Problems and Solutions in Matrix Calculus

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Preface

The manuscript supplies a collection of problems in introductory and advanced matrix problems.

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Contents

Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\|z\|$	modulus of complex number z
	$\|x+iy\| = (x^2+y^2)^{1/2}, \ \ x, y \in \mathbb{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbb{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\\| . \\|$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbb{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbf{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
rank(A)	rank of matrix A
A^T	transpose of matrix A

$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

The Pauli spin matrices are used extensively in the book. They are given by

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In some cases we will also use σ_1, σ_2 and σ_3 to denote σ_x, σ_y and σ_z .

Chapter 1

Basic Operations

Problem 1. Let $\mathbf{x}$ be a column vector in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$. Let

$$A = \frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}$$

where ^{T} denotes the transpose, i.e. $\mathbf{x}^T$ is a row vector. Calculate A^2.

Problem 2. Consider the 8×8 *Hadamard matrix*

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

(i) Do the 8 column vectors in the matrix H form a basis in $\mathbb{R}^8$? Prove or disprove.
(ii) Calculate HH^T, where ^{T} denotes transpose. Compare the results from (i) and (ii) and discuss.

Problem 3. Show that any 2×2 complex matrix has a unique representation of the form

$$a_0 I_2 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3$$

for some $a_0, a_1, a_2, a_3 \in \mathbb{C}$, where I_2 is the 2×2 identity matrix and σ_1, σ_2, σ_3 are the *Pauli spin matrices*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 4. Let A, B be $n \times n$ matrices such that $ABAB = 0_n$. Can we conclude that $BABA = 0_n$?

Problem 5. A square matrix A over $\mathbb{C}$ is called *skew-hermitian* if $A = -A^*$. Show that such a matrix is *normal*, i.e., we have $AA^* = A^*A$.

Problem 6. Let A be an $n \times n$ skew-hermitian matrix over $\mathbb{C}$, i.e. $A^* = -A$. Let U be an $n \times n$ *unitary matrix*, i.e., $U^* = U^{-1}$. Show that $B := U^*AU$ is a skew-hermitian matrix.

Problem 7. Let A, X, Y be $n \times n$ matrices. Assume that

$$XA = I_n, \qquad AY = I_n$$

where I_n is the $n \times n$ unit matrix. Show that $X = Y$.

Problem 8. Let A, B be $n \times n$ matrices. Assume that A is nonsingular, i.e. A^{-1} exists. Show that if $BA = 0_n$, then $B = 0_n$.

Problem 9. Let A, B be $n \times n$ matrices and

$$A + B = I_n, \qquad AB = 0_n.$$

Show that $A^2 = A$ and $B^2 = B$.

Problem 10. Consider the normalized vectors in $\mathbb{R}^2$

$$\begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{pmatrix}, \qquad \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \end{pmatrix}.$$

Find the condition on θ_1 and θ_2 such that

$$\begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{pmatrix} + \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \end{pmatrix}$$

is normalized. A vector $\mathbf{x} \in \mathbb{R}^n$ is called *normalized* if $\|\mathbf{x}\| = 1$, where $\| \ \|$ denotes the Euclidean norm.

Problem 11. Let

$$A := \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix}$$

and $\theta \in \mathbb{R}$. Find $\mathbf{x}^T\mathbf{x}$, $\mathbf{y}^T\mathbf{y}$, $\mathbf{x}^T\mathbf{y}$, $\mathbf{y}^T\mathbf{x}$. Find the matrix A.

Problem 12. Find a 2×2 matrix A over $\mathbb{R}$ such that

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 13. Consider the 2×2 matrix over the complex numbers

$$\Pi(\mathbf{n}) := \frac{1}{2}\left(I_2 + \sum_{j=1}^{3} n_j \sigma_j\right)$$

where $\mathbf{n} := (n_1, n_2, n_3)$ ($n_j \in \mathbb{R}$) is a unit vector, i.e., $n_1^2 + n_2^2 + n_3^2 = 1$. Here σ_1, σ_2, σ_3 are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and I_2 is the 2×2 unit matrix.
(i) Describe the property of $\Pi(\mathbf{n})$, i.e., find $\Pi^*(\mathbf{n})$, $\mathrm{tr}(\Pi(\mathbf{n}))$ and $\Pi^2(\mathbf{n})$, where tr denotes the trace. The trace is the sum of the diagonal elements of a square matrix.
(ii) Find the vector

$$\Pi(\mathbf{n})\begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Discuss.

Problem 14. Let

$$\mathbf{x} = \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

where $\phi, \theta \in \mathbb{R}$.
(i) Find the matrix $\rho := \mathbf{x}\mathbf{x}^*$.
(ii) Find $\mathrm{tr}\rho$.
(iii) Find ρ^2.

Problem 15. Consider the vector space $\mathbb{R}^4$. Find all pairwise orthogonal vectors (column vectors) $\mathbf{x}_1, \ldots, \mathbf{x}_p$, where the entries of the column vectors can only be $+1$ or -1. Calculate the matrix

$$\sum_{j=1}^{p} \mathbf{x}_j \mathbf{x}_j^T$$

and find the eigenvalues and eigenvectors of this matrix.

Problem 16. Let
$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$
(i) Let X be an $m \times n$ matrix. The *column rank* of X is the maximum number of linearly independent columns. The *row rank* is the maximum number of linearly independent rows. The row rank and the column rank of X are equal (called the *rank* of X). Find the rank of A and denote it by k.
(ii) Locate a $k \times k$ submatrix of A having rank k.
(iii) Find 3×3 permutation matrices P and Q such that in the matrix PAQ the submatrix from (ii) is in the upper left portion of A.

Problem 17. Find 2×2 matrices A, B such that $AB = 0_n$ and $BA \neq 0_n$.

Problem 18. Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. Then the *direct sum* of A and B, denoted by $A \oplus B$, is the $(m+p) \times (n+q)$ matrix defined by
$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$
Let A_1, A_2 be $m \times m$ matrices and B_1, B_2 be $n \times n$ matrices. Calculate
$$(A_1 \oplus B_1)(A_2 \oplus B_2).$$

Problem 19. Let A be an $n \times n$ matrix over $\mathbb{R}$. Find all matrices that satisfy the equation $A^T A = 0_n$.

Problem 20. Let π be a permutation on $\{1, 2, \ldots, n\}$. The matrix P_π for which $p_{i*} = e_{\pi(i)*}$ is called the *permutation matrix* associated with π, where p_{i*} is the ith row of P_π and $e_{ij} = 1$ if $i = j$ and 0 otherwise. Let $\pi = (3\ 2\ 4\ 1)$. Find P_π.

Problem 21. A matrix A for which $A^p = 0_n$, where p is a positive integer, is called *nilpotent*. If p is the least positive integer for which $A^p = 0_n$ then A is said to be nilpotent of index p. Find all 2×2 matrices over the real numbers which are nilpotent with $p = 2$, i.e. $A^2 = 0_2$.

Problem 22. A square matrix is called *idempotent* if $A^2 = A$. Find all 2×2 matrices over the real numbers which are idempotent and $a_{ij} \neq 0$ for $i, j = 1, 2$.

Problem 23. A square matrix A such that $A^2 = I_n$ is called *involutory*. Find all 2×2 matrices over the real numbers which are involutory. Assume that $a_{ij} \neq 0$ for $i, j = 1, 2$.

Problem 24. Show that an $n \times n$ matrix A is involutary if and only if $(I_n - A)(I_n + A) = 0_n$.

Problem 25. Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$. Let P be an arbitrary $n \times n$ matrix over $\mathbb{R}$. Show that $P^T AP$ is symmetric.

Problem 26. Let A be an $n \times n$ skew-symmetric matrix over $\mathbb{R}$, i.e. $A^T = -A$. Let P be an arbitrary $n \times n$ matrix over $\mathbb{R}$. Show that $P^T AP$ is skew-symmetric.

Problem 27. Let A be an $m \times n$ matrix. The *column rank* of A is the maximum number of linearly independent columns. The *row rank* is the maximum number of linearly independent rows. The row rank and the column rank of A are equal (called the rank of A). Find the rank of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Problem 28. Let A be an invertible $n \times n$ matrix over $\mathbb{C}$ and B be an $n \times n$ matrix over $\mathbb{C}$. We define the $n \times n$ matrix

$$D := A^{-1}BA.$$

Calculate D^n, where $n = 2, 3, \ldots$.

Problem 29. A *Cartan matrix* A is a square matrix whose elements a_{ij} satisfy the following conditions:
1. a_{ij} is an integer, one of $\{ -3, -2, -1, 0, 2 \}$
2. $a_{jj} = 2$ for all diagonal elements of A
3. $a_{ij} \leq 0$ off of the diagonal
4. $a_{ij} = 0$ iff $a_{ji} = 0$
5. There exists an invertible diagonal matrix D such that DAD^{-1} gives a symmetric and positive definite quadratic form.

Give a 2×2 non-diagonal Cartan matrix.

Problem 30. Let A, B, C, D be $n \times n$ matrices over $\mathbb{R}$. Assume that AB^T and CD^T are symmetric and $AD^T - BC^T = I_n$, where ^{T} denotes transpose. Show that

$$A^T D - C^T B = I_n.$$

Problem 31. Let n be a positive integer. Let A_n be the $(2n+1)\times(2n+1)$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Give A_1 and A_2. Find the rank of A_n.

Problem 32. A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is called a *probability vector* if the components are nonnegative and their sum is 1. Is the vector

$$\mathbf{u} = (1/2,\ 0,\ 1/4,\ 1/4)$$

a probability vector? Can the vector

$$\mathbf{v} = (2,\ 3,\ 5,\ 1,\ 0)$$

be "normalized" so that we obtain a probability vector?

Problem 33. An $n \times n$ matrix $P = (p_{ij})$ is called a *stochastic matrix* if each of its rows is a probability vector, i.e., if each entry of P is nonnegative and the sum of the entries in each row is 1. Let A and B be two stochastic $n \times n$ matrices. Is the matrix product AB also a stochastic matrix?

Problem 34. The *numerical range*, also known as the *field of values*, of an $n \times n$ matrix A over the complex numbers, is defined as

$$F(A) := \{\, \mathbf{z}^* A \mathbf{z} \,:\, \|\mathbf{z}\| = 1,\ \mathbf{z} \in \mathbb{C}^n \,\}.$$

Find the numerical range for the 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Find the numerical range for the 2×2 matrix

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

The *Toeplitz-Hausdorff convexity theorem* tells us that the numerical range of a square matrix is a convex compact subset of the complex plane.

Problem 35. Let A be an $n \times n$ matrix over $\mathbb{C}$. The *field of values* of A is defined as the set

$$F(A) := \{\, \mathbf{z}^* A \mathbf{z} \,:\, \mathbf{z} \in \mathbb{C}^n\ ,\ \mathbf{z}^* \mathbf{z} = 1 \,\}.$$

Let $\alpha \in \mathbb{R}$ and

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha \end{pmatrix}.$$

(i) Show that the set $F(A)$ lies on the real axis.
(ii) Show that

$$|\mathbf{z}^* A \mathbf{z}| \leq \alpha + 16.$$

Problem 36. Let A be an $n \times n$ matrix over $\mathbb{C}$ and $F(A)$ the field of values. Let U be an $n \times n$ unitary matrix.
(i) Show that $F(U^* A U) = F(A)$.
(ii) Apply the theorem to the two matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which are unitarily equivalent.

Problem 37. Can one find a unitary matrix U such that

$$U^* \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} U = \begin{pmatrix} 0 & ce^{i\theta} \\ de^{-i\theta} & 0 \end{pmatrix}$$

where $c, d \in \mathbb{C}$ and $\theta \in \mathbb{R}$?

Problem 38. An $n^2 \times n$ matrix J is called a *selection matrix* such that J^T is the $n \times n^2$ matrix

$$[E_{11}\ E_{22}\ \ldots\ E_{nn}]$$

where E_{ii} is the $n \times n$ matrix of zeros except for a 1 in the (i,i)th position.
(i) Find J for $n = 2$ and calculate $J^T J$.
(ii) Calculate $J^T J$ for arbitrary n.

Problem 39. Consider a symmetric matrix A over $\mathbb{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

and the orthonormal basis (so-called *Bell basis*)

$$\mathbf{x}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{x}^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{y}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{y}^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

The Bell basis forms an orthonormal basis in $\mathbb{R}^4$. Let $\widetilde{A}$ denote the matrix A in the Bell basis. What is the condition on the entries a_{ij} such that the matrix A is diagonal in the Bell basis?

Problem 40. Let A be an $m \times n$ matrix over $\mathbb{C}$. The *Moore-Penrose pseudoinverse matrix* A^+ is the unique $n \times m$ matrix which satisfies

$$\begin{aligned} AA^+A &= A \\ A^+AA^+ &= A^+ \\ (AA^+)^* &= AA^+ \\ (A^+A)^* &= A^+A. \end{aligned}$$

We also have that

$$\mathbf{x} = A^+\mathbf{b} \tag{1}$$

is the shortest length least square solution to the problem

$$A\mathbf{x} = \mathbf{b}. \tag{2}$$

(i) Show that if $(A^*A)^{-1}$ exists, then $A^+ = (A^*A)^{-1}A^*$.
(ii) Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{pmatrix}.$$

Find the Moore-Penrose matrix inverse A^+ of A.

Problem 41. A *Hadamard matrix* is an $n \times n$ matrix H with entries in $\{-1, +1\}$ such that any two distinct rows or columns of H have inner product 0. Construct a 4×4 Hadamard matrix starting from the column vector

$$\mathbf{x}_1 = (1\ \ 1\ \ 1\ \ 1)^T.$$

Problem 42. A *binary Hadamard matrix* is an $n \times n$ matrix M (where n is even) with entries in $\{0, 1\}$ such that any two distinct rows or columns of M have *Hamming distance* $n/2$. The Hamming distance between two vectors is the number of entries at which they differ. Find a 4×4 binary Hadamard matrix.

Problem 43. Let $\mathbf{x}$ be a normalized column vector in $\mathbb{R}^n$, i.e. $\mathbf{x}^T\mathbf{x} = 1$. A matrix T is called a *Householder matrix* if

$$T := I_n - 2\mathbf{x}\mathbf{x}^T.$$

Calculate T^2.

Problem 44. An $n \times n$ matrix P is a *projection matrix* if

$$P^* = P, \qquad P^2 = P.$$

(i) Let P_1 and P_2 be projection matrices. Is $P_1 + P_2$ a projection matrix?
(ii) Let P_1 and P_2 be projection matrices. Is P_1P_2 a projection matrix?
(iii) Let P be a projection matrix. Is $I_n - P$ a projection matrix? Calculate $P(I_n - P)$.
(iv) Is

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

a projection matrix?

Problem 45. Let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

be vectors in $\mathbb{R}^3$. Let $\times$ denote the vector product.
(i) Show that we can find a 3×3 matrix $S(\mathbf{a})$ such that

$$\mathbf{a} \times \mathbf{b} = S(\mathbf{a})\mathbf{b}.$$

(ii) Express the *Jacobi identity*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{0}$$

using the matrices $S(\mathbf{a})$, $S(\mathbf{b})$ and $S(\mathbf{c})$.

Problem 46. Let s (*spin quantum number*)

$$s \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \right\}.$$

Given a fixed s. The indices j,k run over $s, s-1, s-2, \ldots, -s+1, -s$. Consider the $(2s+1)$ unit vectors (standard basis)

$$\mathbf{e}_{s,s} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_{s,s-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \mathbf{e}_{s,-s} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Obviously the vectors have $(2s+1)$ components. The $(2s+1) \times (2s+1)$ matrices s_+ and s_- are defined as

$$s_+ \mathbf{e}_{s,m} := \hbar\sqrt{(s-m)(s+m+1)}\mathbf{e}_{s,m+1}, \quad m = s-1, s-2, \ldots, -s$$
$$s_- \mathbf{e}_{s,m} := \hbar\sqrt{(s+m)(s-m+1)}\mathbf{e}_{s,m-1}, \quad m = s, s-1, \ldots, -s+1$$

and $s_+\mathbf{e}_{ss} = \mathbf{0}$, $s_-\mathbf{e}_{s-s} = \mathbf{0}$, where $\hbar$ is the Planck constant. We have

$$s_+ := \frac{1}{2}(s_x + is_y), \quad s_- := \frac{1}{2}(s_x - is_y).$$

Thus

$$s_x = s_+ + s_-, \qquad s_y = -i(s_+ - s_-).$$

(i) Find the matrix representation of s_+ and s_-.
(ii) The $(2s+1) \times (2s+1)$ matrix s_z is defined as (eigenvalue equation)

$$s_z \mathbf{e}_{s,m} := m\hbar \mathbf{e}_{s,m}, \quad m = s, s-1, \ldots, -s.$$

Let $\mathbf{s} := (s_x, s_y, s_z)$. Find the $(2s+1) \times (2s+1)$ matrix

$$\mathbf{s}^2 := s_x^2 + s_y^2 + s_z^2 \, .$$

(iii) Calculate the expectation values

$$\mathbf{e}_{s,s}^* s_+ \mathbf{e}_{s,s}, \quad \mathbf{e}_{s,s}^* s_- \mathbf{e}_{s,s}, \quad \mathbf{e}_{s,s}^* s_z \mathbf{e}_{s,s}.$$

Problem 47. The *Fibonacci numbers* are defined by the recurrence relation (linear difference equation of second order with constant coefficients)

$$s_{n+2} = s_{n+1} + s_n$$

where $n = 0, 1, \ldots$ and $s_0 = 0$, $s_1 = 1$. Write this recurrence relation in matrix form. Find s_6, s_5, and s_4.

Problem 48. (i) Find four unit (column) vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ in $\mathbb{R}^3$ such that

$$\mathbf{x}_j^T \mathbf{x}_k = \frac{4}{3}\delta_{jk} - \frac{1}{3} = \begin{cases} 1 & \text{for } j = k \\ -1/3 & \text{for } j \neq k. \end{cases}$$

Give a geometric interpretation.
(ii) Calculate the sum

$$\sum_{j=1}^{4} \mathbf{x}_j .$$

(iii) Calculate the sum

$$\sum_{j=1}^{4} \mathbf{x}_j \mathbf{x}_j^T .$$

Problem 49. Assume that

$$A = A_1 + iA_2$$

is a nonsingular $n \times n$ matrix, where A_1 and A_2 are real $n \times n$ matrices. Assume that A_1 is also nonsingular. Find the inverse of A using the inverse of A_1.

Problem 50. Let A and B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A \neq B$, $A^3 = B^3$ and $A^2 B = B^2 A$. Is $A^2 + B^2$ invertible?

Problem 51. Let A be a positive definite $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{x} \in \mathbb{R}$. Show that $A + \mathbf{x}\mathbf{x}^T$ is also positive definite.

Problem 52. Let A, B be $n \times n$ matrices over $\mathbb{C}$. The matrix A is called *similar* to the matrix B if there is an $n \times n$ invertible matrix S such that

$$A = S^{-1} B S.$$

If A is similar to B, then B is also similar to A, since $B = SAS^{-1}$.
(i) Consider the two matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Are the matrices similar?
(ii) Consider the two matrices

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Are the matrices similar?

Problem 53. Normalize the vector in $\mathbb{R}^2$

$$\mathbf{v} = \begin{pmatrix} \sqrt{1 + \sin(\alpha)} \\ \sqrt{1 - \sin(\alpha)} \end{pmatrix}.$$

Then find a normalized vector in $\mathbb{R}^2$ which is orthonormal to this vector.

Chapter 2

Linear Equations

Problem 1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Find the solutions of the system of linear equations $A\mathbf{x} = \mathbf{b}$.

Problem 2. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ \alpha \end{pmatrix}$$

where $\alpha \in \mathbb{R}$. What is the condition on α so that there is a solution of the equation $A\mathbf{x} = \mathbf{b}$?

Problem 3. (i) Find all solutions of the system of linear equations

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ -\sin(\theta) & -\cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

(ii) What type of equation is this?

Problem 4. Let $A \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. Consider the linear equation $A\mathbf{x} = \mathbf{b}$. Show that it can be written as $\mathbf{x} = T\mathbf{x}$, i.e., find $T\mathbf{x}$.

Problem 5. If the system of linear equations $A\mathbf{x} = \mathbf{b}$ admits no solution we call the equations inconsistent. If there is a solution, the equations are called consistent. Let $A\mathbf{x} = \mathbf{b}$ be a system of m linear equations in n unknowns and suppose that the rank of A is m. Show that in this case $A\mathbf{x} = \mathbf{b}$ is consistent.

Problem 6. Show that the *curve fitting problem*

j	0	1	2	3	4
t_j	-1.0	-0.5	0.0	0.5	1.0
y_j	1.0	0.5	0.0	0.5	2.0

by a quadratic polynomial of the form

$$p(t) = a_2 t^2 + a_1 t + a_0$$

leads to an overdetermined linear system.

Problem 7. Consider the overdetermined linear system $A\mathbf{x} = \mathbf{b}$. Find an $\hat{\mathbf{x}}$ such that

$$\|A\hat{\mathbf{x}} - \mathbf{b}\|_2 = \min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2 \equiv \min_{\mathbf{x}} \|\mathbf{r}(\mathbf{x})\|_2$$

with the *residual vector* $\mathbf{r}(\mathbf{x}) := \mathbf{b} - A\mathbf{x}$ and $\| \, . \, \|_2$ denotes the Euclidean norm.

Problem 8. Consider the overdetermined linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 444 \\ 458 \\ 478 \\ 493 \\ 506 \\ 516 \\ 523 \\ 531 \\ 543 \\ 571 \end{pmatrix}.$$

Solve this linear system in the least squares sense (see previous problem) by the normal equations method.

Problem 9. An underdetermined linear system is either inconsistent or has infinitely many solutions. Consider the underdetermined linear system

$$H\mathbf{x} = \mathbf{y}$$

where H is an $n \times m$ matrix with $m > n$ and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

We assume that $H\mathbf{x} = \mathbf{y}$ has infinitely many solutions. Let P be the $n \times m$ matrix

$$P = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}.$$

We define $\hat{\mathbf{x}} := P\mathbf{x}$. Find

$$\min_{\mathbf{x}} \|P\mathbf{x} - \mathbf{y}\|_2^2$$

subject to the constraint $\|H\mathbf{x} - \mathbf{y}\|_2^2 = 0$. We assume that $(\lambda H^T H + P^T P)^{-1}$ exists for all $\lambda > 0$. Apply the *Lagrange multiplier method*.

Problem 10. Show that solving the system of nonlinear equations with the unknowns x_1, x_2, x_3, x_4

$$\begin{aligned} (x_1 - 1)^2 + (x_2 - 2)^2 + x_3^2 &= a^2(x_4 - b_1)^2 \\ (x_1 - 2)^2 + x_2^2 + (x_3 - 2)^2 &= a^2(x_4 - b_2)^2 \\ (x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 &= a^2(x_4 - b_3)^2 \\ (x_1 - 2)^2 + (x_2 - 1)^2 + x_3^2 &= a^2(x_4 - b_4)^2 \end{aligned}$$

leads to a linear underdetermined system. Solve this system with respect to x_1, x_2 and x_3.

Problem 11. Let A be an $m \times n$ matrix over $\mathbb{R}$. We define

$$N_A := \{\, \mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \mathbf{0} \,\}.$$

N_A is called the *kernel* of A and

$$\nu(A) := \dim(N_A)$$

is called the *nullity* of A. If N_A only contains the zero vector, then $\nu(A) = 0$.
(i) Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

Find N_A and $\nu(A)$.

(ii) Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -6 & 3 & -9 \end{pmatrix}.$$

Find N_A and $\nu(A)$.

Problem 12. Let V be a vector space over a field $\mathcal{F}$. Let W be a subspace of V. We define an *equivalence relation* $\sim$ on V by stating that $v_1 \sim v_2$ if $v_1 - v_2 \in W$. The *quotient space* V/W is the set of equivalence classes $[v]$ where $v_1 - v_2 \in W$. Thus we can say that v_1 is equivalent to v_2 modulo W if $v_1 = v_2 + w$ for some $w \in W$. Let

$$V = \mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \; : \; x_1, x_2 \in \mathbb{R} \right\}$$

and the subspace

$$W = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \; : \; x_1 \in \mathbb{R} \right\}.$$

(i) Is

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 1 \end{pmatrix} ?$$

(ii) Give the quotient space V/W.

Problem 13. (i) Let $x_1, x_2, x_3 \in \mathbf{Z}$. Find all solutions of the system of linear equations

$$\begin{aligned} 7x_1 + 5x_2 - 5x_3 &= 8 \\ 17x_1 + 10x_2 - 15x_3 &= -42. \end{aligned}$$

(ii) Find all positive solutions.

Problem 14. Consider the inhomogeneous linear integral equation

$$\int_0^1 (\alpha_1(x)\beta_1(y) + \alpha_2(x)\beta_2(y))\varphi(y)dy + f(x) = \varphi(x) \tag{1}$$

for the unknown function φ, $f(x) = x$ and

$$\alpha_1(x) = x, \quad \alpha_2(x) = \sqrt{x}, \quad \beta_1(y) = y, \quad \beta_2(y) = \sqrt{y}.$$

Thus α_1 and α_2 are continuous in $[0, 1]$ and likewise for β_1 and β_2. We define

$$B_1 := \int_0^1 \beta_1(y)\varphi(y)dy, \qquad B_2 := \int_0^1 \beta_2(y)\varphi(y)dy$$

and

$$a_{\mu\nu} := \int_0^1 \beta_\mu(y)\alpha_\nu(y)dy, \qquad b_\mu := \int_0^1 \beta_\mu(y)f(y)dy$$

where $\mu, \nu = 1, 2$. Show that the integral equation can be cast into a system of linear equations for B_1 and B_2. Solve this system of linear equations and thus find a solution of the integral equation.

Chapter 3

Determinants and Traces

Problem 1. Consider the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Can we find an invertible 2×2 matrix Q such that $Q^{-1}AQ$ is a diagonal matrix?

Problem 2. Let A be a 2×2 matrix over $\mathbb{R}$. Assume that $\mathrm{tr}A = 0$ and $\mathrm{tr}A^2 = 0$. Can we conclude that A is the 2×2 zero matrix?

Problem 3. Consider the $(n-1) \times (n-1)$ matrix

$$A = \begin{pmatrix} 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ 1 & 1 & 1 & 6 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & n+1 \end{pmatrix}.$$

Let D_n be the determinant of this matrix. Is the sequence $\{ D_n/n! \}$ bounded?

Problem 4. For an integer $n \geq 3$, let $\theta := 2\pi/n$. Find the determinant of the $n \times n$ matrix $A + I_n$, where I_n is the $n \times n$ identity matrix and the matrix $A = (a_{jk})$ has the entries $a_{jk} = \cos(j\theta + k\theta)$ for all $j, k = 1, 2, \dots, n$.

Problem 5. Let α, β, γ, δ be real numbers.
(i) Is the matrix

$$U = e^{i\alpha} \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix} \begin{pmatrix} \cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2) \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$$

unitary?
(ii) What the determinant of U?

Problem 6. Let A and B be two $n \times n$ matrices over $\mathbb{C}$. If there exists a non-singular $n \times n$ matrix X such that

$$A = XBX^{-1}$$

then A and B are said to be *similar matrices*. Show that the spectra (eigenvalues) of two similar matrices are equal.

Problem 7. Let U be the $n \times n$ unitary matrix

$$U := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and V be the $n \times n$ unitary diagonal matrix ($\zeta \in \mathbb{C}$)

$$V := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \zeta & 0 & \dots & 0 \\ 0 & 0 & \zeta^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix}$$

where $\zeta^n = 1$. Then the set of matrices

$$\{ \ U^j V^k \ : \ j, k = 0, 1, 2, \dots, n-1 \ \}$$

provide a basis in the Hilbert space for all $n \times n$ matrices with the *scalar product*

$$\langle A, B \rangle := \frac{1}{n} \operatorname{tr}(AB^*)$$

for $n \times n$ matrices A and B. Write down the basis for $n = 2$.

Problem 8. Let A and B be $n \times n$ matrices over $\mathbb{C}$. Show that the matrices AB and BA have the same set of eigenvalues.

Problem 9. An $n \times n$ *circulant matrix* C is given by

$$C := \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}.$$

For example, the matrix

$$P := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is a circulant matrix. It is also called the $n \times n$ *primary permutation matrix.*
(i) Let C and P be the matrices given above. Let

$$f(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1}.$$

Show that $C = f(P)$.
(ii) Show that C is a *normal matrix*, that is, $C^*C = CC^*$.
(iii) Show that the eigenvalues of C are $f(\omega^k)$, $k = 0, 1, \dots, n-1$, where ω is the nth primitive root of unity.
(iv) Show that

$$\det(C) = f(\omega^0)f(\omega^1)\cdots f(\omega^{n-1}).$$

(v) Show that F^*CF is a diagonal matrix, where F is the unitary matrix with (j,k)-entry equal to

$$\frac{1}{\sqrt{n}}\omega^{(j-1)(k-1)}, \quad j,k = 1, \dots, n.$$

Problem 10. An $n \times n$ matrix A is called *reducible* if there is a permutation matrix P such that

$$P^TAP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B and D are square matrices of order at least 1. An $n \times n$ matrix A is called *irreducible* if it is not reducible. Show that the $n \times n$ primary permutation matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is irreducible.

Problem 11. We define a linear *bijection*, h, between $\mathbb{R}^4$ and $\mathbf{H}(2)$, the set of complex 2×2 hermitian matrices, by

$$(t, x, y, z) \to \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}.$$

We denote the matrix on the right-hand side by H.
(i) Show that the matrix can be written as a linear combination of the Pauli spin matrices and the identity matrix I_2.
(ii) Find the inverse map.
(iii) Calculate the determinant of 2×2 hermitian matrix H. Discuss.

Problem 12. Let A be an $n \times n$ invertible matrix over $\mathbb{C}$. Assume that A can be written as $A = B + iB$ where B has only real coefficients. Show that B^{-1} exists and

$$A^{-1} = \frac{1}{2}(B^{-1} - iB^{-1}).$$

Problem 13. Let A be an invertible matrix. Assume that $A = A^{-1}$. What are the possible values for $\det(A)$?

Problem 14. Let A be a skew-symmetric matrix over $\mathbb{R}$, i.e. $A^T = -A$ and of order $2n-1$. Show that $\det(A) = 0$.

Problem 15. Show that if A is hermitian, i.e. $A^* = A$ then $\det(A)$ is a real number.

Problem 16. Let A, B, and C be $n \times n$ matrices. Calculate

$$\det \begin{pmatrix} A & 0_n \\ C & B \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix.

Problem 17. Let A, B are 2×2 matrices over $\mathbb{R}$. Let $H := A + iB$. Express $\det H$ as a sum of determinants.

Problem 18. Let A, B are 2×2 matrices over $\mathbb{R}$. Let $H := A + iB$. Assume that H is hermitian. Show that

$$\det(H) = \det(A) - \det(B).$$

Problem 19. Let A, B, C, D be $n \times n$ matrices. Assume that $DC = CD$, i.e. C and D commute and $\det D \neq 0$. Consider the $(2n) \times (2n)$ matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Show that

$$\det(M) = \det(AD - BC). \tag{1}$$

We know that

$$\det \begin{pmatrix} U & 0_n \\ X & Y \end{pmatrix} = \det(U)\det(Y) \tag{2}$$

and

$$\det \begin{pmatrix} U & V \\ 0_n & Y \end{pmatrix} = \det(U)\det(Y) \tag{3}$$

where U, V, X, Y are $n \times n$ matrices and 0_n is the $n \times n$ zero matrix.

Problem 20. Let A, B be $n \times n$ matrices. We have the identity

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} \equiv \det(A+B)\det(A-B).$$

Use this identity to calculate the determinant of the left-hand side using the right-hand side, where

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 \\ 4 & 6 \end{pmatrix}.$$

Problem 21. Let A, B, C, D be $n \times n$ matrices. Assume that D is invertible. Consider the $(2n) \times (2n)$ matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Show that

$$\det(M) = \det(AD - BD^{-1}CD). \tag{1}$$

Problem 22. Let A, B be $n \times n$ positive definite (and therefore hermitian) matrices. Show that

$$\mathrm{tr}(AB) > 0.$$

Problem 23. Let $P_0(x) = 1$, $P_1(x) = \alpha_1 - x$ and

$$P_k(x) = (\alpha_k - x)P_{k-1}(x) - \beta_{k-1}P_{k-2}(x), \qquad k = 2, 3, \ldots$$

where β_j, $j = 1, 2, \ldots$ are positive numbers. Find a $k \times k$ matrix A_k such that

$$P_k(x) = \det(A_k).$$

Problem 24. Let

$$A = \begin{pmatrix} \dfrac{1}{x_1 + y_1} & \dfrac{1}{x_1 + y_2} \\ \dfrac{1}{x_2 + y_1} & \dfrac{1}{x_2 + y_2} \end{pmatrix}$$

where we assume that $x_i + y_j \neq 0$ for $i, j = 1, 2$. Show that

$$\det(A) = \frac{(x_1 - x_2)(y_1 - y_2)}{(x_1 + y_1)(x_1 + y_2)(x_2 + y_1)(x_2 + y_2)}.$$

Problem 25. For a 3×3 matrix we can use the *rule of Sarrus* to calculate the determinant (for higher dimensions there is no such thing). Let

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Write the first two columns again to the right of the matrix to obtain

$$\left(\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right).$$

Now look at the diagonals. The product of the diagonals sloping down to the right have a plus sign, the ones up to the left have a negative sign. This leads to the determinant

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Use this rule to calculate the determinant of the *rotational matrix*

$$R = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

Problem 26. Let A, S be $n \times n$ matrices. Assume that S is invertible and assume that

$$S^{-1}AS = \rho S$$

where $\rho \neq 0$. Show that A is invertible.

Problem 27. The determinant of an $n \times n$ *circulant matrix* is given by

$$\det \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_3 & a_4 & a_5 & \dots & a_2 \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix} = (-1)^{n-1} \prod_{j=0}^{n-1} \left(\sum_{k=1}^{n} \zeta^{jk} a_k \right) \qquad (1)$$

where $\zeta := \exp(2\pi i/n)$. Find the determinant of the circulant $n \times n$ matrix

$$\begin{pmatrix} 1 & 4 & 9 & \dots & n^2 \\ n^2 & 1 & 4 & \dots & (n-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 9 & 16 & 25 & \dots & 4 \\ 4 & 9 & 16 & \dots & 1 \end{pmatrix}$$

using equation (1).

Problem 28. Let A be a nonzero 2×2 matrix over $\mathbb{R}$. Let B_1, B_2, B_3, B_4 be 2×2 matrices over $\mathbb{R}$ and assume that

$$\det(A + B_j) = \det(A) + \det(B_j) \qquad \text{for} \quad j = 1, 2, 3, 4.$$

Show that there exist real numbers c_1, c_2, c_3, c_4, not all zero, such that

$$c_1 B_1 + c_2 B_2 + c_3 B_3 + c_4 B_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad (1)$$

Problem 29. Let A, B be $n \times n$ matrices. Show that

$$\operatorname{tr}((A + B)(A - B)) = \operatorname{tr}(A^2) - \operatorname{tr}(B^2). \qquad (1)$$

Problem 30. An $n \times n$ matrix Q is *orthogonal* if Q is real and

$$Q^T Q = Q^T Q = I_n$$

i.e. $Q^{-1} = Q^T$.
(i) Find the determinant of an orthogonal matrix.
(ii) Let $\mathbf{u}$, $\mathbf{v}$ be two vectors in $\mathbb{R}^3$ and $\mathbf{u} \times \mathbf{v}$ denotes the *vector product* of $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} \times \mathbf{v} := \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}.$$

Let Q be a 3×3 orthogonal matrix. Calculate

$$(Q\mathbf{u}) \times (Q\mathbf{v}).$$

Problem 31. Calculate the determinant of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

Problem 32. Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

Problem 33. Let A be a 2×2 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $\det(A) \neq 0$. Is $(A^T)^{-1} = (A^{-1})^T$?

Problem 34. Let A be an invertible $n \times n$ matrix. Let $c = 2$. Can we find an invertible matrix S such that

$$SAS^{-1} = cA.$$

Problem 35. Let σ_j ($j = 1, 2, 3$) be one of the Pauli spin matrices. Let M be an 2×2 matrix such that $M^* \sigma_j M = \sigma_j$. Show that $\det(MM^*) = 1$.

Problem 36. Let A be a 2×2 skew-symmetric matrix over $\mathbb{R}$. Then $\det(I_2 - A) = 1 + \det(A) \geq 1$. Can we conclude for a 3×3 skew-symmetric matrix B over $\mathbb{R}$ that

$$\det(I_3 + A) = 1 + \det(A)\,?$$

Problem 37. Consider the symmetric 4×4 matrices

$$A = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad B = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with trace equal to 1. Find the determinant of A and B. Find the rank of A and B. Can one find a permutation matrix P such that $PAP^T = B$?

Problem 38. Find all 2×2 matrices over $\mathbb{C}$ such that

$$\mathrm{tr}(A^2) = (\mathrm{tr}(A))^2.$$

Problem 39. Let $n \geq 2$ and A be an $n \times n$ over $\mathbb{C}$. The determinant of A can be calculated utilizing the traces of A, A^2, $\dots$, A^n as

$$\det(A) = \sum_{k_1, k_2, \dots, k_n} \prod_{\ell=1}^{n} (-1)^{k_\ell+1} \frac{(\mathrm{tr}(A^\ell))^{k_\ell}}{k_\ell! \ell^{k_\ell}}$$

where the sum runs over the sets of nonnegative integers $(k_1, \dots, k_n)$ satisfying the linear *Diophatine equation*

$$\sum_{\ell=1}^{n} \ell k_\ell = n.$$

(i) Apply it to a 2×2 matrix A.
(ii) Give an implementation with SymbolicC++.

Chapter 4

Eigenvalues and Eigenvectors

Problem 1. (i) Find the eigenvalues and normalized eigenvectors of the rotational matrix

$$A = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{pmatrix}.$$

(ii) Are the eigenvectors orthogonal to each other?

Problem 2. (i) An $n \times n$ matrix A such that $A^2 = A$ is called *idempotent*. What can be said about the eigenvalues of such a matrix?

(ii) An $n \times n$ matrix A for which $A^p = 0_n$, where p is a positive integer, is called *nilpotent*. What can be said about the eigenvalues of such a matrix?

(iii) An $n \times n$ matrix A such that $A^2 = I_n$ is called *involutory*. What can be said about the eigenvalues of such a matrix?

Problem 3. Let $\mathbf{x}$ be a nonzero column vector in $\mathbb{R}^n$. Then $\mathbf{x}\mathbf{x}^T$ is an $n \times n$ matrix and $\mathbf{x}^T\mathbf{x}$ is a real number. Show that $\mathbf{x}^T\mathbf{x}$ is an eigenvalue of $\mathbf{x}\mathbf{x}^T$ and $\mathbf{x}$ is the corresponding eigenvector.

Problem 4. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that the eigenvectors corresponding to distinct eigenvalues are linearly independent.

Problem 5. Let A be an $n \times n$ matrix over $\mathbb{C}$. The *spectral radius* of the matrix A is the non-negative number defined by

$$\rho(A) := \max\{\, |\lambda_j(A)| \,:\, 1 \le j \le n \,\}$$

where $\lambda_j(A)$ are the eigenvalues of A. We define the *norm* of A as

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

where $\|A\mathbf{x}\|$ denotes the Euclidean norm of the vector $A\mathbf{x}$. Show that $\rho(A) \le \|A\|$.

Problem 6. Let A be an $n \times n$ hermitian matrix, i.e., $A = A^*$. Assume that all n eigenvalues are different. Then the normalized eigenvectors $\{\, \mathbf{v}_j \,:\, j = 1, 2, \ldots, n \,\}$ form an orthonormal basis in $\mathbb{C}^n$. Consider

$$\beta := (A\mathbf{x} - \mu\mathbf{x}, A\mathbf{x} - \nu\mathbf{x}) \equiv (A\mathbf{x} - \mu\mathbf{x})^*(A\mathbf{x} - \nu\mathbf{x})$$

where $(\ ,\ )$ denotes the scalar product in $\mathbb{C}^n$ and μ, ν are real constants with $\mu < \nu$. Show that if no eigenvalue lies between μ and ν, then $\beta \ge 0$.

Problem 7. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$. Let

$$H := \frac{A + A^*}{2}, \qquad S := \frac{A - A^*}{2i}.$$

Let λ be an eigenvalue of A and $\mathbf{x}$ be the corresponding normalized eigenvector (column vector).
(i) Show that

$$\lambda = \mathbf{x}^* H \mathbf{x} + i\mathbf{x}^* S \mathbf{x}.$$

(ii) Show that the real part λ_r of the eigenvalue λ is given by $\lambda_r = \mathbf{x}^* H \mathbf{x}$ and the imaginary part λ_i is given by $\lambda_i = \mathbf{x}^* S \mathbf{x}$.

Problem 8. Let $A = (a_{jk})$ be a normal nonsymmetric 3×3 matrix over the real numbers. Show that

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_{23} - a_{32} \\ a_{31} - a_{13} \\ a_{12} - a_{21} \end{pmatrix}$$

is an eigenvector of A.

Problem 9. Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Find $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ without calculating the eigenvalues of A or A^2.

Problem 10. Find all solutions of the linear equation

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ -\sin(\theta) & -\cos(\theta) \end{pmatrix} \mathbf{x} = \mathbf{x}, \qquad \theta \in \mathbb{R} \tag{1}$$

with the condition that $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{x}^T\mathbf{x} = 1$, i.e., the vector $\mathbf{x}$ must be normalized. What type of equation is (1)?

Problem 11. Consider the column vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$

$$\mathbf{u} = \begin{pmatrix} \cos(\theta) \\ \cos(2\theta) \\ \vdots \\ \cos(n\theta) \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \sin(\theta) \\ \sin(2\theta) \\ \vdots \\ \sin(n\theta) \end{pmatrix}$$

where $n \geq 3$ and $\theta = 2\pi/n$.
(i) Calculate $\mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v}$.
(ii) Calculate $\mathbf{u}^T\mathbf{u} - \mathbf{v}^T\mathbf{v} + 2i\mathbf{u}^T\mathbf{v}$.
(iii) Calculate the matrix $A = \mathbf{u}\mathbf{u}^T - \mathbf{v}\mathbf{v}^T$, $A\mathbf{u}$ and $A\mathbf{v}$. Give an interpretation of the results.

Problem 12. Let A be an $n \times n$ matrix over $\mathbb{C}$. We define

$$r_j := \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{jk}|, \quad j = 1, 2, \ldots, n.$$

(i) Show that each eigenvalue λ of A satisfies at least one of the following inequalities

$$|\lambda - a_{jj}| \leq r_j, \quad j = 1, 2, \ldots, n.$$

In other words show that all eigenvalues of A can be found in the union of disks

$$\{\, z \,:\, |z - a_{jj}| \leq r_j, \;\; j = 1, 2, \ldots, n \,\}$$

This is *Geršgorin disk theorem.*
(ii) Apply this theorem to the matrix

$$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

(iii) Apply this theorem to the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 9 \\ 1 & 1 & 1 \end{pmatrix}.$$

Problem 13. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let f be an *entire function*, i.e., an analytic function on the whole complex plane, for example $\exp(z)$, $\sin(z)$, $\cos(z)$. An infinite series expansion for $f(A)$ is not generally useful for computing $f(A)$. Using the *Cayley-Hamilton theorem* we can write

$$f(A) = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_2A^2 + a_1A + a_0I_n \tag{1}$$

where the complex numbers a_0, a_1, $\ldots$, a_{n-1} are determined as follows: Let

$$r(\lambda) := a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

which is the right-hand side of (1) with A^j replaced by λ^j, where $j = 0, 1, \ldots, n-1$.
For each distinct eigenvalue λ_j of the matrix A, we consider the equation

$$f(\lambda_j) = r(\lambda_j). \tag{2}$$

If λ_j is an eigenvalue of multiplicity k, for $k > 1$, then we consider also the following equations

$$\begin{aligned} f'(\lambda)|_{\lambda=\lambda_j} &= r'(\lambda)|_{\lambda=\lambda_j} \\ f''(\lambda)|_{\lambda=\lambda_j} &= r''(\lambda)|_{\lambda=\lambda_j} \\ \cdots &= \cdots \\ f^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j} &= r^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j}. \end{aligned}$$

Apply this technique to find $\exp(A)$ with

$$A = \begin{pmatrix} c & c \\ c & c \end{pmatrix}, \qquad c \in \mathbb{R}, \quad c \neq 0.$$

Problem 14. (i) Use the method given above to calculate $\exp(iK)$, where the hermitian 2×2 matrix K is given by

$$K = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}, \qquad a, c \in \mathbb{R}, \quad b \in \mathbb{C}.$$

(ii) Find the condition on a, b and c such that

$$e^{iK} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 15. Let A be a *normal* matrix over $\mathbb{C}$, i.e. $A^*A = AA^*$. Show that if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $\mathbf{x}$ is an eigenvector of A^* with eigenvalue $\bar{\lambda}$.

Problem 16. Show that an $n \times n$ matrix A is singular if and only if at least one eigenvalue is 0.

Problem 17. Let A be an invertible $n \times n$ matrix. Show that if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $\mathbf{x}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

Problem 18. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that A and A^T have the same eigenvalues.

Problem 19. Let A be a symmetric matrix over $\mathbb{R}$. Since A is symmetric over $\mathbb{R}$ there exists a set of orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ which form an orthonormal basis. Let $\mathbf{x} \in \mathbb{R}^n$ be a reasonably good approximation to an eigenvector, say $\mathbf{v}_1$. Calculate

$$R := \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

The quotient is called *Rayleigh quotient*. Discuss.

Problem 20. Let A be an $n \times n$ real symmetric matrix and

$$Q(\mathbf{x}) := \mathbf{x}^T A \mathbf{x}.$$

The following statements hold (*maximum principle*)
1) $\lambda_1 = \max_{\|\mathbf{x}\|=1} Q(\mathbf{x}) = Q(\mathbf{x}_1)$ is the largest eigenvalue of the matrix A and $\mathbf{x}_1$ is the eigenvector corresponding to eigenvalue λ_1.
2) (inductive statement). Let $\lambda_k = \max Q(\mathbf{x})$ subject to the constraints
a) $\mathbf{x}^T \mathbf{x}_j = 0$, $j = 1, 2, \ldots, k-1$.
b) $\|\mathbf{x}\| = 1$.
c) Then $\lambda_k = Q(\mathbf{x}_k)$ is the kth eigenvalue of A, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\mathbf{x}_k$ is the corresponding eigenvectors of A.
Apply the maximum principle to the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Problem 21. Let A be an $n \times n$ matrix. An $n \times n$ matrix can have at most n linearly independent eigenvectors. Now assume that A has $n+1$ eigenvectors (at least one must be linearly dependent) such that any n of them are linearly independent. Show that A is a scalar multiple of the identity matrix I_n.

Problem 22. An $n \times n$ *stochastic matrix* P satisfies the following conditions:

$$p_{ij} \geq 0 \quad \text{for all } \; i, j = 1, 2, \ldots, n$$

and

$$\sum_{i=1}^{n} p_{ij} = 1 \quad \text{for all } \; j = 1, 2, \ldots, n.$$

Show that a stochastic matrix always has at least one eigenvalue equal to one.

Problem 23. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that A is hermitian and unitary. What can be said about the eigenvalues of A?

Problem 24. Consider the $(n+1) \times (n+1)$ matrix

$$A = \begin{pmatrix} 0 & \mathbf{s}^* \\ \mathbf{r} & 0_{n \times n} \end{pmatrix}$$

where $\mathbf{r}$ and $\mathbf{s}$ are $n \times 1$ vectors with complex entries, $\mathbf{s}^*$ denoting the conjugate transpose of $\mathbf{s}$. Find $\det(B - \lambda I_{n+1})$, i.e. find the characteristic polynomial.

Problem 25. The matrix difference equation

$$\mathbf{p}(t+1) = M\mathbf{p}(t), \qquad t = 0, 1, 2, \ldots$$

with the column vector (vector of probabilities)

$$\mathbf{p}(t) = (p_1(t), p_2(t), \ldots, p_n(t))^T$$

and the $n \times n$ matrix

$$M = \begin{pmatrix} (1-w) & 0.5w & 0 & \ldots & 0.5w \\ 0.5w & (1-w) & 0.5w & \ldots & 0 \\ 0 & 0.5w & (1-w) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.5w & 0 & 0 & \ldots & (1-w) \end{pmatrix}$$

plays a role in random walk in one dimension. M is called the *transition matrix* and w denotes the probability $w \in [0, 1]$ that at a given time step the particle jumps to either of its nearest neighbor sites, then the probability that the particle does not jump either to the right of left is $(1 - w)$. The matrix M is of the type known as *circulant matrix*. Such an $n \times n$ matrix is of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}$$

with the normalized eigenvectors

$$e_j = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ e^{2\pi ij/n} \\ \vdots \\ e^{2(n-1)\pi ij/n} \end{pmatrix}$$

for $j = 1, 2, \dots, n$.
(i) Use this result to find the eigenvalues of the matrix C.
(ii) Use (i) to find the eigenvalues of the matrix M.
(iii) Use (ii) to find $\mathbf{p}(t)$ ($t = 0, 1, 2, \dots$), where we expand the initial distribution vector $\mathbf{p}(0)$ in terms of the eigenvectors

$$\mathbf{p}(0) = \sum_{k=1}^{n} a_k \mathbf{e}_k$$

with

$$\sum_{j=1}^{n} p_j(0) = 1.$$

(iv) Assume that

$$\mathbf{p}(0) = \frac{1}{n} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Give the time evolution of $\mathbf{p}(0)$.

Problem 26. Let U be a unitary matrix and $\mathbf{x}$ an eigenvector of U with the corresponding eigenvalue λ, i.e.

$$U\mathbf{x} = \lambda\mathbf{x}.$$

(i) Show that $U^*\mathbf{x} = \overline{\lambda}\mathbf{x}$.
(ii) Let λ, μ be distinct eigenvalues of a unitary matrix U with the corresponding eigenvectors $\mathbf{x}$ and $\mathbf{y}$, i.e.

$$U\mathbf{x} = \lambda\mathbf{x}, \qquad U\mathbf{y} = \mu\mathbf{y}.$$

Show that $\mathbf{x}^*\mathbf{y} = 0$.

Problem 27. Let H, H_0, V be $n \times n$ matrices over $\mathbb{C}$ and $H = H_0 + V$. Let $z \in \mathbb{C}$ and assume that z is chosen so that $(H_0 - zI_n)^{-1}$ and $(H - zI_n)^{-1}$ exist. Show that

$$(H - zI_n)^{-1} = (H_0 - zI_n)^{-1} - (H_0 - zI_n)^{-1}V(H - zI_n)^{-1}.$$

This is called the second *resolvent identity*.

Problem 28. Let $\mathbf{u}$ be a nonzero column vector in $\mathbb{R}^n$. Consider the $n \times n$ matrix

$$A = \mathbf{u}\mathbf{u}^T - \mathbf{u}^T\mathbf{u}I_n.$$

Is $\mathbf{u}$ an eigenvector of this matrix? If so what is the eigenvalue?

Problem 29. An $n \times n$ matrix A is called a *Hadamard matrix* if each entry of A is 1 or -1 and if the rows or columns of A are orthogonal, i.e.,

$$AA^T = nI_n \quad \text{or} \quad A^TA = nI_n.$$

Note that $AA^T = nI_n$ and $A^TA = nI_n$ are equivalent. Hadamard matrices H_n of order 2^n can be generated recursively by defining

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix}$$

for $n \geq 2$. Show that the eigenvalues of H_n are given by $+2^{n/2}$ and $-2^{n/2}$ each of multiplicity 2^{n-1}.

Problem 30. Let U be an $n \times n$ unitary matrix. Then U can be written as

$$U = V\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)V^*$$

where λ_1, λ_2, $\ldots$, λ_n are the eigenvalues of U and V is an $n \times n$ unitary matrix. Let

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the decomposition for U given above.

Problem 31. An $n \times n$ matrix A over the complex numbers is called *positive semidefinite* (written as $A \geq 0$), if

$$\mathbf{x}^*A\mathbf{x} \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{C}^n.$$

Show that for every $A \geq 0$, there exists a unique $B \geq 0$ so that $B^2 = A$.

Problem 32. An $n \times n$ matrix A over the complex numbers is said to be *normal* if it commutes with its conjugate transpose $A^*A = AA^*$. The matrix A can be written

$$A = \sum_{j=1}^{n} \lambda_j E_j$$

where $\lambda_j \in \mathbb{C}$ are the eigenvalues of A and E_j are $n \times n$ matrices satisfying

$$E_j^2 = E_j = E_j^*, \qquad E_j E_k = 0_n \text{ if } j \neq k, \qquad \sum_{j=1}^{n} E_j = I_n.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the decomposition of A given above.

Problem 33. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that A^{-1} exists. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $\mathbf{u}, \mathbf{v}$ are considered as column vectors.
(i) Show that if

$$\mathbf{v}^T A^{-1} \mathbf{u} = -1$$

then $A + \mathbf{u}\mathbf{v}^T$ is not invertible.
(ii) Assume that $\mathbf{v}^T A^{-1} \mathbf{u} \neq -1$. Show that

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1} \mathbf{u}}.$$

Problem 34. The *Denman-Beavers iteration* for the square root of an $n \times n$ matrix A with no eigenvalues on $\mathbb{R}^-$ is

$$Y_{k+1} = \frac{1}{2}(Y_k + Z_k^{-1}), \quad Z_{k+1} = \frac{1}{2}(Z_k + Y_k^{-1})$$

with $k = 0, 1, 2, \ldots$ and $Z_0 = I_n$ and $Y_0 = A$. The iteration has the properties that

$$\lim_{k \to \infty} Y_k = A^{1/2}, \qquad \lim_{k \to \infty} Z_k = A^{-1/2}$$

and, for all k,

$$Y_k = AZ_k, \quad Y_k Z_k = Z_k Y_k, \quad Y_{k+1} = \frac{1}{2}(Y_k + AY_k^{-1}).$$

(i) Can the Denman-Beavers iteration be applied to the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}?$$

(ii) Find Y_1 and Z_1.

Problem 35. Let

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

and I_2 be the 2×2 identity matrix. For $j \geq 1$, let d_j be the greatest common divisor of the entries of $A^j - I_2$. Show that

$$\lim_{j \to \infty} d_j = \infty.$$

Hint. Use the eigenvalues of A and the characteristic polynomial.

Problem 36. (i) Consider the polynomial

$$p(x) = x^2 - sx + d, \quad s, d \in \mathbb{C}.$$

Find a 2×2 matrix A such that its characteristic polynomial is p.
(ii) Consider the polynomial

$$q(x) = -x^3 + sx^2 - qx + d, \quad s, q, d \in \mathbb{C}.$$

Find a 3×3 matrix B such that its characteristic polynomial is q.

Problem 37. Calculate the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

by calculating the eigenvalues of A^2.

Problem 38. Find all 4×4 permutation matrices with the eigenvalues $+1$, -1, $+i$, $-i$.

Problem 39. Let A be an $n \times n$ matrix over $\mathbb{R}$. Let J be the $n \times n$ matrix with 1's in the counter diagonal and 0's otherwise. Assume that

$$\operatorname{tr}(A) = 0, \qquad \operatorname{tr}(JA) = 0.$$

What can be said about the eigenvalues of such a matrix?

Problem 40. Let $\alpha, \beta, \gamma \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors of the 4×4 matrix

$$\begin{pmatrix} 0 & \cos(\alpha) & \cos(\beta) & \cos(\gamma) \\ \cos(\alpha) & 0 & 0 & 0 \\ \cos(\beta) & 0 & 0 & 0 \\ \cos(\gamma) & 0 & 0 & 0 \end{pmatrix}.$$

Problem 41. Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$\begin{pmatrix} \cosh(\alpha) & 0 & 0 & \sinh(\alpha) \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \sinh(\alpha) & 0 & 0 & \cosh(\alpha) \end{pmatrix}.$$

Problem 42. Consider the nonnormal matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

The eigenvalues are 1 and 2. Find the normalized eigenvectors of A and show that they are linearly independent, but not orthonormal.

Problem 43. Find the eigenvalues of the matrices

$$A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Extend to n dimensions.

Problem 44. Lett $x, y \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} x+y & z_1 & z_2 \\ \bar{z}_1 & -x+y & z_3 \\ \bar{z}_2 & \bar{z}_3 & -2y \end{pmatrix}$$

with trace equal to 0.

Problem 45. Let A, B be hermitian matrices. Consider the eigenvalue problem

$$A\mathbf{v}_j = \lambda_j B\mathbf{v}_j, \quad j = 1, \ldots, n.$$

Expanding the eigenvector $\mathbf{v}_j$ with respect to an orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$, $\mathbf{e}_n$, i.e.

$$\mathbf{v}_j = \sum_{k=1}^{n} c_{kj}\mathbf{e}_k.$$

Show that

$$\sum_{k=1}^{n} A_{\ell k} c_{kj} = \lambda_j \sum_{k=1}^{n} B_{\ell k} c_{kj}, \quad \ell = 1, \ldots, n$$

where $A_{k\ell} := \mathbf{e}_k^* A \mathbf{e}_\ell^*$, $B_{k\ell} := \mathbf{e}_k^* B \mathbf{e}_\ell$.

Problem 46. Find the eigenvalues and normalized eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 47. Let H be a hermitian $n \times n$ matrix. Consider the eigenvalue problem $H\mathbf{v} = \lambda\mathbf{v}$.
(i) Find the eigenvalues of $H + iI_n$ and $H - iI_n$.
(ii) Since H is hermitian, the matrices $H + iI_n$ and $H - iI_n$ are invertible. Find $(H + iI_n)\mathbf{v}$. Find $(H - iI_n)(H + iI_n)^{-1}\mathbf{v}$. Discuss.

Problem 48. The matrix

$$A(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

admits the eigenvalues $\lambda_+ = e^{i\alpha}$ and $\lambda_- = e^{-i\alpha}$ with the corresponding normalized eigenvectors

$$\mathbf{v}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad \mathbf{v}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The star product $A(\alpha) \star A(\alpha)$ is given by

$$A(\alpha) \star A(\alpha) = \begin{pmatrix} \cos(\alpha) & 0 & 0 & -\sin(\alpha) \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ \sin(\alpha) & 0 & 0 & \cos(\alpha) \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of $A(\alpha) \star A(\alpha)$.

Problem 49. Let $x_1, x_2, x_3 \in \mathbb{R}$. Find the eigenvalues of the 2×2 matrix

$$\begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}.$$

Problem 50. The Cartan matrix for the Lie algebra g_2 is given by

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Is the matrix nonnormal? Show that the matrix is invertible. Find the inverse. Find the eigenvalues and normalized eigenvectors of A.

Problem 51. Consider the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

Find the eigenvalues.

Problem 52. (i) Let $\ell > 0$. Find the eigenvalues of the matrix

$$\begin{pmatrix} \cos(x/\ell) & \ell \sin(x/\ell) \\ -(1/\ell)\sin(x/\ell) & \cos(x/\ell) \end{pmatrix}.$$

(ii) Let $\ell > 0$. Find the eigenvalues of the matrix

$$\begin{pmatrix} \cosh(x/\ell) & \ell \sinh(x/\ell) \\ (1/\ell)\sinh(x/\ell) & \cos(x/\ell) \end{pmatrix}.$$

Problem 53. Let $a, b, c, d, e \in \mathbb{R}$. Find the eigenvalues of the 4×4 matrix

$$\begin{pmatrix} a & b & c & d \\ b & 0 & e & 0 \\ c & e & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}.$$

Problem 54. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Is the matrix unitary?

Problem 55. Consider the 2×2 matrix over $\mathbb{R}$

$$A = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of A. Find A^2, A^3, A^n. Find

$$\lim_{n \to \infty} A^n$$

applying the spectral theorem.

Problem 56. Find the eigenvalues and eigenvectors of the staircase matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Extend to n-dimensions.

Problem 57. Consider the 3×3 matrix

$$A = \begin{pmatrix} \sqrt{3}e^{i\pi/4}/2 & 0 & 1 \\ 0 & i & e^{i\pi/24}/2 \\ 1 & e^{i\pi/24}/2 & ie^{i\pi/12} \end{pmatrix}.$$

The matrix is not hermitian, but $A = A^T$. Find $H = AA^*$ and the eigenvalues of H.

Problem 58. Let $\alpha, \beta \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors of the 3×3 matrix

$$\begin{pmatrix} \alpha+\beta & 0 & \alpha \\ 0 & \alpha+\beta & 0 \\ \alpha & 0 & \alpha+\beta \end{pmatrix}.$$

Problem 59. Let $x_1, x_2 \in \mathbb{R}$. Show that the eigenvalues of the 2×2 matrix

$$\left(\begin{array}{ccc} 1+x_1^2 & -x_1x_2 & \\ -x_1x_2 & -x_1x_2 & 1+x_2^2 \end{array}\right)$$

are given by $\lambda_1 = 1 + x_1^2 + x_2^2$ and $\lambda_2 = 1$. What curve in the plane is described by

$$\det \left(\begin{array}{ccc} 1+x_1^2 & -x_1x_2 & \\ -x_1x_2 & -x_1x_2 & 1+x_2^2 \end{array}\right) = 0?$$

Problem 60. Consider the skew-symmetric 3×3 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Find the eigenvalues of A. Let 0_3 be the 3×3 zero matrix and A_1, A_2, A_3 be 3×3 skew-symmetric matrices over $\mathbb{R}$. Find the eigenvalues of the 9×9 matrices

$$B = \begin{pmatrix} 0_3 & -A_3 & A_2 \\ A_3 & 0_3 & -A_1 \\ -A_2 & A_1 & 0_3 \end{pmatrix}.$$

Problem 61. Find the inverse matrices of

$$\begin{pmatrix} 1 & \alpha_1 & 0 \\ 0 & 1 & \alpha_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \alpha_1 & 0 & 0 \\ 0 & 1 & \alpha_2 & 0 \\ 0 & 0 & 1 & \alpha_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Extend to n-dimensions.

Problem 62. Let $a, b \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors of the 3×3 matrix

$$M = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & b \\ -a & b & 0 \end{pmatrix}.$$

Problem 63. Find the eigenvalues of the unitary 2×2 matrix

$$U = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}.$$

Problem 64. Find all 4×4 permutation matrices with the eigenvalues $+1$, -1, $+i$, $-i$.

Chapter 5

Commutators and Anticommutators

Problem 1. Let A, B be $n \times n$ matrices. Assume that $[A, B] = 0_n$ and $[A, B]_+ = 0_n$. What can be said about AB and BA?

Problem 2. Let A and B be symmetric $n \times n$ matrices over $\mathbb{R}$. Show that AB is symmetric if and only if A and B commute.

Problem 3. Let A and B be $n \times n$ matrices over $\mathbb{C}$. Show that A and B commute if and only if $A - cI_n$ and $B - cI_n$ commute over every $c \in \mathbb{C}$.

Problem 4. Consider the matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find a nonzero 2×2 matrices A such that

$$[A, e] = 0_n, \quad [A, f] = 0_n, \quad [A, h] = 0_n.$$

Problem 5. Can one find 2×2 matrices A and B such that

$$[A^2, B^2] = 0_n$$

while

$$[A, B] \neq 0_n?$$

Problem 6. Let A, B, C, D be $n \times n$ matrices over $\mathbb{R}$. Assume that AB^T and CD^T are symmetric and $AD^T - BC^T = I_n$, where ^{T} denotes transpose. Show that
$$A^T D - C^T B = I_n.$$

Problem 7. Let A, B, H be $n \times n$ matrices over $\mathbb{C}$ such that
$$[A, H] = 0_n, \qquad [B, H] = 0_n.$$
Find $[[A, B], H]$.

Problem 8. Let A, B be $n \times n$ matrices. Assume that A is invertible. Assume that $[A, B] = 0_n$. Can we conclude that $[A^{-1}, B] = 0_n$?

Problem 9. Let A and B be $n \times n$ hermitian matrices. Suppose that
$$A^2 = I_n, \qquad B^2 = I_n \tag{1}$$
and
$$[A, B]_+ \equiv AB + BA = 0_n \tag{2}$$
where 0_n is the $n \times n$ zero matrix. Let $\mathbf{x} \in \mathbb{C}^n$ be normalized, i.e., $\|\mathbf{x}\| = 1$. Here $\mathbf{x}$ is considered as a column vector.
(i) Show that
$$(\mathbf{x}^* A \mathbf{x})^2 + (\mathbf{x}^* B \mathbf{x})^2 \leq 1. \tag{3}$$
(ii) Give an example for the matrices A and B.

Problem 10. Let A and B be $n \times n$ hermitian matrices. Suppose that
$$A^2 = A, \qquad B^2 = B \tag{1}$$
and
$$[A, B]_+ \equiv AB + BA = 0_n \tag{2}$$
where 0_n is the $n \times n$ zero matrix. Let $\mathbf{x} \in \mathbb{C}^n$ be normalized, i.e., $\|\mathbf{x}\| = 1$. Here $\mathbf{x}$ is considered as a column vector. Show that
$$(\mathbf{x}^* A \mathbf{x})^2 + (\mathbf{x}^* B \mathbf{x})^2 \leq 1. \tag{3}$$

Problem 11. Let A, B be skew-hermitian matrices over $\mathbb{C}$, i.e. $A^* = -A$, $B^* = -B$. Is the commutator of A and B again skew-hermitian?

Problem 12. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let S be an invertible $n \times n$ matrix over $\mathbb{C}$ with
$$\widetilde{A} = S^{-1} A S, \qquad \widetilde{B} = S^{-1} B S.$$

Show that

$$[\widetilde{A}, \widetilde{B}] = S^{-1}[A, B]S.$$

Problem 13. Can we find $n \times n$ matrices A, B over $\mathbb{C}$ such that

$$[A, B] = I_n \tag{1}$$

where I_n denotes the identity matrix?

Problem 14. Can we find 2×2 matrices A and B of the form

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$$

and singular (i.e. $\det A = 0$ and $\det B = 0$) such that $[A, B]_+ = I_2$.

Problem 15. Let A be an $n \times n$ hermitian matrix over $\mathbb{C}$. Assume that the eigenvalues of A, λ_1, λ_2, $\dots$, λ_n are nondegenerate and that the normalized eigenvectors $\mathbf{v}_j$ ($j = 1, 2, \dots, n$) of A form an orthonormal basis in $\mathbb{C}^n$. Let B be an $n \times n$ matrix over $\mathbb{C}$. Assume that $[A, B] = 0_n$, i.e., A and B commute. Show that

$$\mathbf{v}_k^* B \mathbf{v}_j = 0 \quad \text{for} \quad k \neq j. \tag{1}$$

Problem 16. Let A, B be hermitian $n \times n$ matrices. Assume they have the same set of eigenvectors

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j, \quad B\mathbf{v}_j = \mu_j \mathbf{v}_j, \quad j = 1, 2, \dots, n$$

and that the normalized eigenvectors form an orthonormal basis in $\mathbb{C}^n$. Show that

$$[A, B] = 0_n. \tag{1}$$

Problem 17. Let A, B be $n \times n$ matrices. Then we have the expansion

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots$$

(i) Assume that $[A, B] = A$. Calculate $e^A B e^{-A}$.
(ii) Assume that $[A, B] = B$. Calculate $e^A B e^{-A}$.

Problem 18. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$ with $\operatorname{tr}(A) = 0$. Show that A can be written as commutator, i.e., there are $n \times n$ matrices X and Y such that $A = [X, Y]$.

Problem 19. (i) Let A, B be $n \times n$ matrices over $\mathbb{C}$ with $[A, B] = 0_n$. Calculate

$$[A + cI_n, B + cI_n]$$

where $c \in \mathbb{C}$ and I_n is the $n \times n$ identity matrix.
(ii) Let $\mathbf{x}$ be an eigenvector of the $n \times n$ matrix A with eigenvalue λ. Show that $\mathbf{x}$ is also an eigenvector of $A + cI_n$, where $c \in \mathbb{C}$.

Problem 20. Let A, B, C be $n \times n$ matrices. Show that

$$e^A[B, C]e^{-A} \equiv [e^A B e^{-A}, e^A C e^{-A}].$$

Chapter 6

Decomposition of Matrices

Problem 1. Find the LU-decomposition of the 3×3 matrix

$$A = \begin{pmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{pmatrix}.$$

The triangular matrices L and U are not uniquely determined by the matrix equation $A = LU$. These two matrices together contain $n^2 + n$ unknown elements. Thus when comparing elements on the left- and right-hand side of $A = LU$ we have n^2 equations and $n^2 + n$ unknowns. We require a further n conditions to uniquely determine the matrices. There are three additional sets of n conditions that are commonly used. These are *Doolittle's method* with $\ell_{jj} = 1$, $j = 1, 2, \ldots, n$; *Choleski's method* with $\ell_{jj} = u_{jj}$, $j = 1, 2, \ldots, n$; *Crout's method* with $u_{jj} = 1$, $j = 1, 2, \ldots, n$. Apply Crout's method.

Problem 2. Find the QR-decomposition of the 3×3 matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 7 \\ 0 & -1 & -1 \end{pmatrix}.$$

Problem 3. Consider a square non-singular square matrix A over $\mathbb{C}$, i.e. A^{-1} exists. The *polar decomposition theorem* states that A can be written as $A = UP$, where U is a unitary matrix and P is a hermitian positive definite matrix. Show that A has a unique polar decomposition.

Problem 4. Let A be an arbitrary $m \times n$ matrix over $\mathbb{R}$, i.e., $A \in \mathbb{R}^{m \times n}$. Then A can be written as

$$A = U \Sigma V^T$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, Σ is an $m \times n$ diagonal matrix with nonnegative entries and the superscript ^{T} denotes the transpose. This is called the *singular value decomposition*. An algorithm to find the singular value decomposition is as follows.

1) Find the eigenvalues λ_j $(j = 1, 2, \ldots, n)$ of the $n \times n$ matrix $A^T A$. Arrange the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in descending order.

2) Find the number of nonzero eigenvalues of the matrix $A^T A$. We call this number r.

3) Find the orthogonal eigenvectors $\mathbf{v}_j$ of the matrix $A^T A$ corresponding to the obtained eigenvalues, and arrange them in the same order to form the column-vectors of the $n \times n$ matrix V.

4) Form an $m \times n$ diagonal matrix Σ placing on the leading diagonal of it the square root $\sigma_j := \sqrt{\lambda_j}$ of $p = \min(m, n)$ first eigenvalues of the matrix $A^T A$ found in 1) in descending order.

5) Find the first r column vectors of the $m \times m$ matrix U

$$\mathbf{u}_j = \frac{1}{\sigma_j} A \mathbf{v}_j, \quad j = 1, 2, \ldots, r.$$

6) Add to the matrix U the rest of the $m - r$ vectors using the Gram-Schmidt orthogonalization process.

We have

$$A \mathbf{v}_j = \sigma_j \mathbf{u}_j, \qquad A^T \mathbf{u}_j = \sigma_j \mathbf{v}_j$$

and therefore

$$A^T A \mathbf{v}_j = \sigma_j^2 \mathbf{v}_j, \qquad A A^T \mathbf{u}_j = \sigma_j^2 \mathbf{u}_j.$$

Apply the algorithm to the matrix

$$A = \begin{pmatrix} 0.96 & 1.72 \\ 2.28 & 0.96 \end{pmatrix}.$$

Problem 5. Find the singular value decomposition $A = U\Sigma V^T$ of the matrix (row vector) $A = (2\ 1\ -2)$.

Problem 6. Any unitary $2^n \times 2^n$ matrix U can be decomposed as

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} U_3 & 0 \\ 0 & U_4 \end{pmatrix}$$

where U_1, U_2, U_3, U_4 are $2^{n-1} \times 2^{n-1}$ unitary matrices and C and S are the $2^{n-1} \times 2^{n-1}$ diagonal matrices

$$C = \mathrm{diag}(\cos(\alpha_1), \cos\alpha_2, \dots, \cos\alpha_{2^n/2})$$
$$S = \mathrm{diag}(\sin(\alpha_1), \sin\alpha_2, \dots, \sin(\alpha_{2^n/2}))$$

where $\alpha_j \in \mathbb{R}$. This decomposition is called *cosine-sine decomposition*.

Consider the unitary 2×2 matrix

$$U = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Show that U can be written as

$$U = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_3 & 0 \\ 0 & u_4 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ and $u_1, u_2, u_3, u_4 \in U(1)$ (i.e., u_1, u_2, u_3, u_4 are complex numbers with length 1). Find α, u_1, u_2, u_3, u_4.

Problem 7. (i) Find the *cosine-sine decomposition* of the unitary matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) Use the result from (i) to find a 2×2 hermitian matrix K such that $U = \exp(iK)$.

Problem 8. (i) Find the *cosine-sine decomposition* of the unitary matrix (Hadamard matrix)

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 9. For any $n \times n$ matrix A there exists an $n \times n$ unitary matrix ($U^* = U^{-1}$) such that

$$U^* A U = T \tag{1}$$

where T is an $n \times n$ matrix in upper triangular form. Equation (1) is called a *Schur decomposition*. The diagonal elements of T are the eigenvalues of A. Note that such a decomposition is not unique. An iterative algorithm to find a Schur decomposition for an $n \times n$ matrix is as follows.

It generates at each step matrices U_k and T_k ($k = 1, 2, \ldots, n-1$) with the properties: each U_k is unitary, and each T_k has only zeros below its main diagonal in its first k columns. T_{n-1} is in upper triangular form, and $U = U_1 U_2 \cdots U_{n-1}$ is the unitary matrix that transforms A into T_{n-1}. We set $T_0 = A$. The kth step in the iteration is as follows.

Step 1. Denote as A_k the $(n-k+1) \times (n-k+1)$ submatrix in the lower right portion of T_{k-1}.
Step 2. Determine an eigenvalue and the corresponding normalized eigenvector for A_k.
Step 3. Construct a unitary matrix N_k which has as its first column the normalized eigenvector found in step 2.
Step 4. For $k = 1$, set $U_1 = N_1$, for $k > 1$, set

$$U_k = \begin{pmatrix} I_{k-1} & 0 \\ 0 & N_k \end{pmatrix}$$

where I_{k-1} is the $(k-1) \times (k-1)$ identity matrix.
Step 5. Calculate $T_k = U_k^* T_{k-1} U_k$.

Apply the algorithm to the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Problem 10. Let A be an $n \times n$ matrix over $\mathbb{C}$. Then there exists an $n \times n$ unitary matrix Q, such that

$$Q^* A Q = D + N$$

where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix composed of the eigenvalues of A and N is a strictly upper triangular matrix (i.e., N has zero entries on the diagonal). The matrix Q is said to provide a *Schur decomposition* of A.

Let

$$A = \begin{pmatrix} 3 & 8 \\ -2 & 3 \end{pmatrix}, \qquad Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2i & 1 \\ -1 & -2i \end{pmatrix}.$$

Show that Q provides a Schur decomposition of A.

Problem 11. We say that a matrix is upper triangular if all their entries below the main diagonal are 0, and that it is strictly upper triangular if in addition all the entries on the main diagonal are equal to 1. Any invertible real $n \times n$ matrix A can be written as the product of three real $n \times n$ matrices

$$A = ODN$$

where N is strictly upper triangular, D is diagonal with positive entries, and O is orthogonal. This is known as the *Iwasawa decomposition* of the matrix A. The decomposition is unique. In other words, that if $A = O'D'N'$, where O', D' and N' are orthogonal, diagonal with positive entries and strictly upper triangular, respectively, then $O' = O$, $D = D'$ and $N' = N$. Find the Iwasawa decomposition of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

Problem 12. Consider the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Thus M is an element of the Lie group $SL(2, \mathbb{C})$. The *Iwasawa decomposition* is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \delta^{-1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$$

where $\alpha, \beta, \eta \in \mathbb{C}$ and $\delta \in \mathbb{R}^+$. Find α, β, δ and η.

Problem 13. Let A be a unitary $n \times n$ matrix. Let P be an invertible $n \times n$ matrix. Let $B := AP$. Show that PB^{-1} is unitary.

Problem 14. Show that every 2×2 matrix A of determinant 1 is the product of three elementary matrices. This means that matrix A can be written as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}. \qquad (1)$$

Problem 15. Almost any 2×2 matrix A can be factored (*Gaussian decomposition*) as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}.$$

Find the decomposition of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 16. Let A be an $n \times n$ matrix over $\mathbb{R}$. Consider the LU-decomposition $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix. The LDU-decomposition is defined as $A = LDU$, where L is unit lower triangular, D is diagonal and U is unit upper triangular. Let

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}.$$

Find the LDU-decomposition via the LU-decomposition.

Problem 17. Let U be an $n \times n$ unitary matrix. The matrix U can always be diagonalized by a unitary matrix V such that

$$U = V \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{pmatrix} V^*$$

where $e^{i\theta_j}$, $\theta_j \in [0, 2\pi)$ are the eigenvalues of U. Let

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus the eigenvalues are 1 and -1. Find the unitary matrix V such that

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = V \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V^*.$$

Chapter 7

Functions of Matrices

Problem 1. Let A be an $n \times n$ matrix over $\mathbb{C}$ with $A^2 = rA$, where $r \in \mathbb{C}$ and $r \neq 0$.
(i) Calculate e^{zA}, where $z \in \mathbb{C}$.
(ii) Let $U(z) = e^{zA}$. Let $z' \in \mathbb{C}$. Calculate $U(z)U(z')$.

Problem 2. Let A be an $n \times n$ matrix over $\mathbb{C}$. We define $\sin(A)$ as

$$\sin(A) := \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} A^{2j+1}.$$

Can we find a 2×2 matrix B over the real numbers $\mathbb{R}$ such that

$$\sin(B) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}? \tag{1}$$

Problem 3. Consider the unitary matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Can we find an $\alpha \in \mathbb{R}$ such that $U = \exp(\alpha A)$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}?$$

Problem 4. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that $A^2 = cI_n$, where $c \in \mathbb{R}$.

(i) Calculate $\exp(A)$.
(ii) Apply the result to the 2×2 matrix ($z \neq 0$)

$$B = \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix}.$$

Thus B is skew-hermitian, i.e., $\overline{B}^T = -B$.

Problem 5. Let H be a hermitian matrix, i.e., $H = H^*$. It is known that $U := e^{iH}$ is a unitary matrix. Let

$$H = \begin{pmatrix} a & b \\ \overline{b} & a \end{pmatrix}, \qquad a \in \mathbb{R},\ b \in \mathbb{C}$$

with $b \neq 0$.
(i) Calculate e^{iH} using the normalized eigenvectors of H to construct a unitary matrix V such that V^*HV is a diagonal matrix.
(ii) Specify a, b such that we find the unitary matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 6. It is known that any $n \times n$ unitary matrix U can be written as $U = \exp(iK)$, where K is a hermitian matrix. Assume that $\det(U) = -1$. What can be said about the trace of K?

Problem 7. The *MacLaurin series* for $\arctan(z)$ is defined as

$$\arctan(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{2j+1}$$

which converges for all complex values of z having absolute value less than 1, i.e., $|z| < 1$. Let A be an $n \times n$ matrix. Thus the series expansion

$$\arctan(A) = A - \frac{A^3}{3} + \frac{A^5}{5} - \frac{A^7}{7} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j A^{2j+1}}{2j+1}$$

is well-defined for A if all eigenvalues λ of A satisfy $|\lambda| < 1$. Let

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Does $\arctan(A)$ exist?

Problem 8. For every positive definite matrix A, there is a unique positive definite matrix Q such that $Q^2 = A$. The matrix Q is called the *square root* of A. Can we find the square root of the matrix

$$B = \frac{1}{2}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}?$$

Problem 9. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that

$$[A, [A, B]] = [B, [A, B]] = 0_n. \tag{1}$$

Show that

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{2a}$$

$$e^{A+B} = e^B e^A e^{+\frac{1}{2}[A,B]}. \tag{2b}$$

Use the *technique of parameter differentiation*, i.e. consider the matrix-valued function

$$f(\epsilon) := e^{\epsilon A} e^{\epsilon B}$$

where ϵ is a real parameter. Then take the derivative of f with respect to ϵ.

Problem 10. Let

$$J^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_3 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Let $\epsilon \in \mathbb{R}$. Find

$$e^{\epsilon J^+}, \quad e^{\epsilon J^-}, \quad e^{\epsilon (J^+ + J^-)}.$$

(ii) Let $r \in \mathbb{R}$. Show that

$$e^{r(J^+ + J^-)} \equiv e^{J^- \tanh(r)} e^{2J_3 \ln(\cosh(r))} e^{J^+ \tanh(r)}.$$

Problem 11. Let A, B, C_2, ..., C_m, ... be $n \times n$ matrices over $\mathbb{C}$. The *Zassenhaus formula* is given by

$$\exp(A+B) = \exp(A)\exp(B)\exp(C_2)\cdots\exp(C_m)\cdots$$

The left-hand side is called the *disentangled form* and the right-hand side is called the *undisentangled form*. Find C_2, C_3, ..., using the *comparison method*. In the comparison method the disentangled and undisentangled forms are expanded in terms of an ordering scalar α and matrix coefficients of equal powers of α are compared. From

$$\exp(\alpha(A+B)) = \exp(\alpha A)\exp(\alpha B)\exp(\alpha^2 C_2)\exp(\alpha^3 C_3)\cdots$$

we obtain

$$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!}(A+B)^k = \sum_{r_0,r_1,r_2,r_3,\ldots=0}^{\infty} \frac{\alpha^{r_0+r_1+2r_2+3r_3+\ldots}}{r_0!r_1!r_2!r_3!\cdots} A^{r_0}B^{r_1}C_2^{r_2}C_3^{r_3}\cdots$$

(i) Find C_2 and C_3.
(ii) Assume that $[A,[A,B]] = 0_n$ and $[B,[A,B]] = 0_n$. What conclusion can we draw for the Zassenhaus formula?

Problem 12. Calculating $\exp(A)$ we can also use the Cayley-Hamilton theorem and the *Putzer method*. The Putzer method is as follows. Using the Cayley-Hamilton theorem we can write

$$f(A) = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_2A^2 + a_1A + a_0I_n \tag{1}$$

where the complex numbers a_0, a_1, $\ldots$, a_{n-1} are determined as follows: Let

$$r(\lambda) := a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

which is the right-hand side of (1) with A^j replaced by λ^j ($j = 0,1,\ldots,n-1$). For each distinct eigenvalue λ_j of the matrix A, we consider the equation

$$f(\lambda_j) = r(\lambda_j). \tag{2}$$

If λ_j is an eigenvalue of multiplicity k, for $k > 1$, then we consider also the following equations

$$f'(\lambda)|_{\lambda=\lambda_j} = r'(\lambda)|_{\lambda=\lambda_j}, \quad \cdots \quad , f^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j} = r^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j}.$$

Calculate $\exp(A)$ with

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with the Putzer method.

Problem 13. Any unitary matrix U can be written as $U = \exp(iK)$, where K is hermitian. Apply the method of the previous problem to find K for the Hadamard matrix

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 14. Let A, B be $n \times n$ matrices and $t \in \mathbb{R}$. Show that

$$e^{t(A+B)} - e^{tA}e^{tB} = \frac{t^2}{2}(BA - AB) + \text{higher order terms in } t. \tag{1}$$

Problem 15. Let K be an $n \times n$ hermitian matrix. Show that

$$U := \exp(iK)$$

is a unitary matrix.

Problem 16. Let

$$A = \begin{pmatrix} 2 & 3 \\ 7 & -2 \end{pmatrix}.$$

Calculate $\det e^A$.

Problem 17. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that $A^2 = cI_n$, where $c \in \mathbb{R}$. Calculate $\exp(A)$.

Problem 18. Let A be an $n \times n$ matrix with $A^3 = -A$ and $\mu \in \mathbb{R}$. Calculate $\exp(\mu A)$.

Problem 19. Let X be an $n \times n$ matrix over $\mathbb{C}$. Assume that $X^2 = I_n$. Let Y be an arbitrary $n \times n$ matrix over $\mathbb{C}$. Let $z \in \mathbb{C}$.
(i) Calculate $\exp(zX)Y\exp(-zX)$ using the *Baker-Campbell-Hausdorff relation*

$$e^{zX}Ye^{-zX} = Y + z[X,Y] + \frac{z^2}{2!}[X,[X,Y]] + \frac{z^3}{3!}[X,[X,[X,Y]]] + \cdots.$$

(ii) Calculate $\exp(zX)Y\exp(-zX)$ by first calculating $\exp(zX)$ and $\exp(-zX)$ and then doing the matrix multiplication. Compare the two methods.

Problem 20. We consider the *principal logarithm* of a matrix $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$ (the closed negative real axis). This logarithm is denoted by $\log A$ and is the unique matrix B such that $\exp(B) = A$ and the eigenvalues of B have imaginary parts lying strictly between $-\pi$ and π. For $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$ we have the following integral representation

$$\log(s(A - I_n) + I_n) = \int_0^s (A - I_n)(t(A - I_n) + I_n)^{-1} dt.$$

Thus with $s = 1$ we obtain

$$\log A = \int_0^1 (A - I_n)(t(A - I_n) + I_n)^{-1} dt$$

where I_n is the $n \times n$ identity matrix. Let $A = xI_n$ with x a positive real number. Calculate $\log A$.

Problem 21. Let A be a real or complex $n \times n$ matrix with no eigenvalues on $\mathbb{R}^-$ (the closed negative real axis). Then there exists a unique matrix X such that
1) $e^X = A$
2) the eigenvalues of X lie in the strip $\{ z : -\pi < \Im(z) < \pi \}$. We refer to X as the *principal logarithm* of A and write $X = \log A$. Similarly, there is a unique matrix S such that
1) $S^2 = A$
2) the eigenvalues of S lie in the open halfplane: $0 < \Re(z)$. We refer to S as the *principal square root* of A and write $S = A^{1/2}$.
If the matrix A is real then its principal logarithm and principal square root are also real.
The open halfplane associated with $z = \rho e^{i\theta}$ is the set of complex numbers $w = \zeta e^{i\phi}$ such that $-\pi/2 < \phi - \theta < \pi/2$.
Suppose that $A = BC$ has no eigenvalues on $\mathbb{R}^-$ and
1. $BC = CB$
2. every eigenvalue of B lies in the open halfplane of the corresponding eigenvalue of $A^{1/2}$ (or, equivalently, the same condition holds for C).

Show that $\log(A) = \log(B) + \log(C)$.

Problem 22. Let K be a hermitian matrix. Then $U := \exp(iK)$ is a unitary matrix. A method to find the hermitian matrix K from the unitary matrix U is to consider the principal logarithm of a matrix $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$ (the closed negative real axis). This logarithm is denoted by $\log A$ and is the unique matrix B such that $\exp(B) = A$ and the eigenvalues of B have imaginary parts lying strictly between $-\pi$ and π. For $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$ we have the following integral representation

$$\log(s(A - I_n) + I_n) = \int_0^s (A - I_n)(t(A - I_n) + I_n)^{-1} dt.$$

Thus with $s = 1$ we obtain

$$\log(A) = \int_0^1 (A - I_n)(t(A - I_n) + I_n)^{-1} dt$$

where I_n is the $n \times n$ identity matrix. Find $\log U$ of the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

First test whether the method can be applied.

Problem 23. Let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{2^n-1}$ be an orthonormal basis in $\mathbb{C}^{2^n}$. We define

$$U := \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{-i2\pi kj/2^n} \mathbf{x}_k \mathbf{x}_j^*. \tag{1}$$

Show that U is unitary. In other words show that $UU^* = I_{2^n}$, using the *completeness relation*

$$I_{2^n} = \sum_{j=0}^{2^n-1} \mathbf{x}_j \mathbf{x}_j^*.$$

Thus I_{2^n} is the $2^n \times 2^n$ unit matrix.

Problem 24. Consider the unitary matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that we can find a unitary matrix V such that $V^2 = U$. Thus V would be the square root of U. What are the eigenvalues of V?

Problem 25. Let A be an $n \times n$ matrix. Let $\omega, \mu \in \mathbb{R}$. Assume that

$$\|e^{tA}\| \le Me^{\omega t}, \qquad t \ge 0$$

and $\mu > \omega$. Then we have

$$(\mu I_n - A)^{-1} \equiv \int_0^\infty e^{-\mu t} e^{tA} dt. \tag{1}$$

Calculate the left and right-hand side of (1) for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 26. The *Fréchet derivative* of a matrix function $f : \mathbb{C}^{n\times n}$ at a point $X \in \mathbb{C}^{n\times n}$ is a linear mapping $L_X : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ such that for all $Y \in \mathbb{C}^{n\times n}$

$$f(X+Y) - f(X) - L_X(Y) = o(\|Y\|).$$

Calculate the Fréchet derivative of $f(X) = X^2$.

Problem 27. Find the square root of the positive definite 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Chapter 8

Linear Differential Equations

Problem 1. Solve the initial value problem of the linear differential equation

$$\frac{dx}{dt} = 2x + \sin(t).$$

Problem 2. Solve the initial value problem of $d\mathbf{x}/dt = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 3. Solve the initial value problem of $d\mathbf{x}/dt = A\mathbf{x}$, where

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \quad a, b, c \in \mathbb{R}.$$

Problem 4. Show that the n-th order differential equation

$$\frac{d^n x}{dt^n} = c_0 x + c_1 \frac{dx}{dt} + \cdots + c_{n-1} \frac{d^{n-1} x}{dt^{n-1}}, \quad c_j \in \mathbb{R}$$

can be written as a system of first order differential equation.

Problem 5. Let A, X, F be $n \times n$ matrices. Assume that the matrix elements of X and F are differentiable functions of t. Consider the initial-value linear matrix differential equation with an inhomogeneous part

$$\frac{dX(t)}{dt} = AX(t) + F(t), \qquad X(t_0) = C.$$

Find the solution of this matrix differential equation.

Problem 6. Let A, B, C, Y be $n \times n$ matrices. We know that

$$AY + YB = C$$

can be written as

$$((I_n \otimes A) + (B^T \otimes I_n))\mathrm{vec}(Y) = \mathrm{vec}(C)$$

where $\otimes$ denotes the Kronecker product. The *vec operation* is defined as

$$\mathrm{vec}Y := (y_{11}, \ldots, y_{n1}, y_{12}, \ldots, y_{n2}, \ldots, y_{1n}, \ldots, y_{nn})^T.$$

Apply the vec operation to the matrix differential equation

$$\frac{d}{dt}X(t) = AX(t) + X(t)B$$

where A, B are $n \times n$ matrices and the initial matrix $X(t = 0) \equiv X(0)$ is given. Find the solution of this differential equation.

Problem 7. The *motion of a charge* q in an electromagnetic field is given by

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

where m denotes the mass and $\mathbf{v}$ the velocity. Assume that

$$\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{2}$$

are constant fields. Find the solution of the initial value problem.

Problem 8. Consider a system of linear ordinary differential equations with periodic coefficients

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} \tag{1}$$

where $A(t)$ is an $n \times n$ matrix of periodic functions with a period T. From *Floquet theory* we know that any fundamental $n \times n$ matrix $\Phi(t)$, which is defined as a nonsingular matrix satisfying the matrix differential equation

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t)$$

can be expressed as

$$\Phi(t) = P(t)\exp(tR). \tag{2}$$

Here $P(t)$ is a nonsingular $n \times n$ matrix of periodic functions with the same period T, and R, a constant matrix, whose eigenvalues are called the *characteristic exponents* of the periodic system (1). Let

$$\mathbf{y} = P^{-1}(t)\mathbf{x}.$$

Show that $\mathbf{y}$ satisfies the system of linear differential equations with constant coefficients

$$\frac{d\mathbf{y}}{dt} = R\mathbf{y}.$$

Problem 9. Consider the autonomous system of nonlinear first order ordinary differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= a(x_2 - x_1) = f_1(x_1, x_2, x_3) \\
\frac{dx_2}{dt} &= (c-a)x_1 + cx_2 - x_1x_3 = f_2(x_1, x_2, x_3) \\
\frac{dx_3}{dt} &= -bx_3 + x_1x_2 = f_3(x_1, x_2, x_3)
\end{aligned}$$

where $a > 0$, $b > 0$ and c are real constants with $2c > a$.
(i) The *fixed points* are defined as the solutions of the system of equations

$$\begin{aligned}
f_1(x_1^*, x_2^*, x_3^*) &= a(x_2^* - x_1^*) = 0 \\
f_2(x_1^*, x_2^*, x_3^*) &= (c-a)x_1^* + cx_2^* - x_1^*x_3^* = 0 \\
f_3(x_1^*, x_2^*, x_3^*) &= -bx_3^* + x_1^*x_2^* = 0\,.
\end{aligned}$$

Find the fixed points. Obviously $(0,0,0)$ is a fixed point.
(ii) The *linearized equation* (or variational equation) is given by

$$\begin{pmatrix} dy_1/dt \\ dy_2/dt \\ dy_3/dt \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where the 3×3 matrix A is given by

$$A_{\mathbf{x}=\mathbf{x}^*} = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{pmatrix}_{\mathbf{x}=\mathbf{x}^*}$$

where $\mathbf{x} = \mathbf{x}^*$ indicates to insert one of the fixed points into A. Calculate A and insert the first fixed point $(0, 0, 0)$. Calculate the eigenvalues of A. If all eigenvalues have negative real part then the fixed point is stable. Thus study the stability of the fixed point.

Chapter 9

Kronecker Product

Problem 1. (i) Let

$$\mathbf{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus $\{\mathbf{x}, \mathbf{y}\}$ forms an orthonormal basis in $\mathbb{C}^2$ (Hadamard basis). Calculate

$$\mathbf{x} \otimes \mathbf{x}, \quad \mathbf{x} \otimes \mathbf{y}, \quad \mathbf{y} \otimes \mathbf{x}, \quad \mathbf{y} \otimes \mathbf{y}$$

and interpret the result.

Problem 2. Consider the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find $\sigma_1 \otimes \sigma_3$ and $\sigma_3 \otimes \sigma_1$. Is $\sigma_1 \otimes \sigma_3 = \sigma_3 \otimes \sigma_1$?

Problem 3. Every 4×4 unitary matrix U can be written as

$$U = (U_1 \otimes U_2)\exp(i(\alpha\sigma_x \otimes \sigma_x + \beta\sigma_2 \otimes \sigma_2 + \gamma\sigma_3 \otimes \sigma_3))(U_3 \otimes U_4)$$

where $U_j \in U(2)$ $(j = 1,2,3,4)$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Calculate

$$\exp(i(\alpha\sigma_1 \otimes \sigma_1 + \beta\sigma_2 \otimes \sigma_2 + \gamma\sigma_3 \otimes \sigma_3)).$$

Problem 4. Find an orthonormal basis given by hermitian matrices in the Hilbert space $\mathcal{H}$ of 4×4 matrices over $\mathbb{C}$. The *scalar product* in the

Hilbert space $\mathcal{H}$ is given by

$$\langle A, B \rangle := \operatorname{tr}(AB^*), \qquad A, B \in \mathcal{H}.$$

Hint. Start with hermitian 2×2 matrices and then use the Kronecker product.

Problem 5. Consider the 4×4 matrices

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_2$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3.$$

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$, $\mathbf{d} = (d_1, d_2, d_3)$ be elements in $\mathbb{R}^3$ and

$$\mathbf{a} \cdot \boldsymbol{\alpha} := a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3.$$

Calculate the traces

$$\operatorname{tr}((\mathbf{a} \cdot \boldsymbol{\alpha})(\mathbf{b} \cdot \boldsymbol{\alpha})), \quad \operatorname{tr}((\mathbf{a} \cdot \boldsymbol{\alpha})(\mathbf{b} \cdot \boldsymbol{\alpha})(\mathbf{c} \cdot \boldsymbol{\alpha})(\mathbf{d} \cdot \boldsymbol{\alpha})).$$

Problem 6. Given the orthonormal basis

$$\mathbf{x}_1 = \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin\theta \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} -\sin(\theta) \\ e^{-i\phi}\cos\theta \end{pmatrix}$$

in the vector space $\mathbb{C}^2$. Use this orthonormal basis to find an orthonormal basis in $\mathbb{C}^4$.

Problem 7. Let A be an $m \times m$ matrix and B be an $n \times n$ matrix. The underlying field is $\mathbb{C}$. Let I_m, I_n be the $m \times m$ and $n \times n$ unit matrix, respectively.
(i) Show that $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$.
(ii) Show that $\operatorname{tr}(A \otimes I_n + I_m \otimes B) = n\operatorname{tr}(A) + m\operatorname{tr}(B)$.

Problem 8. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$. Show that

$$\exp(A \otimes I_n) \equiv \exp(A) \otimes I_n. \tag{1}$$

Problem 9. Let A, B be arbitrary $n \times n$ matrices over $\mathbb{C}$. Let I_n be the $n \times n$ unit matrix. Show that

$$\exp(A \otimes I_n + I_n \otimes B) \equiv \exp(A) \otimes \exp(B).$$

Problem 10. Let A and B be arbitrary $n \times n$ matrices over $\mathbb{C}$. Prove or disprove the equation

$$e^{A \otimes B} = e^A \otimes e^B.$$

Problem 11. Let A be an $m \times m$ matrix and B be an $n \times n$ matrix. The underlying field is $\mathbb{C}$. The eigenvalues and eigenvectors of A are given by $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$. The eigenvalues and eigenvectors of B are given by $\mu_1, \mu_2, \ldots, \mu_n$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Let ϵ_1, ϵ_2 and ϵ_3 be real parameters. Find the eigenvalues and eigenvectors of the matrix

$$\epsilon_1 A \otimes B + \epsilon_2 A \otimes I_n + \epsilon_3 I_m \otimes B.$$

Problem 12. Let A, B be $n \times n$ matrices over $\mathbb{C}$. A *scalar product* can be defined as

$$\langle A, B \rangle := \operatorname{tr}(AB^*).$$

The scalar product implies a *norm*

$$\|A\|^2 = \langle A, A \rangle = \operatorname{tr}(AA^*).$$

This norm is called the *Hilbert-Schmidt norm.*

(i) Consider the *Dirac matrices*

$$\gamma_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Calculate $\langle \gamma_0, \gamma_1 \rangle$.
(ii) Let U be a unitary $n \times n$ matrix. Find $\langle UA, UB \rangle$.
(iii) Let C, D be $m \times m$ matrices over $\mathbb{C}$. Find $\langle A \otimes C, B \otimes D \rangle$.

Problem 13. Let T be the 4×4 matrix

$$T := \left(I_2 \otimes I_2 + \sum_{j=1}^{3} t_j \sigma_j \otimes \sigma_j \right)$$

where σ_j, $j = 1, 2, 3$ are the Pauli spin matrices and $-1 \leq t_j \leq +1$, $j = 1, 2, 3$. Find T^2.

Problem 14. Let U be a 2×2 unitary matrix and I_2 be the 2×2 identity matrix. Is the 4×4 matrix

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U + \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2, \qquad \alpha \in \mathbb{R}$$

unitary?

Problem 15. Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad x_1 x_1^* + x_2 x_2^* = 1$$

be an arbitrary normalized vector in $\mathbb{C}^2$. Can we construct a 4×4 unitary matrix U such that

$$U \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ? \tag{1}$$

Prove or disprove this equation.

Problem 16. Let A_j $(j = 1, 2, \ldots, k)$ be matrices of size $m_j \times n_j$. We introduce the notation

$$\otimes_{j=1}^{k} A_j = (\otimes_{j=1}^{k-1} A_j) \otimes A_k = A_1 \otimes A_2 \otimes \cdots \otimes A_k.$$

Consider the binary matrices

$$J_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_{01} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(i) Calculate

$$\otimes_{j=1}^{n} (J_{00} + J_{01} + J_{11})$$

for $k = 1$, $k = 2$, $k = 3$ and $k = 8$. Give an interpretation of the result when each entry in the matrix represents a pixel (1 for black and 0 for white). This means we use the Kronecker product for representing images.

(ii) Calculate

$$\left(\otimes_{j=1}^{k}(J_{00}+J_{01}+J_{10}+J_{11})\right)\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

for $k=2$ and give an interpretation as an image, i.e., each entry 0 is identified with a black pixel and an entry 1 with a white pixel. Discuss the case for arbitrary k.

Problem 17. Consider the Pauli spin matrices $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$. Let $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ be unit vectors in $\mathbb{R}^3$. We define

$$Q:=\mathbf{q}\cdot\boldsymbol{\sigma},\quad R:=\mathbf{r}\cdot\boldsymbol{\sigma},\quad S:=\mathbf{s}\cdot\boldsymbol{\sigma},\quad T:=\mathbf{t}\cdot\boldsymbol{\sigma}$$

where $\mathbf{q}\cdot\boldsymbol{\sigma}:=q_1\sigma_1+q_2\sigma_2+q_3\sigma_3$. Calculate

$$(Q\otimes S+R\otimes S+R\otimes T-Q\otimes T)^2.$$

Express the result using commutators.

Problem 18. Let A and X be $n\times n$ matrices over $\mathbb{C}$. Assume that

$$[X,A]=0_n.$$

Calculate the commutator $[X\otimes I_n+I_n\otimes X, A\otimes A]$.

Problem 19. A square matrix is called a *stochastic matrix* if each entry is nonnegative and the sum of the entries in each row is 1. Let A, B be $n\times n$ stochastic matrices. Is $A\otimes B$ a stochastic matrix?

Problem 20. Let X be an $m\times m$ and Y be an $n\times n$ matrix. The *direct sum* is the $(m+n)\times(m+n)$ matrix

$$X\oplus Y=\begin{pmatrix}X&0\\0&Y\end{pmatrix}.$$

Let A be an $n\times n$ matrix, B be an $m\times m$ matrix and C be an $p\times p$ matrix. Then we have the identity

$$(A\oplus B)\otimes C\equiv(A\otimes C)\oplus(B\otimes C).$$

Is

$$A\otimes(B\oplus C)=(A\otimes B)\oplus(A\otimes C)$$

true?

Problem 21. Let A, B be 2×2 matrices, C a 3×3 matrix and D a 1×1 matrix. Find the condition on these matrices such that

$$A \otimes B = C \oplus D$$

where $\oplus$ denotes the *direct sum*. We assume that D is nonzero.

Problem 22. With each $m \times n$ matrix Y we associate the column vector vec$Y$ of length $m \times n$ defined by

$$\text{vec}(Y) := (y_{11}, \ldots, y_{m1}, y_{12}, \ldots, y_{m2}, \ldots, y_{1n}, \ldots, y_{mn})^T.$$

Let A be an $m \times n$ matrix, B an $p \times q$ matrix, and C an $m \times q$ matrix. Let X be an unknown $n \times p$ matrix. Show that the matrix equation

$$AXB = C$$

is equivalent to the system of qm equations in np unknowns given by

$$(B^T \otimes A)\text{vec}(X) = \text{vec}(C).$$

that is, $\text{vec}(AXB) = (B^T \otimes A)\text{vec}X$.

Problem 23. Let A, B, D be $n \times n$ matrices and I_n the $n \times n$ identity matrix. Use the result from the problem above to prove that

$$AX + XB = D$$

can be written as

$$((I_n \otimes A) + (B^T \otimes I_n))\text{vec}X = \text{vec}(D). \tag{1}$$

Problem 24. Let A be an $n \times n$ matrix and I_m be the $m \times m$ identity matrix. Show that

$$\sin(A \otimes I_m) \equiv \sin(A) \otimes I_m. \tag{1}$$

Problem 25. Let A be an $n \times n$ matrix and B be an $m \times m$ matrix. Is

$$\sin(A \otimes I_m + I_n \otimes B) \equiv (\sin(A)) \otimes (\cos(B)) + (\cos(A)) \otimes (\sin(B))? \tag{1}$$

Prove or disprove.

Problem 26. Let σ_1, σ_2, σ_3 be the *Pauli spin matrices*.
(i) Find

$$R_{1x}(\alpha) := \exp(-i\alpha(\sigma_1 \otimes I_2)), \qquad R_{1y}(\alpha) := \exp(-i\alpha(\sigma_2 \otimes I_2))$$

where $\alpha \in \mathbb{R}$ and I_2 denotes the 2×2 unit matrix.
(ii) Consider the special case $R_{1x}(\alpha = \pi/2)$ and $R_{1y}(\alpha = \pi/4)$. Calculate $R_{1x}(\pi/2)R_{1y}(\pi/4)$. Discuss.

Problem 27. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Find a 4×4 matrix A (*flip operator*) such that
$$A(\mathbf{x} \otimes \mathbf{y}) = \mathbf{y} \otimes \mathbf{x}.$$

Problem 28. Let σ_1, σ_2 and σ_3 be the Pauli spin matrices. We define $\sigma_+ := \sigma_1 + i\sigma_2$ and $\sigma_- := \sigma_1 - i\sigma_2$. Let
$$c_k^* := \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \left(\frac{1}{2}\sigma_+\right) \otimes I_2 \otimes I_2 \otimes \cdots \otimes I_2$$
where σ_+ is on the kth position and we have $N-1$ Kronecker products. Thus c_k^* is a $2^N \times 2^N$ matrix.
(i) Find c_k.
(ii) Find the anticommutators $[c_k, c_j]_+$ and $[c_k^*, c_j]_+$.
(iii) Find $c_k c_k$ and $c_k^* c_k^*$.

Problem 29. Using the definitions from the previous problem we define
$$s_{-,j} := \frac{1}{2}(\sigma_{x,j} - i\sigma_{y,j}) = \frac{1}{2}\sigma_{-,j}, \qquad s_{+,j} := \frac{1}{2}(\sigma_{x,j} + i\sigma_{y,j}) = \frac{1}{2}\sigma_{+,j}$$
and
$$c_1 = s_{-,1}$$
$$c_j = \exp\left(i\pi \sum_{\ell=1}^{j-1} s_{+,\ell} s_{-,\ell}\right) s_{-,j} \quad \text{for} \quad j = 2, 3, \ldots$$
(i) Find c_j^*.
(ii) Find the inverse transformation.
(iii) Calculate $c_j^* c_j$.

Problem 30. Let A, B, C, D be symmetric $n \times n$ matrices over $\mathbb{R}$. Assume that these matrices commute with each other. Consider the $4n \times 4n$ matrix
$$H = \begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix}.$$
(i) Calculate HH^T and express the result using the Kronecker product.

(ii) Assume that $A^2 + B^2 + C^2 + D^2 = 4nI_n$.

Problem 31. Can the 4×4 matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

be written as the Kronecker product of two 2×2 matrices A and B, i.e. $C = A \otimes B$?

Problem 32. Let A, B, C be $n \times n$ matrices. Assume that

$$[A, B] = 0_n, \qquad [A, C] = 0_n.$$

Let

$$X := I_n \otimes A + A \otimes I_n, \qquad Y := I_n \otimes B + B \otimes I_n + A \otimes C.$$

Calculate the commutator $[X, Y]$.

Problem 33. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. We define a *wedge product*

$$\mathbf{x} \wedge \mathbf{y} := \mathbf{x} \otimes \mathbf{y} - \mathbf{y} \otimes \mathbf{x}.$$

Show that

$$(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z} + (\mathbf{z} \wedge \mathbf{x}) \wedge \mathbf{y} + (\mathbf{y} \wedge \mathbf{z}) \wedge \mathbf{x} = \mathbf{0}. \tag{1}$$

Problem 34. Let V and W be the unitary matrices

$$V = \exp(i(\pi/4)\sigma_1) \otimes \exp(i(\pi/4)\sigma_1)$$
$$W = \exp(i(\pi/4)\sigma_2) \otimes \exp(i(\pi/4)\sigma_2).$$

Calculate

$$V^*(\sigma_3 \otimes \sigma_3)V, \qquad W^*(\sigma_3 \otimes \sigma_3)W.$$

Chapter 10

Norms and Scalar Products

Problem 1. Consider the vector ($\mathbf{v} \in \mathbb{C}^4$)

$$\mathbf{v} = \begin{pmatrix} i \\ 1 \\ -1 \\ -i \end{pmatrix}.$$

Find the Euclidean norm and then normalize the vector.

Problem 2. Consider the 4×4 matrix (Hamilton operator)

$$\hat{H} = \frac{\hbar\omega}{2}(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2)$$

where ω is the frequency and $\hbar$ is the Planck constant divided by 2π. Find the *norm* of $\hat{H}$, i.e.,

$$\|\hat{H}\| := \max_{\|\mathbf{x}\|=1} \|\hat{H}\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{C}^4$$

applying two different methods. In the first method apply the *Lagrange multiplier method*, where the constraint is $\|\mathbf{x}\| = 1$. In the second method we calculate $\hat{H}^*\hat{H}$ and find the square root of the largest eigenvalue. This is then $\|\hat{H}\|$. Note that $\hat{H}^*\hat{H}$ is positive semi-definite.

Problem 3. Let A be an $n \times n$ matrix over $\mathbb{R}$. The *spectral norm* is

$$\|A\|_2 := \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}.$$

It can be shown that $\|A\|_2$ can also be calculated as

$$\|A\|_2 = \sqrt{\text{largest eigenvalue of } A^T A}.$$

Note that the eigenvalues of $A^T A$ are real and nonnegative. Let

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.$$

Calculate $\|A\|_2$ using this method.

Problem 4. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

in $\mathbb{R}^3$.
(i) Show that the vectors are linearly independent.
(ii) Apply the *Gram-Schmidt orthonormalization process* to these vectors.

Problem 5. Let $\{ \mathbf{v}_j \; : \; j = 1, 2, \ldots, r \}$ be an orthogonal set of vectors in $\mathbb{R}^n$ with $r \leq n$. Show that

$$\| \sum_{j=1}^{r} \mathbf{v}_j \|^2 = \sum_{j=1}^{r} \|\mathbf{v}_j\|^2.$$

Problem 6. Consider the 2×2 matrix over $\mathbb{C}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Find the norm of A implied by the scalar product

$$\langle A, A \rangle = \sqrt{\operatorname{tr}(AA^*)}.$$

Problem 7. Let A, B be $n \times n$ matrices over $\mathbb{C}$. A scalar product is given by

$$\langle A, B \rangle = \operatorname{tr}(AB^*).$$

Let U be a unitary $n \times n$ matrix, i.e. we have $U^{-1} = U^*$.
(i) Calculate $\langle U, U \rangle$. Then find the norm implied by the scalar product.
(ii) Calculate

$$\|U\| := \max_{\|\mathbf{x}\|=1} \|U\mathbf{x}\|.$$

Problem 8. (i) Let $\{\, \mathbf{x}_j \,:\, j = 1, 2, \ldots, n \,\}$ be an orthonormal basis in $\mathbb{C}^n$. Let $\{\, \mathbf{y}_j \,:\, j = 1, 2, \ldots, n \,\}$ be another orthonormal basis in $\mathbb{C}^n$. Show that

$$(U_{jk}) := (\mathbf{x}_j^* \mathbf{y}_k)$$

is a unitary matrix, where $\mathbf{x}_j^* \mathbf{y}_k$ is the scalar product of the vectors $\mathbf{x}_j$ and $\mathbf{y}_k$. This means showing that $UU^* = I_n$.
(ii) Consider the bases in $\mathbb{C}^2$

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$\mathbf{y}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{y}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Use these bases to construct the corresponding 2×2 unitary matrix.

Problem 9. Find the norm $\|A\| = \sqrt{\operatorname{tr}(A^* A)}$ of the skew-hermitian matrix

$$A = \begin{pmatrix} i & 2+i \\ -2+i & 3i \end{pmatrix}$$

without calculating A^*.

Problem 10. Consider the Hilbert space $\mathcal{H}$ of the 2×2 matrices over the complex numbers with the scalar product

$$\langle A, B \rangle := \operatorname{tr}(AB^*), \qquad A, B \in \mathcal{H}.$$

Show that the rescaled Pauli matrices $\mu_j = \frac{1}{\sqrt{2}} \sigma_j$, $j = 1, 2, 3$

$$\mu_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the rescaled 2×2 identity matrix

$$\mu_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $\mathcal{H}$.

Problem 11. Let A and B be 2×2 diagonal matrices over $\mathbb{R}$. Assume that

$$\mathrm{tr}(AA^T) = \mathrm{tr}(BB^T)$$

and

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|B\mathbf{x}\|.$$

Can we conclude that $A = B$?

Problem 12. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let $\|.\|$ be a subordinate matrix norm for which $\|I_n\| = 1$. Assume that $\|A\| < 1$.
(i) Show that the matrix $(I_n - A)$ is nonsingular.
(ii) Show that

$$\|(I_n - A)^{-1}\| \leq (1 - \|A\|)^{-1}.$$

Problem 13. Let A be an $n \times n$ matrix. Assume that $\|A\| < 1$. Show that

$$\|(I_n - A)^{-1} - I_n\| \leq \frac{\|A\|}{1 - \|A\|}.$$

Problem 14. Let A be an $n \times n$ nonsingular matrix and B an $n \times n$ matrix. Assume that $\|A^{-1}B\| < 1$.
(i) Show that $A - B$ is nonsingular.
(ii) Show that

$$\frac{\|A^{-1} - (A - B)^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|}.$$

Problem 15. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the linear system $A\mathbf{x} = \mathbf{b}$. The *condition number* of A is defined as

$$\mathrm{Cond}(A) := \|A\| \, \|A^{-1}\|.$$

Find the condition number for the matrix

$$A = \begin{pmatrix} 1 & 0.9999 \\ 0.9999 & 1 \end{pmatrix}$$

for the infinity norm, 1-norm and 2-norm.

Problem 16. Let A, B be $n \times n$ matrices over $\mathbb{R}$ and $t \in \mathbb{R}$. Let $\| \quad \|$ be a matrix norm. Show that

$$\|e^{tA}e^{tB} - I_n\| \leq \exp(|t|(\|A\| + \|B\|)) - 1.$$

Problem 17. Let $A_1, A_2, \ldots, A_p$ be $m \times m$ matrices over $\mathbb{C}$. Then we have the inequality

$$\| \exp(\sum_{j=1}^{p} A_j) - (e^{A_1/n} \cdots e^{A_p/n})^n \| \leq \frac{2}{n} \left(\sum_{j=1}^{p} \|A_j\| \right)^2 \times \exp\left(\frac{n+2}{n} \sum_{j=1}^{p} \|A_j\| \right)$$

and

$$\lim_{n \to \infty} (e^{A_1/n} e^{A_2/n} \cdots e^{A_p/n})^n = \exp(\sum_{j=1}^{p} A_j).$$

Let $p = 2$. Find the estimate for the 2×2 matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Chapter 11

Groups and Matrices

Problem 1. Find the group generated by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2. (i) Show that the set of matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad C_3^{-1} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

form a group G under matrix multiplication, where C_3^{-1} is the inverse matrix of C_3.
(ii) Find the determinant of all these matrices. Does the set of numbers

$$\{\, \det(E),\ \det(C_3),\ \det(C_3^{-1}),\ \det(\sigma_1),\ \det(\sigma_2),\ \det(\sigma_3)\ \}$$

form a group under multiplication.
(iii) Find two proper subgroups.
(iv) Find the right coset decomposition. Find the left coset decomposition. We obtain the right *coset decomposition* as follows: Let G be a finite group of order g having a proper subgroup $\mathcal{H}$ of order h. Take some element g_2 of G which does not belong to the subgroup $\mathcal{H}$, and make a right coset $\mathcal{H}g_2$. If $\mathcal{H}$ and $\mathcal{H}g_2$ do not exhaust the group G, take some element g_3 of G which

is not an element of $\mathcal{H}$ and $\mathcal{H}g_2$, and make a right coset $\mathcal{H}g_3$. Continue making right cosets $\mathcal{H}g_j$ in this way. If G is a finite group, all elements of G will be exhausted in a finite number of steps and we obtain the right coset decomposition.

Problem 3. We know that the set of matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad C_3^{-1} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

forms a group G under matrix multiplication, where C_3^{-1} is the inverse matrix of C_3. The set of matrices (3×3 *permutation matrices*)

$$I = P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

also forms a group G under matrix multiplication. Are the two groups isomorphic? A homomorphism which is $1-1$ and onto is an *isomorphism*.

Problem 4. (i) Show that the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

form a group under matrix multiplication.
(ii) Show that the matrices

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

form a group under matrix multiplication.
(iii) Show that the two groups (so-called *Vierergruppe*) are isomorphic.

Problem 5. (i) Let $x \in \mathbb{R}$. Show that the 2×2 matrices

$$A(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

form a group under matrix multiplication.
(ii) Is the group commutative?
(iii) Find a group that is isomorphic to this group.

Problem 6. Let $a, b, c, d \in \mathbb{Z}$. Show that the 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$ form a group under matrix multiplication.

Problem 7. The Lie group $SU(2)$ is defined by

$$SU(2) := \{ U \ 2 \times 2 \, \text{matrix} \, : \, UU^* = I_2, \ \det U = 1 \}.$$

Let (3-sphere)

$$S^3 := \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \ \}.$$

Show that $SU(2)$ can be identified as a real manifold with the 3-sphere S^3.

Problem 8. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let

$$U(\alpha, \beta, \gamma) = e^{-i\alpha\sigma_3/2} e^{-i\beta\sigma_2/2} e^{-i\gamma\sigma_3/2}$$

where α, β, γ are the three *Euler angles* with the range $0 \le \alpha < 2\pi$, $0 \le \beta \le \pi$ and $0 \le \gamma < 2\pi$. Show that

$$U(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i\alpha/2} \cos(\beta/2) e^{-i\gamma/2} & -e^{-i\alpha/2} \sin(\beta/2) e^{i\gamma/2} \\ e^{i\alpha/2} \sin(\beta/2) e^{-i\gamma/2} & e^{i\alpha/2} \cos(\beta/2) e^{i\gamma/2} \end{pmatrix}. \qquad (1)$$

Problem 9. The *Heisenberg group* is the set of upper 3×3 matrices of the form

$$H = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c can be taken from some (arbitrary) commutative ring.
(i) Find the inverse of H.
(ii) Given two elements x, y of a group G, we define the *commutator* of x and y, denoted by $[x, y]$ to be the element $x^{-1}y^{-1}xy$. If a, b, c are integers (in the ring $\mathbb{Z}$ of the integers) we obtain the discrete Heisenberg group H_3. It has two generators

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find

$$z = xyx^{-1}y^{-1}.$$

Show that $xz = zx$ and $yz = zy$, i.e., z is the generator of the center of H_3.
(iii) The derived subgroup (or commutator subgroup) of a group G is the subgroup $[G, G]$ generated by the set of commutators of every pair of elements of G. Find $[G, G]$ for the Heisenberg group.
(iv) Let

$$A = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

and $a, b, c \in \mathbb{R}$. Find $\exp(A)$.
(v) The Heisenberg group is a simple connected Lie group whose Lie algebra consists of matrices

$$L = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the commutators $[L, L']$ and $[[L, L'], L']$, where $[L, L'] := LL' - L'L$.

Problem 10. Define

$$\begin{aligned} M \,:\, \mathbb{R}^3 \to V &:= \{\, \mathbf{a} \cdot \boldsymbol{\sigma} \,:\, \mathbf{a} \in \mathbb{R}^3 \,\} \subset \{\, 2 \times 2 \text{ complex matrices} \,\} \\ \mathbf{a} \to M(\mathbf{a}) &= \mathbf{a} \cdot \boldsymbol{\sigma} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3. \end{aligned}$$

This is a linear *bijection* between $\mathbb{R}^3$ and V. Each $U \in SU(2)$ determines a linear map $S(U)$ on $\mathbb{R}^3$ by

$$M(S(U)\mathbf{a}) = U^{-1}M(\mathbf{a})U.$$

The right-hand side is clearly linear in $\mathbf{a}$. Show that $U^{-1}M(\mathbf{a})U$ is in V, that is, of the form $M(\mathbf{b})$.

Problem 11. A *topological group* G is both a group and a topological space, the two structures are related by the requirement that the maps

$x \mapsto x^{-1}$ (of G onto G) and $(x, y) \mapsto xy$ (of $G \times G$ onto G) are continuous. $G \times G$ is given by the product topology.
(i) Given a topological group G, define the maps

$$\phi(x) := xax^{-1}$$

and

$$\psi(x) := xax^{-1}a^{-1} \equiv [x, a].$$

How are the iterates of the maps ϕ and ψ related?
(ii) Consider $G = SO(2)$ and

$$x = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \qquad a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with $x, a \in SO(2)$. Calculate ϕ and ψ. Discuss.

Problem 12. Show that the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

are conjugate in $SL(2, \mathbb{C})$ but not in $SL(2, \mathbb{R})$ (the real matrices in $SL(2, \mathbb{C})$).

Problem 13. (i) Let G be a finite set of real $n \times n$ matrices $\{ A_j \}$, $1 \le i \le r$, which forms a group under matrix multiplication. Suppose that

$$\mathrm{tr}(\sum_{j=1}^{r} A_j) = \sum_{j=1}^{r} \mathrm{tr}(A_j) = 0$$

where tr denotes the trace. Show that

$$\sum_{j=1}^{r} A_j = 0_n.$$

(ii) Show that the 2×2 matrices

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$$

form a group under matrix multiplication, where

$$\omega := \exp(2\pi i/3).$$

(iii) Show that

$$\sum_{j=1}^{6} \operatorname{tr}(B_j) = 0.$$

Problem 14. The unitary matrices are elements of the Lie group $U(n)$. The corresponding Lie algebra $u(n)$ is the set of matrices with the condition

$$X^* = -X.$$

An important subgroup of $U(n)$ is the Lie group $SU(n)$ with the condition that $\det U = 1$. The unitary matrices

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are not elements of the Lie algebra $SU(2)$ since the determinants of these unitary matrices are -1. The corresponding Lie algebra $su(n)$ of the Lie group $SU(n)$ are the $n \times n$ matrices given by

$$X^* = -X, \qquad \operatorname{tr}(X) = 0.$$

Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Then any unitary matrix in $U(2)$ can be represented by

$$U(\alpha, \beta, \gamma, \delta) = e^{i\alpha I_2} e^{-i\beta\sigma_3/2} e^{-i\gamma\sigma_2/2} e^{-i\delta\sigma_3/2}$$

where $0 \le \alpha < 2\pi$, $0 \le \beta < 2\pi$, $0 \le \gamma \le \pi$ and $0 \le \delta < 2\pi$. Calculate the right-hand side.

Problem 15. Given an orthonormal basis (column vectors) in $\mathbb{C}^N$ denoted by

$$\mathbf{x}_0, \ \mathbf{x}_1 \ , \dots, \ \mathbf{x}_{N-1}.$$

(i) Show that

$$U := \sum_{k=0}^{N-2} \mathbf{x}_k \mathbf{x}_{k+1}^* + \mathbf{x}_{N-1}\mathbf{x}_0^*$$

is a unitary matrix.
(ii) Find $\operatorname{tr}(U)$.
(iii) Find U^N.
(iv) Does U depend on the chosen basis? Prove or disprove.
Hint. Consider $N = 2$, the standard basis $(1,0)^T$, $(0,1)^T$ and the basis $\frac{1}{\sqrt{2}}(1,1)^T$, $\frac{1}{\sqrt{2}}(1,-1)^T$.
(v) Show that the set

$$\{\, U, U^2, \dots, U^N \,\}$$

forms a *commutative group* (*abelian group*) under matrix multiplication. The set is a subgroup of the group of all permutation matrices.
(vi) Assume that the set given above is the standard basis. Show that the matrix U is given by

$$U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Problem 16. (i) Let

$$M := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{pmatrix}.$$

Is the matrix M unitary?
(ii) Let

$$U_H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad U_S := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

and

$$U_{CNOT2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Show that the matrix M can be written as

$$M = U_{CNOT2}(I_2 \otimes U_H)(U_S \otimes U_S).$$

(iii) Let $SO(4)$ be the special orthogonal Lie group. Let $SU(2)$ be the special unitary Lie group. Show that for every real orthogonal matrix $U \in SO(4)$, the matrix MUM^{-1} is the Kronecker product of two 2-dimensional special unitary matrices, i.e.,

$$MUM^{-1} \in SU(2) \otimes SU(2).$$

Problem 17. Sometimes we parametrize the group elements of the three parameter group $SO(3)$ in terms of the *Euler angles* ψ, θ, ϕ

$$A(\psi, \theta, \phi) =$$

$$\begin{pmatrix} \cos(\phi)\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos(\phi)\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta \\ \sin(\phi)\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin(\phi)\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin(\theta)\cos(\psi) & \sin\theta\sin\psi & \cos(\theta) \end{pmatrix}$$

with the parameters falling in the intervals

$$-\pi \le \psi < \pi, \qquad 0 \le \theta \le \pi, \qquad -\pi \le \phi < \pi.$$

Describe the shortcomings this parametrization suffers.

Problem 18. The *octonion algebra* $\mathcal{O}$ is an 8-dimensional non-associative algebra. It is defined in terms of the basis elements e_μ ($\mu = 0, 1, \dots, 7$) and their multiplication table. e_0 is the unit element. We use greek indices ($\mu, \nu, \dots$) to include the 0 and latin indices ($i, j, k, \dots$) when we exclude the 0. We define

$$\hat{e}_k := e_{4+k} \quad \text{for} \quad k = 1, 2, 3.$$

The multiplication rules among the basis elements of octonions e_μ are given by

$$e_i e_j = -\delta_{ij} e_0 + \sum_{k=1}^{3} \epsilon_{ijk} e_k, \quad i, j, k = 1, 2, 3 \tag{1}$$

and

$$-e_4 e_i = e_i e_4 = \hat{e}_i, \quad e_4 \hat{e}_i = -\hat{e}_i e_4 = e_i, \quad e_4 e_4 = -e_0$$

$$\hat{e}_i \hat{e}_j = -\delta_{ij} e_0 - \sum_{k=1}^{3} \epsilon_{ijk} e_k, \quad i, j, k = 1, 2, 3$$

$$-\hat{e}_j e_i = e_i \hat{e}_j = -\delta_{ij} e_4 - \sum_{k=1}^{3} \epsilon_{ijk} \hat{e}_k, \quad i, j, k = 1, 2, 3$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is $+1$ if (ijk) is an even permutation of (123), -1 if (ijk) is an odd permutation of (123) and 0 otherwise. We can formally summarize the multiplications as

$$e_\mu e_\nu = g_{\mu\nu} e_0 + \sum_{k=1}^{7} \gamma_{\mu\nu}^k e_k$$

where

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1, -1, -1, -1, -1), \qquad \gamma_{ij}^k = -\gamma_{ji}^k$$

with $\mu, \nu = 0, 1, \dots, 7$, and $i, j, k = 1, 2, \dots, 7$.
(i) Show that the set $\{ e_0, e_1, e_2, e_3 \}$ is a closed associative subalgebra.
(ii) Show that the octonian algebra $\mathcal{O}$ is non-associative.

Problem 19. Consider the set

$$\left\{ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Then under matrix multiplication we have a group. Consider the set

$$\{ e \otimes e, \quad e \otimes a, \quad a \otimes e, \quad a \otimes a \}.$$

Does this set form a group under matrix multiplication, where $\otimes$ denotes the Kronecker product?

Problem 20. Let

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(i) Find all 2×2 matrices A over $\mathbb{R}$ such that

$$A^T J A = J.$$

(ii) Do these 2×2 matrices form a group under matrix multiplication?

Problem 21. Let J be the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and 0_n is the $n \times n$ zero matrix. Show that the $2n \times 2n$ matrices A satisfying

$$A^T J A = J$$

form a group under matrix multiplication. This group is called the *symplectic group* $Sp(2n)$.

Problem 22. We consider the following subgroups of the Lie group $SL(2, \mathbb{R})$. Let

$$K := \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

$$A := \left\{ \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix} : r > 0 \right\}$$

$$N := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

It can be shown that any matrix $m \in SL(2, \mathbb{R})$ can be written in a unique way as the product $m = kan$ with $k \in K$, $a \in A$ and $n \in N$. This decomposition is called *Iwasawa decomposition* and has a natural generalization to

$SL(n,\mathbb{R})$, $n \geq 3$. The notation of the subgroups comes from the fact that K is a compact subgroup, A is an abelian subgroup and N is a nilpotent subgroup of $SL(2,\mathbb{R})$. Find the Iwasawa decomposition of the matrix

$$\begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Problem 23. Let $GL(m,\mathbb{C})$ be the general linear group over $\mathbb{C}$. This Lie group consists of all nonsingular $m \times m$ matrices. Let G be a Lie subgroup of $GL(m,\mathbb{C})$. Suppose $u_1, u_2, \ldots, u_n$ is a coordinate system on G in some neighborhood of I_m, the $m \times m$ identity matrix, and that $X(u_1, u_2, \ldots, u_m)$ is a point in this neighborhood. The matrix dX of differential one-forms contains n linearly independent differential one-forms since the n-dimensional Lie group G is smoothly embedded in $GL(m,\mathbb{C})$. Consider the matrix of differential one forms

$$\Omega := X^{-1}dX, \qquad X \in G.$$

The matrix Ω of differential one forms contains n-linearly independent ones.
(i) Let A be any fixed element of G. The *left-translation* by A is given by

$$X \to AX.$$

Show that $\Omega = X^{-1}dX$ is left-invariant.
(ii) Show that

$$d\Omega + \Omega \wedge \Omega = 0$$

where $\wedge$ denotes the exterior product for matrices, i.e. we have matrix multiplication together with the *exterior product*. The exterior product is linear and satisfies

$$du_j \wedge du_k = -du_k \wedge du_j.$$

Therefore $du_j \wedge du_j = 0$ for $j = 1, 2, \ldots, n$. The exterior product is also associative.
(iii) Find dX^{-1} using $XX^{-1} = I_m$.

Problem 24. Consider $GL(m,\mathbb{R})$ and a Lie subgroup of it. We interpret each element X of G as a linear transformation on the vector space $\mathbb{R}^m$ of row vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Thus

$$\mathbf{v} \to \mathbf{w} = \mathbf{v}X.$$

Show that $d\mathbf{w} = \mathbf{w}\Omega$.

Problem 25. Consider the Lie group $SO(2)$ consisting of the matrices

$$X = \begin{pmatrix} \cos(u) & -\sin(u) \\ \sin(u) & \cos(u) \end{pmatrix}.$$

Calculate dX and $X^{-1}dX$.

Problem 26. Let n be the dimension of the Lie group G. Since the vector space of differential one-forms at the identity element is an n-dimensional vector space, there are exactly n linearly independent left invariant differential one-forms in G. Let σ_1, σ_2, ..., σ_n be such a system. Consider the Lie group

$$G := \left\{ \begin{pmatrix} u_1 & u_2 \\ 0 & 1 \end{pmatrix} \; : \; u_1, u_2 \in \mathbb{R}, \;\; u_1 > 0 \right\}.$$

Let

$$X = \begin{pmatrix} u_1 & u_2 \\ 0 & 1 \end{pmatrix}.$$

(i) Find X^{-1} and $X^{-1}dX$. Calculate the left-invariant differential one-forms. Calculate the left-invariant volume element.
(ii) Find the right-invariant forms.

Problem 27. Consider the Lie group consisting of the matrices

$$X = \begin{pmatrix} u_1 & u_2 \\ 0 & u_1 \end{pmatrix}, \;\; u_1, u_2 \in \mathbb{R}, \quad u_1 > 0.$$

Calculate X^{-1} and $X^{-1}dX$. Find the left-invariant differential one-forms and the left-invariant volume element.

Problem 28. Find the group generated by the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

under matrix multiplication.

Problem 29. Find the group generated by the two permutation matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Chapter 12

Lie Algebras and Matrices

Problem 1. Consider the $n \times n$ matrices E_{ij} having 1 in the (i,j) position and 0 elsewhere, where $i, j = 1, 2, \ldots, n$. Calculate the commutator. Discuss.

Problem 2. Show that the matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are the generators for a Lie algebra.

Problem 3. Consider the matrices

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that the matrices form a basis of a Lie algebra.

Problem 4. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Calculate $\mathrm{tr}([A, B])$. Discuss.

Problem 5. An $n \times n$ matrix X over $\mathbb{C}$ is *skew-hermitian* if $X^* = -X$. Show that the commutator of two skew-hermitian matrices is again skew-hermitian. Discuss.

Problem 6. The Lie algebra $su(m)$ consists of all $m \times m$ matrices X over $\mathbb{C}$ with the conditions $X^* = -X$ (i.e. X is skew-hermitian) and $\mathrm{tr} X = 0$. Note that $\exp(X)$ is a unitary matrix. Find a basis for $su(3)$.

Problem 7. Any fixed element X of a Lie algebra L defines a linear transformation

$$\mathrm{ad}(X) : Z \to [X, Z] \quad \text{for any} \quad Z \in L.$$

Show that for any $K \in L$ we have

$$[\mathrm{ad}(Y), \mathrm{ad}(Z)]K = \mathrm{ad}([Y, Z])K.$$

The linear mapping ad gives a representation of the Lie algebra known as *adjoint representation.*

Problem 8. There is only one non-commutative Lie algebra L of dimension 2. If x, y are the generators (basis in L), then

$$[x, y] = x.$$

(i) Find the *adjoint representation* of this Lie algebra. Let v, w be two elements of a Lie algebra. Then we define

$$\mathrm{ad}v(w) := [v, w]$$

and $w\mathrm{ad}v := [v, w]$.
(ii) The *Killing form* is defined by

$$\kappa(x, y) := \mathrm{tr}(\mathrm{ad}x\, \mathrm{ad}y)$$

for all $x, y \in L$. Find the Killing form.

Problem 9. Consider the Lie algebra $L = s\ell(2, \mathbb{F})$ with $\mathrm{char}\mathbb{F} \neq 2$. Take as the standard basis for L the three matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Find the multiplication table, i.e. the commutators.
(ii) Find the adjoint representation of L with the ordered basis $\{\, x \ h \ y \,\}$.

(iii) Show that L is *simple*. If L has no ideals except itself and 0, and if moreover $[L, L] \neq 0$, we call L simple. A subspace I of a Lie algebra L is called an *ideal* of L if $x \in L$, $y \in I$ together imply $[x, y] \in I$.

Problem 10. Consider the Lie algebra $gl_2(\mathbb{R})$. The matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis of $gl_2(\mathbb{R})$. Find the adjoint representation.

Problem 11. Let $\{ e, f \}$ with

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

a basis for a Lie algebra. We have $[e, f] = e$. Is $\{ e \otimes I_2, f \otimes I_2 \}$ a basis of a Lie algebra? Here I_2 denotes the 2×2 unit matrix and $\otimes$ the Kronecker product.

Problem 12. Let $\{ e, f \}$ with

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

a basis for a Lie algebra. We have $[e, f] = e$. Is $\{ e \otimes e, e \otimes f, f \otimes e, f \otimes f \}$ a basis of a Lie algebra?

Problem 13. The elements (generators) $Z_1, Z_2, \ldots, Z_r$ of an r-dimensional Lie algebra satisfy the conditions

$$[Z_\mu, Z_\nu] = \sum_{\tau=1}^{r} c^\tau_{\mu\nu} Z_\tau$$

with $c^\tau_{\mu\nu} = -c^\tau_{\nu\mu}$, where the $c^\tau_{\mu\nu}$'s are called the *structure constants*. Let A be an arbitrary linear combination of the elements

$$A = \sum_{\mu=1}^{r} a^\mu Z_\mu.$$

Suppose that X is some other linear combination such that

$$X = \sum_{\nu=1}^{r} b^\nu Z_\nu$$

and

$$[A, X] = \rho X.$$

This equation has the form of an eigenvalue equation, where ρ is the corresponding eigenvalue and X the corresponding eigenvector. Assume that the Lie algebra is represented by matrices. Find the secular equation for the eigenvalues ρ.

Problem 14. Let $c^{\tau}_{\sigma\lambda}$ be the structure constants of a Lie algebra. We define

$$g_{\sigma\lambda} = g_{\lambda\sigma} = \sum_{\rho=1}^{r}\sum_{\tau=1}^{r} c^{\tau}_{\sigma\rho} c^{\rho}_{\lambda\tau}$$

and

$$g^{\sigma\lambda} g_{\sigma\lambda} = \delta^{\lambda}_{\sigma}.$$

A Lie algebra L is called *semisimple* if and only if $\det |g_{\sigma\lambda}| \neq 0$. We assume in the following that the Lie algebra is semisimple. We define

$$C := \sum_{\rho=1}^{r}\sum_{\sigma=1}^{r} g^{\rho\sigma} X_\rho X_\sigma.$$

The operator C is called *Casimir operator*. Let X_τ be an element of the Lie algebra L. Calculate the commutator $[C, X_\tau]$.

Problem 15. Show that the matrices

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form generators of a Lie algebra. Is the Lie algebra simple? A Lie algebra is *simple* if it contains no ideals other than L and 0.

Problem 16. The *roots* of a semisimple Lie algebra are the Lie algebra weights occurring in its adjoint representation. The set of roots forms the root system, and is completely determined by the semisimple Lie algebra. Consider the semisimple Lie algebra $s\ell(2, \mathbb{R})$ with the generators

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find the roots.

Problem 17. The Lie algbra $sl(2, \mathbb{R})$ is spanned by the matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(i) Find the commutators $[h, e]$, $[h, f]$ and $[e, f]$.
(ii) Consider

$$C = \frac{1}{2}h^2 + ef + fe.$$

Find C. Calculate the commutators $[C, h]$, $[C, e]$, $[C, f]$. Show that C can be written in the form

$$C = \frac{1}{2}h^2 + h + 2fe.$$

(iii) Consider the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Calculate $h\mathbf{v}$, $e\mathbf{v}$, $f\mathbf{v}$ and $C\mathbf{v}$. Give an interpretation.

Problem 18. Let L be a finite dimensional Lie algebra. Let $C^\infty(S^1)$ be the set of all infinitely differentiable functions, where S^1 is the unit circle manifold. In the product space $L \otimes C^\infty(S^1)$ we define the Lie bracket ($g_1, g_2 \in L$ and $f_1, f_2 \in C^\infty(S^1)$)

$$[g_1 \otimes f_1, g_2 \otimes f_2] := [g_1, g_2] \otimes (f_1 f_2).$$

Calculate

$$[g_1 \otimes f_1, [g_2 \otimes f_2, g_3 \otimes f_3]] + [g_3 \otimes f_3, [g_1 \otimes f_1, g_2 \otimes f_2]] + [g_2 \otimes f_2, [g_3 \otimes f_3, g_1 \otimes f_1]].$$

Problem 19. A basis for the Lie algebra $su(N)$, for odd N, may be built from two unitary unimodular $N \times N$ matrices

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where ω is a primitive Nth root of unity, i.e. with period not smaller than N, here taken to be $\exp(4\pi i/N)$. We obviously have

$$hg = \omega gh. \tag{1}$$

(i) Find g^N and h^N.
(ii) Find $\mathrm{tr}(g)$.
(iii) Let $\mathbf{m} = (m_1, m_2)$, $\mathbf{n} = (n_1, n_2)$ and define

$$\mathbf{m} \times \mathbf{n} := m_1 n_2 - m_2 n_1$$

where $m_1 = 0, 1, \ldots, N-1$ and $m_2 = 0, 1, \ldots, N-1$. The complete set of unitary unimodular $N \times N$ matrices

$$J_{m_1,m_2} := \omega^{m_1 m_2/2} g^{m_1} h^{m_2}$$

suffice to span the Lie algebra $su(N)$, where $J_{0,0} = I_N$. Find J^*.
(iv) Calculate $J_{\mathbf{m}} J_{\mathbf{n}}$.
(v) Find the commutator $[J_{\mathbf{m}}, J_{\mathbf{n}}]$.

Problem 20. Consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with the commutator $[A, B] = A$. Thus we have a basis of a two-dimesnional non-abelian Lie algebra. Do the three 8×8 matrices

$$V_1 = A \otimes B \otimes I_2, \quad V_2 = A \otimes I_2 \otimes B, \quad V_3 = I_2 \otimes A \otimes B$$

form a basis of Lie algebra under the commutator?

Problem 21. Let $\alpha \in \mathbb{R}$. Find the Lie algebra generated by the 2×2 matrices

$$A(\alpha) = \begin{pmatrix} 0 & \cosh(\alpha) \\ \sinh(\alpha) & 0 \end{pmatrix}, \quad B(\alpha) = \begin{pmatrix} 0 & \sinh(\alpha) \\ \cosh(\alpha) & 0 \end{pmatrix}.$$

Chapter 13

Graphs and Matrices

Problem 1. A walk of length k in a digraph is a succession of k arcs joining two vertices. A trail is a walk in which all the arcs (but not necessarily all the vertices) are distinct. A path is a walk in which all the arcs and all the vertices are distinct. Show that the number of walks of length k from vertex i to vertex j in a digraph D with n vertices is given by the ijth element of the matrix A^k, where A is the adjacency matrix of the digraph.

Problem 2. Consider a digraph. The out-degree of a vertex v is the number of arcs incident from v and the in-degree of a vertex V is the number of arcs incident to v. Loops count as one of each.

Determine the in-degree and the out-degree of each vertex in the digraph given by the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and hence determine if it is an Eulerian graph. Display the digraph and determine an Eulerian trail.

Problem 3. A digraph is strongly connected if there is a path between every pair of vertices. Show that if A is the adjacency matrix of a digraph

D with n vertices and B is the matrix

$$B = A + A^2 + A^3 + \cdots + A^{n-1}$$

then D is strongly connected iff each non-diagonal element of B is greater than 0.

Problem 4. Write down the adjacency matrix A for the digraph shown. Calculate the matrices A^2, A^3 and A^4. Consequently find the number of walks of length 1, 2, 3 and 4 from w to u. Is there a walk of length 1, 2, 3, or 4 from u to w? Find the matrix $B = A + A^2 + A^3 + A^4$ for the digraph and hence conclude whether it is strongly connected. This means finding out whether all off diagonal elements are nonzero.

Chapter 14

Hadamard Product

Problem 1. Let A and B be $m \times n$ matrices. The *Hadamard product* $A \circ B$ is defined as the $m \times n$ matrix

$$A \bullet B := (a_{ij} b_{ij}).$$

(i) Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ 7 & 1 \end{pmatrix}.$$

Calculate $A \bullet B$.
(ii) Let C, D be $m \times n$ matrices. Show that

$$\text{rank}(A \bullet B) \leq (\text{rank} A)(\text{rank} B).$$

Problem 2. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$$

be symmetric matrices over $\mathbb{R}$. The *Hadamard product* $A \bullet B$ is defined as

$$A \bullet B := \begin{pmatrix} a_1 b_1 & a_2 b_2 \\ a_2 b_2 & a_3 b_3 \end{pmatrix}.$$

Assume that A and B are positive definite. Show that $A \bullet B$ is positive definite using the trace and determinant.

Problem 3. Let A be an $n \times n$ matrix over $\mathbb{C}$. The *spectral radius* $\rho(A)$ is the radius of the smallest circle in the complex plane that contains all its eigenvalues. Every characteristic polynomial has at least one root. For any two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, the *Hadamard product* of A and B is the $n \times n$ matrix

$$A \bullet B := (a_{ij} b_{ij}).$$

Let A, B be nonnegative matrices. Then

$$\rho(A \bullet B) \leq \rho(A)\rho(B).$$

Apply this inequality to the nonnegative matrices

$$A = \begin{pmatrix} 1/4 & 0 \\ 0 & 3/4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 4. Let A, B be $m \times n$ matrices. The *Hadamard product* of A and B is defined by the $m \times n$ matrix

$$A \bullet B := (a_{ij} b_{ij}).$$

We consider the case $m = n$. There exists an $n^2 \times n$ *selection matrix* J such that

$$A \bullet B = J^T (A \otimes B) J$$

where J^T is defined as the $n \times n^2$ matrix

$$[E_{11} \;\; E_{22} \; \ldots \; E_{nn}]$$

with E_{ii} the $n \times n$ matrix of zeros except for a 1 in the (i,i)th position. Prove this identity for the special case $n = 2$.

Chapter 15

Differentiation

Problem 1. Let Q and P be $n \times n$ symmetric matrices over $\mathbb{R}$, i.e., $Q = Q^T$ and $P = P^T$. Assume that P^{-1} exists. Find the maximum of the function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x}$$

subject to $\mathbf{x}^T P \mathbf{x} = 1$. Use the *Lagrange multiplier method*.

Chapter 16

Integration

Problem 1. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$, i.e. $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Calculate

$$\int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A\mathbf{x})d\mathbf{x}.$$

Problem 2. Let V be an $N \times N$ unitary matrix, i.e. $VV^* = I_N$. The eigenvalues of V lie on the unit circle; that is, they may be expressed in the form $\exp(i\theta_n)$, $\theta_n \in \mathbb{R}$. A function $f(V) = f(\theta_1, \ldots, \theta_N)$ is called a *class function* if f is symmetric in all its variables. Weyl gave an explicit formula for averaging class functions over the circular unitary ensemble

$$\int_{U(N)} f(V)dV =$$

$$\frac{1}{(2\pi)^N N!}\int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_N) \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_N.$$

Thus we integrate the function $f(V)$ over $U(N)$ by parametrizing the group by the θ_i and using *Weyl's formula* to convert the integral into an N-fold integral over the θ_i. By definition the Haar measure dV is invariant under $V \to \widetilde{U}V\widetilde{U}^*$, where $\widetilde{U}$ is any $N \times N$ unitary matrix. The matrix V can always be diagonalized by a unitary matrix, i.e.

$$V = W \begin{pmatrix} e^{i\theta_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{i\theta_N} \end{pmatrix} W^*$$

where W is an $N \times N$ unitary matrix. Thus the integral over V can be written as an integral over the matrix elements of W and the eigenphases θ_n. Since the measure is invariant under unitary transformations, the integral over the matrix elements of U can be evaluated straightforwardly, leaving the integral over the eigenphases. Show that for f a class function we have

$$\int_{U(N)} f(V)dV = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_N) \det(e^{i\theta_n(n-m)}) d\theta_1 \cdots d\theta_N.$$

Chapter 17

Numerical Methods

Problem 1. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the system of linear equation $A\mathbf{x} = \mathbf{b}$ or

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, n.$$

Let $A = C - R$. This is called a *splitting* of the matrix A and R is the defect matrix of the splitting. Consider the iteration

$$C\mathbf{x}^{(k+1)} = R\mathbf{x}^{(k)} + \mathbf{b}, \qquad k = 0, 1, 2, \ldots .$$

Let

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The iteration converges if $\rho(C^{-1}R) < 1$, where $\rho(C^{-1}R)$ denotes the *spectral radius* of $C^{-1}R$. Show that $\rho(C^{-1}R) < 1$. Perform the iteration.

Problem 2. Let A be an $n \times n$ matrix over $\mathbb{R}$ and let $\mathbf{b} \in \mathbb{R}^n$. Consider the linear equation $A\mathbf{x} = \mathbf{b}$. Assume that $a_{jj} \neq 0$ for $j = 1, 2, \ldots, n$. We define the diagonal matrix $D = \mathrm{diag}(a_{jj})$. Then the linear equation $A\mathbf{x} = \mathbf{b}$ can be written as

$$\mathbf{x} = B\mathbf{x} + \mathbf{c}$$

with $B := -D^{-1}(A - D)$, $\mathbf{c} := D^{-1}\mathbf{b}$. The *Jacobi method* for the solution of the linear equation $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c}, \qquad k = 0, 1, \ldots$$

where $\mathbf{x}^{(0)}$ is any initial vector in $\mathbb{R}^n$. The sequence converges if

$$\rho(B) := \max_{j=1,\ldots,n} |\lambda_j(B)| < 1$$

where $\rho(B)$ is the *spectral radius* of B. Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(i) Show that the Jacobi method can applied for this matrix.
(ii) Find the solution of the linear equation with $\mathbf{b} = (1\ 1\ 1)^T$.

Problem 3. Let A be an $n \times n$ matrix over $\mathbb{R}$. The (p, q) *Padé approximation* to $\exp(A)$ is defined by

$$R_{pq}(A) := (D_{pq}(A))^{-1} N_{pq}(A)$$

where

$$N_{pq}(A) = \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} A^j$$

$$D_{pq}(A) = \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} (-A)^j.$$

Nonsingularity of $D_{pq}(A)$ is assured if p and q are large enough or if the eigenvalues of A are negative. Find the Padé approximation for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $p = q = 2$. Compare with the exact solution.

Problem 4. Let A be an $n \times n$ matrix. We define the $j - k$ approximant of $\exp(A)$ by

$$f_{j,k}(A) := \left(\sum_{\ell=0}^{k} \frac{1}{\ell!} \left(\frac{A}{j} \right)^{\ell} \right)^{j}. \tag{1}$$

We have the inequality

$$\| e^A - f_{j,k}(A) \| \le \frac{1}{j^k (k+1)!} \|A\|^{k+1} e^{\|A\|} \tag{2}$$

and $f_{j,k}(A)$ converges to e^A, i.e.

$$\lim_{j \to \infty} f_{j,k}(A) = \lim_{k \to \infty} f_{j,k}(A) = e^A.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Find $f_{2,2}(A)$ and e^A. Calculate the right-hand side of the inequality (2).

Problem 5. The *Denman-Beavers iteration* for the *square root* of an $n \times n$ matrix A with no eigenvalues on $\mathbb{R}^-$ is

$$Y_{k+1} = \frac{1}{2}(Y_k + Z_k^{-1}), \quad Z_{k+1} = \frac{1}{2}(Z_k + Y_k^{-1})$$

with $k = 0, 1, 2, \ldots$ and $Z_0 = I_n$ and $Y_0 = A$. The iteration has the properties that

$$\lim_{k\to\infty} Y_k = A^{1/2}, \qquad \lim_{k\to\infty} Z_k = A^{-1/2}$$

and, for all k,

$$Y_k = AZ_k, \quad Y_k Z_k = Z_k Y_k, \quad Y_{k+1} = \frac{1}{2}(Y_k + AY_k^{-1}).$$

(i) Can the Denman-Beavers iteration be applied to the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}?$$

(ii) Find Y_1 and Z_1.

Problem 6. Write a C++ program that implements Gauss elimination to solve linear equations. Apply it to the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \end{pmatrix}.$$

Problem 7. Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$. Since A is symmetric over $\mathbb{R}$ there exists a set of orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ which form an orthonormal basis in $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ be a reasonably good approximation to an eigenvector, say $\mathbf{v}_1$. Calculate

$$R := \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

The quotient is called *Rayleigh quotient*. Discuss.

Problem 8. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the linear system $A\mathbf{x} = \mathbf{b}$. The *condition number* of A is defined as

$$\mathrm{Cond}(A) := \|A\| \, \|A^{-1}\|.$$

Find the condition number for the matrix

$$A = \begin{pmatrix} 1 & 0.9999 \\ 0.9999 & 1 \end{pmatrix}$$

for the infinity norm, 1-norm and 2-norm.

Problem 9. The *collocation polynomial* $p(x)$ for unequally-spaced arguments x_0, x_1, $\dots$, x_n can be found by the determinant method

$$\det \begin{pmatrix} p(x) & 1 & x & x^2 & \dots & x^n \\ y_0 & 1 & x_0 & x_0^2 & \dots & x_0^n \\ y_1 & 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_n & 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} = 0$$

where $p(x_k) = y_k$ for $k = 0, 1, \dots, n$. Apply it to $(n = 2)$

$$p(x_0 = 0) = 1 = y_0, \qquad p(x_1 = 1/2) = 9/4 = y_1, \qquad p(x_2 = 1) = 4 = y_2.$$

Supplementary Problems

Problem 1. Consider the hermitian 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Find A^2 and A^3. We know that

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3, \quad \mathrm{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad \mathrm{tr}(A^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3.$$

Use Newton's method to solve this system of three equations to find the eigenvalues of A.

Problem 2. (i) Given an $m \times n$ matrix over $\mathbb{R}$. Write a C++ program that finds the maximum value in each row and then the minimum value of these values.

(ii) Given an $m \times n$ matrix over $\mathbb{R}$. Write a C++ program that finds the minimum value in each row and then the maximum value of these values.

Problem 3. Given an $m \times n$ matrix over $\mathbb{C}$. Find the elements with the largest absolute values and store the entries (j,k) $(j = 0,1,\ldots,m-1)$; $k = 0,1,\ldots,n-1)$ which contain the elements with the largest absolute value.

Problem 4. Let A be an $n \times n$ matrix over $\mathbb{C}$. Then any eigenvalue of A satisfies the inequality

$$|\lambda| \leq \max_{1 \leq j \leq n} \sum_{k=1}^{n} |a_{jk}|.$$

Write a C++ program that calculates the right-hand side of the inequality for a given matrix. Apply the complex class of STL. Apply it to the matrix

$$A = \begin{pmatrix} i & 0 & 0 & i \\ 0 & 2i & 2i & 0 \\ 0 & 3i & 3i & 0 \\ 4i & 0 & 0 & 4i \end{pmatrix}.$$

Problem 5. The Leverrier's method finds the characteristic polynomial of an $n \times n$ matrix. Find the characteristic polynomial for

$$A \otimes B, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

using this method. How are the coefficients c_i of the polynomial related to the eigenvalues? The eigenvalues of $A \otimes B$ are given by 0 (twice) and $\pm\sqrt{2}$.

Problem 6. Consider an $n \times n$ permutation matrix P. Obviously $+1$ is always an eigenvalue since the column vector with all n entries equal to $+1$ is an eigenvector. Apply a *brute force method* and give a C++ implementation to figure out whether -1 is an eigenvalue. We run over all column vectors $\mathbf{v}$ of length n, where the entries can only be $+1$ or -1, where of course the cases with all entries $+1$ or all entries -1 can be omitted. Thus the number of column vectors we have to run through are $2^n - 2$. The condition then to be checked is $P\mathbf{v} = -\mathbf{v}$. If true we have an eigenvalues -1 with the corresponding eigenvector $\mathbf{v}$.

Problem 7. Consider an $n \times n$ symmetric tridiagonal matrix over $\mathbb{R}$. Let

$f_n(\lambda) := \det(A - \lambda I_n)$ and

$$f_k(\lambda) = \det \begin{pmatrix} \alpha_1 - \lambda & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 - \lambda & \beta_2 & \cdots & 0 \\ 0 & \beta_2 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \alpha_{k-1} - \lambda & \beta_{k-1} \\ 0 & \cdots & 0 & \beta_{k-1} & \alpha_k - \lambda \end{pmatrix}$$

for $k = 1, 2, \ldots, n$ and $f_0(\lambda) = 1$, $f_{-1}(\lambda) = 0$. Then

$$f_k(\lambda) = (\alpha - \lambda) f_{k-1}(\lambda) - \beta_{k-1}^2 f_{k-2}(\lambda)$$

for $k = 2, 3, \ldots, n$. Find $f_4(\lambda)$ for the 4×4 matrix

$$\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

Problem 8. The *power method* is the simplest algorithm for computing eigenvectors and eigenvalues. Consider the vector space $\mathbb{R}^n$ with the Euclidean norm $\|\mathbf{x}\|$ of a vector $\mathbf{x} \in \mathbb{R}^n$. The iteration is as follows: Given a nonsingular $n \times n$ matrix M and a vector $\mathbf{x}_0$ with $\|\mathbf{x}_0\| = 1$. One defines

$$\mathbf{x}_{t+1} = \frac{M\mathbf{x}_t}{\|M\mathbf{x}_t\|}, \quad t = 0, 1, \ldots$$

This defines a dynamical system on the sphere S^{n-1}. Since M is invertible we have

$$\mathbf{x}_t = \frac{M^{-1}\mathbf{x}_{t+1}}{\|M^{-1}\mathbf{x}_{t+1}\|}, \quad t = 0, 1, \ldots$$

(i) Apply the power method to the nonnormal matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) Apply the power method to the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(iii) Consider the 3×3 symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Find the largest eigenvalue and the corresponding eigenvector using the power method. Start from the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Note that

$$A\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad A^2\mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

and the largest eigenvalue is $\lambda = 2 + \sqrt{2}$ with the corresponding eigenvector

$$(1 \quad -\sqrt{2} \quad 1)^T.$$

Problem 9. Let A be an $n \times n$ matrix over $\mathbb{R}$. Then we have the *Taylor expansion*

$$\sin(A) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1}, \qquad \cos(A) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} A^{2k}.$$

To calculate $\sin(A)$ and $\cos(A)$ from a truncated Taylor series approximation is only worthwhile near the origin. We can use the repeated application of the *double angle formula*

$$\cos(2A) \equiv 2\cos^2(A) - I_n, \qquad \sin(2A) \equiv 2\sin(A)\cos(A).$$

We can find $\sin(A)$ and $\cos(A)$ of a matrix A from a suitably truncated Taylor series approximates as follows

$S_0 =$ Taylor approximate to $\sin(A/2^k)$, $\quad C_0 =$ Taylor approximate to $\cos(A/2^k)$

and the recursion

$$S_j = 2S_{j-1}C_{j-1}, \qquad C_j = 2C_{j-1}^2 - I_n$$

where $j = 1, 2, \ldots$. Here k is a positive integer chosen so that, say $\|A\|_\infty \approx 2^k$. Apply this recursion to calculate sine and cosine of the 2×2 matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Use $k = 2$.

Chapter 18

Miscellaneous

Problem 1. Can one find a unitary matrix U such that

$$U^* \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} U = \begin{pmatrix} 0 & ce^{i\theta} \\ de^{-i\theta} & 0 \end{pmatrix}$$

where $c, d \in \mathbb{C}$ and $\theta \in \mathbb{R}$?

Problem 2. Let

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}.$$

Find U^*U. Show that U^*JU is a diagonal matrix.

Problem 3. Given four points $\mathbf{x}_i$, $\mathbf{x}_j$, $\mathbf{x}_k$, $\mathbf{x}_\ell$ (pairwise different) in $\mathbb{R}^2$. One can define their *cross-ratio*

$$r_{ijk\ell} := \frac{|\mathbf{x}_i - \mathbf{x}_j||\mathbf{x}_k - \mathbf{x}_\ell|}{|\mathbf{x}_i - \mathbf{x}_\ell||\mathbf{x}_k - \mathbf{x}_j|}.$$

Show that the cross-rations are invariant under *conformal transformation.*

Problem 4. Consider

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{v}_0 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find

$$\mathbf{v}_1 = A\mathbf{v}_0 - \frac{\mathbf{v}_0^T A\mathbf{v}_0}{\mathbf{v}_0^T\mathbf{v}_0}\mathbf{v}_0, \quad \mathbf{v}_2 = A\mathbf{v}_1 - \frac{\mathbf{v}_1^T A\mathbf{v}_1}{\mathbf{v}_1^T\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{v}_1^T\mathbf{v}_1}{\mathbf{v}_0^T\mathbf{v}_0}\mathbf{v}_0.$$

Are the vectors $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ linearly independent?

Problem 5. Let A, B be invertible $n \times n$ matrices. We define $A \circ B := AB - A^{-1}B^{-1}$. Find matrices A, B such that $A \circ B$ is invertible.

Problem 6. Let A be an $n \times n$ matrix. Let

$$B = \begin{pmatrix} A & I_n & 0_n \\ I_n & A & I_n \\ 0_n & I_n & A \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix. Calculate B^2 and B^3.

Problem 7. Let $a > b > 0$ and integers. Find the rank of the 4×4 matrix

$$M(a,b) = \begin{pmatrix} a & a & b & b \\ a & b & a & b \\ b & a & b & a \\ b & b & a & a \end{pmatrix}.$$

Problem 8. Let $0 \leq \theta < \pi/4$. Note that $\sec(x) := 1/\cos(x)$. Consider the 2×2 matrix

$$A(\theta) = \begin{pmatrix} \sec(2\theta) & -i\tan(2\theta) \\ i\tan(2\theta) & \sec(2\theta) \end{pmatrix}.$$

Show that the matrix is hermitian and the determinant is equal to 1. Show that the matrix is not unitary.

Problem 9. Show that the inverse of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 10. Let B be the *Bell matrix*

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find B^{-1} and B^T.
(ii) Show that

$$(I_2 \otimes B)(B \otimes I_2)(I_2 \otimes B) \equiv \frac{1}{\sqrt{2}}(I_2 \otimes B^2 + B^2 \otimes I_2).$$

Problem 11. Consider the two normalized vectors

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

in the Hilbert space $\mathbb{R}^8$. Show that the vectors obtained by applying

$$(\sigma_1 \otimes I_2 \otimes I_2), \quad (I_2 \otimes \sigma_1 \otimes I_2), \quad (I_2 \otimes I_2 \otimes \sigma_1)$$

together with the two original ones form an orthonormal basis in $\mathbb{R}^8$.

Problem 12. Let $\phi \in \mathbb{R}$. Consider the $n \times n$ matrix

$$H = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & & 1 \\ e^{i\phi} & & & & & 0 \end{pmatrix}.$$

(i) Show that the matrix is unitary.
(ii) Find the eigenvalues of H.
(iii) Consider the $n \times n$ diagonal matrix

$$G = \mathrm{diag}(1,\, \omega,\, \omega^2,\, \cdots \omega^{n-1})$$

where $\omega := \exp(i2\pi/n$. Find $\omega GH - HG$.

Problem 13. Let I_n be the $n \times n$ identity matrix and 0_n be the $n \times n$ zero matrix. Find the eigenvalues and eigenvectors of the $2n \times 2n$ matrices

$$A = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \qquad B = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}.$$

Problem 14. Let $a_j \in \mathbb{R}$ with $j = 1, 2, 3$. Consider the 4×4 matrices

$$A = \frac{1}{2} \begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_1 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & a_3 & 0 \end{pmatrix}, \qquad B = \frac{1}{2i} \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}.$$

Find the spectrum of A and B. Find the spectrum of $[A, B]$.

Problem 15. Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Show that $P^4 = I_4$.
(ii) Using this information find the eigenvalues.

Problem 16. Consider the skew-symmetric 3×3 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Find the eigenvalues. Let 0_3 be the 3×3 zero matrix. Let A_1, A_2, A_3 be skew-symmetric 3×3 matrices over $\mathbb{R}$. Find the eigenvalues of the 9×9 matrix

$$B = \begin{pmatrix} 0_3 & -A_3 & A_2 \\ A_3 & 0_3 & -A_1 \\ -A_2 & A_1 & 0_3 \end{pmatrix}.$$

Problem 17. (i) Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the 4×4 permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 18. Let $z \in \mathbb{C}$ and A, B, C be $n \times n$ matrices over $\mathbb{C}$. Calculate the commutator

$$[I_n \otimes A + A \otimes e^{zC}, e^{-zC} \otimes B + B \otimes I_n].$$

The commutator plays a role for Hopf algebras.

Problem 19. Consider the 4×4 matrix

$$A(\alpha,\beta,\gamma) = \begin{pmatrix} \cosh(\alpha) & 0 & 0 & \sinh(\alpha) \\ -\sin(\beta)\sinh(\alpha) & \cos(\beta) & 0 & -\sin(\beta)\cosh(\alpha) \\ \sin(\gamma)\cos(\beta)\sinh(\alpha) & \sin(\gamma)\sin(\beta) & \cos(\gamma) & \sin(\gamma)\cos(\beta)\cosh(\alpha) \\ \cos(\gamma)\cos(\beta)\sinh(\alpha) & \cos(\gamma)\sin(\beta) & -\sin(\gamma) & \cos(\gamma)\cos(\beta)\cosh(\alpha) \end{pmatrix}.$$

(i) Is each column a normalized vector in $\mathbb{R}^4$?
(ii) Calculate the scalar product between the column vectors. Discuss.

Problem 20. Consider the 2×2 matrices

$$S = \begin{pmatrix} r & t \\ t & r \end{pmatrix}, \qquad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Calculate RSR^T. Discuss.

Problem 21. Let $\phi_1, \phi_2 \in \mathbb{R}$. Consider the vector in $\mathbb{R}^3$

$$\mathbf{v}(\phi_1,\phi_2) = \begin{pmatrix} \cos(\phi_1)\cos(\phi_2) \\ \sin(\phi_2)\cos(\phi_1) \\ \sin(\phi_1) \end{pmatrix}$$

(i) Find the 3×3 matrix $\mathbf{v}(\phi_1,\phi_2)\mathbf{v}^T(\phi_1,\phi_2)$. What type of matrix do we have?
(ii) Find the eigenvalues of the 3×3 matrix $\mathbf{v}(\phi_1,\phi_2)\mathbf{v}^T(\phi_1,\phi_2)$. Compare with $\mathbf{v}^T(\phi_1,\phi_2)\mathbf{v}(\phi_1,\phi_2)$.

Problem 22. Can any skew-hermitian matrix K be written as $K = iH$, where H is a hermitian matrix?

Problem 23. Can one find a (column) vector in $\mathbb{R}^2$ such that $\mathbf{v}\mathbf{v}^T$ is an invertible 2×2 matrix?

Problem 24. (i) Consider the normalized vectors in $\mathbb{R}^4$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

(i) Do the vectors form a basis in $\mathbb{R}^4$?
(ii) Find the 4×4 matrix $\mathbf{v}_1\mathbf{v}_1^*+\mathbf{v}_2\mathbf{v}_2^*+\mathbf{v}_3\mathbf{v}_3^*+\mathbf{v}_4\mathbf{v}_4^*$ and then the eigenvalues.
(iii) Find the 4×4 matrix $\mathbf{v}_1\mathbf{v}_2^* + \mathbf{v}_2\mathbf{v}_3^* + \mathbf{v}_3\mathbf{v}_4^* + \mathbf{v}_4\mathbf{v}_1^*$ and then the eigenvalues.
(iv) Consider the normalized vectors in $\mathbb{R}^4$

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(v) Do the vectors form a basis in $\mathbb{R}^4$?
(vi) Find the 4×4 matrix $\mathbf{w}_1\mathbf{w}_1^* + \mathbf{w}_2\mathbf{w}_2^* + \mathbf{w}_3\mathbf{w}_3^* + \mathbf{w}_4\mathbf{w}_4^*$ and then the eigenvalues.
(vii) Find the 4×4 matrix $\mathbf{w}_1\mathbf{w}_2^* + \mathbf{w}_2\mathbf{w}_3^* + \mathbf{w}_3\mathbf{w}_4^* + \mathbf{w}_4\mathbf{w}_1^*$ and then the eigenvalues.

Problem 25. Let

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be elements of $\mathbb{C}^2$. Solve the equation $\mathbf{z}^*\mathbf{w} = \mathbf{w}^*\mathbf{z}$.

Problem 26. Let S be an invertible $n \times n$ matrix. Find the inverse of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0_n & S^{-1} \\ S & 0_n \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix.

Problem 27. Consider the normalized vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

and the vector

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

in the Hilbert space $\mathbb{C}^2$. Show that

$$\sum_{j=1}^{3} |\mathbf{v}_j^* \mathbf{w}|^2 = \sum_{j=1}^{3} |\mathbf{v}_j^* \mathbf{w}|^2 = \frac{3}{2} \|\mathbf{w}\|^2.$$

Problem 28. Find the determinant of the 4×4 matrices

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

Problem 29. Let R be an nonsingular $n \times n$ matrix over $\mathbb{C}$. Let A be an $n \times n$ matrix over $\mathbb{C}$ of rank one.
(i) Show that the matrix $R+A$ is nonsingular if and only if $\mathrm{tr}(R^{-1}A) \neq -1$.
(ii) Show that in this case we have

$$(R+A)^{-1} = R^{-1} - (1 + \mathrm{tr}(R^{-1}A))^{-1} R^{-1} A R^{-1}.$$

(iii) Simplify to the case that $R = I_n$.

Problem 30. Find the determinant of the matrices

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}.$$

Extend to $n \times n$ matrices. Then consider the limit $n \to \infty$.

Problem 31. Consider the 2×2 matrix

$$A(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ -\sinh(\alpha) & -\cosh(\alpha) \end{pmatrix}.$$

Find the maxima and minima of the function $f(\alpha) = \mathrm{tr}(A^2(\alpha)) - (\mathrm{tr}(A(\alpha)))^2$.

Problem 32. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}.$$

Extend to the $n \times n$ case.

Problem 33. Find the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

Problem 34. Let $\phi_k \in \mathbb{R}$. Consider the matrices

$$A(\phi_1, \phi_2, \phi_3, \phi_4) = \begin{pmatrix} 0 & e^{i\phi_1} & 0 & 0 \\ e^{i\phi_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi_3} \\ 0 & 0 & e^{i\phi_4} & 0 \end{pmatrix},$$

$$B(\phi_5, \phi_6, \phi_7, \phi_8) = \begin{pmatrix} 0 & 0 & e^{i\phi_5} & 0 \\ 0 & 0 & 0 & e^{i\phi_6} \\ e^{i\phi_7} & 0 & 0 & 0 \\ 0 & e^{i\phi_8} & 0 & 0 \end{pmatrix}$$

and $A(\phi_1, \phi_2, \phi_3, \phi_4)B(\phi_5, \phi_6, \phi_7, \phi_8)$. Find the eigenvalues of these matrices.

Problem 35. Consider the 4×4 orthogonal matrices

$$V_{12}(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V_{34}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Find the eigenvalues of $V(\theta) = V_{12}(\theta)V_{23}(\theta)V_{34}(\theta)$.

Problem 36. We know that any $n \times n$ unitary matrix has only eigenvalues λ with $|\lambda| = 1$. Assume that a given $n \times n$ matrix has only eigenvalues with $|\lambda| = 1$. Can we conclude that the matrix is unitary?

Problem 37. Consider the 4×4 matrices over $\mathbb{R}$

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{11} & a_{12} & 0 \\ 0 & a_{12} & a_{11} & a_{12} \\ 0 & 0 & a_{12} & a_{11} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{12} & a_{11} & a_{12} & 0 \\ 0 & a_{12} & a_{11} & a_{12} \\ a_{14} & 0 & a_{12} & a_{11} \end{pmatrix}.$$

Find the eigenvalues of A_1 and A_2.

Problem 38. Find the eigenvalues and normalized eigenvectors of the hermitian 3×3 matrix

$$H = \begin{pmatrix} \epsilon_1 & 0 & v_1 \\ 0 & \epsilon_2 & v_2 \\ v_1^* & v_2^* & \epsilon_3 \end{pmatrix}$$

with $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{C}$.

Problem 39. Consider the *Bell matrix*

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

which is a unitary matrix. Each column vector of the matrix is a fully entangled state. Are the normalized eigenvectors of B are also fully entangled states?

Problem 40. Let $x \in \mathbb{R}$. Is the 4×4 matrix

$$A(x) = \begin{pmatrix} \cos(x) & 0 & \sin(x) & 0 \\ 0 & \cos(x) & 0 & \sin(x) \\ -\sin(x) & 0 & \cos(x) & 0 \\ 0 & -\sin(x) & 0 & \cos(x) \end{pmatrix}$$

an orthogonal matrix?

Problem 41. Let $z \in \mathbb{C}$. Can one find a 4×4 permutation matrix P such that

$$P \begin{pmatrix} z & 0 & z \\ 0 & z & 0 \\ z & 0 & z \end{pmatrix} P^T = \begin{pmatrix} z & 0 & 0 \\ 0 & z & z \\ 0 & z & z \end{pmatrix} ?$$

Problem 42. (i) Consider the 3×3 permutation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the condition on a 3×3 matrix A such that $CAC^T = A$. Note that $C^T = C^{-1}$.

(ii) Consider the 4×4 permutation matrix

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find the condition on a 4×4 matrix B such that $DBD^T = B$. Note that $C^T = C^{-1}$.

Problem 43. (i) Let $x \in \mathbb{R}$. Show that the 4×4 matrix

$$A(x) = \begin{pmatrix} \cos(x) & 0 & -\sin(x) & 0 \\ 0 & \cos(x) & 0 & -\sin(x) \\ \sin(x) & 0 & \cos(x) & 0 \\ 0 & \sin(x) & 0 & \cos(x) \end{pmatrix}$$

is invertible. Find the inverse. Do these matrices form a group under matrix multiplication?
(ii) Let $x \in \mathbb{R}$. Show that the matrix

$$B(x) = \begin{pmatrix} \cosh(x) & 0 & \sinh(x) & 0 \\ 0 & \cosh(x) & 0 & \sinh(x) \\ \sinh(x) & 0 & \cosh(x) & 0 \\ 0 & \sinh(x) & 0 & \cosh(x) \end{pmatrix}$$

is invertible. Find the inverse. Do these matrices form a group under matrix multiplication.

Problem 44. Let $\alpha, \beta \in \mathbb{R}$. Do the 3×3 matrices

$$A(\alpha,\beta) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\beta) & \sinh(\beta) \\ 0 & \sinh(\beta) & \cosh(\beta) \end{pmatrix}$$

form a group under matrix multiplication? For $\alpha = \beta = 0$ we have the identity matrix.

Problem 45. Is the invertible matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

an element of the Lie group $SO(4)$? The matrix is unitary and we have $U^T = U$.

Problem 46. Let

$$J_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_3 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\epsilon \in \mathbb{R}$. Find $e^{\epsilon J_+}$, $e^{\epsilon J_-}$, $e^{\epsilon(J_+ + J_-)}$. Let $r \in \mathbb{R}$. Show that

$$e^{r(J_+ + J_-)} \equiv e^{J_- \tanh(r)} e^{2J_3 \ln(\cosh(r))} e^{J_+ \tanh(r)}.$$

Problem 47. Let $t_j \in \mathbb{R}$ for $j = 1, 2, 3, 4$. Find the eigenvalues and eigenvectors of

$$\hat{H} = \begin{pmatrix} 0 & t_1 & 0 & t_4 e^{i\phi} \\ t_1 & 0 & t_2 & 0 \\ 0 & t_2 & 0 & t_3 \\ t_4 e^{-i\phi} & 0 & t_3 & 0 \end{pmatrix}.$$

Problem 48. (i) Study the eigenvalue problem for the symmetric matrices over $\mathbb{R}$

$$A_3 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

Extend to n dimensions

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

Problem 49. Find the eigenvalues and eigenvectors of the 6×6 matrix

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & b_{16} \\ 0 & b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & 0 \\ b_{61} & 0 & 0 & 0 & 0 & b_{66} \end{pmatrix}.$$

Problem 50. Find the eigenvalues and eigenvectors of 4×4 matrix

$$A(z) = \begin{pmatrix} 1 & 1 & 1 & z \\ 1 & 1 & 1 & z \\ 1 & 1 & 1 & z \\ \bar{z} & \bar{z} & \bar{z} & 1 \end{pmatrix}.$$

Problem 51. Find the eigenvalues of the matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{pmatrix}.$$

Problem 52. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Consider the map

$$\tau(A, B) := A \otimes B - A \otimes I_n - I_n \otimes B.$$

Find the commutator $[\tau(A, B), \tau(B, A)]$.

Problem 53. Let A be an $n \times n$ matrix over $\mathbb{C}$. Consider the matrices

$$B_{12} = A \otimes A \otimes I_n \otimes I_n, \quad B_{13} = A \otimes I_n \otimes A \otimes I_n, \quad B_{14} = A \otimes I_n \otimes I_n \otimes A,$$

$$B_{23} = I_n \otimes A \otimes A \otimes I_n, \quad B_{24} = I_n \otimes A \otimes I_n \otimes A, \quad B_{34} = I_n \otimes I_n \otimes A \otimes A.$$

Find the commutators $[B_{jk}, B_{\ell m}]$.

Problem 54. We know that for any $n \times n$ matrix A over $\mathbb{C}$ the matrix $\exp(A)$ is invertible with the inverse $\exp(-A)$. What about $\cos(A)$ and $\cosh(A)$?

Problem 55. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Find $A \otimes B$, $B \otimes A$.
(ii) Find $\mathrm{tr}(A \otimes B)$, $\mathrm{tr}(B \otimes A)$. Find $\det(A \otimes B)$, $\det(B \otimes A)$.
(iii) Find the eigenvalues of A and B.
(iv) Find the eigenvalues of $A \otimes B$ and $B \otimes A$.
(v) Find $\mathrm{rank}(A)$, $\mathrm{rank}(B)$ and $\mathrm{rank}(A \otimes B)$.

Problem 56. Consider the hermitian 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Show that the rank of the matrix is 2. The trace and determinant are equal to 0 and thus two of the eigenvalues are 0. The other two eigenvalues are $\pm\sqrt{3}$.

Problem 57. Let A, B be $n \times n$ matrices. Let

$$X := A \otimes I_n \otimes I_n + I_n \otimes A \otimes I_n + I_n \otimes I_n \otimes A, \quad Y := B \otimes I_n \otimes I_n + I_n \otimes B \otimes I_n + I_n \otimes I_n \otimes B.$$

Find the commutator $[X, Y]$.

Problem 58. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $[A, B] = -A$. Let

$$\Delta A := B \otimes A + A \otimes I_2, \quad \Delta B := B \otimes B.$$

Find the commutator $[\Delta A, \Delta B]$.

Problem 59. (i) Let A, B be $n \times n$ matrices with $[A, B] = 0_n$. Find the commutators

$$[A \otimes A, B \otimes B], \quad [A \otimes B, B \otimes A].$$

Find the anti-commutators

$$[A \otimes A, B \otimes B]_+, \quad [A \otimes B, B \otimes A]_+.$$

(ii) Let A, B be $n \times n$ matrices with $[A, B]_+ = 0_n$. Find the commutators

$$[A \otimes A, B \otimes B], \quad [A \otimes B, B \otimes A].$$

Find the anti-commutators

$$[A \otimes A, B \otimes B]_+, \quad [A \otimes B, B \otimes A]_+.$$

(iii) Let A, B be $n \times n$ matrices. We define

$$\Delta(A) := A \otimes B + B \otimes A, \qquad \Delta(B) := B \otimes B - A \otimes A.$$

Find the commutator $[\Delta(A), \Delta(B)]$ and anticommutator $[\Delta(A), \Delta(B)]_+$.

Problem 60. (i) Let $\alpha, \beta \in \mathbb{C}$ and

$$M(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}.$$

Calculate $\exp(M(\alpha, \beta))$.
(ii) Let $\alpha, \beta \in \mathbb{C}$. Consider the 2×2 matrix

$$N(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}.$$

Calculate $\exp(M(\alpha, \beta))$.

Problem 61. Consider the six 3×3 permutation matrices. Which two of the matrices generate all the other ones.

Problem 62. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Extend to the general case n odd.

Problem 63. Let A be a real or complex $n \times n$ matrix with no eigenvalues on $\mathbb{R}^-$ (the closed negative real axis). Then there exists a unique matrix X such that
1) $e^X = A$
2) the eigenvalues of X lie in the strip $\{ z : -\pi < \Im(z) < \pi \}$. We refer to X as the *principal logarithm* of A and write $X = \log(A)$. Similarly, there is a unique matrix S such that
1) $S^2 = A$
2) the eigenvalues of S lie in the open halfplane: $0 < \Re(z)$. We refer to S as the *principal square root* of A and write $S = A^{1/2}$.
If the matrix A is real then its principal logarithm and principal square root are also real.
The open halfplane associated with $z = \rho e^{i\theta}$ is the set of complex numbers $w = \zeta e^{i\phi}$ such that $-\pi/2 < \phi - \theta < \pi/2$.
Suppose that $A = BC$ has no eigenvalues on $\mathbb{R}^-$ and
1. $BC = CB$

2. every eigenvalue of B lies in the open halfplane of the corresponding eigenvalue of $A^{1/2}$ (or, equivalently, the same condition holds for C).

Show that $\log(A) = \log(B) + \log(C)$.

Problem 64. Let $a, b \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$M = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}.$$

Problem 65. Let $n \geq 2$. Consider the Hilbert space $\mathcal{H} = \mathbb{C}^{2^n}$. Let A, B be nonzero $n \times n$ hermitian matrices and I_n the identity matrix. Consider the Hamilton operator $\hat{H}$ in this Hilbert space

$$\hat{H} = A \otimes I_n + I_n \otimes B + \epsilon A \otimes B$$

in this Hilbert space, where $\epsilon \in \mathbb{R}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A and let $\mu_1, \ldots, \mu_n$ be the eigenvalues of B. Then the eigenvalues of $\hat{H}$ are given by

$$\lambda_j + \mu_k + \epsilon \lambda_j \mu_k.$$

Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the eigenvectors of A and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the eigenvectors of B. Then the eigenvectors of $\hat{H}$ are given by

$$\mathbf{u}_j \otimes \mathbf{v}_k.$$

Thus all the eigenvectors of this Hamilton operator are not entangled. Consider now the Hamilton operator

$$\hat{K} = A \otimes I_n + I_n \otimes B + \epsilon B \otimes A.$$

Can we find hermitian matrices B, A such that the eigenvectors of $\hat{K}$ cannot be written as a product state, i.e. they are entangled? Note that $A \otimes B$ and $B \otimes A$ have the same eigenvalues with the eigenvectors $\mathbf{u}_j \otimes \mathbf{v}_k$ and $\mathbf{v}_j \otimes \mathbf{u}_k$, respectively.

Problem 66. Let A be an $m \times n$ matrix over $\mathbb{C}$ and B be a $s \times t$ matrix over $\mathbb{C}$. Show that

$$A \otimes B = \mathrm{vec}^{-1}_{ms \times nt} \left(L_{A,s \times t} \left(\mathrm{vec}_{s \times t}(B) \right) \right)$$

where

$$L_{A,s \times t} := (I_n \otimes I_t \otimes A \otimes I_s) \sum_{j=1}^{n} \mathbf{e}_{j,n} \otimes I_t \otimes \mathbf{e}_{j,n} \otimes I_s.$$

Problem 67. Consider the 2×2 hermitian matrices A and B with $A \neq B$ with the eigenvalues λ_1, λ_2; μ_1, μ_2; and the corresponding normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$; $\mathbf{v}_1$, $\mathbf{v}_2$, respectively. Form from the normalized eigenvectors the 2×2 matrix

$$\begin{pmatrix} \mathbf{u}_1^* \mathbf{v}_1 & \mathbf{u}_1^* \mathbf{v}_2 \\ \mathbf{u}_2^* \mathbf{v}_1 & \mathbf{u}_2^* \mathbf{v}_2 \end{pmatrix}.$$

Is this matrix unitary? Find the eigenvalues of this matrix and the corresponding normalized eigenvectors of the 2×2 matrix. How are the eigenvalues and eigenvectors are linked to the eigenvalues and eigenvectors of A and B?

Problem 68. Let $\alpha, \beta \in \mathbb{R}$. Are the 4×4 matrices

$$U = \begin{pmatrix} e^{i\alpha}\cosh(\beta) & 0 & 0 & \sinh\beta \\ 0 & e^{-i\alpha}\cosh\beta & \sinh\beta & 0 \\ 0 & \sinh\beta & e^{i\alpha}\cosh\beta & 0 \\ \sinh\beta & 0 & 0 & e^{-i\alpha}\cosh\beta \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & e^{i\alpha}\cosh\beta & -e^{i\alpha}\sinh\beta & 0 \\ -e^{-i\alpha}\cosh\beta & 0 & 0 & e^{-i\alpha}\sinh\beta \\ e^{i\alpha}\sinh\beta & 0 & 0 & -e^{i\alpha}\cosh\beta \\ 0 & -e^{-i\alpha}\sinh\beta & e^{-i\alpha}\cosh\beta & 0 \end{pmatrix}$$

unitary?

Problem 69. Find the conditions on $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$ such that the 4×4 matrix

$$A(\epsilon_1, \epsilon_2, \epsilon_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ 0 & \epsilon_2 & -1 & 0 \\ 0 & \epsilon_3 & 0 & -1 \end{pmatrix}$$

is invertible.

Problem 70. Given a normal 5×5 matrix which provides the characteristic equation

$$-\lambda^5 + 4\lambda^3 - 3\lambda = 0$$

with the eigenvalues $\lambda_1 = -\sqrt{3}$, $\lambda_2 = -1$, $\lambda_3 = 0$, $\lambda_4 = 1$, $\lambda_5 = \sqrt{3}$ and the corresponding normalized eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} \sqrt{3}/6 \\ -1/2 \\ 1/\sqrt{3} \\ -1/2 \\ \sqrt{3}/6 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ -1/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} \sqrt{3}/6 \\ 1/2 \\ 1/\sqrt{3} \\ 1/2 \\ \sqrt{3}/6 \end{pmatrix}.$$

Apply the spectral theorem and show that the matrix is given by

$$A = \sum_{j=1}^{5} \lambda_j \mathbf{v}_j \mathbf{v}_j^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 71. Let $\alpha \in \mathbb{R}$. Consider the 2×2 matrix

$$A(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ -\sin(\alpha) & -\cos(\alpha) \end{pmatrix}.$$

(i) Find the matrices

$$X = \left. \frac{dA(\alpha)}{d\alpha} \right|_{\alpha=0}, \qquad B(\alpha) = \exp(\alpha X).$$

Compare $A(\alpha)$ and $B(\alpha)$. Discuss.
(ii) All computer algebra programs (except one) provide the correct eigenvalues $+1$ and -1 and then the corresponding eigenvectors

$$\begin{pmatrix} 1 \\ (1+\cos(\alpha))/\sin(\alpha) \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -(1-\cos(\alpha))/\sin(\alpha) \end{pmatrix}.$$

Discuss. Then find the correct eigenvectors.

Problem 72. Let A be an $m \times n$ matrix and B a $p \times q$ matrix. Show that

$$A \otimes B = (A \otimes I_p)\mathrm{diag}(B, B, \ldots, B).$$

Problem 73. Let $z \in \mathbb{C}$. Find the eigenvalues and eigenvectors of the 3×3 matrix

$$A = \begin{pmatrix} 0 & z & \bar{z} \\ \bar{z} & 0 & z \\ z & \bar{z} & 0 \end{pmatrix}.$$

Discuss the dependence of the eigenvalues on z. The matrix is hermitian. Thus the eigenvalues must be real and since $\mathrm{tr}(A) = 0$ we have $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Set $z = re^{i\phi}$.

Problem 74. (i) Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the symmetric 3×3 and 4×4 matrices, respectively

$$A_3(\alpha) = \begin{pmatrix} \alpha & -1 & 0 \\ -1 & \alpha & -1 \\ 0 & -1 & \alpha \end{pmatrix}, \quad A_4(\alpha) = \begin{pmatrix} \alpha & -1 & 0 & 0 \\ -1 & \alpha & -1 & 0 \\ 0 & -1 & \alpha & -1 \\ 0 & 0 & -1 & \alpha \end{pmatrix}.$$

Extend to n dimensions.
(ii) Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the symmetric 4×4 matrix, respectively

$$B_4(\alpha) = \begin{pmatrix} \alpha & -1 & 0 & -1 \\ -1 & \alpha & -1 & 0 \\ 0 & -1 & \alpha & -1 \\ -1 & 0 & -1 & \alpha \end{pmatrix}.$$

Extend to n dimensions.

Problem 75. Let $a, b \in \mathbb{R}$. Consider the 2×2

$$K = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Find $\exp(iK)$. Use the result to find a, b such that

$$\exp(iK) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 76. Let

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X := S_+ \otimes S_- + S_- \otimes S_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find $R(\gamma) = \exp(\gamma X)$, where $\gamma \in \mathbb{R}$. Is the matrix unitary?

Problem 77. Let $z \in \mathbb{C}$ and $z \neq 0$.
(i) Do the 2×2 matrices

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$$

form a group under matrix multiplication?
(ii) Do the 3×3 matrices

$$\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & z \\ 0 & 1 & 0 \\ z^{-1} & 0 & 0 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 78. The $(1+1)$ Poincaré Lie algebra $iso(1,1)$ is generated by one boost generator K and the translation generators along the light-cone P_+ and P_-. The commutators are

$$[K, P_+] = 2P_+, \quad [K, P_-] = -2P_-, \quad [P_+, P_-] = 0.$$

Can one find 2×2 matrices K, P_+, P_- which satisfy these commutation relations?

Problem 79. Let $\epsilon \in \mathbb{R}$. Is the matrix

$$T(\epsilon) = \begin{pmatrix} 1-\epsilon & 1+\epsilon \\ -(1+\epsilon) & 1-\epsilon \end{pmatrix}$$

invertible for all ϵ?

Problem 80. Show that the $2n \times 2n$ matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ iI_n & -iI_n \end{pmatrix}$$

invertible. Find the inverse.

Problem 81. Consider the map $\mathbf{f} : \mathbb{C}^2 \mapsto \mathbb{R}^3$

$$\begin{pmatrix} \cos(\theta) \\ e^{i\phi}\sin(\theta) \end{pmatrix} \mapsto \begin{pmatrix} \sin(2\theta)\cos(\phi) \\ \sin(2\theta)\sin(\phi) \\ \cos(2\theta) \end{pmatrix}.$$

(i) Consider the map for the special case $\theta = 0$, $\phi = 0$.
(ii) Consider the map for the special case $\theta = \pi/4$, $\phi = \pi/4$.

Problem 82. Let A, B be $n \times n$ hermitian matrices. Show that $A \otimes I_n + I_n \otimes B$ is also a hermitian matrix. Apply

$$(A \otimes I_n + I_n \otimes B)^* = (A \otimes I_n)^* + (I_n \otimes B)^* = A^* \otimes I_n^* + I_n^* \otimes B^* = A \otimes I_n + I_n \otimes B.$$

Problem 83. (i) Find the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the 4×4 matrices

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & -a_{34} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Problem 84. A classical 3×3 matrix representation of the algebra $iso(1,1)$ is given by

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the commutators and anticommutators.

Problem 85.

Problem 86. (i) Let A be an invertible $n \times n$ matrix over $\mathbb{C}$. Assume we know the eigenvalues and eigenvectors of A. What can be said about the eigenvalues and eigenvectors of $A + A^{-1}$?
(ii) Apply the result from (i) to the 5×5 permutation matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 87. Let $n \geq 2$ and $j,k = 1,\ldots,n$. Let E_{jk} be the $n \times n$ elementary matrices with 1 at the position jk and 0 otherwise. Find the eigenvalues of

$$T = \sum_{k<j,j=1}^{n} (E_{jk} \otimes E_{kj} - E_{kj} \otimes E_{jk}).$$

Problem 88. Consider the 2×2 matrices

$$J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using the Kronecker product we form the 16×16 matrices

$$U_1 = \frac{1}{\sqrt{2}}(J \otimes I_2 \otimes I_2 \otimes I_2), \qquad U_2 = \frac{1}{\sqrt{2}}(I_2 \otimes J \otimes I_2 \otimes I_2),$$

$$U_3 = \frac{1}{\sqrt{2}}(I_2 \otimes I_2 \otimes J \otimes I_2), \qquad U_4 = \frac{1}{\sqrt{2}}(I_2 \otimes I_2 \otimes I_2 \otimes J)$$

and

$$V_{12} = \sqrt{2}(K \otimes K \otimes I_2 \otimes I_2), \; V_{13} = \sqrt{2}(K \otimes I_2 \otimes K \otimes I_2), \; V_{14} = \sqrt{2}(K \otimes I_2 \otimes I_2 \otimes K),$$

$V_{23} = \sqrt{2}(I_2 \otimes K \otimes K \otimes I_2)$, $\quad V_{24} = \sqrt{2}(I_2 \otimes K \otimes I_2 \otimes K)$, $\quad V_{34} = \sqrt{2}(I_2 \otimes I_2 \otimes K \otimes K)$.

Find the 16×16 matrices $U_j V_{k\ell} U_j$ and $V_{k\ell} U_j V_{k\ell}$ for $j = 1,2,3,4$, $k = 1,2,3$ and $\ell > k$. Find all the commutators between the 16×16 matrices.

Problem 89. Let $a, b \in \mathbb{R}$ and $a \neq 0$. Show that

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ 1 \end{pmatrix}.$$

Problem 90. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \equiv \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$, where $\cdot$ denotes the scalar product and $\times$ the vector product.

Problem 91. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ considered as column vectors. Show that $\mathbf{v}^* A \mathbf{u} = \mathbf{u}^* A^* \mathbf{v}$.

Problem 92. Let A be an $n \times n$ matrix over $\mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$. Show that

$$\Re(\mathbf{x}^* A \mathbf{x}) \equiv \frac{1}{2} \mathbf{x}^* (A + A^*) \mathbf{x}$$

where $\Re$ denotes the real part of a complex number.

Problem 93. Let ω be the solutions of the quadratic equation $\omega^2 + \omega + 1 = 0$. Consider the normal and invertible matrix M

$$M(\omega) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad M^{-1}(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \bar{\omega} & 0 \end{pmatrix}.$$

Let E_{jk} ($j, k = 1,2,3$) be the nine 3×3 elementary matrices. Calculate the 9×9 matrix

$$T = \sum_{j,k=1}^{3} (E_{jk} \otimes M^{n_j - n_k})$$

where $n_1 - n_2 = 2$, $n_1 - n_3 = 1$, $n_2 - n_3 = -1$ and $M^0 = I_3$.

Problem 94. Find the 6×6 matrix

$$I_3 \otimes \begin{pmatrix} a_{11} & a_{12} \\ a_{16} & a_{11} \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a_{13} & a_{14} \\ a_{12} & a_{13} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} a_{15} & a_{16} \\ a_{14} & a_{15} \end{pmatrix}.$$

Problem 95. Find all 4×4 matrices Y such that

$$Y \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) Y.$$

Problem 96. Let M be a 4×4 matrix over $\mathbb{R}$. Can M be written as

$$M = A \otimes B + C \otimes D$$

where A, B, C, D be 2×2 matrices over $\mathbb{R}$? Note that

$$\mathrm{tr}(M) = \mathrm{tr}(A)\mathrm{tr}(B) + \mathrm{tr}(C)\mathrm{tr}(D).$$

Problem 97. (i) Let $x \in \mathbb{R}$. Find the determinant and the inverse of the matrix

$$\begin{pmatrix} e^x \cos(x) & e^x \sin(x) \\ -e^{-x}\sin(x) & e^{-x}\cos(x) \end{pmatrix}.$$

(ii) Let $\alpha \in \mathbb{R}$. Find the determinant of the matrices

$$A(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} \cos(\alpha) & i\sin(\alpha) \\ i\sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

(iii) Let $\alpha \in \mathbb{R}$. Find the determinant of the matrices

$$C(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}, \qquad D(\alpha) = \begin{pmatrix} \cosh(\alpha) & i\sinh(\alpha) \\ -i\sinh(\alpha) & \cosh(\alpha) \end{pmatrix}.$$

Problem 98. Consider a symmetric matrix over $\mathbb{R}$. We impose the following conditions. The diagonal elements are all zero. The non-diagonal elements can only be $+1$ or -1. Show that such a matrix can only have integer values as eigenvalues. An example would be

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

with eigenvalues 3 and -1 (three times).

Problem 99. Let A, B be real symmetric and block tridiagonal 4×4 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{12} & b_{22} & b_{23} & 0 \\ 0 & b_{23} & b_{33} & b_{34} \\ 0 & 0 & b_{34} & b_{44} \end{pmatrix}.$$

Assume that B is positive definite. Solve the eigenvalue problem $A\mathbf{v} = \lambda B\mathbf{v}$.

Problem 100. Find the determinant and eigenvalues of the matrices

$$A_2 = \begin{pmatrix} 0 & a_{12} \\ 1 & a_{22} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & a_{13} \\ 1 & 0 & a_{23} \\ 0 & 1 & a_{33} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

Extend to the n-dimensional case. Note that $\det(A_2) = -a_{12}$, $\det(A_3) = a_{13}$, $\det(A_4) = -a_{14}$. For A_n we find $\det(A_n) = (-1)^{n+1} a_{1n}$.

Problem 101. Find the eigenvalues of the nonnormal matrices

$$A_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix}.$$

Extend to the $n \times n$ case. Owing to the structure of the matrices 0 is always an eigenvalue and the multiplicity depends on n. For A_2 we find $\lambda = 0$ and $\lambda = 3$.

Problem 102. Let λ, $\mathbf{u}$ be an eigenvalue and normalized eigenvector of the $n \times n$ matrix A, respectively. Let μ, $\mathbf{v}$ be an eigenvalue and normalized eigenvector of the $n \times n$ matrix B, respectively. Find an eigenvalue and normalized eigenvector of $A \otimes I_n \otimes B$.

Problem 103. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Consider the eigenvalue equations $A\mathbf{u} = \lambda\mathbf{u}$, $B\mathbf{v} = \mu\mathbf{v}$. Show that (identity)

$$(A \otimes B - \lambda I_n \otimes \mu I_n)(\mathbf{u} \otimes \mathbf{v}) = ((A - \lambda I_n) \otimes B + A \otimes (B - \mu I_n))(\mathbf{u} \otimes \mathbf{v}).$$

Problem 104. Show that the condition on a_{11}, a_{12}, b_{11}, b_{12} such that

$$\begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{12} & b_{11} & 0 \\ a_{12} & 0 & 0 & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(i.e. we have an eigenvalue equation of the matrix) is given by $\lambda = a_{11} + a_{12} = b_{11} + b_{12}$.

Problem 105. Let A, B, C, D $n \times n$ matrices over $\mathbb{C}$. Assume that $[A, C] = 0_n$ and $[B, D] = 0_n$. Find the commutators

$$[A \otimes I_n, C \otimes D], \quad [I_n \otimes B, C \otimes D], \quad [A \otimes B, C \otimes D].$$

Problem 106. Let A, B be $n \times n$ matrices. What is condition on A and B so that the commutator

$$[A \otimes I_n + I_n \otimes B + A \otimes B, B \otimes I_n + I_n \otimes A + B \otimes A]$$

vanishes?

Problem 107. Given the 4×4 matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$. Show that $A^2 = 0_4$, $B^2 = 0_4$ and $[A, B]_+ = (A + B)^2$.

Problem 108. Let A be a nonnormal matrix. Show that $\operatorname{tr}(AA^*) = \operatorname{tr}(A^*A)$. Show that $\det(AA^*) = \det(A^*A)$.

Problem 109. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Calculate the commutators of the three matrices

$$A \otimes B \otimes I_n, \qquad I_n \otimes A \otimes B, \qquad A \otimes I_n \otimes B.$$

Assume that A, B satisfy $[A, B] = A$, i.e. A, B form a basis of a noncommutative Lie algebra. Discuss the commutators found above from a Lie algebra point of view.

Problem 110. Let $j, k \in \{1, 2, 3\}$ and E_{jk} be the (nine) *elementary matrices*. Find the eigenvalues and eigenvectors of the 9×9 matrix

$$Q = \sum_{j=1}^{3} \sum_{k=1}^{3} (E_{jk} \otimes E_{kj}).$$

Does Q satisfy the braid-like relation

$$(I_2 \otimes Q)(Q \otimes I_2)(I_2 \otimes Q) = (Q \otimes I_2)(I_2 \otimes Q)(Q \otimes I_2)\,?$$

Problem 111. All real symmetric matrices are diagonalizable. Show that not all complex symmetric matrices are diagonalizable.

Problem 112. Find all 2×2 matrices T_1, T_2 such that

$$T_1 = [T_2, [T_2, T_1]], \qquad T_2 = [T_1, [T_1, T_2]].$$

Problem 113. Let $\alpha \in \mathbb{R}$.
(i) Consider the 2×2 matrix

$$A(\alpha) = \begin{pmatrix} e^{\alpha} & e^{-\alpha} \\ e^{-\alpha} & e^{\alpha} \end{pmatrix}.$$

Calculate $dA(\alpha)/d\alpha$, $X = dA(\alpha)/d\alpha|_{\alpha=0}$ and $B(\alpha) = e^{\alpha X}$. Compare $A(\alpha)$ and $B(\alpha)$. Discuss.
(ii) Consider the 2×2 matrix

$$C(\alpha) = \begin{pmatrix} e^{\alpha} & e^{-\alpha} \\ -e^{\alpha} & e^{-\alpha} \end{pmatrix}.$$

Calculate $dC(\alpha)/d\alpha$, $Y = dC(\alpha)/d\alpha|_{\alpha=0}$ and $D(\alpha) = e^{\alpha Y}$. Compare $C(\alpha)$ and $D(\alpha)$. Discuss.

Problem 114. Find all nonzero 2×2 matrices H, A such that

$$[H, A] = A.$$

Then find $[H \otimes H, A \otimes A]$.

Problem 115. Consider the 3×3 real symmetric matric A and the normalized vector $\mathbf{v}$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \mathbf{v} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find

$$\mu_1 = \mathbf{v}^T A \mathbf{v}, \quad \mu_2 = \mathbf{v}^T A^2 \mathbf{v}, \quad \mu_3 = \mathbf{v}^T A^3 \mathbf{v}.$$

Can the matrix A be uniquely reconstructed from μ_1, μ_2, μ_3? It can be assumed that A is real symmetric.

Problem 116. Consider the 3×3 real symmetric matrix A and the orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}3$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Calculate

$$\mu = \mathbf{v}_1^* A \mathbf{v}_2 + \mathbf{v}_2^* A \mathbf{v}_3 + \mathbf{v}_3^* A \mathbf{v}_1.$$

Discuss.

Problem 117. Let E_{jk} $(j, k = 1, \ldots, n)$ be the $n \times n$ elementary matrices, i.e. E_{jk} is the matrix with 1 at the j-th row and the k-th column and 0 otherwise. Let $n = 3$. Find $E_{12}E_{23}E_{31}$. Find $E_{12} \otimes E_{23} \otimes E_{31}$.

Problem 118. Let $f \in L_2(\mathbb{R})$. The Fourier transform is given by

$$=$$

Problem 119. Consider the 5×5 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix}.$$

Find the norm

$$\|A\| = \max_{\|\mathbf{x}\|} \|A\mathbf{x}\|.$$

(i) Apply the Lagrange multiplier method.
(ii) Calculate AA^T and then find the square root of the largest eigenvalue of AA^T. This is then the norm of A.
(iii) Is the matrix A normal? Find the rank of A and AA^T.

Problem 120. Find all 2×2 invertible matrices S over $\mathbb{R}$ with $\det(S) = 1$ such that

$$S \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} S \; S \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus we have to solve the three equations

$$s_{21} = 0, \quad s_{11} + s_{12} = s_{22}, \quad s_{11}s_{22} = 1.$$

Problem 121. Consider the standard basis in $\mathbb{C}^6$

$$\mathbf{e}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \ldots \quad \mathbf{e}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Can one find a unitary matrix U such that

$$U\mathbf{e}_0 = \mathbf{e}_1, \; U\mathbf{e}_1 = \mathbf{e}_2, \quad \ldots \quad U\mathbf{e}_4 = \mathbf{e}_5, \; U\mathbf{e}_5 = \mathbf{e}_0?$$

If so find the inverse of U.

Problem 122. (i) Do the three vectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

form an orthonormal basis in $\mathbb{R}^3$?
(ii) Is the 3×3 matrix

$$(\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3)$$

an orthonormal matrix?

Problem 123. Let A be an $n \times n$ matrix over $\mathbb{C}$ and B be an $m \times m$ matrix over $\mathbb{C}$. Show that

$$\|A \otimes B\|_F = \|A\|_F \cdot \|B\|_F$$

where $\|.\|_F$ denotes the Frobenius norm.

Problem 124. Let E_{jk} be the elementary matrices with 1 at position (entry) (j,k) and 0 otherwise. Let $n = 3$. Find $E_{12}E_{23}E_{31}$.

Problem 125. Let A, B 2×2 matrices with $\det(A) = 1$, $\det(B) = 1$. Let $\star$ be the star product. Show that

$$\det(A \star B) = 1.$$

Problem 126. Let A, B be 2×2 hermitian matrices over $\mathbb{C}$. Assume that

$$\mathrm{tr}(A) = \mathrm{tr}(B), \quad \mathrm{tr}(A^2) = \mathrm{tr}(B^2).$$

Are the eigenvalues of A and B are the same?

Problem 127. Let A, B be $n \times n$ matrices. Show that

$$e^{A+B} = \int_0^\infty d\alpha_1 e^{\alpha_1 A}\delta(1-\alpha_1) + \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 e^{\alpha_1 A} B e^{\alpha_2 A}\delta(1-\alpha_1-\alpha_2)$$
$$+ \int_0^\infty \int_0^\infty \int_0^\infty e^{\alpha_1 A} B e^{\alpha_2 A} B e^{\alpha_3 A}\delta(1-\alpha_1-\alpha_2-\alpha_3) + \cdots$$

Problem 128. Consider a vector $\mathbf{a}$ in $\mathbb{C}^4$ and the corresponding 2×2 matrix A via the vec^{-1} operator

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$$

and analogously

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}.$$

Show that

$$\mathbf{a}^* \mathbf{b} = \mathrm{tr}(A^* B).$$

Problem 129. Find $n \times n$ matrices A, B such that

$$\|[A, B] - I_n\| \to \min$$

where $\|.\|$ denotes the norm and $[,]$ denotes the commutator.

Problem 130. Let

$$D = \{(x_1, x_2, x_3) \, : \, x_1^2 + x_2^2 + x_3^2 = 1, \ \ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

Let $\mathbf{v}$ be a normalized vector in $\mathbb{R}^3$ with nonnegative entries and A be a 3×3 matrix over $\mathbb{R}$ with strictly positive entries. Show that the map $f : D \to D$

$$f(\mathbf{v}) = \frac{A\mathbf{v}}{\|A\mathbf{v}\|}$$

has fixed point, i.e. there is a $\mathbf{v}_0$ such that

$$\frac{A\mathbf{v}_0}{\|A\mathbf{v}_0\|} = \mathbf{v}_0.$$

Problem 131. Let A be an $n \times n$ matrix over $\mathbb{R}$. The matrix A is called symmetric if $A = A^T$. Let B be an $n \times n$ matrix over $\mathbb{R}$. If B is symmetric about the northeast-to-southwest diagonal then B is called persymmetric. Let J be the $n \times n$ counter identity matrix. Note that $J^T = J$ and $J^2 = I_n$. Then persymmetric can be written as

$$JAJ = A^T.$$

(i) Show that the power A^k of a symmetric persymmetric matrix over $\mathbb{R}$ is again symmetric persymmetric.
(ii) Show that the Kronecker product of two symmetric persymmetric matrices X and Y is again symmetric persymmetric.

Problem 132. Find all 2×2 matrices A, B that satisfy

$$ABA = BAB \quad \text{and} \quad A \otimes B \otimes A = B \otimes A \otimes B.$$

Problem 133. Find all 2×2 matrices S over $\mathbb{R}$ such that $SS^T = I_2$.

Problem 134. Show that the group S_4 has five inequivalent irreducible representations, namely two 1-dimensional representations, one 2-dimensional representation and two 3-dimensional representations.

Problem 135. Let R_{ij} denote the generators of an $SO(n)$ rotation in the $x_i - x_j$ plane of the n-dimensional Euclidean space. Give an n-dimensional matrix representation of these generators and use it to derive the Lie algebra $so(n)$ of the compact Lie group $SO(n)$.

Problem 136. Consider the vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Find the conditions such that

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{v} \otimes \mathbf{u}.$$

Find solutions to these conditions.

Problem 137. Consider the 4×4 matrices

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes I_2 \;\; \Rightarrow \;\; X^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I_2$$

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;\; \Rightarrow \;\; Y^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Find the anti-commutators

$$[X, X]_+, \;\; [Y, Y]_+, \;\; [X, X^T]_+, \;\; [Y, Y^T], \;\; [X, Y]_+, \;\; [X, Y^T]_+.$$

Problem 138. Show that the square roots of the 2×2 unit matrix I_2 are given by I_2 and

$$S\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}S^{-1}, \quad S\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}S^{-1}, \quad S\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}S^{-1}$$

where S is an arbitrary invertible matrix.

Problem 139. Let S_1, S_2, S_3 be the spin-$\frac{1}{2}$ matrices. Show that

$$[S_1 \otimes S_2, S_2 \otimes S_3] = 0_4, \quad [S_2 \otimes S_3, S_3 \otimes S_1] = 0_4, \quad [S_3 \otimes S_1, S_1 \otimes S_2] = 0_4.$$

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