tag{3}$$

(i) Show that (1) is invariant under the 10-parameter Galilean group $G(1, 3)$.
(ii) Show that the Lie algebra of this group has a basis consisting of the following infinitesimal generators

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3},$$

$$X_3 = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}, \quad X_4 = t\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right),$$

$$X_5 = t\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right), \quad X_6 = t\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}\right),$$

$$X_7 = y_k\frac{\partial}{\partial z_k} - z_k\frac{\partial}{\partial y_k}, \quad X_8 = z_k\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial z_k}, \quad X_9 = x_k\frac{\partial}{\partial y_k} - y_k\frac{\partial}{\partial x_k}. \tag{4}$$

Remark. Ten integrals of motion of the spatial three-body problem were known already to Lagrange

Problem 23. The *Kepler problem* is the paradigm of the two-body problem in mechanics. Kepler proposed three empirical laws governing the motion of planets: (1) the orbit is an ellipse, (2) the area velocity of the orbit is a constant, (3) the period of revolution and the semimajor axis of the orbit are related according to $T \propto R^{3/2}$. In reduced coordinates the equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu\mathbf{r}}{r^3} \tag{1}$$

in the standard notation, where $r = \|\mathbf{r}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
(i) Show that the equation of motion (1) can be derived from the Lagrange function

$$L(\dot{\mathbf{r}}, \mathbf{r}) = \frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\mu}{r} \tag{2}$$

where $\cdot$ denotes the scalar product.
(ii) Show that the system admits the first integrals

$$E = \frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{\mu}{r}$$
$$\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$$
$$\mathbf{J} = \dot{\mathbf{r}} \times \mathbf{L} - \mu\hat{\mathbf{r}}$$
$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu\hat{\omega}}{L}$$

where $\times$ denotes the vector product and $\hat{\boldsymbol{\omega}}$ is the unit vector in the direction of the angular velocity ($\hat{\boldsymbol{\omega}} := \hat{\mathbf{L}} \times \hat{\mathbf{r}}$). In plane polar coordinates, it coincides with $\hat{\theta}$.
(iii) Are the first integrals independent?
(iv) Give the Hamilton function H.

Problem 24. Consider the Hamilton system

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2)$$

where the potential V is given by

$$V(q_1, q_2) = \frac{k}{(q_1 - q_2)^2} + \frac{1}{2}(Aq_1^2 + Bq_2^2), \qquad k, A, B > 0.$$

Show that the system is integrable if $A = B$.

Problem 25. (i) Find the first integral of the differential equation

$$\frac{d^2x}{dt^2} + \alpha + \beta x + \gamma x^2 + \epsilon x^3 = 0 \tag{1}$$

where α, β, γ and ϵ are constants.
(ii) Show that the differential equation can be derived from a Hamilton function.
(ii) Find the general solution to the differential equation.

Problem 26. The *Emden equation* is given by

$$\frac{d^2u}{dt^2} + \frac{2}{t}\frac{du}{dt} + u^n = 0 \tag{1}$$

which represents in general an anharmonic oscillator subject to damping dependent upon the velocity.
(i) Show that in the case $n = 5$ the Emden equation can be derived from the variational integral

$$J = \int_{t_0}^{t_1} t^2 \left(\frac{1}{2} \left(\frac{du}{dt} \right)^2 - \frac{1}{6} u^6 \right) dt. \tag{2}$$

Remark. Let

$$L(\dot{u}, u, t) = t^2 \left(\frac{1}{2}\dot{u}^2 - \frac{1}{6}u^6 \right). \tag{3}$$

Then the equation of motion (1) follows from the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0. \tag{4}$$

(ii) Use Noether's theorem to show that (1) admits the first integral

$$I = \frac{1}{6}t^3 u^6 + \frac{1}{2}t^3 \left(\frac{du}{dt} \right)^2 + \frac{1}{2}t^2 u \frac{du}{dt}. \tag{5}$$

(iii) Show that the generalized equation

$$\frac{d^2u}{dt^2} + \beta(t)\frac{du}{dt} + \alpha(t) u^m = 0, \qquad m \neq -1 \tag{6}$$

admits the first integral

$$I = \left(\left(\frac{du}{dt} \right)^2 + \frac{2\alpha}{m+1} u^{m+1} \right) \exp\left(2 \int^t \beta(t') dt' \right)$$

$$\times\left(C+C_4\int^t \exp\left(-\int^{t'}\beta(t'')dt''\right)dt'\right)-C_4u\frac{du}{dt}\exp\left(\int^t\beta(t')dt'\right) \tag{7}$$

exists when $\alpha(t)$ and $\beta(t)$ satisfy the relation

$$\alpha^{-2/(m+3)}\exp\left(-\frac{4}{m+3}\int^t\beta(t')dt'\right)-C_4\int^t\exp\left(-\int^{t'}\beta(t'')dt''\right)dt'=C \tag{8}$$

and C and C_4 are constants.

Problem 27. Consider *Emden's equation*

$$\frac{d^2q}{dt^2}+\frac{2}{t}\frac{dq}{dt}+q^5=0 \tag{1}$$

which is of special significance in astrophysics.
(i) Show that this equation can be obtained from Lagrange's equation with a Lagrangian given as

$$L(t,q,\dot{q})=\left(\frac{1}{2}\dot{q}^2-\frac{1}{6}q^6\right)t^2. \tag{2}$$

(ii) Let $p:=\partial L/\partial\dot{q}$. Show that the Hamilton function for (1) is found to be

$$H(p,q,t)=\frac{1}{2}\frac{p^2}{t^2}+\frac{1}{6}q^6t^2. \tag{3}$$

Hint: The *Euler-Lagrange equation* is given by

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}-\frac{\partial L}{\partial q}=0 \tag{4}$$

and the Hamilton function takes the form

$$H(p,q,t)=\frac{\partial L}{\partial\dot{q}}\dot{q}-L \tag{5}$$

Problem 28. Show that the equation of motion

$$\frac{dp}{dt}=-\sin q+\frac{h^2}{24}(\sin 2q-p^2\sin q),\qquad \frac{dq}{dt}=p-\frac{h^2}{12}p\cos q \tag{1}$$

can be derived from the Hamilton function

$$H(p,q)=\frac{1}{2}p^2+(1-\cos q)+\frac{h^2}{48}(-2p^2\cos q+\cos 2q-1). \tag{2}$$

Here h is a (small) positive constant.

Problem 29. Consider the system

$$\frac{dp}{dt} = -\sin q - \frac{h^2}{24}(p^2 \sin q + 2\sin 2q) \qquad (1a)$$

$$\frac{dq}{dt} = p + \frac{h^2}{6} p \cos q \qquad (1b)$$

where h is a (small) positive constant.
(i) Show that this system is not Hamiltonian, but has the reversibility property of being invariant under the change of p into $-p$ and t into $-t$.
(ii) Show that it has the first integral

$$F(p,q) = \frac{p^2}{2(1+(h^2/6)\cos q)^{1/2}} + \int_0^q \frac{\sin s + (h^2/3)\sin 2s}{(1+(h^2/6)\cos s)^{3/2}} ds. \qquad (2)$$

Problem 30. Consider the second order differential equation

$$t\frac{d^2q}{dt^2} = \left(\frac{dq}{dt}\right)^3 + \frac{dq}{dt}. \qquad (1)$$

Show that it has the Hamilton function

$$H(p,q,t) = -4t(q^3 p/2)^{\frac{1}{2}} - 4tq, \qquad p = 2q^2 - 2\frac{t^2}{\dot{q}^2} \qquad (2)$$

and first integrals

$$I_1(p,q,t) = 2(q^2 - p/2)^{\frac{1}{2}} + 2q$$

$$I_2(p,q,t) = t^2 - q^2 - 2q(q^2 - p/2)^{\frac{1}{2}}$$

$$I_3(p,q,t) = \frac{t^2 - q^2 - 2q(q^2 - p/2)^{\frac{1}{2}}}{2q + 2(q^2 - p/2)^{\frac{1}{2}}}. \qquad (3)$$

Problem 31. Consider a pendulum with varying length $r(t)$. The total length is $l = r(t) + y(t)$.
(i) Show that the governing equation is

$$(l-y)\frac{d^2\theta}{dt^2} + g\sin\theta - 2\frac{dy}{dt}\frac{d\theta}{dt} - \frac{d^2y}{dt^2}\sin\theta = 0. \qquad (1)$$

(ii) Show that linearizing the equation yields

$$\frac{d^2\theta}{dt^2} + \frac{g - d^2y/dt^2}{l-y}\theta - 2\frac{dy/dt d\theta}{l-y} = 0. \qquad (2)$$

Problem 32. (i) Consider a pendulum attached to a rotating base. Show that the governing equation is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} - \frac{1}{2}\Omega^2 \sin(2\theta) = 0. \tag{1}$$

(ii) Let

$$\Omega(t) = \Omega_0(1 + \epsilon \cos(\omega t))$$

where $\epsilon \ll 1$. Linearize (1) and show that

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{l} - \Omega_0^2 - 2\Omega_0^2\epsilon\cos(\omega t) - \Omega_0^2\epsilon^2\cos^2(\omega t)\right)\theta = 0. \tag{2}$$

Problem 33. Consider the system

$$\frac{d^2u_1}{dt^2} + u_1 + 3au_1^2 + 2bu_1u_2 + cu_2^2 = 0 \tag{1}$$

$$\frac{d^2u_2}{dt^2} + u_2 + bu_1^2 + 2cu_1u_2 + 3du_2^2 = 0 \tag{2}$$

which represents a class of Hamiltonian systems. Find the first integrals.

Problem 34. Find the Hamilton function $H(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2)$ of two pendulums coupled by a massless spring. Write down the Hamilton equations of motion. Show that

$$\frac{d}{dt}H = 0.$$

Problem 35. Let

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}k(q_1 - q_2)^2$$

where k is a positive constant. Find the equations of motion and solve them for the initial values

$$q_1(t=0) = q_{10}, \quad q_2(t=0) = q_{20}, \quad p_1(t=0) = p_{10}, \quad p_2(t=0) = p_{20}.$$

Draw the phase portrait $(q_1(t), p_1(t))$ and $(q_2(t), p_2(t))$ for $k = m = 1$ and $q_{10} = 1$, $q_{20} = 1$, $p_{10} = 1$, $p_{20} = 2$.

Problem 36. We consider systems of $(2n + 1)$ ordinary nonlinear differential equations. These are the multiple three-wave interaction system describing triads (a_j, b_j, u), $j = 1, \ldots, n$, evolving in time alone and interacting with each other through the single common member u. These systems can be derived from a Hamilton function

$$H(\mathbf{b}, \mathbf{c}, u) = \frac{1}{2} i \sum_{j=1}^{n} \epsilon_j (c_j c_j^* - b_j b_j^*) + i \sum_{j=1}^{n} \alpha_j (u b_j^* c_j + u^* b_j c_j^*) \tag{1}$$

and with *Poisson bracket* defined as

$$\{f,\, g\} := \frac{\partial f}{\partial u}\frac{\partial g}{\partial u^*} - \frac{\partial f}{\partial u^*}\frac{\partial g}{\partial u} + \sum_{j=1}^{n} \left(\frac{\partial f}{\partial b_j}\frac{\partial g}{\partial b_j^*} - \frac{\partial f}{\partial b_j^*}\frac{\partial g}{\partial b_j} + \frac{\partial f}{\partial c_j}\frac{\partial g}{\partial c_j^*} - \frac{\partial f}{\partial c_j^*}\frac{\partial g}{\partial c_j} \right). \tag{2}$$

Thus (u, u^*), (b_j, b_j^*), (c_j, c_j^*) are pairs of canonical variables (where * means complex conjugate). The arbitrary real parameters α_j, ϵ_j play the role of frequencies.
(i) Show that from (1) and (2), with the $\alpha_j = 1$, it follows that the Hamilton's equations of motion are

$$\frac{du}{dt} = i \sum_{j=1}^{n} b_j c_j^*, \qquad \frac{db_j}{dt} = -\frac{1}{2} i \epsilon_j b_j + i u c_j, \qquad \frac{dc_j}{dt} = \frac{1}{2} i \epsilon_j c_j + i u^* b_j, \quad \text{and} \quad \text{c.c.} \tag{3}$$

where c.c. stands for complex conjugate. (ii) Show that they have the Lax representation

$$\frac{dL}{dt} = \{L, H\}(t) \equiv [A, L](t) \tag{4}$$

in which L and A are the $(2n+2) \times (2n+2)$ matrices

$$L := \frac{1}{2} \begin{pmatrix} \pi & \sigma_1 & \cdots & \sigma_n \\ \tau_1 & \epsilon_1 I & & 0 \\ \vdots & & \ddots & \\ \tau_n & 0 & & \epsilon_n I \end{pmatrix}, \qquad A := \frac{1}{2} i \begin{pmatrix} 0 & \omega\sigma_1 & \cdots & \omega\sigma_n \\ \tau_1 \omega & & & \\ \vdots & & 0 & \\ \tau_n \omega & & & \end{pmatrix}. \tag{5}$$

The π, σ_j and τ_j are the 2×2 matrices

$$\pi := \begin{pmatrix} 0 & -2u \\ 2u^* & 0 \end{pmatrix}, \qquad \tau_j := \begin{pmatrix} b_j^* & -c_j^* \\ -c_j & -b_j \end{pmatrix}, \qquad \sigma_j := \begin{pmatrix} -b_j & -c_j^* \\ -c_j & b_j^* \end{pmatrix} \tag{6}$$

and $\omega = \mathrm{diag}(-1, 1)$. I is the 2×2 identity matrix.
(iv) Show that Hamilton function (1) is given by

$$H = \frac{2}{3} i \,\mathrm{tr}(L^3) + \text{ const.} \tag{7}$$

where tr denotes the trace.

Hint. From (2) we find

$$\frac{du}{dt} = \{u, H\} = \frac{\partial H}{\partial u^*}. \tag{8}$$

Problem 37. The motion of the N particle *Toda lattice* is described by the Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} \exp(q_j - q_{j+1})$$

where (q_j, p_j) are coordiantes and momenta, and $q_{N+1} = q_1$.
(i) Write down the Hamilton equations of motion.
(ii) We define variables $a_j(t)$, $b_j(t)$ $(j = 1, 2, \ldots, N)$ with

$$a_j := \frac{1}{2}\exp\left(\frac{1}{2}(q_j - q_{j+1})\right), \qquad b_j := \frac{1}{2}p_j.$$

Find the equations of motion for these variables.
(iii) Show that for the variables a_j, b_j we can find a Lax representation

$$\frac{dL}{dt} = [A, L](t) \equiv (AL - LA)(t)$$

where L and A are $N \times N$ matrices.
(iv) Show that

$$\mathrm{tr}(L^k)(t) = \mathrm{const}, \qquad k = 1, 2, \ldots, N$$

where tr denotes the trace.

Problem 38. Consider the Lax representation

$$\frac{dL}{dt} = [A, L](t)$$

where $L := AJ + JA$ with

$$A = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$$

where $c_j \neq 0$. Note that A is skew-symmetric. Find the equations of motion for u_1, u_2, u_3.

Problem 39. Consider the autonomous system

$$\frac{du_1}{dt} = u_1(u_2 - u_n)$$
$$\frac{du_2}{dt} = u_2(u_3 - u_1)$$
$$\vdots$$
$$\frac{du_n}{dt} = u_n(u_1 - u_{n-1}).$$

(i) Find a Lax representation for the case $n = 3$ and the first integrals using $\mathrm{tr}L^n$.
(ii) Find the Lax representation for the case $n = 4$ and the first integrals.
(iii) Find the Lax representation for arbitrary n.

Problem 40. Consider the Hamilton function

$$H(q_1, p_1, q_2, p_2) = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) - \frac{1}{3}q_2^3.$$

Find the the Hamilton equations of motion and show that they admit the periodic orbits

$$\Gamma_0 : \ \ \gamma_0(t) = (q_1(t), p_1(t), q_2(t), p_2(t)) = (k\cos t, -k\sin t, 0, 0)$$

$$\Gamma_1 : \ \ \gamma_1(t) = (q_1(t), p_1(t), q_2(t), p_2(t)) = (\sqrt{k^2 - 1/3}\cos t, -\sqrt{k^2 - 1/3}\sin t, 1, 0)$$

which lie on the surface

$$q_1^2 + p_1^2 + q_2^2 + p_2^2 - \frac{2}{3}q_2^3 = k^2$$

for $k^2 > 1/3$.

Problem 41. Given the Lagrange function

$$L(x, dx/dt, t) = \frac{1}{2}e^{\gamma t}\left(\frac{dx}{dt}\right)^2 - e^{\gamma t}V(x, t)$$

describing a dynamical system with damping.
(i) Find the equations of motion.
(ii) Find the corresponding Hamilton function.

Problem 42. The motion of the N particle *Toda lattice* is described by the Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} \exp(q_j - q_{j+1})$$

where (q_j, p_j) are coordiantes and momenta, and $q_{N+1} = q_1$.
(i) Write down the Hamilton equations of motion.
(ii) We define variables $a_j(t)$, $b_j(t)$ $(j = 1, 2, \ldots, N)$ with

$$a_j := \frac{1}{2} \exp\left(\frac{1}{2}(q_j - q_{j+1})\right), b_j := \frac{1}{2} p_j.$$

Find the equations of motion for these variables.
(iii) Show that for the variables a_j, b_j we can find a Lax representation

$$\frac{dL}{dt} = [A, L](t) \equiv (AL - LA)(t)$$

where L and A are $N \times N$ matrices.
(iv) Show that

$$\mathrm{tr}(L^k)(t) = \mathrm{const}, \qquad k = 1, 2, \ldots, N$$

where tr denotes the trace.

Problem 43. Consider the Lax representation

$$\frac{dL}{dt} = [A, L](t)$$

where $L := AJ + JA$ with

$$A = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$$

Find the equations of motion for u_1, u_2, u_3.

Problem 44. Let

$$L = \begin{pmatrix} 0 & 1 & 0 & u_1 \\ u_2 & 0 & 1 & 0 \\ 0 & u_3 & 0 & 1 \\ 1 & 0 & u_4 & 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} u_1 + u_2 & 0 & 1 & 0 \\ 0 & u_2 + u_3 & 0 & 1 \\ 1 & 0 & u_3 + u_4 & 0 \\ 0 & 1 & 0 & u_4 + u_1 \end{pmatrix}.$$

(i) Show that L and A are the Lax representation

$$\frac{dL}{dt} = [A, L](t) \equiv [A(t), L(t)]$$

of the autonomous first order system

$$\frac{du_1}{dt} = u_1(u_2 - u_4)$$
$$\frac{du_2}{dt} = u_2(u_3 - u_1)$$
$$\frac{du_3}{dt} = u_3(u_4 - u_2)$$
$$\frac{du_4}{dt} = u_4(u_1 - u_3).$$

(ii) Show that

$$I_1(\mathbf{u}) = u_1 + u_2 + u_3 + u_4$$
$$I_2(\mathbf{u}) = u_1 u_2 u_3 u_4$$
$$I_3(\mathbf{u}) = u_1 u_3 + u_2 u_4$$

are first integrals of the system.
(iii) Which first integrals do we find from L^k, where $k = 2, 3, \ldots$?

Problem 45. The motion of a charged particle in the plane perpendicular to the uniform constant magnetic field is described in the classical case by system of second order ordinary differential equations

$$\frac{d^2x}{dt^2} = \omega\frac{dy}{dt}, \qquad \frac{d^2y}{dt^2} = -\omega\frac{dx}{dt}$$

where ω is a constant frequency. Show that this system of differential equations can be derived from the Hamilton function

$$H(p_x, p_y, x, y) = \frac{1}{2}\left(p_x + \frac{1}{2}\omega y\right)^2 + \frac{1}{2}\left(p_y - \frac{1}{2}\omega x\right)^2.$$

Problem 46. In case of linear dissipation the Lagrangian of a particle moving in a one-dimensional potential $V(x)$ is given by

$$L(x, dx/dt, t) = e^{\gamma t}\left(\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right)$$

Find the associated Hamilton function. Find the equation of motion.

Problem 47. Consider the Hamilton function for N particles

$$H(\mathbf{p}, \mathbf{q}) = H_{kin}(\mathbf{p}) + V(\mathbf{q})$$

where

$$H_{kin}(\mathbf{q}) = \sum_{k=1}^{N}\sum_{j=1}^{3}\frac{p_{kj}^2}{2m_k}$$

and $\mathbf{p} = (p_{11}, p_{12}, p_{13}, p_{21}, \ldots, p_{N3})$, $\mathbf{q} = (q_{11}, q_{12}, q_{13}, q_{21}, \ldots, q_{N3})$ with $\mathbf{p}$, $\mathbf{q}$ are the corresponding momenta and positions. Here $V(\mathbf{q})$ is a differentiable potential. The Hamilton equations of motion are

$$\frac{dp_{kj}}{dt} = -\frac{\partial V}{\partial q_{kj}}, \qquad \frac{dq_{kj}}{dt} = \frac{p_{kj}}{m_k}.$$

The formal solution of these system of ordinary differential equation is $\Phi^t(\mathbf{p}(0), \mathbf{q}(0))$, where Φ^t denotes the flow and $(\mathbf{p}(0), \mathbf{q}(0)$ the initial conditions. Let R denote the momentum reversion, i.e. $R(\mathbf{p}, \mathbf{q}) = (-\mathbf{p}, \mathbf{q})$. Show that the flow Φ^t is R-reversible, i.e.

$$R \circ \Phi^{-t} \circ R = \Phi^t.$$

Problem 48. Consider the Hamilton function

$$H(\mathbf{p}_1, \ldots, \mathbf{p}_N, \mathbf{q}_1, \ldots, \mathbf{q}_N) = \sum_{\alpha=1}^{N}\sum_{k=1}^{3}\frac{p_{\alpha,k}^2}{2m_\alpha} + \sum_{\alpha<\beta}^{N}\sum_{k=1}^{3} U_{\alpha\beta}\left(|q_{\alpha k} - q_{\beta k}|\right)$$

where α denotes the particle and k is the component of the vectors $\mathbf{p}_\alpha$, $\mathbf{q}_\alpha$ with $k = 1, 2, 3$. N is the number of particles. Show that the Hamilton function admits the first integrals

$$P_k = \sum_{\alpha=1}^{N} p_{\alpha k}, \quad k = 1, 2, 3 \qquad \text{total momentum}$$

$$I_i = \sum_{\alpha=1}^{N}\sum_{k,\ell=1}^{3} \varepsilon_{ik\ell}\, q_{\alpha k} p_{\alpha \ell}, \quad i = 1, 2, 3 \qquad \text{total angular momentum}$$

$$G_k = \sum_{\alpha=1}^{N}(p_{\alpha k} t - m_\alpha q_{\alpha k}), \quad k = 1, 2, 3 \qquad \text{centre of mass}$$

and the Hamilton function. Here

$$\varepsilon_{ij\ell} := \begin{cases} 1 & \text{even permutation of } (1,2,3) \\ -1 & \text{odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

The total number of first integrals is given by $3 + 3 + 3 + 1 = 10$.

Problem 49. From the previous problem consider the case $N = 2$ (*Kepler problem*). Assume that the potential U depends only on q, where $\mathbf{q} := \mathbf{q}_1 - \mathbf{q}_2$, $q = \|\mathbf{q}\|$ and

$$U(q) = -\gamma \frac{m_1 m_2}{q}.$$

with the gravitational constant $\gamma = 6.685 \cdot 10^{-8} cm^3 g sec^{-2}$. Find Newton's equation of motion.

Problem 50. Consider the Newton's equations of motion from the previous problem. The *centre of mass* with N mass points with masses m_j ($j = 1, 2, \dots, N$) and vectors $\mathbf{q}_j$ is defined as

$$\mathbf{R} := \frac{\sum_{j=1}^{N} m_j \mathbf{q}_j}{\sum_{j=1}^{N} m_j}.$$

The *centre of mass system* is defined as $\mathbf{R} = \mathbf{0}$. Find the equations of motion for this case.

Problem 51. Consider the equation of motion

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2 \mathbf{q}}{dt^2} = m \frac{d^2 \mathbf{q}}{dt^2} = -\nabla_{\mathbf{q}} U = -\frac{\delta}{q^3} \mathbf{q}.$$

Show that the *Lenz vector* defined by

$$\mathbf{L} := \frac{\mathbf{p} \times \mathbf{J}}{\delta m} - \frac{\mathbf{q}}{q}$$

is a first integral of this equation, where $\mathbf{J} := \mathbf{q} \times \mathbf{p}$ and $\delta = \gamma m_1 m_2$.

Problem 52. Let $\mathbf{q} = (q_1, q_2, q_3)^T$, $\mathbf{p} = (p_1, p_2, p_3)^T$ be the coordinates and associated momenta of a Hamilton system in a 6-dimensional phase space with Hamilton function

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{p}) + V(\mathbf{q}) \equiv \frac{1}{2} \mathbf{p}^T \mathbf{p} + V(\mathbf{q}).$$

A fictive time τ is introduced through the ordinary differential equation

$$\frac{dt}{d\tau} = g(\mathbf{q}, \mathbf{p})$$

defining a Sundman transformation, where g is a positive scalar monitor function which is taken to be small if the solution of the Hamilton system is evolving rapidly and τ is the fictive time which is used for all computation. Two new conjugate coordinates are introduced

$$q^t := H(\mathbf{q}_0, \mathbf{p}_0), \qquad p^t := t.$$

To preserve the Hamiltonian structure of the system after rescaling in terms of the fictive time τ, one applies a Poincaré transformation. Using the Poincareé transformation the system $(\mathbf{q}, q^t, \mathbf{p}, p^t)$ is Hamiltonian and evolves in the fictive time, τ, with Hamilton function

$$K(\mathbf{q}, q^t, \mathbf{p}, p^t) = g(\mathbf{q}, \mathbf{p})(H(\mathbf{q}, \mathbf{p}) - q^t).$$

Let

$$g(\mathbf{q}) = \sqrt{q_1^2 + q_2^2 + q_3^2}.$$

Find the Hamilton function

$$K(\mathbf{q}, q^t, \mathbf{p}, p^t) = g(\mathbf{q}, \mathbf{p})(H(\mathbf{q}, \mathbf{p}) - q^t)$$

and the equations of motion.

Problem 53. Consider the Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}k(q_1 - q_2)^2$$

where k is a positive constant. Find the equations of motion. Solve the initial value problem

$$q_1(t=0) = q_{10}, \quad q_2(t=0) = q_{20}, \quad p_1(t=0) = p_{10}, \quad p_2(t=0) = p_{20}.$$

Draw the phase portrait $(q_1(t), p_1(t))$ and $(q_2(t), p_2(t))$ for $k = m = 1$ and $q_{10} = 1$, $q_{20} = 1$, $p_{10} = 1$, $p_{20} = 2$.

Problem 54. Consider the Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(p_1^2 + p_2^2) + U_\varepsilon(\mathbf{q})$$

where $0 \le \varepsilon \le 1$ and the potential is given by

$$U_\varepsilon(\mathbf{q}) = \frac{1-\varepsilon}{12}(q_1^4 + q_2^4) + \frac{1}{2}q_1^2 q_2^2.$$

Show that the potential $U_\varepsilon(\mathbf{q})$ admits the C_{4v} point group.

Problem 55. The Hamilton function for a linear chain with cyclic boundary condition ($N \equiv 0$) is given by

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2m}\sum_{j=1}^{N} p_j^2 + \frac{k}{2}\sum_{j=1}^{N}(q_j - q_{j-1} - a)^2.$$

Introducing the transformation $q_j \to q_j - ja$ we obtain the Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2m} \sum_{j=1}^{N} p_j^2 + k \sum_{j=1}^{N} (q_j^2 - q_1 q_{j-1}).$$

The Hamilton equations of motions are given by

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \ldots, N.$$

Let $N = 3$. Then the equations of motion can be written in matrix form

$$\begin{pmatrix} dq_1/dt \\ dq_2/dt \\ dq_3/dt \\ dp_1/dt \\ dp_2/dt \\ dp_3/dt \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/m \\ -2k & k & k & 0 & 0 & 0 \\ k & -2k & k & 0 & 0 & 0 \\ k & k & -2k & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of the 6×6 matrix on the right-hand side. Use the normalized eigenvectors to rotate this matrix into diagonal form.

Problem 56. Consider the Hamilton function $H : \mathbb{R}^2 \to \mathbb{R}$

$$H(p, q) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{q^4}{4}.$$

Show that there is saddle and centers. Show that there are two homoclinic orbits.

Problem 57. Consider N vortices with the strengths (velocity circulation around the vortex), κ_j $(j = 1, 2, \ldots, N)$. We denote the cartesian coordinates of the vortices in a flow plane by (x_j, y_j), $(j = 1, 2, \ldots, N)$. Then the dyanamics of the vortices is given by the system

$$\kappa_j \frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}$$
$$\kappa_j \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j}$$

with the Hamilton function

$$H(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \sum_{j<k} \kappa_j \kappa_k \ln(a_{jk}), \qquad a_{jk} := \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2}.$$

Find the first integrals.

Problem 58. Let u be a solution of the *Painlevé equation* of the first kind

$$\frac{d^2u}{dt^2} = 6u^2 + t.$$

Then $u(t)$ is meromorphic on the complex plane $\mathbb{C}$ and the function $\tau(t)$ defined by

$$f(z) = -\left(\frac{d^2}{dt^2}\right)\log\tau(t) = \frac{(d\tau/dt)^2 - \tau(t)d^2\tau/dt^2}{\tau(t)^2}$$

is holomorphic on $\mathbb{C}$. Show that the Painlevé equation of first kind is equivalent to the Hamiltonion system

$$\frac{du}{dt} = \frac{\partial H}{\partial v}, \qquad \frac{dv}{dt} = -\frac{\partial H}{\partial u}$$

i.e. find the Hamilton function $H(u,v)$.

Problem 59. (i) Show that the second Painlevé equation $P_{II}(\alpha)$ ($\alpha \in \mathbb{C}$) is the Hamilton system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

where the Hamilton function is

$$H(p,q,t,\alpha) = \frac{p^2}{2} - \left(q^2 + \frac{t}{2}\right)p - \alpha q.$$

(ii) Let (p,q) be a solution to $P_{II}(\alpha)$. Show that birational canonical transformations defined by

$$s(p,q) = (q + \frac{\alpha}{p}, p), \qquad \pi(p,q) = (-q, -p + 2q^2 + t)$$

give solutions to $P_{II}(-\alpha)$ and $P_{II}(1-\alpha)$, respectively.

Problem 60. The equation of motion for a particle of unit mass moving in a conservative central force field is given by the second order differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = f(r)\mathbf{r}$$

where $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$.
(i) Show that the angular momentum

$$\mathbf{L} := \mathbf{r} \times \frac{d\mathbf{r}}{dt}$$

is a constant of motion, where $\times$ denotes the vector product.
(ii) Show that the total energy given by

$$H = \frac{1}{2}\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt} - \frac{1}{2}\int^r f(\epsilon)d\epsilon$$

is a constant of motion, where $\cdot$ denotes the scalar product.

Problem 61. (i) Show that the second Painlevé equation $P_{II}(\alpha)$ ($\alpha \in \mathbb{C}$) is the Hamilton system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

where

$$H(q,p,t,\alpha) = \frac{p^2}{2} - \left(q^2 + \frac{t}{2}\right)p - \alpha q.$$

(ii) Let (q,p) be a solution to $P_{II}(\alpha)$. Show that the birational canonical transformations defined by

$$s(q,p) = (q + \frac{\alpha}{p}, p), \qquad \pi(q,p) = (-q, -p + 2q^2 + t)$$

give solutions to $P_{II}(-\alpha)$, $P_{II}(1-\alpha)$, respectively.

Problem 62. Show that the non-relativistic Coulomb Hamilton function

$$H = \frac{1}{2m}\mathbf{p}^2 + \frac{\alpha}{r}$$

possesses the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and the Lenz vector

$$\mathbf{A} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} + \mathbf{p}) + \alpha\hat{\mathbf{r}}$$

as vector invariants, where $\hat{\mathbf{r}} := \mathbf{r}/r$.

Problem 63. Let g_1, g_2, g_3 be positive constants. Consider the Hamilton function (Calogero potential)

$$H_C(p_1,p_2,p_3,q_1,q_2,q_3) =$$

$$\frac{1}{2}(p_1^2+p_2^2+p_3^2)+\frac{1}{6}((q_1-q_2)^2+(q_2-q_3)^2+(q_3-q_1)^2)+\frac{g_1}{(q_1-q_2)^2}+\frac{g_2}{(q_2-q_3)^2}+\frac{g_3}{(q_3-q_1)^2}.$$

Transform the Hamilton function to the centre of mass and Jacobi coordinates R, x, y

$$R(q_1,q_2,q_3) = \frac{1}{3}(q_1+q_2+q_3)$$

$$x(q_1,q_2,q_3) = \frac{1}{\sqrt{2}}(q_1 - q_2)$$
$$y(q_1,q_2,q_3) = \frac{1}{\sqrt{6}}(q_1 + q_2 - 2q_3).$$

Show that the centre of mass only executes free motion, and the (x, y) dynamics is described by the reduced Hamilton function

$$H(p_x,p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{g_1}{2x^2} + \frac{2g_2}{(x - \sqrt{3}y)^2} + \frac{2g_3}{(x + \sqrt{3}y)^2}.$$

Problem 64. Consider the Hamilton function

$$H = \frac{1}{2}(p_1^2+p_2^2+p_3^2)+\frac{1}{6}((q_1-q_2)^2+(q_2-q_3)^2+(q_3-q_1)^2)+\frac{g_1}{(x_1-x_2)^2}+\frac{g_2}{(q_2-q_3)^2}+\frac{g_3}{(q_3-q_1)^2}$$

where g_1, g_2, g_3 are positive constants. Transform the Hamilton function to the centre of mass and *Jacobi coordinates*

$$R(q_1,q_2,q_3) = \frac{1}{3}(q_1 + q_2 + q_3)$$
$$x(q_1,q_2,q_3) = \frac{1}{\sqrt{2}}(q_1 - q_2)$$
$$y(q_1,q_2,q_3) = \frac{1}{\sqrt{6}}(q_1 + q_2 - 2q_3).$$

Problem 65. Consider the symmetry operation

$$\text{Inversion } P : (q_1,q_2,p_1,p_2) \mapsto (-q_1,-q_2,-p_1,-p_2)$$
$$\text{Time reversal } T : (q_1,q_2,p_1,p_2) \mapsto (q_1,q_2,-p_1,-p_2)$$
$$\text{Reflection } S_1 : (q_1,q_2,p_1,p_2) \mapsto (-q_1,q_2,-p_1,p_2).$$

Does the Hamilton function

$$H(q_1,q_2,p_1,p_2) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + c\frac{1}{4}q_1^4 q_2^4$$

satisfies these symmetries?

Problem 66. Consider the Hamilton function H for a three body problem

$$H(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) = \frac{1}{2M}\mathbf{p}_1^2+\frac{1}{2M}\mathbf{p}_2^2+\frac{1}{2m}\mathbf{p}_3^2+V(\mathbf{q}_1-\mathbf{q}_2,\mathbf{q}_2-\mathbf{q}_3,\mathbf{q}_3-\mathbf{q}_1).$$

Show that the Hamilton function and equations of motion can be simplified by introducing *Jacobi variables*

$$\mathbf{R} = \frac{M(\mathbf{q}_1 + \mathbf{q}_2) + m\mathbf{q}_3}{2M + m}, \quad \mathbf{x} = \mathbf{q}_2 - \mathbf{q}_1, \quad \mathbf{y} = \frac{\sqrt{m}}{\sqrt{2M + m}}(2\mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_2).$$

Problem 67. Assume that a autonomous system of ordinary differential equations can be written as

$$\frac{d\mathbf{x}}{dt} = J(\mathbf{x})\nabla H(\mathbf{x})$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, J, H are differentiable functions and J an antisymmetric $n \times n$ matrix. Show that $dH/dt = 0$.

Problem 68. Consider the Hamilton function $H : \mathbb{R}^4 \to \mathbb{R}$

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2) + \frac{1}{2}g^2(q_1 q_2)^2$$

with $H = H_0 + H_1$, where

$$H_0(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2), \qquad H_1(\mathbf{p}, \mathbf{q}) = \frac{1}{2}g^2(q_1 q_2)^2.$$

(i) Introduce the new variables J_i, φ_i $(i = 1, 2)$

$$q_i = (2J_i/\omega)^{1/2} \sin\varphi_i, \qquad p_i = (2J_i\omega)^{1/2} \cos\varphi_i$$

and write down the Hamilton function.
(ii) Introduce the variables

$$j_1 = J_1 + J_2, \qquad \phi_1 = \frac{1}{2}(\varphi_1 + \varphi_2)$$

$$j_2 = J_1 - J_2, \qquad \phi_2 = \frac{1}{2}(\varphi_1 - \varphi_2)$$

and write down the Hamilton equations of motion. Find the Hamilton equations of motion for H_0. Discuss.
(iii) Calculate

$$\overline{H}_1(\mathbf{j}, \phi_2) = \frac{1}{2\pi}\int_0^{2\pi} H_1(\mathbf{j}, \boldsymbol{\phi}) d\phi_1.$$

This means we average the Hamilton function $H_1(\mathbf{j}, \boldsymbol{\phi})$ over the fast variable and extracting the secular part of H_1. Discuss the Hamilton equations of motion for the Hamilton function $H = H_0 + \overline{H}_1$.

Problem 69. The equations of motion of a solid in an ideal fluid have the form

$$\frac{dp_1}{dt} = p_2\frac{\partial H}{\partial \ell_3} - p_3\frac{\partial H}{\partial \ell_2}$$
$$\frac{dp_2}{dt} = p_3\frac{\partial H}{\partial \ell_1} - p_1\frac{\partial H}{\partial \ell_3}$$

$$\frac{dp_3}{dt} = p_1\frac{\partial H}{\partial \ell_2} - p_2\frac{\partial H}{\partial \ell_1}$$
$$\frac{d\ell_1}{dt} = p_2\frac{\partial H}{\partial p_3} - p_3\frac{\partial H}{\partial p_2} + \ell_2\frac{\partial H}{\partial \ell_3} - \ell_3\frac{\partial H}{\partial \ell_2}$$
$$\frac{d\ell_2}{dt} = p_3\frac{\partial H}{\partial p_1} - p_1\frac{\partial H}{\partial p_3} + \ell_3\frac{\partial H}{\partial \ell_1} - \ell_1\frac{\partial H}{\partial \ell_3}$$
$$\frac{d\ell_3}{dt} = p_1\frac{\partial H}{\partial p_2} - p_2\frac{\partial H}{\partial p_1} + \ell_1\frac{\partial H}{\partial \ell_2} - \ell_2\frac{\partial H}{\partial \ell_1}$$

with the Hamilton function

$$H(\mathbf{p},\ell) = \frac{1}{2}\sum_{j,k=1}^{3}(a_{jk}\ell_j\ell_k + 2b_{jk}\ell_j p_k + c_{jk}p_j p_k).$$

Show that besides $H = I_1$ we have the first integrals

$$I_2(\mathbf{p},\ell) = p_1^2 + p_2^2 + p_3^2, \qquad I_3(\mathbf{p},\ell) = p_1\ell_1 + p_2\ell_2 + p_3\ell_3.$$

Problem 70. Show that

$$G_j(t) := tP_j^{kin} - MR_j, \qquad j = 1,2,3$$

where

$$P_j^{kin} := \sum_{k=1}^{N} p_{kj}, \quad M := \sum_{k=1}^{N} m_k, \quad R_j := \frac{1}{M}\sum_{k=1}^{N} m_k q_{kj}$$

are explictly time-dependent first integrals for the Hamilton system

$$H(\mathbf{p},\mathbf{q}) = \sum_{k=1}^{N}\sum_{j=1}^{3}\frac{p_{kj}^2}{2m_k} + \frac{1}{2}\sum_{\substack{k,\ell=1\\k\neq\ell}}^{N}\sum_{j=1}^{3} V_{k\ell}(|\mathbf{q}_k - \mathbf{q}_\ell|).$$

Problem 71. Given a smooth Hamilton function

$$H(\mathbf{p},\mathbf{q}) = \sum_{j=1}^{n}\frac{p_j^2}{2} + U(\mathbf{q})$$

with n degrees of freedom ($\mathbf{p} = (p_1,\ldots,p_n)$, $\mathbf{q} = (q_1,\ldots,q_n)$). Let $V(E)$ be the classical phase space volume at energy E of a smooth Hamilton function is given by

$$V(E) = \int_{\mathbb{R}^{2n}} \Theta(E - H(\mathbf{p},\mathbf{q}))d^n\mathbf{p}d^n\mathbf{q}$$

where Θ is the step function. Assume that $U(\epsilon\mathbf{q}) = \epsilon^m U(\mathbf{q})$.
(i) Consider the transformation

$$\mathbf{p} = E^{1/2}\mathbf{p}', \qquad \mathbf{q} = E^{1/n}\mathbf{q}'$$

with the inverse transformation

$$\mathbf{p}' = E^{-1/2}\mathbf{p}, \qquad \mathbf{q}' = E^{-1/n}\mathbf{q}.$$

Find $d^n\mathbf{p}'d^n\mathbf{q}'$ and $H(\mathbf{p}',\mathbf{q}')$.
(ii) Calculate $V(E)$ with the assumption that $E > 0$. Find the asymptotic behaviour.

Problem 72. Consider the three-particle nonrelativistic Schrödinger eigenvalue equation (MKSA-system)

$$\left(-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{R}_0} - \frac{\hbar^2}{2m}\nabla^2_{\mathbf{R}_1} - \frac{\hbar^2}{2m}\nabla^2_{\mathbf{R}_2} - \frac{Ze^2}{4\pi\epsilon_0|\mathbf{R}_0-\mathbf{R}_1|} - \frac{Ze^2}{4\pi\epsilon_0|\mathbf{R}_0-\mathbf{R}_2|}\right.$$
$$\left.+\frac{e^2}{4\pi\epsilon_0|\mathbf{R}_1-\mathbf{R}_2|}\right)u(\mathbf{R}_0,\mathbf{R}_1,\mathbf{R}_2) = Eu(\mathbf{R}_0,\mathbf{R}_1,\mathbf{R}_2)$$

where $\mathbf{R}_0 = (R_{01},R_{02},R_{03})^T$ is the position vector of the nucleus of mass M, and $\mathbf{R}_1 = (R_{11},R_{12},R_{13})^T$ and $\mathbf{R}_2 = (R_{21},R_{22},R_{23})^T$ are the position vectors of the two electrons of mass m and $Z = 2$. The *Jacobi coordinates* are given by

$$\begin{aligned}\mathbf{r} &= (\mathbf{R}_1 - \mathbf{R}_0)/a_\mu\\ \mathbf{x} &= \Lambda(\mathbf{R}_2 - \mathbf{R}_0 - y(\mathbf{R}_1 - \mathbf{R}_0))/a_\mu\\ \mathbf{X} &= \Lambda(\mathbf{R}_0 + y(\mathbf{R}_1 + \mathbf{R}_2 - \mathbf{R}_0))/a_\mu\end{aligned}$$

where $\mathbf{r} = (r_1,r_2,r_3)^T$, $\mathbf{x} = (x_1,x_2,x_3)^T$, $\mathbf{X} = (X_1,X_2,X_3)^T$,

$$\mu = \frac{mM}{m+M}, \qquad y := \mu/M, \qquad \Lambda := 1/(1-y^2)$$

and $a_\mu = (m/\mu)a_0$ is the reduced Bohr radius with $a_0 = (4\pi\epsilon_0\hbar^2)/(me^2)$. Thus $\mathbf{r}$, $\mathbf{x}$, $\mathbf{X}$ are dimensionless.
(i) Find the inverse of this transformation.
(ii) Express the Hamilton operator

$$\hat{H} = \left(-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{R}_0} - \frac{\hbar^2}{2m}\nabla^2_{\mathbf{R}_1} - \frac{\hbar^2}{2m}\nabla^2_{\mathbf{R}_2}\right.$$
$$\left.-\frac{Ze^2}{4\pi\epsilon_0|\mathbf{R}_0-\mathbf{R}_1|} - \frac{Ze^2}{4\pi\epsilon_0|\mathbf{R}_0-\mathbf{R}_2|} + \frac{e^2}{4\pi\epsilon_0|\mathbf{R}_1-\mathbf{R}_2|}\right)$$

in this coordinates.

Problem 73. Consider a bead of mass m slides frictionless upon a smooth circular wire of radius r. The wire rotates with constant frequency ω about a vertical axis parallel to the earth's gravitational field. Consider the bifurcation parameter $\mu := \omega^2 r/g$. Find the value μ_c of the bifurcation parameter for which there is a bifurcation. The kinetic energy is

$$T(\theta, \dot{\theta}) = \frac{mr^2}{2} \left(\left(\frac{d\theta}{dt} \right)^2 + \omega^2 \sin^2 \theta \right)$$

The potential energy is

$$V(\theta) = mgr(1 - \cos\theta)$$

with the Lagrange function $L = T - V$.

Problem 74. (i) Consider the Hamilton function

$$H(\theta, p, t) = \frac{1}{2I} p^2 + mB_0 \cos(\theta) \sin(\omega t)$$

which depends explicitly on time and $B = B_0 \sin(\omega t)$ is a time periodic magnetic field. I is the moment of inertia of the dipole and m the dipole moment. Find the equation of motion. Does the system show chaotic behaviour depending on B_0?

Problem 75. (i) Show that the second order ordinary differential equation

$$1 + \left(\frac{du}{dt} \right)^2 - u \frac{d^2u}{dt^2} = 0$$

can be derived from the Lagrange function

$$L(u, \dot{u}) = u\sqrt{(\dot{u})^2 + 1}.$$

(ii) Setting $u(t) = \exp(v(t))$ show that the differential equation takes the form

$$\frac{d^2v}{dt^2} = e^{-2v(t)}.$$

Problem 76. (i) Consider the damped anharmonic oscillator

$$\frac{d^2x}{dt^2} + c_1 \frac{dx}{dt} + c_2 x + x^3 = 0$$

where c_1, c_2 are constants. Show that the equation of motion can be derived from the explicitly time-dependent Lagrange function

$$L(t, x(t), \dot{x}(t)) = \frac{1}{2} e^{c_1 t} \dot{x}^2 - e^{c_1 t} V(x)$$

where

$$V(x) = \int_0^x f(s)ds$$

where the function f is given by $f(x) = c_2 x + x^3$.
(ii) Show that the corresponding Hamilton function is given by

$$H(t, x(t), p(t)) = \frac{1}{2} e^{-c_1 t} p^2 + e^{c_1 t} V(x).$$

Problem 77. Consider three masses m_A, m_B, m_C and the Hamilton function

$$H = \frac{\mathbf{p}_A^2}{2m_A} + \frac{\mathbf{p}_B^2}{2m_B} + \frac{\mathbf{p}_C^2}{2m_C} + V_A(\mathbf{q}_B - \mathbf{q}_C) + V_B(\mathbf{q}_C - \mathbf{q}_A) + V_C(\mathbf{q}_A - \mathbf{q}_B).$$

Let $M = m_A + m_B + m_C$.
(i) Are the total momentum

$$\mathbf{P} = \mathbf{p}_A + \mathbf{p}_B + \mathbf{p}_C$$

and the position of the centre of mass

$$\mathbf{S} = \frac{1}{M}(m_A \mathbf{q}_A + m_B \mathbf{q}_B + m_C \mathbf{q}_C)$$

are constants of motions?
(ii) Show that the centre of mass can be separated out.

Problem 78. Consider the Lagrange function

$$L(\phi(t), \theta(t), \dot{\phi}(t), \dot{\theta}(t)) = r\sqrt{(\dot{\phi})^2 \sin^2(\theta) + (\dot{\theta})^2}.$$

The Euler-Lagrange equation takes the form

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0, \quad \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0.$$

Write down the Euler-Lagrange equation and solve the initial value problem.

Bibliography

Books

Arrowsmith D. K. and Place C. M.
An introduction to Dynamical Systems, Cambridge University Press, 1990

Arrowsmith D. K. and Place C. M.
Dynamical Systems: Differential equations, maps and chaotic behaviour, Chapman and Hall, 1992

Collatz L *The Numerical Treatment of Differential Equations*, Springer, Berlin, 1960

Davis H. T.
Introduction to Nonlinear Differential and Integral Equations, Dover, New York, 1962

Hairer E., Norsett S. P. and Wanner G.
Solving Ordinary Differential Equations I, Nonstiff Problems, second revised edition, Springer, 1993

Ince E. L.
Ordinary Differential Equations, Dover, New York, 1926

Steeb W.-H.
The Nonlinear Workbook, fifth edition, World Scientific Publishing, Singapore, 2011

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume I: Introductory Level
World Scientific Publishing, Singapore (2009)

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume II: Adanced Level
World Scientific Publishing, Singapore (2009)

Index