

Problems and Solutions  
in  
Hilbert space theory,  
Fourier transform,  
wavelets  
and  
generalized functions

by  
Willi-Hans Steeb  
International School for Scientific Computing  
at  
University of Johannesburg, South Africa

# Preface

The purpose of this book is to supply a collection of problems in Hilbert space theory, wavelets and generalized functions.

## **Prescribed books for problems.**

1) Hilbert Spaces, Wavelets, Generalized Functions and Modern Quantum Mechanics

by Willi-Hans Steeb  
Kluwer Academic Publishers, 1998  
ISBN 0-7923-5231-9

2) Classical and Quantum Computing with C++ and Java Simulations

by Yorick Hardy and Willi-Hans Steeb  
Birkhauser Verlag, Boston, 2002  
ISBN 376-436-610-0

3) Problems and Solutions in Quantum Computing and Quantum Information, second edition

by Willi-Hans Steeb and Yorick Hardy  
World Scientific, Singapore, 2006  
ISBN 981-256-916-2  
<http://www.worldscibooks.com/physics/6077.html>

The International School for Scientific Computing (ISSC) provides certificate courses for this subject. Please contact the author if you want to do this course or other courses of the ISSC.

e-mail addresses of the author:

[steebwilli@gmail.com](mailto:steebwilli@gmail.com)  
[steeb\\_wh@yahoo.com](mailto:steeb_wh@yahoo.com)

Home page of the author:

<http://issc.uj.ac.za>



# Contents

<b>Notation</b>	<b>x</b>
<b>1 General</b>	<b>1</b>
1.1 Solved Problems . . . . .	1
1.2 Supplementary Problems . . . . .	9
<b>2 Finite Dimensional Hilbert Spaces</b>	<b>10</b>
2.1 Introduction . . . . .	10
2.2 Solved Problems . . . . .	11
2.3 Supplementary Problems . . . . .	20
<b>3 Hilbert Space <math>L_2(\Omega)</math></b>	<b>22</b>
3.1 Solved Problems . . . . .	23
3.2 Supplementary Problems . . . . .	49
<b>4 Hilbert Space <math>\ell_2(\mathbb{N})</math></b>	<b>54</b>
4.1 Solved Problems . . . . .	54
4.2 Supplementary Problems . . . . .	57
<b>5 Fourier Transform</b>	<b>58</b>
5.1 Solved Problems . . . . .	58
5.2 Supplementary Problems . . . . .	61
<b>6 Wavelets</b>	<b>63</b>
6.1 Solved Problems . . . . .	63
6.2 Supplementary Problems . . . . .	66
<b>7 Linear Operators</b>	<b>68</b>
7.1 Solved Problems . . . . .	68
7.2 Supplementary Problems . . . . .	71

<b>8 Generalized Functions</b>	<b>73</b>
8.1 Solved Problems . . . . .	73
8.2 Supplementary Problems . . . . .	85
<b>Bibliography</b>	<b>90</b>
<b>Index</b>	<b>93</b>



# Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbf{N}$	set of natural numbers
$\mathbf{Z}$	set of integers
$\mathbf{Q}$	set of rational numbers
$\mathbf{R}$	set of real numbers
$\mathbf{R}^+$	set of nonnegative real numbers
$\mathbf{C}$	set of complex numbers
$\mathbf{R}^n$	$n$ -dimensional Euclidean space
$\mathbf{C}^n$	space of column vectors with $n$ real components
	$n$ -dimensional complex linear space
	space of column vectors with $n$ complex components
$\mathcal{H}$	Hilbert space
$i$	$\sqrt{-1}$
$\Re z$	real part of the complex number $z$
$\Im z$	imaginary part of the complex number $z$
$ z $	modulus of complex number $z$
	$ x + iy  = (x^2 + y^2)^{1/2}$ , $x, y \in \mathbf{R}$
$T \subset S$	subset $T$ of set $S$
$S \cap T$	the intersection of the sets $S$ and $T$
$S \cup T$	the union of the sets $S$ and $T$
$f(S)$	image of set $S$ under mapping $f$
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbf{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\ \cdot\ $	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbf{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbf{R}^3$
$A, B, C$	$m \times n$ matrices
$\det(A)$	determinant of a square matrix $A$
$\text{tr}(A)$	trace of a square matrix $A$
$\text{rank}(A)$	rank of matrix $A$
$A^T$	transpose of matrix $A$

$\bar{A}$	conjugate of matrix $A$
$A^*$	conjugate transpose of matrix $A$
$A^\dagger$	conjugate transpose of matrix $A$ (notation used in physics)
$A^{-1}$	inverse of square matrix $A$ (if it exists)
$I_n$	$n \times n$ unit matrix
$I$	unit operator
$0_n$	$n \times n$ zero matrix
$AB$	matrix product of $m \times n$ matrix $A$ and $n \times p$ matrix $B$
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices $A$ and $B$
$[A, B] := AB - BA$	commutator for square matrices $A$ and $B$
$[A, B]_+ := AB + BA$	anticommutator for square matrices $A$ and $B$
$A \otimes B$	Kronecker product of matrices $A$ and $B$
$A \oplus B$	Direct sum of matrices $A$ and $B$
$\delta_{jk}$	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
$\delta$	delta function
$\Theta$	Heaviside's function
$\lambda$	eigenvalue
$\epsilon$	real parameter
$t$	time variable
$\hat{H}$	Hamilton operator



# Chapter 1

## General

---

### 1.1 Solved Problems

**Problem 1.** Consider a Hilbert space  $\mathcal{H}$  with scalar product  $\langle \cdot, \cdot \rangle$ . The scalar product implies a norm via  $\|f\|^2 := \langle f, f \rangle$ , where  $f \in \mathcal{H}$ .

(i) Show that

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

Start with

$$\|f + g\|^2 + \|f - g\|^2 = \langle f + g, f + g \rangle + \langle f - g, f - g \rangle.$$

(ii) Assume that  $\langle f, g \rangle = 0$ , where  $f, g \in \mathcal{H}$ . Show that

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2.$$

Start with

$$\|f + g\|^2 = \langle f + g, f + g \rangle.$$

**Problem 2.** Show that

$$\langle f, g \rangle = \frac{1}{4}\|f + g\|^2 - \frac{1}{4}\|f - g\|^2$$

or

$$\langle f, g \rangle = \frac{1}{4}\|f + g\|^2 - \frac{1}{4}\|f - g\|^2 + \frac{i}{4}\|f + ig\|^2 - \frac{i}{4}\|f - ig\|^2$$

## 2 Problems and Solutions

depending on whether we are dealing with a real or complex Hilbert space.

**Problem 3.** Let  $\mathcal{H}$  be a Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$ . Let  $u, v \in \mathcal{H}$ . Let  $\|\cdot\|$  be the norm induced by the scalar product, i.e.  $\|u\|^2 = \langle u, u \rangle$ . Show that (*Schwarz-Cauchy inequality*)

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|.$$

Obviously for  $u = 0$  or  $v = 0$  the inequality is an equality. So we can assume that  $u \neq 0$  and  $v \neq 0$  in the following.

**Problem 4.** (i) Let  $\alpha \in \mathbb{C}$  and  $u, v \in \mathcal{H}$ . Find

$$\langle u + \alpha v, u + \alpha v \rangle.$$

(ii) Let  $u, v \in \mathcal{H}$  and  $v \neq 0$ . Show that

$$\langle u, v \rangle \langle v, u \rangle \leq \langle u, u \rangle \langle v, v \rangle.$$

**Problem 5.** Let  $f, g \in \mathcal{H}$ . Show that

$$\langle f|f \rangle \langle g|g \rangle \geq \frac{1}{4} (\langle f|g \rangle + \langle g|f \rangle)^2.$$

**Problem 6.** Let  $f, g \in \mathcal{H}$ . Use the *Schwarz inequality*

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \langle g, g \rangle = \|f\|^2 \|g\|^2$$

to prove the *triangle inequality*

$$\|f + g\| \leq \|f\| + \|g\|.$$

**Problem 7.** Consider a Hilbert space  $\mathcal{H}$  and  $\|\cdot\|$  be the norm implied by the scalar product. Let  $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ .

(i) Show that

$$\|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v}\| \geq \|\mathbf{u}\|.$$

(ii) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle \leq 2\|\mathbf{u}\| \cdot \|\mathbf{v}\|.$$

**Problem 8.** Let  $P$  be a nonzero projection operator in a Hilbert space  $\mathcal{H}$ . Show that  $\|P\| = 1$ .

**Problem 9.** Let  $|\psi\rangle, |s\rangle, |\phi\rangle$  be normalized states in a Hilbert space  $\mathcal{H}$ . Let  $U$  be a unitary operator, i.e.  $U^{-1} = U^*$  in the Hilbert space  $\mathcal{H} \otimes \mathcal{H}$  such that

$$\begin{aligned} U(|\psi\rangle \otimes |s\rangle) &= |\psi\rangle \otimes |\psi\rangle \\ U(|\phi\rangle \otimes |s\rangle) &= |\phi\rangle \otimes |\phi\rangle. \end{aligned}$$

Show that  $\langle \phi|\psi\rangle = \langle \phi|\psi\rangle^2$ . Find solutions to this equation.

**Problem 10.** A family,  $\{\psi_j\}_{j \in J}$  of vectors in the Hilbert space,  $\mathcal{H}$ , is called a *frame* if for any  $f \in \mathcal{H}$  there exist two constants  $k_1 > 0$  and  $0 < k_2 < \infty$ , such that

$$k_1 \|f\|^2 \leq \sum_{j \in J} |\langle \psi_j | f \rangle|^2 \leq k_2 \|f\|^2.$$

Consider the Hilbert space  $\mathcal{H} = \mathbb{R}^2$  and the family of vectors

$$\left\{ \psi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Show that we have a tight frame.

**Problem 11.** Let  $T : X \rightarrow Y$  be a linear map between linear spaces (vector spaces)  $X, Y$ . The *null space* or *kernel* of the linear map  $T$ , denoted by  $\ker(T)$ , is the subset of  $X$  defined by

$$\ker(T) := \{x \in X : Tx = 0\}.$$

The *range* of  $T$ , denoted by  $\text{ran}T$ , is the subset of  $Y$  defined by

$$\text{ran}(T) := \{y \in Y : \text{there exists } x \in X \text{ such that } Tx = y\}.$$

Let  $P$  be a projection operator in a Hilbert space  $\mathcal{H}$ . Show that  $\text{ran}(P)$  is closed and

$$\mathcal{H} = \text{ran}(P) \oplus \ker(P)$$

is the orthogonal direct sum of  $\text{ran}(P)$  and  $\ker(P)$ .

**Problem 12.** Let  $\mathcal{H}$  be an arbitrary Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$ . Show that if  $\varphi$  is a bounded linear functional on the Hilbert space  $\mathcal{H}$ , then there is a unique vector  $u \in \mathcal{H}$  such that

$$\varphi(x) = \langle u, x \rangle \quad \text{for all } x \in \mathcal{H}.$$

**Problem 13.** Let  $\mathcal{H}$  be an arbitrary Hilbert space. A bounded linear operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  satisfies the *Fredholm alternative* if one of the following

4 Problems and Solutions

two alternatives holds:

- (i) either  $Ax = 0$ ,  $A^*x = 0$  have only the zero solution, and the linear equations  $Ax = y$ ,  $A^*x = y$  have a unique solution  $x \in \mathcal{H}$  for every  $y \in \mathcal{H}$ ;  
(ii) or  $Ax = 0$ ,  $A^*x = 0$  have nontrivial, finite-dimensional solution spaces of the same dimension,  $Ax = y$  has a (nonunique) solution if and only if  $y \perp u$  for every solution  $u$  of  $A^*u = 0$ , and  $A^*x = y$  has a (nonunique) solution if and only if  $y \perp u$  for every solution  $u$  of  $Au = 0$ .

Give an example of a bounded linear operator that satisfies the Fredholm alternative.

**Problem 14.** Let  $(M, d)$  be a complete metric space (for example a Hilbert space) and let  $f : M \rightarrow M$  be a mapping such that

$$d(f^{(m)}(x), f^{(m)}(y)) \leq kd(x, y), \quad \forall x, y \in M$$

for some  $m \geq 1$ , where  $0 \leq k < 1$  is a constant. Show that the map  $f$  has a unique fixed point in  $M$ .

**Problem 15.** Let  $\mathcal{H}$  be a Hilbert space and let  $f : \mathcal{H} \rightarrow \mathcal{H}$  be a monotone mapping such that for some constant  $\beta > 0$

$$\|f(u) - f(v)\| \leq \beta\|u - v\| \quad \forall u, v \in \mathcal{H}.$$

Show that for any  $w \in \mathcal{H}$ , the equation  $u + f(u) = w$  has a unique solution  $u$ .

**Problem 16.** Consider a Hilbert space  $\mathcal{H}$  with scalar product  $\langle \cdot, \cdot \rangle$ . Let  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  be elements of the Hilbert space with  $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$ . Show that

$$\sqrt{1 - |\langle \mathbf{u}, \mathbf{v} \rangle|^2} \leq \sqrt{1 - |\langle \mathbf{u}, \mathbf{w} \rangle|^2} + \sqrt{1 - |\langle \mathbf{w}, \mathbf{v} \rangle|^2}.$$

**Problem 17.** Given a Hilbert space  $\mathcal{H}$  and a Hilbert subspace  $\mathcal{G}$  of  $\mathcal{H}$ . The Hilbert space *projection theorem* states that for every  $f \in \mathcal{H}$ , there exists a unique  $g \in \mathcal{G}$  such that

$$(i) \quad f - g \in \mathcal{G}^\perp$$

$$(ii) \quad \|f - g\| = \inf_{h \in \mathcal{G}} \|f - h\|$$

where the space  $\mathcal{G}^\perp$  is defined by

$$\mathcal{G}^\perp := \{ k \in \mathcal{H} : \langle k | u \rangle = 0 \text{ for all } u \in \mathcal{G} \}.$$

Show that if  $g$  is the minimizer of  $\|f - h\|$  over all  $h \in \mathcal{G}$ , then it is true that  $f - g \in \mathcal{G}^\perp$ .

**Problem 18.** Let  $\{\phi_n\}_{n \in \mathbb{Z}}$  be an orthonormal basis in a Hilbert space  $\mathcal{H}$ . Then any vector  $f \in \mathcal{H}$  can be written as

$$f = \sum_{n \in \mathbb{Z}} \langle f, \phi_n \rangle \phi_n.$$

Now suppose that  $\{\psi_n\}_{n \in \mathbb{Z}}$  is also a basis for  $\mathcal{H}$ , but it is not orthonormal. Show that if we can find a so-called dual basis  $\{\chi_n\}_{n \in \mathbb{Z}}$  satisfying

$$\langle \psi_n | \chi_m \rangle = \delta(n - m)$$

then for any vector  $f \in \mathcal{H}$ , we have

$$f = \sum_{n \in \mathbb{Z}} \langle f | \chi_n \rangle \psi_n.$$

Here  $\delta(n - m)$  denotes the Kronecker delta with  $\delta(n - m) = 0$  if  $n \neq m$  and 1 otherwise.

**Problem 19.** Let  $(X_1, \|\cdot\|_1)$  and  $(X_2, \|\cdot\|_2)$  be two normed spaces. Show that the product vector spaces  $X = X_1 \times X_2$  is also a normed vector space if we define

$$\|x\| := \max(\|x_1\|_1, \|x_2\|_2)$$

with  $x = (x_1, x_2)$ .

**Problem 20.** Let  $A$  be a linear bounded self-adjoint operator in a Hilbert space  $\mathcal{H}$ . Let  $u, v \in \mathcal{H}$  and  $\lambda \in \mathbb{C}$ . Consider the equation

$$Au - \lambda u = v.$$

(i) Show that for  $\lambda$  nonreal (i.e. it has an imaginary part)  $v$  cannot vanish unless  $u$  vanishes.

(ii) Show that for  $\lambda$  nonreal we have

$$\|(A - \lambda I)^{-1}v\| \leq \frac{1}{|\Im(\lambda)|} \|v\|.$$

**Problem 21.** Let  $\mathbb{E}$  be the exterior of the unit disc

$$\{z \in \mathbb{C} : |z| > 1\}$$

and  $\mathbb{T}$  the unit circle

$$\{z \in \mathbb{C} : |z| = 1\}.$$

6 Problems and Solutions

Let  $\mathcal{H}_2(\mathbb{E})$  be the *Hardy space* of square integrable functions on  $\mathbb{T}$ , analytic in the region  $\mathbb{E}$ . The inner product for  $f(z), g(z) \in \mathcal{H}_2(\mathbb{E})$  is defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\omega})^* g(e^{i\omega}) d\omega = \frac{1}{2\pi i} \oint_{\mathbb{T}} f^*(1/z^*) g(z) \frac{dz}{z}.$$

Let  $f(z) = z^2$  and  $g(z) = z + 1$ . Find the scalar product  $\langle f, g \rangle$ .

**Problem 22.** Let  $\mathcal{O} = \{u_1, u_2, \dots\}$  be an orthonormal set in a infinite dimensional Hilbert space. Show that if

$$x = \sum_{j=1}^{\infty} c_j u_j$$

then

$$\|x\|^2 = \sum_{j=1}^{\infty} |c_j|^2.$$

**Problem 23.** Two Cauchy sequences  $\{x_k\}$  and  $\{y_k\}$  are said to be equivalent if for all  $\epsilon > 0$ , there is a  $k(\epsilon)$  such that for all  $j \geq k(\epsilon)$  we have  $d(x_j, y_j) < \epsilon$ . One writes  $\{x_k\} \sim \{y_k\}$ . Obviously,  $\sim$  is an equivalence relationship. Show that equivalent Cauchy sequences have the same limit.

**Problem 24.** Consider the sequence  $\{x_k\}$ ,  $k = 1, 2, \dots$  in  $\mathbb{R}$  defined by  $x_k = 1/k^2$  for all  $k = 1, 2, \dots$ . Show that this sequence is a Cauchy sequence.

**Problem 25.** Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{S}$  be a sub Hilbert space. Show that any element  $u$  of  $\mathcal{H}$  can be decomposed uniquely

$$u = v + w$$

where  $v$  is in  $\mathcal{S}$  and  $w$  is in  $\mathcal{S}^\perp$ .

**Problem 26.** Let  $u, v_1, v_2$  be elements of a Hilbert space. Show that

$$2\|u - v_1\|^2 + 2\|u - v_2\|^2 = \|2\left(u - \frac{v_1 + v_2}{2}\right)\|^2 + \|v_1 - v_2\|^2.$$

**Problem 27.** Let  $P$  be the set of prime numbers. We define the set

$$S := \{(p, q) : p, q \in P, p \leq q\}.$$

Show that

$$d((p_1, q_1), (p_2, q_2)) := |p_1 q_1 - p_2 q_2|$$

defines a metric.

**Problem 28.** Consider the vector space of all continuous functions defined on  $[a, b]$ . We define a metric

$$d(f, g) := \max_{a \leq x \leq b} |f(x) - g(x)|.$$

Let  $a = \pi$ ,  $b = \pi$ ,  $f(x) = \sin(x)$  and  $g(x) = \cos(x)$ . Find  $d(f, g)$ .

**Problem 29.** The  $n \times n$  matrices over  $\mathbb{R}$  form a vector space. Show that

$$d(A, B) := \sum_{j=1}^n \sum_{k=1}^n |a_{jk} - b_{jk}|$$

defines a metric.

**Problem 30.** Let  $n \geq 1$ . Consider the continuous function

$$f_n(t) = \begin{cases} 0 & 0 \leq t < 1/2 - 1/n \\ 1/2 + \frac{n}{2}(t - 1/2) & 1/2 - 1/n \leq t \leq 1/2 + 1/n \\ 1 & 1/2 + 1/n \leq t \leq 1 \end{cases}$$

Show that the sequence  $\{f_n(t)\}$  is not a Cauchy sequence for the uniform norm, but with any of the  $L^p$  norms ( $1 \leq p < \infty$ ) it is a Cauchy sequence.

**Problem 31.** The sequence space consists of the set of all (bounded or unbounded) sequences of complex

$$x = (\chi_1, \chi_2, \dots)$$

Thus we have a vector space. Can we define a metric in this vector space which is implied by a norm?

**Problem 32.** A sequence  $\{f_n\}$  ( $n \in \mathbb{N}$ ) of elements in a normed space  $E$  is called a *Cauchy sequence* if, for every  $\epsilon > 0$ , there exists a number  $M_\epsilon$ , such that  $\|f_p - f_q\| < \epsilon$  for  $p, q > M_\epsilon$ . Consider the Hilbert space  $\mathbb{R}$ . Show that

$$s_n = \sum_{j=1}^n \frac{1}{(j-1)!}, \quad n \geq 1$$

is a Cauchy sequence.

**Problem 33.** Two Cauchy sequences  $\{x_k\}$  and  $\{y_k\}$  are said to be equivalent if for all  $\epsilon > 0$ , there is a  $k(\epsilon)$  such that for all  $j \geq k(\epsilon)$  we have  $d(x_j, y_j) < \epsilon$ . One writes  $\{x_k\} \sim \{y_k\}$ . Obviously,  $\sim$  is an equivalence relationship. Show that equivalent Cauchy sequences have the same limit.

**Problem 34.** Consider the sequence  $\{x_k\}$ ,  $k = 1, 2, \dots$  in  $\mathbb{R}$  defined by  $x_k = 1/k^2$  for all  $k = 1, 2, \dots$ . Show that this sequence is a Cauchy sequence.

**Problem 35.** The sequence space consists of the set of all (bounded or unbounded) sequences of complex numbers

$$x = (x_1, x_2, \dots)$$

Thus we have a vector space. Can we define a metric in this vector space which is not implied by a norm?

**Problem 36.** Consider a complex Hilbert space  $\mathcal{H}$  and  $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$ . Let  $c_1, c_2 \in \mathbb{C}$ . An *antilinear operator*  $K$  in this Hilbert space  $\mathcal{H}$  is characterized by

$$K(c_1|\phi_1\rangle + c_2|\phi_2\rangle) = c_1^*K|\phi_1\rangle + c_2^*K|\phi_2\rangle.$$

A *comb* is an antilinear operator  $K$  with zero expectation value for all states  $|\psi\rangle$  of a certain complex Hilbert space  $\mathcal{H}$ . This means

$$\langle\psi|K|\psi\rangle = \langle\psi|LC|\psi\rangle = \langle\psi|L|\psi^*\rangle = 0$$

for all states  $|\psi\rangle \in \mathcal{H}$ , where  $L$  is a linear operator and  $C$  is the complex conjugation.

(i) Consider the two-dimensional Hilbert space  $\mathcal{H} = \mathbb{C}^2$ . Find a unitary  $2 \times 2$  matrix such that

$$\langle\psi|UC|\psi\rangle = 0.$$

(ii) Consider the Pauli spin matrices  $\sigma_1, \sigma_2, \sigma_3$  with  $\sigma_0 = I_2$ . Find

$$\sum_{\mu=0}^3 \sum_{\nu=0}^3 \langle\psi|\sigma_\mu C|\psi\rangle g^{\mu,\nu} \langle\psi|\sigma_\nu C|\psi\rangle$$

where  $g^{\mu,\nu} = \text{diag}(-1, 1, 0, 1)$ .



## 1.2 Supplementary Problems

**Problem 1.** (i) Let  $f, g \in \mathcal{H}$ . Find all solutions to the equations

$$\langle f, g \rangle \langle g, f \rangle = 1.$$

(i) Let  $f, g \in \mathcal{H}$ . Find all solutions to the equations

$$\langle f, g \rangle \langle g, f \rangle = i.$$

**Problem 2.** Let  $\mathcal{H}$  be a Hilbert space and  $f \in \mathcal{H}$ . Let  $\{e_k : k \in I\}$  ( $I$  index set) be an orthonormal sequence in the Hilbert space  $\mathcal{H}$ . Then (*Bessel inequality*)

$$\sum_{k \in I} |\langle f, e_k \rangle|^2 \leq \|f\|^2.$$

Consider the Hilbert space  $\mathbb{R}^3$  and

$$f = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Find the left-hand side and right-hand side of the inequality.

**Problem 3.** Let  $x, y \in \mathbb{R}$ . Is

$$d(x, y) := \frac{|x - y|}{2 + |x - y|}$$

a metric on  $\mathbb{R}$ ?

**Problem 4.** Show that each Cauchy sequence is bounded.

## Chapter 2

# Finite Dimensional Hilbert Spaces

---

### 2.1 Introduction

We consider the Hilbert space  $\mathbb{C}^n$  with the scalar product

$$\mathbf{v}^* \mathbf{v} = \sum_{j=1}^n \bar{v}_j v_j$$

and the Hilbert space for  $n \times n$  matrices over  $\mathbb{C}$  with the scalar product

$$\langle A, B \rangle = \text{tr}(AB^*).$$

The standard basis in  $\mathbb{C}^n$  is given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

## 2.2 Solved Problems

**Problem 1.** Consider the Hilbert space  $\mathbb{R}^4$  and the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

- (i) Show that the vectors are linearly independent.  
 (ii) Use the *Gram-Schmidt orthogonalization process* to find mutually orthogonal vectors.

**Problem 2.** Consider the Hilbert space  $\mathbb{R}^4$ . Show that the vectors (*Bell basis*)

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

are linearly independent. Show that they form an orthonormal basis in the Hilbert space  $\mathbb{R}^4$ .

**Problem 3.** Consider the Hilbert space  $\mathbb{R}^4$ . Find all pairwise orthogonal vectors (column vectors)  $\mathbf{x}_1, \dots, \mathbf{x}_p$ , where the entries of the column vectors can only be +1 or -1. Calculate the matrix

$$\sum_{i=1}^p \mathbf{x}_i \mathbf{x}_i^T$$

and find the eigenvalues and eigenvectors of this matrix

**Problem 4.** Consider the Hilbert space  $\mathbb{C}^2$  and the vectors

$$|0\rangle = \begin{pmatrix} i \\ i \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Normalize these vectors and then calculate the probability  $|\langle 0|1\rangle|^2$ .

**Problem 5.** Consider the Hilbert space  $\mathbb{R}^n$ . Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Show that

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 \equiv 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

Note that

$$\|\mathbf{x}\|^2 := \langle \mathbf{x}, \mathbf{x} \rangle.$$

**Problem 6.** Let  $|0\rangle, |1\rangle$  be an orthonormal basis in the Hilbert space  $\mathbb{C}^2$ . Let

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi} \sin(\theta/2)|1\rangle$$

where  $\theta, \phi \in \mathbb{R}$ .

- (i) Find  $\langle\psi|\psi\rangle$ .
- (ii) Find the probability  $|\langle 0|\psi\rangle|^2$ . Discuss  $|\langle 0|\psi\rangle|^2$  as a function of  $\theta$ .
- (iii) Assume that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the  $2 \times 2$  matrix  $|\psi\rangle\langle\psi|$  and calculate the eigenvalues.

**Problem 7.** Consider the Hilbert space  $\mathbb{R}^2$ . Show that the vectors

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

are linearly independent. Find

$$\begin{aligned} & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\ & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Show that these four vectors form a basis in  $\mathbb{R}^4$ . Consider the  $4 \times 4$  matrix  $Q$  which is constructed from the four vectors given above, i.e. the columns of the  $4 \times 4$  matrix are the four vectors. Find  $Q^T$ . Is  $Q$  invertible? If so find the inverse  $Q^{-1}$ . What is the use of the matrix  $Q$ ?

**Problem 8.** Consider the Hilbert space  $\mathbb{R}^4$ . Let  $A$  be a symmetric  $4 \times 4$  matrix over  $\mathbb{R}$ . Assume that the eigenvalues are given by  $\lambda_1 = 0$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = 2$  and  $\lambda_4 = 3$  with the corresponding normalized eigenfunctions

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Find the matrix  $A$  by means of the *spectral theorem*.

**Problem 9.** Show that the  $2 \times 2$  matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space  $M_2(\mathbb{C})$ .

**Problem 10.** Consider the Hilbert space  $\mathcal{H}$  of the  $2 \times 2$  matrices over the complex numbers with the scalar product

$$\langle A, B \rangle := \operatorname{tr}(AB^*), \quad A, B \in \mathcal{H}.$$

Show that the rescaled Pauli matrices  $\mu_j := \frac{1}{\sqrt{2}}\sigma_j$ ,  $j = 1, 2, 3$

$$\mu_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the rescaled  $2 \times 2$  identity matrix

$$\mu_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space  $\mathcal{H}$ .

**Problem 11.** Let  $A, B$  be two  $n \times n$  matrices over  $\mathbb{C}$ . We introduce the scalar product

$$\langle A, B \rangle := \frac{\operatorname{tr}(AB^*)}{\operatorname{tr}I_n} = \frac{1}{n}\operatorname{tr}(AB^*).$$

This provides us with a Hilbert space.

The Lie group  $SU(N)$  is defined by the complex  $n \times n$  matrices  $U$

$$SU(N) := \{ U : U^*U = UU^* = I_n, \det(U) = 1 \}.$$

The dimension is  $N^2 - 1$ . The Lie algebra  $su(N)$  is defined by the  $n \times n$  matrices  $X$

$$su(N) := \{ X : X^* = -X, \operatorname{tr}(X) = 0 \}.$$

(i) Let  $U \in SU(N)$ . Calculate  $\langle U, U \rangle$ .

(ii) Let  $A$  be an arbitrary complex  $n \times n$  matrix. Let  $U \in SU(N)$ . Calculate  $\langle UA, UA \rangle$ .

(iii) Consider the Lie algebra  $su(2)$ . Provide a basis. The elements of the basis should be orthogonal to each other with respect to the scalar product given above. Calculate the commutators of these matrices.

**Problem 12.** Let  $\hat{H} = \omega S_1$  be a Hamilton operator, where

$$S_1 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and  $\omega$  is the frequency.

- (i) Find  $\exp(-i\hat{H}t/\hbar)\psi(0)$ , where  $\psi(0) = (1, 1, 1)^T/\sqrt{3}$ .  
 (ii) Calculate the time evolution of

$$S_3 := \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

using the Heisenberg equation of motion. The matrices  $S_x, S_y, S_z$  are the spin-1 matrices, where

$$S_2 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

**Problem 13.** Consider the Hilbert space  $\mathbb{R}^4$ . Show that the *Bell basis*

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

forms an orthonormal basis in this Hilbert space.

**Problem 14.** (i) Consider the Hilbert space  $\mathbb{C}^4$ . Show that the matrices

$$\Pi_1 = \frac{1}{2}(I_2 \otimes I_2 + \sigma_1 \otimes \sigma_1), \quad \Pi_2 = \frac{1}{2}(I_2 \otimes I_2 - \sigma_1 \otimes \sigma_1)$$

are projection matrices in  $\mathbb{C}^4$ .

- (ii) Find  $\Pi_1\Pi_2$ . Discuss.  
 (iii) Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$  be the standard basis in  $\mathbb{C}^4$ . Calculate

$$\Pi_1\mathbf{e}_j, \quad \Pi_2\mathbf{e}_j, \quad j = 1, 2, 3, 4$$

and show that we obtain 2 two-dimensional Hilbert spaces under these projections.

**Problem 15.** Consider the  $3 \times 3$  matrix

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- (i) The matrix  $A$  can be considered as an element of the Hilbert space of the  $3 \times 3$  matrices with the scalar product  $\langle A, B \rangle := \text{tr}(AB^T)$ . Find the norm of  $A$  with respect to this Hilbert space.

(ii) On the other hand  $A$  can be considered as a linear operator in the Hilbert space  $\mathbb{R}^3$ . Find die norm

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|, \quad \mathbf{x} \in \mathbb{R}^3.$$

(iii) Find the eigenvalues of  $A$  and  $AA^T$ . Compare the result with (i) and (ii).

**Problem 16.** Consider the Hilbert space  $M_4(\mathbb{C})$  of all  $4 \times 4$  matrices over  $\mathbb{C}$  with the scalar product  $\langle A, B \rangle := \text{tr}(AB^*)$ , where  $A, B \in M_4(\mathbb{C})$ . The  $\gamma$ -matrices are given by

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ \gamma_3 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \gamma_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

and

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We define the  $4 \times 4$  matrices

$$\sigma_{jk} := \frac{i}{2}[\gamma_j, \gamma_k], \quad j < k$$

where  $j = 1, 2, 3, k = 2, 3, 4$  and  $[\cdot, \cdot]$  denotes the commutator.

(i) Calculate  $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{34}$ .

(ii) Do the 16 matrices

$$I_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_5\gamma_1, \gamma_5\gamma_2, \gamma_5\gamma_3, \gamma_5\gamma_4, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{34}$$

form a basis in the Hilbert space  $M_4(\mathbb{C})$ ? If so is the basis orthogonal?

**Problem 17.** Find the spectrum (eigenvalues and normalized eigenvectors) of matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find  $\|A\|$ , where  $\|\cdot\|$  denotes the norm.

**Problem 18.** Let  $A$  and  $B$  be two arbitrary matrices. Give the definition of the Kronecker product. Let  $\mathbf{u}_j$  ( $j = 1, 2, \dots, m$ ) be an orthonormal basis in the Hilbert space  $\mathbb{R}^m$ . Let  $\mathbf{v}_k$  ( $k = 1, 2, \dots, n$ ) be an orthonormal basis in the Hilbert space  $\mathbb{R}^n$ . Show that  $\mathbf{u}_j \otimes \mathbf{v}_k$  ( $j = 1, 2, \dots, m$ ), ( $k = 1, 2, \dots, n$ ) is an orthonormal basis in  $\mathbb{R}^{m+n}$ .

**Problem 19.** Show that the  $2 \times 2$  matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space  $M^2(\mathbb{C})$ .

**Problem 20.** Show that the  $2 \times 2$  matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis in the Hilbert space  $M^2(\mathbb{R})$ . Apply the *Gram-Schmidt technique* to obtain an orthonormal basis.

**Problem 21.** Consider the  $3 \times 3$  matrices over the real numbers

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) The matrix  $A$  can be considered as an element of the Hilbert space of the  $3 \times 3$  matrices over the real numbers with the scalar product

$$\langle B, C \rangle := \text{tr}(BC^T).$$

Find the norm of  $A$  with respect to this Hilbert space.

(ii) On the other hand the matrix  $A$  can be considered as a linear operator in the Hilbert space  $\mathbb{R}^3$ . Find the norm

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|, \quad \mathbf{x} \in \mathbb{R}^3.$$

(iii) Find the eigenvalues of  $A$  and  $A^T A$ . Compare the result with (i) and (ii).



**Problem 22.** Consider the Hilbert space  $\mathbb{C}^2$ . The Pauli spin matrices  $\sigma_x, \sigma_y, \sigma_z$  act as linear operators in this Hilbert space. Let

$$\hat{H} = \hbar\omega\sigma_3$$

be a Hamilton operator, where

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and  $\omega$  is the frequency. Calculate the time evolution (initial value problem) of

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i.e.

$$i\hbar \frac{d\sigma_1}{dt} = [\sigma_1, \hat{H}](t).$$

**Problem 23.** Consider the Hilbert space  $\mathbb{C}^4$ . Consider the Hamilton operator

$$\hat{H} := \hbar\omega \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Find the time-evolution of the operator

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

using the Heisenberg equation of motion

$$i\hbar \frac{d\gamma_3}{dt} = [\gamma_3, \hat{H}](t).$$

**Problem 24.** Let  $M$  be any  $n \times n$  matrix. Let  $\mathbf{x} = (x_1, x_2, \dots)^T$ . The linear operator  $A$  is defined by

$$A\mathbf{x} = (w_1, w_2, \dots)^T$$

where

$$w_j := \sum_{k=1}^n M_{jk}x_k, \quad j = 1, 2, \dots, n$$

$$w_j := x_j, \quad j > n$$

and  $\mathcal{D}(A) = \ell_2(\mathbf{N})$ . Show that  $A$  is self-adjoint if the  $n \times n$  matrix  $M$  is hermitian. Show that  $A$  is unitary if  $M$  is unitary.

**Problem 25.** Consider the Hilbert space  $\mathbb{C}^n$ . Let  $\mathbf{u}_j, j = 1, 2, \dots, n$ , and  $\mathbf{v}_j, j = 1, 2, \dots, n$  be orthonormal bases in  $\mathbb{C}^n$ , where  $\mathbf{u}_j, \mathbf{v}_j$  are considered as column vectors. Show that

$$U = \sum_{j=1}^n \mathbf{u}_j \mathbf{v}_j^*$$

is a unitary  $n \times n$  matrix.

**Problem 26.** Consider the Hilbert space  $\mathbb{R}^2$ . Given the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}.$$

The three vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  are at 120 degrees of each other and are normalized, i.e.  $\|\mathbf{u}_j\| = 1$  for  $j = 1, 2, 3$ . Every given two-dimensional vector  $\mathbf{v}$  can be written as

$$\mathbf{v} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3, \quad c_1, c_2, c_3 \in \mathbb{R}$$

in many different ways. Given the vector  $\mathbf{v}$  minimize

$$\frac{1}{2}(c_1^2 + c_2^2 + c_3^2)$$

subject to the constraint  $\mathbf{v} - c_1 \mathbf{u}_1 - c_2 \mathbf{u}_2 - c_3 \mathbf{u}_3 = \mathbf{0}$ .

**Problem 27.** Let  $A, H$  be  $n \times n$  hermitian matrices, where  $H$  plays the role of the Hamilton operator. The Heisenberg equations of motion is given by

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[H, A(t)].$$

with  $A = A(t=0) = A(0)$ . Let  $E_j$  ( $j = 1, 2, \dots, n^2$ ) be an orthonormal basis in the Hilbert space  $\mathcal{H}$  of the  $n \times n$  matrices with scalar product

$$\langle X, Y \rangle := \text{tr}(XY^*), \quad X, Y \in \mathcal{H}.$$

Now  $A(t)$  can be expanded using this orthonormal basis as

$$A(t) = \sum_{j=1}^{n^2} c_j(t) E_j$$

and  $H$  can be expanded as

$$H = \sum_{j=1}^{n^2} h_j E_j.$$

Find the time evolution for the coefficients  $c_j(t)$ , i.e.  $dc_j/dt$ , where  $j = 1, 2, \dots, n^2$ .

**Problem 28.** Consider the Hilbert space  $M_2(\mathbb{C})$  of all  $2 \times 2$  matrices over  $\mathbb{C}$  with scalar product

$$\langle A, B \rangle := \text{tr}(AB^*), \quad A, B \in M_2(\mathbb{C}).$$

The standard basis is

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

A *mutually unbiased basis* is

$$\begin{aligned} \mu_0 &= \frac{1}{\sqrt{2}}\sigma_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \mu_1 &= \frac{1}{\sqrt{2}}\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mu_2 &= \frac{1}{\sqrt{2}}\sigma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \mu_3 &= \frac{1}{\sqrt{2}}\sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

(i) Express the Hadamard matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with this mutually unbiased basis.

(ii) Express the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

with the basis (sixteen dimensional) given by

$$\mu_j \otimes \mu_k, \quad j, k = 0, 1, 2, 3.$$

## 2.3 Supplementary Problems

**Problem 1.** Consider the eigenvalue problem

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Show that

$$\lambda = \frac{\mathbf{v}^* A \mathbf{v}}{\mathbf{v}^* \mathbf{v}}.$$

**Problem 2.** Consider the Hilbert space  $\mathbb{R}^4$ . Let  $x \in \mathbb{R}$ . Do the vectors

$$\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2x \\ 3x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6x \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \end{pmatrix}$$

form a basis in this Hilbert space?

**Problem 3.** Consider the linear operator

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in the Hilbert space  $\mathbb{R}^3$ . Find

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

using the method of the Lagrange multiplier.

**Problem 4.** Consider the Hilbert space  $\mathbb{R}^3$ . Let  $\mathbf{x} \in \mathbb{R}^3$ , where  $\mathbf{x}$  is considered as a column vector and  $\mathbf{x} \neq \mathbf{0}$ . Find the matrix  $\mathbf{x}\mathbf{x}^T$  and the real number  $\mathbf{x}^T\mathbf{x}$ . Show that the matrix  $\mathbf{x}\mathbf{x}^T$  admits only one nonzero eigenvalue given by  $\mathbf{x}^T\mathbf{x}$ .

**Problem 5.** Consider the Hilbert space  $\mathbb{R}^3$ . Find the spectrum (eigenvalues and normalized eigenvectors) of matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Find  $\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ , where  $\|\cdot\|$  denotes the norm and  $\mathbf{x} \in \mathbb{R}^3$ .

**Problem 6.** Find the spectrum (eigenvalues and normalized eigenvectors) of the  $3 \times 3$  matrix

$$A = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

Find  $\|A\|$ , where  $\|\cdot\|$  denotes the norms

$$\|A\|_1 := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

$$\|A\|_2 := \sqrt{\operatorname{tr}(AA^*)}.$$

Compare the norms with the eigenvalues. Find  $\exp(A)$ .

**Problem 7.** Let

$$\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

be an orthonormal basis in the Hilbert space  $\mathbb{C}^4$ . Show that the vectors

$$\mathbf{u}_0 = \frac{1}{2}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \quad \mathbf{u}_1 = \frac{1}{2}(\mathbf{v}_0 - \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3),$$

$$\mathbf{u}_2 = \frac{1}{2}(\mathbf{v}_0 + \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3), \quad \mathbf{u}_3 = \frac{1}{2}(\mathbf{v}_0 - \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3),$$

also form an orthonormal basis in  $\mathbb{C}^4$ .

## Chapter 3

# Hilbert Space $L_2(\Omega)$

---

Let  $\Omega$  be a convex domain in  $\mathbb{R}^n$ . We consider the Hilbert space of the square integrable function with scalar product

$$\langle f, g \rangle = \int_{\Omega} f(x)g^*(x)dx.$$

The scalar product implies the norm

$$\|f\|^2 = \langle f, f \rangle = \int_{\Omega} f(x)f^*(x)dx \geq 0.$$

Consider the vector space

$$\mathcal{H}(\mathbb{C}^{n \times N}) := \left\{ f : \mathbb{C}^{n \times N} \rightarrow \mathbb{C} \mid f \text{ holomorphic } \int_{\mathbb{C}^{n \times N}} |f(z)|^2 d\mu(z) < \infty \right\}$$

where  $z = (z_{jk})$  with  $z_{jk} = x_{jk} + iy_{jk}$  ( $j = 1, \dots, n; k = 1, \dots, N$ ) and

$$d\mu(z) = \frac{1}{\pi^{nN}} \exp(-\text{tr}(zz^*)), \quad dz = \prod_{j=1}^n \prod_{k=1}^N dx_{jk} dy_{jk}.$$

Then  $\mathcal{H}(\mathbb{C}^{n \times N})$  is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}^{n \times N}} f(z)\overline{g(z)}d\mu(z).$$

### 3.1 Solved Problems

**Problem 1.** Consider the Hilbert space  $L_2[a, b]$ , where  $a, b \in \mathbb{R}$  and  $b > a$ . Find the condition on  $a$  and  $b$  such that

$$\langle \cos(x), \sin(x) \rangle = 0$$

where  $\langle, \rangle$  denotes the scalar product in  $L_2[a, b]$ .

Hint. Since  $b > a$ , we can write  $b = a + \epsilon$ , where  $\epsilon > 0$ .

**Problem 2.** Consider the Hilbert space  $L_2[-1, 1]$ . Normalize the function  $f(x) = x$  in this Hilbert space.

**Problem 3.** Consider the Hilbert space  $L_2[-\pi, \pi]$ . Obviously  $\cos(x) \in L_2[-\pi, \pi]$ . Find the norm  $\|\cos(x)\|$ . Find nontrivial functions  $f, g \in L_2[-\pi, \pi]$  such that

$$\langle f(x), \cos(x) \rangle = 0, \quad \langle g(x), \cos(x) \rangle = 0$$

and

$$\langle f(x), g(x) \rangle = 0.$$

**Problem 4.** Consider the Hilbert space  $L_2[0, \pi]$ . Let  $\|\cdot\|$  be the norm induced by the scalar product of  $L_2[0, \pi]$ . Find the constants  $a, b$  such that

$$\|\sin(x) - (ax^2 + bx)\|$$

is a minimum.

**Problem 5.** Consider the Hilbert space  $L_2[0, 1]$ . Find a non-trivial function

$$f(x) = ax^3 + bx^2 + cx + d.$$

such that

$$\langle f(x), x \rangle = 0, \quad \langle f(x), x^2 \rangle = 0, \quad \langle f(x), x^3 \rangle = 0$$

where  $\langle, \rangle$  denotes the scalar product

**Problem 6.** (i) Consider the functions

$$f(x) = \frac{1}{1+x^2}, \quad g(x) = \frac{x}{1+x^2}.$$

Obviously  $f, g \in L_2(\mathbb{R})$ . Calculate the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx.$$

(ii) Let  $\omega > 0$ . Consider the functions

$$f(t) = \frac{\sin(\omega t)}{\omega t}, \quad g(t) = \frac{1 - \cos(\omega t)}{\omega t}.$$

Obviously  $f(0) = 1$ ,  $g(0) = 0$  and  $f, g \in L_2(\mathbb{R})$ . Calculate the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt.$$

**Problem 7.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $j, k = 1, 2, \dots$ . Consider the functions

$$f_j(x) = x^j e^{-|x|}, \quad f_k(x) = x^k e^{-|x|}.$$

Find the scalar product

$$(f_j(x), f_k(x)) := \int_{-\infty}^{\infty} f_j(x)\bar{f}_k(x)dx = \int_{-\infty}^{\infty} f_j(x)f_k(x)dx.$$

Note that ( $a > 0$  and  $n \in \mathbb{N}$ )

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}}.$$

Discuss.

**Problem 8.** A basis in the Hilbert space  $L_2[0, 1]$  is given by

$$B := \{e^{2\pi i x n} : n \in \mathbb{Z}\}.$$

Let

$$f(x) = \begin{cases} 2x & 0 \leq x < 1/2 \\ 2(1-x) & 1/2 \leq x < 1 \end{cases}$$

Is  $f \in L_2[0, 1]$ ? Find the first two expansion coefficients of the Fourier expansion of  $f$  with respect to the basis given above.

**Problem 9.** (i) Consider the Hilbert space  $L_2[-1, 1]$ . Consider the sequence

$$f_n(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq -1/n \\ nx & \text{if } -1/n \leq x \leq 1/n \\ +1 & \text{if } 1/n \leq x \leq 1 \end{cases}$$

where  $n = 1, 2, \dots$ . Show that  $\{f_n(x)\}$  is a sequence in  $L_2[-1, 1]$  that is a Cauchy sequence in the norm of  $L_2[-1, 1]$ .



(ii) Show that  $f_n(x)$  converges in the norm of  $L_2[-1, 1]$  to

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ +1 & \text{if } 0 < x \leq 1 \end{cases}.$$

(iii) Use this sequence to show that the space  $C[-1, 1]$  is a subspace of  $L_2[-1, 1]$  that is not closed.

**Problem 10.** Consider the Hilbert space  $L_2[0, 1]$ . The *Legendre polynomials* are defined as

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1).$$

Show that the first four elements are given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Normalize the four elements. Show that the four elements are pairwise orthonormal.

**Problem 11.** Let  $R$  be a bounded region in  $n$ -dimensional space. Consider the eigenvalue problem

$$-\Delta u = \lambda u, \quad u|_{\partial R} = 0$$

where  $\partial R$  denotes the boundary of  $R$ .

(i) Show that all eigenvalues are real and positive

(ii) Show that the eigenfunctions which belong to different eigenvalues are orthogonal.

**Problem 12.** Consider the inner product space

$$C[a, b] = \{ f(x) : f \text{ is continuous on } x \in [a, b] \}$$

with the inner product

$$\langle f, g \rangle := \int_a^b f(x)g^*(x)dx.$$

This implies a norm

$$\langle f, f \rangle = \int_a^b f(x)f^*(x)dx = \|f\|^2.$$

Show that  $C[a, b]$  is incomplete. This means find a Cauchy sequence in the space  $C[a, b]$  which converges to an element which is not in the space  $C[a, b]$ .

**Problem 13.** Consider the Hilbert space  $L_2[-\pi, \pi]$ . Given the function

$$f(x) = \begin{cases} 1 & 0 < x \leq \pi \\ 0 & x = 0 \\ -1 & -\pi \leq x < 0 \end{cases}$$

Obviously  $f \in L_2[-\pi, \pi]$ . Find the Fourier expansion of  $f$ . The orthonormal basis  $\mathcal{B}$  is given by

$$\mathcal{B} := \left\{ \phi_k(x) = \frac{1}{\sqrt{2\pi}} \exp(ikx) \quad k \in \mathbb{Z} \right\}.$$

Find the approximation  $a_0\phi_0(x) + a_1\phi_1(x) + a_{-1}\phi_{-1}(x)$ , where  $a_0, a_1, a_{-1}$  are the Fourier coefficients.

**Problem 14.** Consider the linear operator  $A$  in the Hilbert space  $L_2[0, 1]$  defined by  $Af(x) := xf(x)$ . Find the matrix elements

$$\langle P_i, AP_j \rangle$$

for  $i, j = 0, 1, 2, 3$ , where  $P_i$  are the (normalized) Legendre polynomials. Is the matrix  $A_{ij}$  symmetric?

**Problem 15.** Consider the Hilbert space  $L_2[0, 2\pi]$ . Let

$$g(x) = \cos(x), \quad f(x) = x.$$

Find the conditions on the coefficients of the polynomial

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that

$$\langle g(x), p(x) \rangle = 0, \quad \langle f(x), p(x) \rangle = 0.$$

Solve the equations for  $a_3, a_2, a_1, a_0$ .

**Problem 16.** Let  $b > a$ . Consider the Hilbert space  $L_2([a, b])$  and the functions

$$\phi_n(x) := \sin\left(n\pi \frac{x-a}{b-a}\right), \quad n = 1, 2, \dots$$

which form an orthonormal basis in  $L_2([a, b])$ . Find

$$\langle \phi_m(x), x\phi_n(x) \rangle \equiv \int_a^b \phi_m(x)x\phi_n(x)dx, \quad m, n = 0, 1, 2, \dots$$

**Problem 17.** Let  $m, n \in \mathbb{N}$ . Consider the Hilbert space  $L_2([-1, 1])$  the functions

$$f_n(x) = \frac{1}{1 + nx^2}, \quad n = 1, 2, \dots$$

which are elements in this Hilbert space. Find

$$\|f_n(x) - f_m(x)\|.$$

**Problem 18.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Give the definition and an example of an even function in  $L_2(\mathbb{R})$ . Give the definition and an example of an odd function in  $L_2(\mathbb{R})$ . Show that any function  $f \in L_2(\mathbb{R})$  can be written as a combination of an even and an odd function.

**Problem 19.** The Chebyshev polynomials  $T_n(x)$  of the 1-st kind are defined for  $x \in [-1, 1]$  and given by

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, 2, \dots$$

The Chebyshev polynomials  $U_n(x)$  of the 2-nd kind are defined for  $x \in [-1, 1]$  and given by

$$U_n(x) = \frac{\sin((n + 1) \arccos x)}{\sqrt{1 - x^2}}, \quad n = 0, 1, 2, \dots$$

Consider the Hilbert spaces

$$\mathcal{H}_1 = L_2 \left( [-1, 1], \frac{dx}{\pi\sqrt{1 - x^2}} \right), \quad \mathcal{H}_2 = L_2 \left( [-1, 1], \frac{2\sqrt{1 - x^2}dx}{\pi} \right)$$

which bases are formed by the Chebyshev polynomials of the 1-st and 2-nd type

$$\begin{aligned} \Phi_n^{(1)}(x) &= \sqrt{2}T_n(x), \quad n \geq 1, & \Phi_0^{(1)} &= T_0(x) = 1 \\ \Phi_n^{(2)}(x) &= U_n(x), \quad n \geq 0 \end{aligned}$$

Find a recursion relation for  $\Phi_n^{(1)}$  and  $\Phi_n^{(2)}$ .

**Problem 20.** Consider the Hilbert space  $L_2[0, 1]$ . Find a non-trivial polynomial  $p$

$$p(x) = ax^3 + bx^2 + cx + d$$

such that

$$\langle p, 1 \rangle = 0, \quad \langle p, x \rangle = 0, \quad \langle p, x^2 \rangle = 0.$$

**Problem 21.** Consider the set of polynomials

$$\{1, x, x^2, \dots, x^n, \dots\}.$$

Use the Gram-Schmidt procedure and the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\omega(x)dx, \quad \omega(x) > 0$$

to obtain the first four orthogonal polynomials when

- (i)  $a = -1, b = 1, \omega(x) = 1$  (Legendre polynomials)
- (ii)  $a = -1, b = 1, \omega(x) = (1 - x^2)^{-1/2}$  (Chebyshev polynomials)
- (iii)  $a = 0, b = +\infty, \omega(x) = e^{-x}$  (Laguerre polynomials)
- (iv)  $a = -\infty, b = +\infty, \omega(x) = e^{-x^2}$  (Hermite polynomials)

**Problem 22.** Consider the function

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \cos(jx).$$

Is  $f$  an element of  $L_2[-\pi, \pi]$ ?

**Problem 23.** Consider the Hilbert space  $L_2([0, 1])$ . The *shifted Legendre polynomials*, defined on the interval  $[0, 1]$ , are obtained from the Legendre polynomial by the transformation  $y = 2x - 1$ . The shifted Legendre polynomials are given by the recurrence formula

$$P_j(x) = \frac{(2j+1)(2x-1)}{j+1}P_j(x) - \frac{j}{j+1}P_{j-1}(x) \quad j = 1, 2, \dots$$

and  $P_0(x) = 1, P_1(x) = 2x - 1$ . They are elements of the Hilbert space  $L_2([0, 1])$ . A function  $u$  in the Hilbert space  $L_2([0, 1])$  can be approximated in the form of a series with  $n + 1$  terms

$$u(x) = \sum_{j=0}^n c_j P_j(x)$$

where the coefficients  $c_j \in \mathbb{R}, j = 0, 1, \dots, n$ . Consider the Volterra integral equation of first kind

$$\lambda \int_0^x \frac{y(t)}{(x-t)^\alpha} dt = f(x), \quad 0 \leq t \leq x \leq 1$$

with  $0 < \alpha < 1$  and  $f \in L_2([0, 1])$ . Consider the ansatz

$$y_n(x) = a_0 x^\alpha + \sum_{j=0}^n c_j P_j(x).$$

to find an approximate solution to the *Volterra integral equation* of first kind ( $\alpha = 1/2$ )

$$\lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x)$$

where

$$f(x) = \frac{2}{105} \sqrt{x}(105 - 56x^2 + 48x^3).$$

**Problem 24.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let

$$f_n(x) = \frac{x}{1 + nx^2}, \quad n = 1, 2, \dots$$

(i) Find  $\|f_n(x)\|$  and

$$\lim_{n \rightarrow \infty} \|f_n(x)\|.$$

(ii) Does the sequence  $f_n(x)$  converge uniformly on the real line?

**Problem 25.** Let  $n = 1, 2, \dots$ . We define the functions  $f_n \in L_2[0, \infty)$  by

$$f_n(x) = \begin{cases} \sqrt{n} & \text{for } n \leq x \leq n + 1/n \\ 0 & \text{otherwise} \end{cases}$$

(i) Calculate the norm  $\|f_n - f_m\|$  implied by the scalar product. Does the sequence  $\{f_n\}$  converge in the  $L_2[0, \infty)$  norm?

(ii) Show that  $f_n(x)$  converges pointwise in the domain  $[0, \infty)$  and find the limit. Does the sequence converge pointwise uniformly?

(iii) Show that  $\{f_n\}$  ( $n = 1, 2, \dots$ ) is an orthonormal system. Is it a basis in the Hilbert space  $L_2[0, \infty)$ ?

**Problem 26.** Consider the function  $f \in L_2[0, 1]$

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1/2 \\ 1 - x & \text{for } 1/2 \leq x \leq 1 \end{cases}$$

A basis in the Hilbert space is given by

$$\mathcal{B} := \left\{ 1, \sqrt{2} \cos(\pi nx) \quad : \quad n = 1, 2, \dots \right\}.$$

Find the Fourier expansion of  $f$  with respect to this basis. From this expansion show that

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

**Problem 27.** A particle is enclosed in a rectangular box with impenetrable walls, inside which it can move freely. The Hilbert space is

$$L_2([0, a] \times [0, b] \times [0, c])$$

where  $a, b, c > 0$ . Find the eigenfunctions and the eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions?

**Problem 28.** Consider the Hilbert space  $L_2[0, 1]$  and the polynomials

$$1, x, x^2, x^3, x^4.$$

Apply the Gram-Schmidt orthogonalization process to these polynomials.

**Problem 29.** Consider the Hilbert space  $L_2(\mathbb{T})$ . Let  $f \in L_2(\mathbb{T})$ . Give an example of a bounded linear functional.

**Problem 30.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Show that the Hilbert space is the direct sum of the Hilbert space  $\mathcal{M}$  of even functions and the Hilbert space  $\mathcal{N}$  of odd functions. Give an example of such functions in this Hilbert space.

**Problem 31.** Let  $a > 0$ . Consider the Hilbert space  $L_2[0, a]$ . Let

$$Af(x) := xf(x)$$

for  $f \in L_2[0, a]$ . Find the norm of the operator  $A$ . We define

$$\|A\| := \sup_{\|f\|=1} \|Af\|.$$

**Problem 32.** Consider the Hilbert space  $L_2[0, 2\pi]$ . Let

$$g(x) = \cos(x), \quad f(x) = x.$$

Find the conditions on the coefficients of the polynomial

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that

$$\langle g(x), p(x) \rangle = 0, \quad \langle f(x), p(x) \rangle = 0.$$

Solve the equations for  $a_3, a_2, a_1, a_0$ .

**Problem 33.** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{1 - \cos(2\pi x)}{x}.$$

Using L'Hospital rules we have  $f(0) = 0$ . Is  $f \in L_2(\mathbb{R})$ ?

**Problem 34.** Consider the Hilbert space  $L_2[-1, 1]$ . The *Legendre polynomials* are given by

$$P_j(x) := \frac{1}{2^j j!} \frac{d^j}{dx^j} (x^2 - 1)^j.$$

Find the scalar product

$$\langle P_j(x), P_k(x) \rangle.$$

**Problem 35.** Consider the Hilbert space  $\mathcal{H} = L_2(\mathbb{T})$ . This is the vector space of  $2\pi$ -periodic functions. Then

$$u(x) = \frac{1}{\sqrt{2}}$$

is a constant function which is normalized, i.e.  $\|u\| = 1$ . Show that the projection operator  $P_u$  defined by

$$P_u f := \langle u, f \rangle u$$

maps a function  $f$  to its mean. This means

$$P_u f = \langle f \rangle, \quad \langle f \rangle = \int_0^{2\pi} f(x) dx.$$

**Problem 36.** Consider the Hilbert space  $L_2[-\pi, \pi]$  and the vector space of continuous real-valued functions  $C[-\pi, \pi]$  on the interval  $[-\pi, \pi]$ . Let  $k > 0$  and

$$f_k(x) = \begin{cases} 0 & \text{if } -\pi \leq x \leq 0 \\ kx & \text{if } 0 \leq x \leq 1/k \\ 1 & \text{if } 1/k \leq x \leq \pi \end{cases}$$

The sequence of functions  $f_k$  belongs to the vector space  $C[-\pi, \pi]$ .

(i) Show that  $f_n \rightarrow \chi$  in the norm of the Hilbert space  $L_2[-\pi, \pi]$ , where

$$\chi(x) := \begin{cases} 0 & \text{if } -\pi \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq \pi \end{cases}$$

so that the sequence  $\{f_k\}$  is a Cauchy sequence in the Hilbert space  $L_2[-\pi, \pi]$ .

(ii) Show that  $\|\chi - g\| > 0$  for every  $g \in C[-\pi, \pi]$ . Conclude that  $C[-\pi, \pi]$  is not a Hilbert space.

**Problem 37.** The *Legendre polynomials* are defined on the interval  $[-1, 1]$  and defined by the recurrence formula

$$L_j(x) = \frac{2j+1}{j+1}xL_j(x) - \frac{j}{j+1}L_{j-1}(x) \quad j = 1, 2, \dots$$

and  $L_0(y) = 1$ ,  $L_1(x) = x$ . They are elements of the Hilbert space  $L_2([-1, 1])$ . Calculate the scalar product

$$\langle L_j(x), L_k(x) \rangle$$

for  $j, k = 0, 1, \dots$ . Discuss.

**Problem 38.** Let  $f_n : [-1, 1] \rightarrow [-1, 1]$  be defined by

$$f_n(x) = \begin{cases} 1 & \text{for } -1 \leq x \leq 0 \\ \sqrt{1-nx} & \text{for } 0 \leq x \leq 1/n \\ 0 & \text{for } 1/n \leq x \leq 1 \end{cases}$$

Show that  $f_n \in L_2[-1, 1]$ . Show that  $f_n$  is a Cauchy sequence.

**Problem 39.** Consider the Hilbert space  $L_2([-1, 1])$ . The *Chebyshev polynomials* are defined by

$$T_n(x) := \cos(n \cos^{-1} x), \quad n = 0, 1, 2, \dots$$

They are elements of the Hilbert space  $L_2([-1, 1])$ . We have

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

Calculate the scalar products

$$\langle T_0, T_1 \rangle, \quad \langle T_1, T_2 \rangle, \quad \langle T_2, T_3 \rangle.$$

Calculate the integrals

$$\int_{-1}^1 \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx$$

for  $(m, n) = (0, 1), (m, n) = (1, 2), (m, n) = (2, 3)$ .

**Problem 40.** (i) Consider the Hilbert space  $L_2[0, 1]$  with the scalar product  $\langle \cdot, \cdot \rangle$ . Let  $f : [0, 1] \rightarrow [0, 1]$

$$f(x) := \begin{cases} 2x & \text{if } x \in [0, 1/2) \\ 2(1-x) & \text{if } x \in [1/2, 1] \end{cases}$$



Thus  $f \in L_2[0, 1]$ . Calculate the moments  $\mu_k$ ,  $k = 0, 1, 2, \dots$  defined by

$$\mu_k := \langle f(x), x^k \rangle \equiv \int_0^1 f(x)x^k dx.$$

(ii) Show that

$$\sum_{k=0}^{\infty} |\mu_k|^2 < \pi \int_0^1 |f(x)|^2 dx.$$

**Problem 41.** Let  $a, b \in \mathbb{R}$  and  $-\infty < a < b < +\infty$ . Let  $f$  be a function in the class  $C^1$  (i.e., the derivative  $df/dt$  exists and is continuous) on the interval  $[a, b]$ . Thus  $f$  is also an element of the Hilbert space  $L_2([a, b])$ . Show that

$$\lim_{\omega \rightarrow \infty} \int_a^b f(t) \sin(\omega t) dt = 0. \tag{1}$$

**Problem 42.** Consider the problem of a particle in a one-dimensional box. The underlying Hilbert space is  $L_2(-a, a)$ . Solve the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

as follows: The formal solution is given by

$$\psi(t) = \exp(-i\hat{H}t/\hbar)\psi(0).$$

Expand  $\psi(0)$  with respect to the eigenfunctions of the operator  $\hat{H}$ . The eigenfunctions form a basis of the Hilbert space. Then apply  $\exp(-i\hat{H}t/\hbar)$ . Calculate the probability

$$P = |\langle \phi, \psi(t) \rangle|^2$$

where

$$\phi(q) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right)$$

and

$$\psi(q, 0) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right).$$

**Problem 43.** Let  $f \in L_2(\mathbb{R}^n)$ . Consider the following operators

$$\begin{aligned} T_{\mathbf{y}}f(\mathbf{x}) &= f(\mathbf{x} - \mathbf{y}), && \text{translation operator} \\ M_{\mathbf{k}}f(\mathbf{x}) &= e^{i\mathbf{x} \cdot \mathbf{k}} f(\mathbf{x}), && \text{modulation operator} \\ D_s f(\mathbf{x}) &= |s|^{-n/2} f(s^{-1}\mathbf{x}), \quad s \in \mathbb{R} \setminus \{0\} && \text{dilation operator} \end{aligned}$$

where  $\mathbf{x} \cdot \mathbf{k} = k_1 x_1 + \cdots + x_n k_n$ .

- (i) Find  $\|T_{\mathbf{y}} f\|$ ,  $\|M_{\mathbf{k}}\|$ ,  $\|D_s f\|$ , where  $\|\cdot\|$  denotes the norm in  $L_2(\mathbb{R}^n)$ .  
 (ii) Find the adjoint operators of these three operators.

**Problem 44.** Consider the vector space

$$H_1(a, b) := \{ f(x) \in L_2(a, b) : f'(x) \in L_2(a, b) \}$$

with the norm  $g \in H_1(a, b)$

$$\|g\|_1 := \sqrt{\|g\|_0^2 + \|\partial g / \partial x\|_0^2}.$$

Consider the Hilbert space  $L_2(-\pi, \pi)$  and  $f(x) = \sin(x)$ . Find the norm  $\|f\|_1$ .

**Problem 45.** Let  $f \in H_1(a, b)$ . Then for  $a \leq x < y \leq b$  we have

$$f(y) = f(x) + \int_x^y f'(s) ds.$$

- (i) Show that  $f \in C[a, b]$ .  
 (ii) Show that

$$|f(y) - f(x)| \leq \|f\|_1 \sqrt{|y - x|}.$$

**Problem 46.** Consider the Hilbert space  $L_2[0, \infty)$ . The *Laguerre polynomials* are defined by

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}), \quad n = 0, 1, 2, \dots$$

The first five Laguerre polynomials are given by

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= 1 - x \\ L_2(x) &= 2 - 4x + x^2 \\ L_3(x) &= 6 - 18x + 9x^2 - x^3 \\ L_4(x) &= 24 - 96x + 72x^2 - 16x^3 + x^4. \end{aligned}$$

Show that the function

$$\phi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x)$$

form an orthonormal system in the Hilbert space  $L_2[0, \infty)$ .

**Problem 47.** Consider the Hilbert space  $L_2[-\pi, \pi]$ . A basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} : k \in \mathbb{Z} \right\}.$$

Find the Fourier expansion of

$$f(x) = 1.$$

**Problem 48.** Consider the Hilbert space  $L_2[0, 1]$ . Let  $\mathcal{P}^n$  be the  $n + 1$ -dimensional real linear space of all polynomial of maximal degree  $n$  in the variable  $x$ , i.e.

$$\mathcal{P}^n = \text{span}\{1, x, x^2, \dots, x^n\}.$$

The linear space  $\mathcal{P}^n$  can be spanned by various systems of basis functions. An important basis is formed by the *Bernstein polynomials*

$$\{B_0^n(x), B_1^n(x), \dots, B_n^n(x)\}$$

of degree  $n$  with

$$B_i^n(x) := x^i(1-x)^{n-i}, \quad i = 0, 1, \dots, n.$$

The Bernstein polynomials have a unique dual basis

$$\{D_0^n(x), D_1^n(x), \dots, D_n^n(x)\}$$

which consists of the  $n + 1$  dual basis functions

$$D_i^n(x) = \sum_{j=0}^n c_{ij} B_j^n(x).$$

The dual basis functions satisfy

$$\langle D_i^n(x), B_j^n(x) \rangle = \delta_{ij}.$$

(i) Calculate the scalar product

$$\langle B_i^m(x), B_j^n(x) \rangle.$$

(ii) Find the coefficients  $c_{ij}$ .

**Problem 49.** Consider Fourier series and analytic (harmonic) functions on the disc

$$\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}.$$

A Fourier series can be viewed as the boundary values of a Laurent series

$$\sum_{n=-\infty}^{\infty} c_n z^n.$$

Suppose we are given a function  $f$  on  $\mathbb{T}$ . Find the harmonic extension  $u$  of  $f$  into  $\mathbb{D}$ . This means

$$\Delta u = 0 \quad \text{and} \quad u = f \quad \text{on} \quad \partial\mathbb{D} = \mathbb{T}$$

where  $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$ .

**Problem 50.** Consider the compact abelian Lie group  $U(1)$

$$U(1) = \{ e^{2\pi i\theta} : 0 \leq \theta < 1 \}.$$

The Hilbert space  $L_2(U(1))$  is the space  $L_2([0, 1])$  consisting of all measurable functions  $f(\theta)$  with period 1 such that

$$\int_0^1 |f(\theta)|^2 d\theta < \infty.$$

Now the set of functions

$$\{ e^{2\pi im\theta} : m \in \mathbb{Z} \}$$

form an orthonormal basis for the Hilbert space  $L_2([0, 1])$ . Thus every  $f \in L_2([0, 1])$  can be expressed uniquely as

$$f(\theta) = \sum_{m=-\infty}^{+\infty} c_m e^{2\pi im\theta}, \quad c_m = \int_0^1 f(\theta) e^{-2\pi im\theta} d\theta.$$

Calculate

$$\int_0^1 |f(\theta)|^2.$$

**Problem 51.** The Hilbert space  $L_2(\mathbb{R})$  is the vector space of measurable functions defined almost everywhere on  $\mathbb{R}$  such that  $|f|^2$  is integrable.  $H_1(\mathbb{R})$  is the vector space of functions with first derivatives in  $L_2(\mathbb{R})$ . Give two examples of such a function.

**Problem 52.** Consider the Hilbert space  $L_2[-\pi, \pi]$ . The set of functions

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{-inx} \right\}_{n \in \mathbb{Z}}$$

is an orthonormal basis for  $L_2[-\pi, \pi]$ . Let

$$K(x, t) = \frac{1}{\sqrt{2\pi}} e^{itx}.$$

For  $t$  fixed find the Fourier expansion of this function.

**Problem 53.** Consider the vector space  $C([0, 1])$  of continuous functions. We define the *triangle function*

$$\Lambda(x) := \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2 - 2x & 1/2 < x \leq 1 \end{cases}.$$

Let  $\Lambda_0(x) := x$  and

$$\Lambda_n(x) := \Lambda(2^j x - k)$$

where  $j = 0, 1, 2, \dots, n = 2^j + k$  and  $0 \leq k < 2^j$ . The functions

$$\{1, \Lambda_0, \Lambda_1, \dots\}$$

are the *Schauder basis* for the vector space  $C([0, 1])$ . Let  $f \in C([0, 1])$ . Then

$$f(x) = a + bx + \sum_{n=1}^{\infty} c_n \Lambda_n(x).$$

- (i) Find the Schauder coefficients  $a, b, c_n$ .
- (ii) Consider  $g : [0, 1] \rightarrow [0, 1]$

$$g(x) = 4x(1 - x).$$

Find the Schauder coefficients for this function.

**Problem 54.** Let  $s$  be a nonnegative integer. Let  $x \in \mathbb{R}$  and  $h_n$  ( $n = 0, 1, 2, \dots$ ) be

$$h_n(x) = \frac{(-1)^n}{2^{n/2} \sqrt{n!} \sqrt[4]{\pi}} \exp(x^2/2) \frac{d^n e^{-x^2}}{dx^n}.$$

Thus  $h_n$  for an orthonormal basis in the Hilbert space  $L_2(\mathbb{R})$ . Consider the sequence

$$f_s(x) = \frac{1}{\sqrt{s+1}} \sum_{n=0}^s e^{in\theta} h_n(x)$$

where  $s = 0, 1, 2, \dots$ . Show that the sequence converges weakly but not strongly to 0.

**Problem 55.** Consider the Hilbert space  $L_2[0, 2\pi]$  with the scalar product

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{i\theta}) \overline{f_2(e^{i\theta})} d\theta.$$

- (i) Let  $f_1(z) = z$  and  $f_2(z) = z^2$ . Find  $\langle f_1, f_2 \rangle$ .  
 (ii) Let  $f_1(z) = z^2$  and  $f_2(z) = \sin(z)$ . Find  $\langle f_1, f_2 \rangle$ .

**Problem 56.** Consider the Hilbert space  $L_2(\mathbb{R}^2)$  with the basis

$$\psi_{mn}(x_1, x_2) = NH_m(x_1)H_n(x_2)e^{-(x_1^2+x_2^2)/2}$$

where  $m, n = 0, 1, \dots$  and  $N$  is the normalization factor. Consider the two-dimensional potential

$$V(x_1, x_2) = \frac{a}{4}(x_1^4 + x_2^4) + cx_1x_2.$$

- (i) Find all linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$V(T\mathbf{x}) = V(\mathbf{x}).$$

- (ii) Show that these  $2 \times 2$  matrices form a group. Is the group abelian.  
 (iii) Find the conjugacy classes and the irreducible representations.  
 (iv) Consider the Hilbert space  $L_2(\mathbb{R}^2)$  with the orthogonal basis

$$\psi_{mn}(x_1, x_2) = H_m(x_1)e^{-x_1^2/2}H_n(x_2)e^{-x_2^2/2}$$

where  $m, n = 0, 1, 2, \dots$ . Find the invariant subspaces from the projection operators of the irreducible representations.

**Problem 57.** Consider the Hilbert space  $L_2[0, 1]$ . Let  $\mathcal{P}^n$  be the  $n + 1$ -dimensional real linear space of all polynomial of maximal degree  $n$  in the variable  $x$ , i.e.

$$\mathcal{P}^n = \text{span}\{1, x, x^2, \dots, x^n\}.$$

The linear space  $\mathcal{P}^n$  can be spanned by various systems of basis functions. An important basis is formed by the *Bernstein polynomials*

$$\{B_0^n(x), B_1^n(x), \dots, B_n^n(x)\}$$

of degree  $n$  with

$$B_i^n(x) := x^i(1-x)^{n-i}, \quad i = 0, 1, \dots, n.$$

The Bernstein polynomials have a unique dual basis  $\{D_0^n(x), D_1^n(x), \dots, D_n^n(x)\}$  which consists of the  $n + 1$  dual basis functions

$$D_i^n(x) = \sum_{j=0}^n c_{ij} B_j^n(x).$$

The dual basis functions satisfy

$$\langle D_i^n(x), B_j^n(x) \rangle = \delta_{ij}.$$

(i) Find the scalar product

$$\langle B_i^m(x), B_j^n(x) \rangle.$$

(ii) Find the coefficients  $c_{ij}$ .

**Problem 58.** Let  $V$  be a metric vector space. A *reproducing kernel Hilbert space* on  $V$  is a Hilbert space  $\mathcal{H}$  of functions on  $V$  such that for each  $x \in V$ , the point evaluation functional

$$\delta_x(f) := f(x), \quad f \in \mathcal{H}$$

is continuous. A reproducing kernel Hilbert space  $\mathcal{H}$  possesses a unique reproducing kernel  $K$  which is a function on  $V \times V$  characterized by the properties that for all  $f \in \mathcal{H}$  and  $x \in V$ ,  $K(x, \cdot) \in \mathcal{H}$  and

$$f(x) = \langle f, K(x, \cdot) \rangle_{\mathcal{H}}$$

where  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  denotes the inner product on  $\mathcal{H}$ . The reproducing kernel  $K$  uniquely determines the reproducing kernel Hilbert space  $\mathcal{H}$ . The reproducing kernel Hilbert space of a reproducing kernel  $K$  is denoted by  $\mathcal{H}_K$ . The *Paley-Wiener space* is defined by

$$S := \{ f \in C(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) : \text{supp} \hat{f} \subseteq [-\pi, \pi]^d \}$$

is a reproducing kernel Hilbert space. The Fourier transform of  $f \in L_1(\mathbb{R}^d)$  is given by

$$\hat{f}(\mathbf{k}) := \frac{1}{(\sqrt{2\pi})^{2d}} \int_{\mathbb{R}^{2d}} f(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{k}} d\mathbf{x}, \quad \mathbf{k} \in \mathbb{R}^d$$

where  $\mathbf{x} \cdot \mathbf{k} = x_1 k_1 + \dots + x_d k_d$  is the inner product in  $\mathbb{R}^d$ . The norm on the vector space  $S$  inherits from that in  $L_2(\mathbb{R}^d)$ . Show that the reproducing kernel for the Paley-Wiener space  $S$  is the *sinc function*

$$\text{sinc}(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^d \frac{\sin(\pi(x_j - y_j))}{\pi(x_j - y_j)}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

**Problem 59.** Show that (Mehler's formula)

$$\exp(-(u^2 + v^2 - 2uvz)/(1 - z^2)) = (1 - z^2)^{1/2} \exp(-(u^2 + v^2)) \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(u) H_n(v)$$

where  $H_n$  are the Hermite polynomials.

**Problem 60.** Consider the Hilbert space  $L_2(\mathbb{R})$  and the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2} + V(x)\right)u(x) = Eu(x)$$

where the potential  $V$  is given by

$$V(x) = x^2 + \frac{ax^2}{1+bx^2}$$

where  $b > 0$ . Insert the ansatz

$$u(x) = e^{-x^2/2}v(x)$$

and find the differential equation for  $v$ . Discuss. Make a polynomial ansatz for  $v$ .

**Problem 61.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $g > 0$ . Consider the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2} + x^2 + \frac{\lambda x^2}{1+gx^2}\right)u(x) = Eu(x).$$

Find a solution of the second order differential equation by making the ansatz

$$u(x) = A(1+gx^2)\exp(-x^2/2).$$

**Problem 62.** (i) Consider the Hilbert space  $L_2[-1/2, 1/2]$ . Show that the following sets

$$\begin{aligned}\mathcal{B}_1 &:= \{ \phi_k(x) = \exp(2\pi i k x), k \in \mathbb{Z} \} \\ \mathcal{B}_2 &:= \{ \psi_k(x) = \sqrt{2} \sin(2\pi k x), k \in \mathbb{N} \}\end{aligned}$$

each form an orthonormal basis in this Hilbert space.

(ii) Expand the step function

$$f(x) = \begin{cases} -1 & \text{for } x \in [-1/2, 0] \\ 1 & \text{for } x \in [0, 1/2] \end{cases}$$

with respect to the basis  $\mathcal{B}_1$  and with respect to the basis  $\mathcal{B}_2$ . Show that the two expansions are equivalent. Recall that

$$2 \sin(x) \sin(y) \equiv \cos(x-y) - \cos(x+y).$$



**Problem 63.** Consider the problem of a free particle in a one-dimensional box  $[-a, a]$ . The underlying Hilbert space is  $L_2[-a, a]$ . An orthonormal basis in  $L_2[-a, a]$  is given by

$$\mathcal{B} = \{ u_k^{(+)}(q), u_k^{(-)}(q) : k \in \mathbb{N} \}$$

where

$$u_k^{(+)} = \frac{1}{\sqrt{a}} \cos\left(\frac{(k-1/2)\pi q}{a}\right), \quad u_k^{(-)} = \frac{1}{\sqrt{a}} \sin\left(\frac{k\pi q}{a}\right).$$

The formal solution of the initial value problem of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

is given by

$$\psi(t) = \exp(-i\hat{H}t/\hbar)\psi(0).$$

Let

$$\psi(q, 0) = \frac{1}{\sqrt{a}} \sin(\pi q/a), \quad \phi(q) = \frac{1}{\sqrt{a}} \sin(\pi q/a).$$

Find  $\exp(-i\hat{H}t/\hbar)$  and  $P = |\langle \phi, \psi(t) \rangle|^2$ .

**Problem 64.** Let  $n$  be a positive integer. Consider the Hilbert space  $L_2[0, n]$  and the function

$$f(x) = e^{-x}.$$

Find  $a, b \in \mathbb{R}$  such that

$$\|f(x) - (ax^2 + bx)\|$$

is a minimum. The norm in the Hilbert space  $L_2[0, n]$  is induced by the scalar product.

**Problem 65.** Give a function  $f \in L_2([0, \infty))$  such that

$$\int_0^\infty f(x)dx = 1, \quad \int_0^\infty xf(x)dx = 1.$$

**Problem 66.** Consider the Hilbert space  $L_2[0, 2\pi]$ . The linear operator  $Lf(x) := df(x)/dx$  acts on a dense subset of  $L_2[0, 2\pi]$ . Show that this linear operator is not bounded.

**Problem 67.** Consider the Hilbert space  $L_2(\mathbb{R}^3, d\mathbf{x})$  and let

$$S^2 = \{ (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1 \}.$$

In spherical coordinates this Hilbert space has the decomposition

$$L_2(\mathbb{R}^3, d\mathbf{x}) = L_2(\mathbb{R}^+, r^2 dr) \otimes L_2(S^2, \sin(\theta)d\theta d\phi).$$

Let  $\hat{I}$  be the identity operator in the Hilbert space  $L_2(S^2, \sin(\theta)d\theta d\phi)$ . Then the radial momentum operator

$$\hat{P}_r := -i\hbar \frac{1}{r} \left( \frac{\partial}{\partial r} \right) r$$

is identified with the closure of the operator  $\hat{P}_r \otimes \hat{I}$  defined on  $D(\hat{P}_r) \otimes L_2(S^2, \sin(\theta)d\theta d\phi)$  where

$$D(\hat{P}_r) = \left\{ f \in L_2(\mathbb{R}^+, r^2 dr) : f \in AC(\mathbb{R}^+), \frac{1}{r} \frac{d}{dr} r f(r) \in L_2(\mathbb{R}^+, r^2 dr) \lim_{r \rightarrow 0} r |f(r)| = 0 \right\}$$

and for each  $f \in D(\hat{P}_r)$

$$\hat{P}_r f(r) = -i\hbar \frac{1}{r} \frac{d}{dr} (r f(r))$$

where  $\hat{P}_r$  is maximal symmetric in  $L_2(\mathbb{R}^+, r^2 dr)$ . Show that  $\hat{P}_r$  is not self-adjoint.

**Problem 68.** Consider the Hilbert space  $L_2[-\pi, \pi]$  and the functions

$$f(x) = |\sin(x)| \quad g(x) = |\cos(x)|.$$

Find the distance

$$\|f(x) - g(x)\|$$

in this Hilbert space.

**Problem 69.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Show that the spectrum of the *position operator*  $\hat{x}$  is the real line denoted by  $\mathbb{R}$ .

**Problem 70.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Is

$$\phi_n(x) = \frac{1}{\sqrt{\pi(1+x^2)}} e^{2in \arctan(x)}, \quad n \in \mathbb{Z}$$

an orthonormal basis in  $L_2(\mathbb{R})$ ?

**Problem 71.** Consider the Hilbert space  $L_2[0, \infty)$ . Show that the functions

$$\phi_n(x) = e^{-x/2} L_n(x), \quad n = 0, 1, 2, \dots$$

form an orthonormal basis in  $L_2[0, \infty)$ , where  $L_n$  are the *Laguerre polynomials* defined by

$$L_n(x) = \frac{x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} x^k.$$

**Problem 72.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Show that the functions

$$\phi_n(x) = \frac{1}{2^{n/2} \sqrt{n!} (\pi)^{1/4}} H_n(x) e^{-x^2/2}, \quad n = 0, 1, 2, \dots$$

form an orthonormal basis in the Hilbert space  $L_2(\mathbb{R})$ , where  $H_n$  are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \quad n = 0, 1, 2, \dots$$

**Problem 73.** Let  $b > a$  and  $n = 1, 2, \dots$ . Consider the Hilbert space  $L_2[a, b]$ . Find

$$\int_a^b \sin^2 \left( \frac{n\pi(x-a)}{b-a} \right) dx.$$

The functions

$$\phi_n(x) = \sqrt{\frac{2}{b-a}} \sin \left( \frac{n\pi(x-a)}{b-a} \right)$$

form an orthonormal basis in the Hilbert space  $L_2[a, b]$ .

**Problem 74.** Consider the Hilbert space  $L_2[0, \infty)$  and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) \cos(yx) dy, \quad g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos(yx) dy.$$

Let  $g: \mathbb{R}^+ \rightarrow \mathbb{R}$

$$g(y) = e^{-y}$$

Find  $f(x)$ .

**Problem 75.** Consider the Hilbert space  $L_2[-1, 1]$ . An orthonormal basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} : |x| \leq \pi, k \in \mathbb{Z} \right\}.$$

Consider the function  $f(x) = e^{iax}$  in this Hilbert space, where the constant  $a$  is real but not an integer. Apply *Parseval's relation*

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} |\langle f, \phi_k \rangle|^2, \quad \phi_k(x) = \frac{1}{\sqrt{2}} e^{ikx}$$

to show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(a-k)^2} = \frac{\pi^2}{\sin^2(ax)}.$$

**Problem 76.** Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{x}{\sinh(x)}$$

with  $f(0) = 1$ . Show that

$$\frac{x}{\sinh(x)} = 1 + 2 \sum_{j=1}^{\infty} (-1)^j \frac{x^2}{x^2 + (j\pi)^2}.$$

**Problem 77.** Let  $\mu > 0$  and

$$\mathbf{R} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{R}' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

(i) Show that

$$\frac{\exp(-|\mathbf{R} - \mathbf{R}'|/\mu)}{|\mathbf{R} - \mathbf{R}'|} =$$

$$\sum_{n=0}^{\infty} \varepsilon_n \cos(n(\phi - \phi')) \int_0^{\infty} \frac{1}{k^2 + 1/\mu^2} J_n(kr) J_n(kr') \exp(-\sqrt{k^2 + 1/\mu^2} |x_3 - x'_3|) k dk$$

where  $\varepsilon_n = 1$  for  $n = 0$  and  $\varepsilon_n = 2$  for  $n > 0$  and  $J_m(kr)$  is ordinary *Bessel function* of order  $m$ .

(ii) Consider the functions

$$f_{s,k,n}(\mathbf{R}) = \frac{\sqrt{k}}{2\pi} J_n(kr) e^{in\phi + isx_3}$$

where  $0 < k < \infty$ ,  $-\infty < s < \infty$ , and  $n = 0, \pm 1, \pm 2, \dots$ . Show that

$$\int_0^{2\pi} \int_{-\infty}^{+\infty} dx_3 \int_0^{\infty} f_{s,k,n}(\mathbf{R}) \bar{f}_{s',k',n'}(\mathbf{R}) r dr = \delta_{nn'} \delta(s - s') \delta(k - k')$$

**Problem 78.** Let  $a > 0$ . Consider the Hilbert space  $L_2([0, a])$  and the function  $f_n \in L_2([0, a])$

$$f_n(x) = \frac{1}{\sqrt{a}} e^{2\pi i x n / a}, \quad n = 1, 2, \dots$$

Find  $\|f_n(x)\|$  and  $\|\frac{df_n(x)}{dx}\|$ .

**Problem 79.** Let  $a > 0$ . Consider the Hilbert space  $L_2([0, a])$  and the linear bounded operator

$$Tf(x) = xf(x), \quad f \in L_2([0, a]).$$

Find  $\|A\|$ .

**Problem 80.** Consider the Hilbert space  $L_2([0, 2\pi])$ . For any  $2\pi$  periodic function  $k(\tau)$  in  $L_2([0, 2\pi])$  we define

$$K(u) := \int_0^{2\pi} k(x - \tau)u(\tau)d\tau.$$

Find the eigenfunctions and eigenvalues of  $K$ . Note that the functions  $\{e^{inx} : n \in \mathbb{Z}\}$  form an orthonormal basis in  $L_2([0, 2\pi])$ .

**Problem 81.** Let  $b > a$ . Consider the Hilbert space  $L_2([a, b])$  and the second order ordinary differential equations

$$\frac{d^2u}{dx^2} + \lambda u = 0$$

with the boundary conditions  $u(a) = u(b) = 0$ . Solve the differential equation with this boundary condition. Discuss.

**Problem 82.** Let  $a, b > 0$ . Consider the linear partial differential equation

$$\frac{\partial^2 u}{\partial x_1^4} + 2 \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4} + c^2 \frac{\partial^2 u}{\partial t^2} = 0.$$

For the space coordinates  $x_1, x_2$  we have the domain  $0 \leq x_1 \leq a, 0 \leq x_2 \leq b$  and the boundary conditions that  $u(0) = u(a) = u(b) = 0$ . We consider the Hilbert space  $L_2([0, a] \times [0, b])$  for the space coordinates. Find a solution of the partial differential equation.

**Problem 83.** The *Anosov map* is defined as follows:  $\Omega = [0, 1]^2$ ,

$$\phi(x, y) = (x + y, x + 2y).$$

In matrix form we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod{1}.$$

- (i) Show that the map preserves Lebesgue measure.
- (ii) Show that  $\phi$  is invertible. Show that the entire sequence can be recovered from one term.
- (iii) Show that  $\phi$  is mixing utilizing the Hilbert space  $L_2([0, 1] \times [0, 1])$ .

**Problem 84.** Let  $\Omega$  be the unit disk. A Hilbert space of analytic functions can be defined by

$$\mathcal{H} := \left\{ f(z) \text{ analytic, } |z| < 1 : \sup_{a < 1} \int_{|z|=a} |f(z)|^2 ds < \infty \right\}$$

and the scalar product

$$\langle f, g \rangle := \lim_{a \rightarrow 1} \int_{|z|=a} \overline{f(z)} g(z) ds.$$

Let  $c_n$  ( $n = 0, 1, 2, \dots$ ) be the coefficients of the power-series expansion of the analytic function  $f$ . Find the norm of  $f$ .

**Problem 85.** Let  $\mathbb{C}^n$  denote the complex Euclidean space. Let  $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$  and  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{C}^n$  then the scalar product (inner product) is given by

$$\mathbf{z} \cdot \mathbf{w} := \mathbf{z} \mathbf{w}^* = \mathbf{z} \overline{\mathbf{w}}^T$$

where  $\overline{\mathbf{z}} = (\overline{z_1}, \dots, \overline{z_n})$ . Let  $E_n$  denote the set of entire functions in  $\mathbb{C}^n$ . Let  $F_n$  denote the set of  $f \in E_n$  such that

$$\|f\|^2 := \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(\mathbf{z})|^2 \exp(-|\mathbf{z}|^2) dV$$

is finite. Here  $dV$  is the volume element (Lebesgue measure)

$$dV = \prod_{j=1}^n dx_j dy_j = \prod_{j=1}^n r_j dr_j d\theta_j$$

with  $z_j = r_j e^{i\theta_j}$ . The norm follows from the scalar product of two functions  $f, g \in F_n$

$$\langle f, g \rangle := \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(\mathbf{z}) \overline{g(\mathbf{z})} \exp(-|\mathbf{z}|^2) dV.$$

Let

$$\mathbf{z}^m := z_1^{m_1} \cdots z_n^{m_n}$$

where the multiindex  $m$  is defined by  $m! = m_1! \cdots m_n!$  and  $|m| = \sum_{j=1}^n m_j$ . Find the scalar product

$$\langle \mathbf{z}^m, \mathbf{z}^p \rangle.$$

**Problem 86.** Let  $\Psi$  be a complex-valued differentiable function of  $\phi$  in the interval  $[0, 2\pi]$  and  $\Psi(0) = \Psi(2\pi)$ , i.e.  $\Psi$  is an element of the Hilbert space  $L_2([0, 2\pi])$ . Assume that (normalization condition)

$$\int_0^{2\pi} \Psi^*(\phi)\Psi(\phi)d\phi = 1.$$

Calculate

$$\Im \frac{\hbar}{i} \int_0^{2\pi} \Psi^*(\phi)\phi \frac{d}{d\phi} \Psi(\phi)d\phi$$

where  $\Im$  denotes the imaginary part.

**Problem 87.** The *Fock space*  $\mathcal{F}$  is the Hilbert space of entire functions with inner product given by

$$\langle f|g \rangle := \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2} dx dy, \quad z = x + iy$$

where  $\mathbb{C}$  denotes the complex numbers. Therefore the growth of functions in the Hilbert space  $\mathcal{F}$  is dominated by  $\exp(|z|^2/2)$ . Let  $f, g \in \mathcal{F}$  with Taylor expansions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad g(z) = \sum_{j=0}^{\infty} b_j z^j.$$

- (i) Find  $\langle f|g \rangle$  and  $\|f\|^2$ .
- (ii) Consider the special that  $f(z) = \sin(z)$  and  $g(z) = \cos(z)$ . Calculate  $\langle f|g \rangle$ .
- (iii) Let

$$\mathcal{K}(z, w) := e^{z\bar{w}}, \quad z, w \in \mathbb{C}.$$

Calculate  $\langle f(z)|\mathcal{K}(z, w) \rangle$ .

**Problem 88.** Let  $\mathbb{C}^{n \times N}$  be the vector space of all  $n \times N$  complex matrices. Let  $Z \in \mathbb{C}^{n \times N}$ . Then  $Z^* \equiv \bar{Z}^T$ , where  $^T$  denotes transpose. One defines a *Gaussian measure*  $\mu$  on  $\mathbb{C}^{n \times N}$  by

$$d\mu(Z) := \frac{1}{\pi^{nN}} \exp(-\text{tr}(ZZ^*))dZ$$

where  $dZ$  denotes the Lebesgue measure on  $\mathbb{C}^{n \times N}$ . The *Fock space*  $\mathcal{F}(\mathbb{C}^{n \times N})$  consists of all entire functions on  $\mathbb{C}^{n \times N}$  which are square integrable with respect to the Gaussian measure  $d\mu(Z)$ . With the scalar product

$$\langle f|g \rangle := \int_{\mathbb{C}^{n \times N}} f(Z)\overline{g(Z)}d\mu(Z), \quad f, g \in \mathbb{F}(\mathbb{C}^{n \times N})$$

one has a Hilbert space. Show that this Hilbert space has a *reproducing kernel*  $K$ . This means a continuous function  $K(Z, Z') : \mathbb{C}^{n \times N} \times \mathbb{C}^{n \times N} \rightarrow \mathbb{C}$  such that

$$f(Z) = \int_{\mathbb{C}^{n \times N}} K(Z, Z')f(Z')d\mu(Z')$$

for all  $Z \in \mathbb{C}^{n \times N}$  and  $f \in \mathcal{F}(\mathbb{C}^{n \times N})$ .

**Problem 89.** Consider the *Lie group*

$$SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

The elements of this Lie group act as analytic automorphism of the disk

$$\Omega := \{ |z| < 1 \}$$

under

$$z \rightarrow zg = \frac{\bar{\alpha}z + \beta}{\bar{\beta}z + \alpha}$$

where  $(zg)h = z(gh)$ . Let  $n \geq 2$ . We define

$$\mathcal{H}_n := \{ f(z) \text{ analytic in } \Omega, \|f\|^2 = \int_{\Omega} |f(z)|^2 (1 - |z|^2)^{n-2} dx dy < \infty \}$$

and

$$U_n(g)f(z) := \frac{1}{(\bar{\beta}z + \alpha)^n} f((\bar{\alpha}z + \beta)/(\bar{\beta}z + \alpha)).$$

Then  $\mathcal{H}_n$  is a Hilbert space, i.e. the analytic functions in

$$L_2(\Omega, (1 - |z|^2)^{n-2} dx dy)$$

form a closed subspace.  $U_n$  is a representation, i.e.,

$$U_n(gh) = U_n(g)U_n(h)$$

and  $U_n(e) = I$ , where  $e$  is the identity element in  $SU(1, 1)$  ( $2 \times 2$  unit matrix).

Show that

$$\frac{1}{(1 - |z|^2)^2} dx \wedge dy$$

is invariant  $z \rightarrow zg$ .



### 3.2 Supplementary Problems

**Problem 1.** Consider the Hilbert space  $L_2[0, \infty)$  and the function  $f \in L_2[0, \infty)$

$$f(x) = \exp(-u^{1/4}) \sin(u^{1/4}).$$

Show that

$$\langle f, x^n \rangle = \int_0^\infty f(x)x^n dx = 0 \quad \text{for } n = 0, 1, 2, \dots$$

**Problem 2.** Let  $0 \leq r < 1$ . Consider the Hilbert space  $L_2[0, 2\pi]$  and  $f(\theta) \in L_2[0, 2\pi]$ . Show that

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta + \frac{1}{\pi} \int_0^{2\pi} \sum_{j=1}^{\infty} r^j f(\theta) \cos(j(\phi - \theta)) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{1 - r^2}{1 - 2r \cos(\phi - \theta) + r^2} d\theta. \end{aligned}$$

**Problem 3.** Consider the Hilbert space  $L_2[-\pi, \pi]$ . Find the series

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

where  $\theta \in [-\pi, \pi]$  and

$$c_0 = 1, \quad c_1 = c_{-1} = 1, \quad c_2 = c_{-2} = \frac{1}{2}, \dots, c_n = c_{-n} = \frac{1}{n}.$$

**Problem 4.** Consider the real axis  $\mathbb{R}$  and a cover

$$\mathbb{R} = \cup_{j=-\infty}^{j=\infty} I_j, \quad I_j = [\epsilon_j, \epsilon_{j+1}), \quad \epsilon_j < \epsilon_{j+1}$$

with  $d_j := \epsilon_{j+1} - \epsilon_j = |I_j|$ . Let  $f_j$  be a window function supported in the interval  $[\epsilon_j - d_{j-1}/2, \epsilon_{j+1} + d_{j+1}/2)$  such that

$$\sum_{j=-\infty}^{j=\infty} f_j^2(x) = 1$$

and  $f_j^2(x) = 1 - f_j^2(2\epsilon_{j+1} - x)$  for  $x$  near  $\epsilon_{j+1}$ . Show that the functions

$$g_{j,k}(x) = \frac{2}{\sqrt{2d_j}} f_j(x) \sin\left(\frac{\pi}{2d_j} (2k+1)(x - \epsilon_j)\right), \quad k = 0, 1, 2, \dots$$

form an orthonormal basis of the Hilbert space  $L_2(\mathbb{R})$  subordinate to the partition  $f_j$ .

**Problem 5.** Let

$$S = L_{2,\text{real}}(\mathbb{R}, dx/(1 + |x|^3))$$

and  $q \in S$ . Consider the operator

$$L(q) := -\frac{d^2}{dx^2} + q.$$

Show that the operator  $L(q)$  defines a selfadjoint operator in the Hilbert space  $L_2(\mathbb{R}, dx)$ .

**Problem 6.** Let  $a > 0$ . Consider the Hilbert space  $L_2([0, a] \times [0, a] \times [0, a])$ . Let  $n_j \in \mathbb{Z}$  with  $j = 1, 2, 3$ . Show that the functions

$$\phi_{n_1, n_2, n_3}(\mathbf{x}) = \frac{1}{a^{3/2}} \exp(i(n_1 x_1 + n_2 x_2 + n_3 x_3))$$

form an orthonormal basis in this Hilbert space.

**Problem 7.** Consider the Hilbert space  $L_2[0, 2\pi]$ , the curve in the plane expressed in polar coordinates

$$r(\theta) = 1 + r \cos(2\theta)$$

and

$$r(\theta) = \sum_{n=-\infty}^{+\infty} C_n \exp(in\theta), \quad C_n = \frac{1}{2\pi} \int_0^{2\pi} r(\theta) \exp(-in\theta) d\theta.$$

Find the coefficients  $C_n$ .

**Problem 8.** Let  $\ell_1 > 0$ ,  $\ell_2 > 0$ ,  $\ell_3 > 0$  with dimension length. Consider the Hilbert space  $L_2([0, \ell_1] \times [0, \ell_2] \times [0, \ell_3])$ . Let  $n_1, n_2, n_3 \in \mathbb{N}_0$ .

(i) Show that the functions

$$\begin{aligned} u(x_1, x_2, x_3) &= \cos(k_1 x_1) \sin(k_2 x_2) \sin(k_3 x_3) \\ v(x_1, x_2, x_3) &= \sin(k_1 x_1) \cos(k_2 x_2) \sin(k_3 x_3) \\ w(x_1, x_2, x_3) &= \sin(k_1 x_1) \sin(k_2 x_2) \cos(k_3 x_3) \end{aligned}$$

in this Hilbert space, where

$$k_1 = \frac{n_1 \pi}{\ell_1}, \quad k_2 = \frac{n_2 \pi}{\ell_2}, \quad k_3 = \frac{n_3 \pi}{\ell_3}.$$

- (ii) Show that  $u(x_1, x_2, x_3), v(x_1, x_2, x_3), w(x_1, x_2, x_3)$  are 0 at the boundaries of the box (rectangular cavity)  $[0, \ell_1] \times [0, \ell_2] \times [0, \ell_3]$ .  
 (iii) Show that the functions ( $A, B, C$  are constants)

$$\begin{aligned}\tilde{u}(x_1, x_2, x_3, t) &= Au(x_1, x_2, x_3)e^{i\omega t} \\ \tilde{v}(x_1, x_2, x_3, t) &= Bv(x_1, x_2, x_3)e^{i\omega t} \\ \tilde{w}(x_1, x_2, x_3, t) &= Cw(x_1, x_2, x_3)e^{i\omega t}.\end{aligned}$$

satisfy the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2}$$

with (*dispersion relation*)

$$\omega_{\mathbf{k}} = c(k_1^2 + k_2^2 + k_3^2)$$

**Problem 9.** Let  $\ell = 0, 1, 2, \dots$ . Show that

$$\sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta, \phi) = \frac{2\ell + 1}{4\pi}.$$

**Problem 10.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $T$  be the linear operator of pointwise multiplication on  $L_2(\mathbb{R})$  given by

$$(Tf)x = xf(x) \quad \text{for } f \in L_2(\mathbb{R}).$$

Find the spectrum of  $T$ .

**Problem 11.** Let  $Ai(x)$  be the *Airy function*,  $dAi(x)/dx$  be the derivative and  $a_n$  ( $n = 1, 2, \dots$ ) be the zeros of the Airy functions. Show that the functions

$$\frac{Ai(x + a_n)}{dAi(x = a_n)/dx}, \quad n = 1, 2, \dots$$

form an orthonormal basis in the Hilbert space  $L_2([0, \infty))$ .

**Problem 12.** Let  $b > a$ . Consider the Hilbert space  $L_2([a, b])$ . Do the functions (*Chebyshev polynomial*)

$$T_n(x) = \frac{(b-a)^n}{2^{2n-1}} \cos \left( n \arccos \left( \frac{2x}{b-a} - \frac{b+a}{b-a} \right) \right)$$

form an orthonormal basis in  $L_2([a, b])$ ?

**Problem 13.** Let  $a > 0$ . Consider the Hilbert space  $L_2([0, a])$ .

(i) Show that the functions

$$\phi_n(x) = \frac{1}{\sqrt{a}} e^{2\pi i n x/a}, \quad n \in \mathbb{Z}$$

form an orthonormal basis in  $L_2([0, a])$ .

(ii) Show that the functions

$$\frac{1}{\sqrt{a}}, \quad \sqrt{2/a} \cos(2\pi n x/a), \quad \sqrt{2/a} \sin(2\pi n x/a), \quad n = 1, 2, 3, \dots$$

form an orthonormal basis in  $L_2([0, a])$ .

(i) Show that the functions

$$\phi_n(x) = \sqrt{2/a} \sin(\pi n x/a), \quad n = 1, 2, 3, \dots$$

form an orthonormal basis in  $L_2([0, a])$ .

(i) Show that the functions

$$1/\sqrt{a}, \quad \sqrt{2/a} \cos(\pi n x/a), \quad n = 1, 2, 3, \dots$$

form an orthonormal basis in  $L_2([0, a])$ .

**Problem 14.** Consider the Hilbert space  $L_2[-\pi, \pi]$ . A basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} : k \in \mathbb{Z} \right\}.$$

Find the Fourier expansion of  $f(x) = 1$ .

**Problem 15.** Can one construct an orthonormal basis in the Hilbert space  $L_2(\mathbb{R})$  starting from  $(\sigma > 0)$

$$e^{-|x|/\sigma}, \quad x e^{-|x|/\sigma}, \quad x^2 e^{-|x|/\sigma}, \quad x^3 e^{-|x|/\sigma}, \dots?$$

**Problem 16.** Consider the Hilbert space  $L_2([-1, 1])$  and the function  $f_n \in L_2([-1, +1])$  ( $n \in \mathbb{N}$ )

$$f_n(x) \begin{cases} 1 & \text{for } -1 \leq x \leq 0 \\ \sqrt{1-nx} & \text{for } 0 \leq x \leq 1/n \\ 0 & \text{for } 1/n \leq x \leq 1. \end{cases}$$

Show that

$$\|f_n(x) - f_m(x)\| \leq \frac{1}{n} + \frac{1}{m}.$$

Note that if  $n \rightarrow \infty, m \rightarrow \infty$ , then

$$\|f_n(x) - f_m(x)\|^2 \rightarrow 0.$$

**Problem 17.** Starting with the set of polynomials  $\{1, x, x^2, \dots, x^n, \dots\}$  use the Gram-Schmidt procedure the scalar product (inner product)

$$\langle f, g \rangle = \int_{-1}^1 (f(x)g(x) + f'(x)g'(x))dx$$

to find the first five orthogonal polynomials, where  $f'$  denotes derivative.

**Problem 18.** Let  $I = [0, 1)$ ,  $\mu$  the Lebesgue measure,  $f_0(x) = 1$  and for  $n \geq 1$

$$f_n(x) = \begin{cases} +1 & \text{for } 2^{n-1}x = y \in [0, 1/2) \pmod{1} \\ -1 & \text{for } 2^{n-1}x = y \in [1/2, 1) \pmod{1} \end{cases}$$

Show that the functions  $f_n : [0, 1) \rightarrow \mathbb{R}$  form an orthonormal sequence in the Hilbert space  $L_2[0, 1)$ . The functions are called Rademacher functions.

**Problem 19.** Apply Parseval's equality to show that

$$\frac{\pi^2}{8} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2}, \quad \frac{\pi^6}{945} = \sum_{j=1}^{\infty} \frac{1}{j^6}.$$

**Problem 20.** Consider the Hilbert space  $L_2([0, 1])$ . Let  $n \geq 2$  and consider the function  $f_n : [0, 1] \rightarrow \mathbb{R}$

$$f_n(x) = \begin{cases} n^2x & \text{for } 0 \leq x \leq 1/n \\ -n^2(x - 2/n) & \text{for } 1/n \leq x \leq 2/n \\ 0 & \text{for } 2/n \leq x \leq 1. \end{cases}$$

Show that

$$\int_0^1 f_n(x)dx = 1.$$

Study

$$\lim_{n \rightarrow \infty} f_n.$$

## Chapter 4

# Hilbert Space $\ell_2(\mathbb{N})$

---

### 4.1 Solved Problems

**Problem 1.** Consider the Hilbert space  $\ell_2(\mathbb{N})$ . Let  $\mathbf{x} = (x_1, x_2, \dots)^T$  be an element of  $\ell_2(\mathbb{N})$ . We define the linear operator  $A$  in  $\ell_2(\mathbb{N})$  as

$$A\mathbf{x} = (x_2, x_3, \dots)^T$$

i.e.  $x_1$  is omitted and the  $n+1$ st coordinate replaces the  $n$ th for  $n = 1, 2, \dots$ . Then for the domain we have  $\mathcal{D}(A) = \ell_2(\mathbb{N})$ . Find  $A^*\mathbf{y}$  and the domain of  $A^*$ , where  $\mathbf{y} = (y_1, y_2, \dots)$ . Is  $A$  unitary?

**Problem 2.** Consider the Hilbert space  $\ell_2(\mathbb{N})$  and  $\mathbf{x} = (x_1, x_2, \dots)^T$ . The linear bounded operator  $A$  is defined by

$$A(x_1, x_2, x_3, \dots, x_{2n}, x_{2n+1}, \dots)^T = \\ (x_2, x_4, x_1, x_6, x_3, x_8, x_5, \dots, x_{2n+2}, x_{2n-1}, \dots)^T.$$

Show that the operator  $A$  is unitary. Show that the point spectrum of  $A$  is empty and the continuous spectrum is the entire unit circle in the  $\lambda$ -plane.

**Problem 3.** We have

$$A^* =$$

**Problem 4.** Consider the Hilbert space  $\ell_2(\mathbb{N})$ . Suppose that  $S$  and  $T$  are the right and left shift linear operators on this sequence space, defined by

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots), \quad T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

Show that  $T = S^*$ .

**Problem 5.** Find the spectrum of the infinite dimensional matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ & & \ddots & & \ddots & \\ & & & & & \ddots \end{pmatrix}.$$

In other words

$$a_{ij} = \begin{cases} 1 & \text{if } i = j + 1 \\ 1 & \text{if } i = j - 1 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 6.** Let  $P_j$  ( $j = 0, 1, 2, \dots$ ) be the *Legendre polynomials*

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \dots$$

Calculate the infinite dimensional matrix  $A = (A_{jk})$

$$A_{jk} = \int_{-1}^{+1} P_j(x) \frac{dP_k(x)}{dx} dx$$

where  $j, k = 0, 1, \dots$ . Consider the matrix  $A$  as a linear operator in the Hilbert space  $\ell_2(\mathbb{N}_0)$ . Is  $\|A\| < \infty$ ?

**Problem 7.** Let  $\mathbb{Z}$  be the set of integers. Consider the Hilbert space  $\ell_2(\mathbb{Z}^2)$ . Let  $(m_1, m_2) \in \mathbb{Z}^2$ . Let  $f(m_1, m_2)$  be an element of  $\ell_2(\mathbb{Z}^2)$ . Consider the unitary operators

$$Uf(m_1, m_2) := e^{-2\pi i \alpha m_2} f(m_1 + 1, m_2),$$

$$Vf(m_1, m_2) := e^{-2\pi i \beta m_1} f(m_1, m_2 + 1).$$

They are the so-called magnetic translation operators with phase  $\alpha$  and  $\beta$ , respectively. Find the spectrum of  $U$  and  $V$ . Find the commutator  $[U, V]$ . The so-called *Harper operator* which is self-adjoint is defined by

$$\hat{H} := U + U^* + V + V^*.$$

Find the spectrum of  $\hat{H}$ . Consider the case  $\alpha, \beta$  irrational and  $\alpha, \beta$  rational.

**Problem 8.** The spectrum  $\sigma(\hat{H})$  of a linear operator  $\hat{H}$  is defined as the set of all  $\lambda$  for which the *resolvent*

$$R(\lambda) = (\lambda I - \hat{H})^{-1}$$

does not exist. If the linear operator  $\hat{H}$  is self-adjoint, the spectrum is a subset of the real axis. The Lebesgue decomposition theorem states that

$$\sigma = \sigma_{pp} \cup \sigma_{ac} \cup \sigma_{sing}$$

where  $\sigma_{pp}$  is the countable union of points (the pure point spectrum),  $\sigma_{ac}$  is absolutely continuous with respect to Lebesgue measure and  $\sigma_{sing}$  is singular with respect to Lebesgue measure, i.e. it is supported on a set of measure zero. Consider the Hilbert space  $\ell_2(\mathbb{Z})$  and the linear operator

$$\hat{H} = \cdots \otimes I_2 \otimes I_2 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_3 \otimes I_2 \otimes I_2 \otimes \cdots$$

where  $\sigma_1$  is at position 0. Find the spectrum of this linear operator.

**Problem 9.** Let  $\Omega$  be the unit disk. A Hilbert space of analytic functions can be defined by

$$\mathcal{H} := \left\{ f(z) \text{ analytic } |z| < 1 : \sup_{a < 1} \int_{|z|=a} |f(z)|^2 ds < \infty \right\}$$

and the scalar product

$$\langle f, g \rangle := \lim_{a \rightarrow 1} \int_{|z|=a} \overline{f(z)} g(z) ds.$$

Let  $c_n$  ( $n = 0, 1, 2, \dots$ ) be the coefficients of the power-series expansion of the analytic function  $f$ . Find the norm of  $f$ .

**Problem 10.** Let  $|n\rangle$  be the number states ( $n = 0, 1, \dots$ ). Let  $k = 0, 1, \dots$ . Define the operators

$$T_k := \sum_{n=0}^{\infty} |n\rangle \langle 2n+k|.$$

- (i) Show that  $T_k T_{k'}^\dagger = \delta_{kk'} I$ .
- (ii) Show that  $T_k^\dagger T_k = P_k$  is a projection operator.
- (iii) Show that  $\sum_{k=0}^{\infty} P_k = I$ .



(iv) Is the operator

$$\sum_{k=0}^{\infty} T_k \otimes T_k^\dagger$$

unitary?

**Problem 11.** Let  $A$  be a square finite-dimensional matrix over  $\mathbb{R}$  such that

$$AA^T = I_n. \quad (1)$$

Show that

$$A^T A = I_n. \quad (2)$$

Does (2) also hold for infinite dimensional matrices ?

## 4.2 Supplementary Problems

**Problem 1.** Let  $M$  be any  $n \times n$  matrix. Let  $\mathbf{x} = (x_1, x_2, \dots)^T$ . The linear operator  $A$  is defined by

$$A\mathbf{x} = (w_1, w_2, \dots)^T$$

where

$$\begin{aligned} w_j &= \sum_{k=1}^n M_{jk} x_k, & j &= 1, 2, \dots, n \\ w_j &= x_j, & j &> n \end{aligned}$$

and  $\mathcal{D}(A) = \ell_2(\mathbb{N})$ . Show that  $A$  is self-adjoint if the  $n \times n$  matrix  $M$  is hermitian. Show that  $A$  is unitary if  $M$  is unitary.

## Chapter 5

# Fourier Transform

---

### 5.1 Solved Problems

**Problem 1.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0 \\ e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 2.** (i) Find the Fourier transform for

$$f_\alpha(x) = \frac{\alpha}{2} \exp(-\alpha|x|), \quad \alpha > 0.$$

Discuss  $\alpha$  large and  $\alpha$  small.

(ii) Calculate

$$\int_{-\infty}^{\infty} f_\alpha(x) dx.$$

**Problem 3.** Find the Fourier transform of the *hat function*

$$f(t) = \begin{cases} 1 - |t| & \text{for } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 4.** Let  $f \in L_2(\mathbb{R})$  and  $f \in L_1(\mathbb{R})$ . Assume that  $f(x) = f(-x)$ . Can we conclude that  $\hat{f}(k) = \hat{f}(-k)$ ?

**Problem 5.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Find the Fourier transform of

$$f(x) = e^{-a|x|}, \quad a > 0.$$

**Problem 6.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $a > 0$ . Define

$$f_a(x) = \begin{cases} \frac{1}{2a} & |x| < a \\ 0 & |x| > a \end{cases}$$

Calculate

$$\int_{\mathbb{R}} f_a(x) dx$$

and the Fourier transform of  $f_a$ . Discuss the result in dependence of  $a$ .

**Problem 7.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let

$$\hat{\psi}(\omega) = \begin{cases} 1 & \text{if } 1/2 \leq |\omega| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\phi}(\omega) = e^{-\alpha|\omega|}, \quad \alpha > 0.$$

(i) Calculate the inverse Fourier transform of  $\hat{\psi}(\omega)$  and  $\hat{\phi}(\omega)$ , i.e.

$$\begin{aligned} \psi(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \hat{\psi}(\omega) \\ \phi(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \hat{\phi}(\omega). \end{aligned}$$

(ii) Calculate the scalar product  $\langle \psi(t) | \phi(t) \rangle$  by utilizing the identity

$$2\pi \langle \psi(t) | \phi(t) \rangle = \langle \hat{\psi}(\omega) | \hat{\phi}(\omega) \rangle.$$

**Problem 8.** Consider the Hilbert space  $L_2(\mathbb{R})$  and the function  $f \in L_2(\mathbb{R})$

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Calculate  $f * f$  and verify the *convolution theorem*

$$\widehat{f * f} = \hat{f} \hat{f}.$$

**Problem 9.** Let

$$\hat{f}(\omega) = \begin{cases} (1 - \omega^2) & \text{for } |\omega| \leq 1 \\ 0 & \text{for } |\omega| > 1 \end{cases}$$

Find  $f(t)$ .

**Problem 10.** Let  $a > 0$ . Find the Fourier transform of the function  $f_a : \mathbb{R} \rightarrow \mathbb{R}$

$$f_a(x) = \begin{cases} x/a^2 + 1/a & \text{for } -a \leq x \leq 0 \\ -x/a^2 + 1/a & \text{for } 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

**Problem 11.** Let  $a > 0$ . Find the Fourier transform of

$$f_a(t) = \frac{1}{\sqrt{a}} e^{-a|t|}.$$

Discuss the cases  $a$  large and  $a$  small. Is  $f_a \in L_2(\mathbb{R})$ .

**Problem 12.** Show that the Fourier transform of the rectangular window of size  $N$

$$w_n = \begin{cases} 1 & \text{for } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

is

$$W(e^{i\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-i\omega(N-1)/2}.$$

**Problem 13.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $T > 0$ . Consider the function in  $L_2(\mathbb{R})$

$$f(t) = \begin{cases} A \cos(\Omega t) & \text{for } -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

where  $A$  is a positive constant. Calculate the Fourier transform.

**Problem 14.** Let  $\sigma > 0$ . Show that the Fourier transform of the *Gaussian function*

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

is again a Gaussian function

$$\hat{g}_\sigma(k) = e^{-\sigma^2 k^2/2}.$$

We have  $\int_{-\infty}^{\infty} g_{\sigma}(x) dx = 1$ . Is

$$\int_{-\infty}^{\infty} \hat{g}_{\sigma_k}(k) dk = 1?$$

**Problem 15.** Show that the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \operatorname{sech}(\pi x) \equiv \frac{1}{\cosh(\pi x)}$$

is an element of  $L_2(\mathbb{R})$  and  $L_1(\mathbb{R})$ . Find the Fourier transform of the function.

**Problem 16.** Let  $a > 0$ . Find the Fourier transform of

$$\sqrt{2\pi} f_a(x) + \frac{\sin(ax)}{ax}$$

where  $f_a$  is the function with 1 for  $|x| \leq a$  and 0 otherwise.

**Problem 17.** Consider the Hermite-Gauss functions

$$f_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi}x) \exp(-\pi x^2), \quad n = 0, 1, 2, \dots$$

where  $H_n$  is the  $n$ th Hermite polynomial. They form an orthonormal basis in the Hilbert space  $L_2(\mathbb{R})$ . Do the Fourier transform of the functions form an orthonormal basis in the Hilbert space  $L_2(\mathbb{R})$ ?

**Problem 18.** Let  $\omega_0 > 0$  be a fixed frequency and  $t$  the time. Calculate

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\omega_0 t|} e^{-i\omega t} dt.$$

## 5.2 Supplementary Problems

**Problem 1.** The Hilbert transform  $h(t)$  of the function  $f(t)$  is the principal value of the convolution of  $f(t)$  with the kernel function  $k(t) = 1/(\pi t)$

$$h(t) = \int_{-\infty}^{\infty} f(s)k(t-s)ds = \int_{-\infty}^{\infty} f(s)\frac{1}{t-s}ds.$$

Let

$$G(\omega) = \int_{-\infty}^{\infty} g(t) \exp(i\omega t) dt$$

be the Fourier transform of  $g$ . Show that the Hilbert transform can be written as

$$H(\omega) = F(\omega)K(\omega) = -i\operatorname{sgn}(\omega)F(\omega).$$

**Problem 2.** Let  $f \in L_2(\mathbb{R})$  and let us call the Fourier transformed function  $\hat{f}$ . Is  $\hat{f} \in L_2(\mathbb{R})$ ? What is preserved under the Fourier transform?

# Chapter 6

## Wavelets

---

### 6.1 Solved Problems

**Problem 1.** Consider the Hilbert space  $L_2[0, 1]$  and the function  $f(x) = x^2$  in this Hilbert space. Project the function  $f$  onto the subspace of  $L_2[0, 1]$  spanned by the functions  $\phi(x)$ ,  $\psi(x)$ ,  $\psi(2x)$ ,  $\psi(2x - 1)$ , where

$$\phi(x) := \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$
$$\psi(x) := \begin{cases} 1 & \text{for } 0 \leq x < 1/2 \\ -1 & \text{for } 1/2 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

This is related to the Haar wavelet expansion of  $f$ . The function  $\phi$  is called the father wavelet and  $\psi$  is called the mother wavelet.

**Problem 2.** Consider the function  $H \in L_2(\mathbb{R})$

$$H(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$H_{mn}(x) := 2^{-m/2} H(2^{-m}x - n)$$

where  $m, n \in \mathbb{Z}$ . Draw a picture of  $H_{11}$ ,  $H_{21}$ ,  $H_{12}$ ,  $H_{22}$ . Show that

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \delta_{mk} \delta_{nl}, \quad k, l \in \mathbb{Z}$$

where  $\langle \cdot \rangle$  denotes the scalar product in  $L_2(\mathbb{R})$ . Expand the function

$$f(x) = \exp(-|x|)$$

with respect to  $H_{mn}$ . The functions  $H_{mn}$  form an orthonormal basis in  $L_2(\mathbb{R})$ .

**Problem 3.** Consider the Hilbert space  $L_2[0, 1]$  and the *Haar scaling function* (father wavelet)

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let  $n$  be a positive integer. We define

$$g_k(x) := \sqrt{n}\phi(nx - k), \quad k = 0, 1, \dots, n-1.$$

(i) Show that the set of functions  $\{g_0, g_1, \dots, g_{n-1}\}$  is an orthonormal set in the Hilbert space  $L_2[0, 1]$ .

(ii) Let  $f$  be a continuous function on the unit interval  $[0, 1]$ . Thus  $f \in L_2[0, 1]$ . Form the projection  $f_n$  on the subspace  $S_n$  of the Hilbert space  $L_2[0, 1]$  spanned by  $\{g_0, g_1, \dots, g_{n-1}\}$ , i.e.

$$f_n = \sum_{k=0}^{n-1} \langle f, g_k \rangle g_k.$$

Show that  $f_n(x) \rightarrow f(x)$  pointwise in  $x$  as  $n \rightarrow \infty$ .

**Problem 4.** The *continuous wavelet transform*

$$Wf(a, b) = \frac{1}{a} \int_{-\infty}^{+\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad (a, b \in \mathbb{R}, a > 0)$$

decomposes the function  $f \in L_2(\mathbb{R})$  hierarchically in terms of elementary components  $\psi((t-b)/a)$ . They are obtained from a single *analyzing wavelet*  $\psi$  applying *dilations* and *translations*. Here  $\bar{\psi}$  denotes the complex conjugate of  $\psi$  and  $a$  is the scale and  $b$  the shift parameter. The function  $\psi$  has to be chosen so that it is well localized both in physical and Fourier space. The signal  $f(t)$  can be uniquely recovered by the *inverse wavelet transform*

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} Wf(a, b) \psi\left(\frac{t-b}{a}\right) \frac{da}{a} db$$

if  $\psi(t)$  (respectively its Fourier transform  $\hat{\psi}(\omega)$ ) satisfies the *admissibility condition*

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty.$$



Consider the analytic function

$$\psi(t) = te^{-t^2/2}.$$

Does  $\psi$  satisfies the admissibility condition?

**Problem 5.** Consider the function  $H \in L_2(\mathbb{R})$

$$H(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$H_{mn}(x) := 2^{-m/2}H(2^{-m}x - n)$$

where  $m, n \in \mathbb{Z}$ . Draw a picture of  $H_{11}$ ,  $H_{21}$ ,  $H_{12}$ ,  $H_{22}$ . Show that

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \delta_{mk}\delta_{nl}, \quad k, l \in \mathbb{Z}$$

where  $\langle \cdot \rangle$  denotes the scalar product in  $L_2(\mathbb{R})$ . Expand the function

$$f(x) = \exp(-|x|)$$

with respect to  $H_{mn}$ . The functions  $H_{mn}$  form an orthonormal basis in  $L_2(\mathbb{R})$ .

**Problem 6.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $\phi \in L_2(\mathbb{R})$  and assume that  $\phi$  satisfies

$$\int_{\mathbb{R}} \phi(t)\overline{\phi(t-k)}dt = \delta_{0,k}$$

i.e. the integral equals 1 for  $k = 0$  and vanishes for  $k = 1, 2, \dots$ . Show that for any fixed integer  $j$  the functions

$$\phi_{jk}(t) := 2^{j/2}\phi(2^j t - k), \quad k = 0, \pm 1, \pm 2, \dots$$

form an orthonormal set.

**Problem 7.** Consider the function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$

$$\psi(x) := \begin{cases} 1 & \text{for } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Find  $\psi(x) := \phi(2x) - \phi(2x - 1)$ . Calculate

$$\int_{-\infty}^{\infty} \psi(x)dx.$$

**Problem 8.** Consider the Littlewood-Paley orthonormal basis of wavelets. The mother wavelet of this set is

$$L(x) := \frac{1}{\pi x} (\sin(2\pi x) - \sin(\pi x)).$$

Show that

$$L_{mn}(x) = \frac{1}{2^{m/2}} L(2^{-m}x - n), \quad m, n \in \mathbb{Z}$$

generates an orthonormal basis in the Hilbert space  $L_2(\mathbb{R})$ . Apply the rule of L'Hospital to find  $L(0)$ .

## 6.2 Supplementary Problems

**Problem 1.** (i) Consider the Hilbert space  $L_2(\mathbb{R})$  and  $\phi \in L_2(\mathbb{R})$ . The basic scaling function (father wavelet) satisfies a scaling relation of the form

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x - k).$$

Show that the *Hilbert transform* of  $\phi$

$$H(\phi)(y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\phi(x)}{x - y} dx$$

is a solution of the same scaling relation. Note that the scaling function  $\phi$  may have compact support, the Hilbert transform has support on the real line and decays as  $y^{-1}$ .

(ii) Show that the Hilbert transform of the related mother wavelet  $\psi$  is also noncompact and decays like  $y^{-p-1}$  where

$$\int_{\mathbb{R}} x^m \psi(x) dx = 0$$

for  $m = 0, 1, \dots, p - 1$ .

**Problem 2.** Is

$$f(x) = e^{-x^2/2} \cos(x)$$

a mother wavelet for the Hilbert space  $L_2(\mathbb{R})$ .

**Problem 3.** The linear spline function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined as

$$f(x) = \begin{cases} 1 - |x| & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that the function generates an orthonormal wavelet basis in the Hilbert space  $L_2(\mathbb{R})$  using the technique of multiresolution analysis.

A wavelet basis of the Hilbert space  $L_2(\mathbb{R})$  is an unconditional basis of  $L_2(\mathbb{R})$  generated by the dilations and translations of a wavelet function  $\psi$  (mother wavelet)

$$\{\psi_{j,k}\}, \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

**Problem 4.** Show that for the *Shannon wavelet* the projection operators are given by

$$(P_j)(x) = \int_{\mathbb{R}} \frac{\sin(2^j \pi(y-x))}{\pi(y-x)} f(y) dy.$$

# Chapter 7

## Linear Operators

---

### 7.1 Solved Problems

**Problem 1.** Show that an isometric operator need not be a unitary operator.

**Problem 2.** Consider the Hilbert space  $L_2[0, 1]$ . Show that the linear operator  $T : L_2[0, 1] \rightarrow L_2[0, 1]$  defined by

$$Tf(x) = xf(x)$$

is a bounded self-adjoint linear operator without eigenvalues.

**Problem 3.** Show that if two bounded self-adjoint linear operators  $S$  and  $T$  on a Hilbert space  $\mathcal{H}$  are positive semi-definite and commute ( $ST = TS$ ), then their product  $ST$  is positive semi-definite. We have to show that  $\langle STf, f \rangle \geq 0$  for all  $f \in \mathcal{H}$ .

**Problem 4.** Let  $a > 0$ . Consider the Hilbert space  $L_2[-a, a]$ . Consider the Hamilton operator

$$\hat{H} = \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

where

$$V(x) = \begin{cases} 0 & \text{for } |x| \leq a \\ \infty & \text{otherwise} \end{cases}$$

Solve the Schrödinger equation, where the initial function  $\psi(t = 0) = \phi(x)$  is given by

$$\phi(x) = \begin{cases} x/a^2 + 1/a & \text{for } -a \leq x \leq 0 \\ -x/a^2 + 1/a & \text{for } 0 \leq x \leq a \end{cases}$$

Normalize  $\phi$ . Calculate the probability to find the particle in the state

$$\chi(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right)$$

after time  $t$ . A basis in the Hilbert space  $L_2[-a, a]$  is given by

$$\left\{ \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{a}\right), \frac{1}{\sqrt{a}} \cos\left(\frac{(n-1/2)\pi x}{a}\right) \quad n = 1, 2, \dots \right\}.$$

**Problem 5.** Show that in one dimensional problems the energy spectrum of the bound state is always non-degenerate. Hint. Suppose that the opposite is true. Let  $u_1$  and  $u_2$  be two linearly independent eigenfunctions with the same energy eigenvalues  $E$ , i.e.

$$\frac{d^2 u_1}{dx^2} + \frac{2m}{\hbar^2}(E - V)u_1 = 0, \quad \frac{d^2 u_2}{dx^2} + \frac{2m}{\hbar^2}(E - V)u_2 = 0.$$

**Problem 6.** A particle is enclosed in a rectangular box with impenetrable walls, inside which it can move freely. The Hilbert space is  $L_2([0, a] \times [0, b] \times [0, c])$ . Find the eigenfunctions and eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions.

**Problem 7.** Consider the Hilbert space  $L_2[0, 1]$  and the linear operator  $T : L_2[0, 1] \rightarrow L_2[0, 1]$  defined by

$$(Tf)(x) := xf(x).$$

Show that  $T$  is self-adjoint and positive definite. Find its positive square root.

**Problem 8.** Consider the Hilbert space  $\ell_2(\mathbb{N})$  and the linear operator  $T$  defined by

$$T : (x_1, x_2, x_3, \dots) \mapsto (0, 0, x_3, x_4, \dots).$$

Is  $T$  bounded? Is  $T$  self-adjoint? If so is  $T$  positive?

**Problem 9.** In classical mechanics we have

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{T} = \mathbf{r} \times \mathbf{F}$$

where  $\mathbf{T}$  is the torque,  $\mathbf{F} = -\nabla V$  ( $V$  potential depending only on  $\mathbf{r}$ ) and

$$\frac{d\mathbf{L}}{dt} = \mathbf{T}.$$

In quantum mechanics with  $\mathbf{p} \rightarrow -i\hbar\nabla$ ,  $\mathbf{r} \rightarrow \mathbf{r}$  and wave function  $\psi$  we have

$$\mathbf{L} = -i\hbar \int_{\mathbb{R}^3} d^3\mathbf{x} \psi^* (\mathbf{r} \times \nabla) \psi$$

and

$$\mathbf{T} = - \int_{\mathbb{R}^3} d^3\mathbf{x} \psi^* (\mathbf{r} \times \nabla V) \psi$$

since  $\mathbf{F} = -\nabla V$ .  $\psi$  and  $\psi^*$  obey the Schrödinger equation

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \\ -i\hbar \frac{\partial \psi^*}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 \psi^* + V\psi^*. \end{aligned}$$

Show that

$$\frac{d\mathbf{L}}{dt} = \mathbf{T}.$$

**Problem 10.** Let  $\hat{H}$  be a bounded self-adjoint Hamilton operator with normalized eigenfunctions  $\phi_j$  ( $j \in I$ ) which form an orthonormal basis in the underlying Hilbert space. We can write

$$\psi(t) = \sum_{j \in I} c_j e^{-iE_j t/\hbar} \phi_j$$

where  $E_j$  are the eigenvalues of  $\hat{H}$ . Find  $P(t) = \langle \psi(t=0) | \psi(t) \rangle$ .

**Problem 11.** Consider the Hamilton operator

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + D(1 - e^{-\alpha x})^2 + eEx \cos(\omega t)$$

where  $\alpha > 0$ . So the third term is a driving force. Find the quantum Liouville equation for this Hamilton operator.

**Problem 12.** Consider the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{1}{2m} \Delta + V(x) \right) \psi$$

Find the coupled system of partial differential equations for

$$\rho := \psi^* \psi, \quad v := \Im \left( \frac{\nabla \psi}{\psi} \right).$$

**Problem 13.** Let  $a > 0$ . Consider the Hilbert space  $L_2([0, a])$  and the linear bounded operator  $A$  defined by

$$Af(x) := xf(x), \quad f \in L_2([0, a])$$

Find  $\|A\|$ .

## 7.2 Supplementary Problems

**Problem 1.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $k \in \mathbb{Z}$ . For  $k = 0$  we define  $s_0 = 0$ , for  $k \geq 1$  we define

$$s_k := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

and for  $k < 0$  we define  $s_k = -s_{-k}$ . Let  $\epsilon > 0$ . Define the indicator functions  $W_k$  as

$$W_k(x) := \begin{cases} 1 & \text{for } s_k < x/\epsilon \leq s_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

Let  $u \in L_2(\mathbb{R})$ . Define the linear operator  $O$  as

$$(Ou)(x) := g(x)u(x)$$

where

$$g(x) = -\frac{x}{\epsilon} + \sum_{k \in \mathbb{Z}} \left( \frac{s_k + s_{k+1}}{2} \right) W_k(x).$$

- (i) Show that  $O$  is a bounded self-adjoint operator for any  $\epsilon > 0$ .
- (ii) Show that the norm of  $O$

$$\|O\| = \sup_{\|u\|=1} \|Ou\|$$

is given by  $1/2$ .

**Problem 2.** Consider the *momentum operator*

$$\hat{p} = -i \frac{d}{dx}$$

defined on  $C_0^\infty[0, \infty)$ . Show that the operator has no self-adjoint extensions on the Hilbert space  $L_2[0, \infty)$ .

**Problem 3.** Consider the Hilbert space  $L_2(\mathbb{R})$  and the linear operator

$$\hat{p} = -i \frac{d}{dx} \quad \text{on} \quad D = \left\{ f : f, \frac{df}{dx} \in L_2(\mathbb{R}) \right\}.$$

Show that the spectrum is the whole real axis.

**Problem 4.** Consider the Hilbert space  $L_2(\mathbb{R})$ . Let  $f \in L_2(\mathbb{R})$  and  $\theta \in \mathbb{R}$ . We define the operator  $U(\theta)$  as

$$U(\theta)f(x) := e^{i\theta/2} f(xe^{i\theta}).$$

Is the operator  $U(\theta)$  unitary?



## Chapter 8

# Generalized Functions

---

### 8.1 Solved Problems

**Problem 1.** Consider the function  $H : \mathbb{R} \rightarrow \mathbb{R}$

$$H(x) := \begin{cases} 1 & 0 \leq x \leq 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the derivative of  $H$  in the sense of generalized functions. Obviously  $H$  can be considered as a regular functional

$$\int_{\mathbb{R}} H(x)\phi(x)dx.$$

Find the Fourier transform of  $H$ . Draw a picture of the Fourier transform.

**Problem 2.** Let  $C^m[a, b]$  be the vector space of  $m$ -times differentiable functions and the  $m$ -th derivative is continuous over the interval  $[a, b]$  ( $b > a$ ). We define an inner product (scalar product) of such two functions  $f$  and  $g$  as

$$\langle f, g \rangle_m := \int_a^b \left( fg + \frac{df}{dx} \frac{dg}{dx} + \cdots + \frac{d^m f}{dx^m} \frac{d^m g}{dx^m} \right) dx.$$

Given (Legendre polynomials)

$$f(x) = \frac{1}{2}(3x^2 - 1), \quad g(x) = \frac{1}{2}(5x^3 - 3x)$$

and the interval  $[-1, 1]$ , i.e.  $a = -1$  and  $b = 1$ . Show that  $f$  and  $g$  are orthogonal with respect to the inner product  $\langle f, g \rangle_0$ . Are they orthogonal with respect to  $\langle f, g \rangle_1$ ?

**Problem 3.** Let  $P$  be the parity operator, i.e.

$$P\mathbf{r} := -\mathbf{r}.$$

Obviously,  $P = P^{-1}$ . We define

$$O_P u(\mathbf{r}) := u(P^{-1}\mathbf{r}) \equiv u(-\mathbf{r}).$$

The vector  $\mathbf{r}$  can be expressed in spherical coordinates as

$$\mathbf{r} = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

where

$$0 \leq \phi < 2\pi \quad 0 \leq \theta < \pi.$$

- (i) Calculate  $P(r, \theta, \phi)$ .  
(ii) Let

$$Y_{lm}(\theta, \phi) = \frac{(-1)^{l+m}}{2^l l!} \left( \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right)^{1/2} (\sin \theta)^m \frac{d^{l+m}}{d(\cos \theta)^{l+m}} (\sin \theta)^{2l} e^{im\phi}$$

be the *spherical harmonics*. Find  $O_P Y_{lm}$ .

**Problem 4.** In the Hilbert space  $\mathcal{H} = \ell_2(\mathbb{N}_0)$  Bose annihilation and creation operators denoted by  $b$  and  $b^\dagger$  are defined as follows: They have a common domain

$$\mathcal{D}(b) = \mathcal{D}(b^\dagger) = \left\{ \xi = (x_0, x_1, x_2, \dots)^T : \sum_{j=0}^{\infty} j|x_j|^2 < \infty \right\}.$$

Then  $b\eta$  is given by

$$b(x_0, x_1, x_2, \dots)^T = (x_1, \sqrt{2}x_2, \sqrt{3}x_3, \dots)^T$$

and  $b^\dagger\eta$  is given by

$$b^\dagger(x_0, x_1, x_2, \dots) = (0, x_0, \sqrt{2}x_1, \sqrt{3}x_2, \dots).$$

The infinite dimensional vectors

$$u_n = (0, 0, \dots, 0, 1, 0, \dots)^T$$

where the 1 is at the  $n$  position ( $n = 0, 1, 2, \dots$ ) form the standard basis in  $\mathcal{H} = \ell_2(\mathbb{N}_0)$ . Is

$$\xi = (1, 1/2, 1/3, \dots, 1/n, \dots)$$

an element of  $\mathcal{D}(a)$ ?

**Problem 5.** Given a function (signal)  $f(\mathbf{t}) = f(t_1, t_2, \dots, t_n) \in L_2(\mathbb{R}^n)$  of  $n$  real variables  $\mathbf{t} = (t_1, t_2, \dots, t_n)$ . We define the *symplectic tomogram* associated with the square integrable function  $f$

$$w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{k=1}^n \frac{1}{2\pi|\nu_k|} \left| \int_{\mathbb{R}^n} dt_1 dt_2 \cdots dt_n f(\mathbf{t}) \exp \left( \sum_{j=1}^n \left( \frac{i\mu_j}{2\nu_j} t_j^2 - \frac{iX_j}{\nu_j} t_j \right) \right) \right|^2$$

where ( $\nu_j \neq 0$  for  $j = 1, 2, \dots, n$ )

$$\mathbf{X} = (X_1, X_2, \dots, X_n), \quad \boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n), \quad \boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_n).$$

(i) Prove the equality

$$\int_{\mathbb{R}^n} w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) d\mathbf{X} = \int_{\mathbb{R}^n} |f(\mathbf{t})|^2 dt \tag{1}$$

for the special case  $n = 1$ . The tomogram is the probability distribution function of the random variable  $\mathbf{X}$ . This probability distribution function depends on  $2n$  extra real parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\nu}$ .

(ii) The map of the function  $f(\mathbf{t})$  onto the tomogram  $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$  is invertible. The square integrable function  $f(\mathbf{t})$  can be associated to the density matrix

$$\rho_f(\mathbf{t}, \mathbf{t}') = f(\mathbf{t})f(\mathbf{t}').$$

This density matrix can be mapped onto the *Ville-Wigner function*

$$W(\mathbf{q}, \mathbf{p}) = \int_{\mathbb{R}^n} \rho_f \left( \mathbf{q} + \frac{\mathbf{u}}{2}, \mathbf{q} - \frac{\mathbf{u}}{2} \right) e^{-i\mathbf{p} \cdot \mathbf{u}} d\mathbf{u}.$$

Show that this map is invertible.

(iii) How is the tomogram  $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$  related to the Ville-Wigner function?

(iv) Show that the Ville-Wigner function can be reconstructed from the function  $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ .

(v) Show that the density matrix  $f(\mathbf{t})f^*(\mathbf{t}')$  can be found from  $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ .

**Problem 6.** Describe the one-dimensional scattering of a particle incident on a Dirac delta function, i.e.

$$U(q) = U_0 \delta(q)$$

where  $u_0 > 0$ . Find the transmission and reflection coefficient.

**Problem 7.** (i) Give the definition of the current density, transmission coefficient, and reflection coefficient.

(ii) Calculate the transmission and the reflection coefficients of a particle having total energy  $E$ , at the potential barrier given by

$$V(x) = a\delta(x), \quad a > 0$$

**Problem 8.** Show that

$$\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} = \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$$

in the sense of generalized functions

Hint. Expand the  $2\pi$  periodic function

$$f(x) = \frac{1}{2} - \frac{x}{2\pi}$$

into a Fourier series.

**Problem 9.** (i) Give the definition of a generalized function.

(ii) Calculate the first and second derivative in the sense of generalized function of

$$f(x) = \begin{cases} 0 & x < 0 \\ 4x(1-x) & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

(iii) Calculate the Fourier transform of  $f(x) = 1$  in the sense of generalized functions.

**Problem 10.** Consider the generalized function

$$f(x) = |\cos(x)|.$$

Find the derivative in the sense of generalized functions.

**Problem 11.** Consider the generalized function

$$f(x) := \begin{cases} \cos(x) & \text{for } x \in [0, 2\pi) \\ 0 & \text{otherwise} \end{cases}$$

Find the first and second derivative of  $f$  in the sense of generalized functions.

**Problem 12.** Find the derivative of  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = |x|$$

in the sense of generalized functions.

**Problem 13.** Find the first three derivatives of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = e^{-|x|}$$

in the sense of generalized functions.

**Problem 14.** The *Sobolev space* of order  $m$ , denoted by  $H^m(\Omega)$ , is defined to be the space consisting of those functions in the Hilbert space  $L_2(\Omega)$  that, together with all their weak partial derivatives up to and including those of order  $m$ , belong to the Hilbert space  $L_2(\Omega)$ , i.e.

$$H^m(\Omega) := \{ u : D^\alpha u \in L_2(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| \leq m \}.$$

We consider real-valued functions only, and make  $H^m(\Omega)$  an inner product space by introducing the Sobolev inner product  $\langle \cdot, \cdot \rangle_{H^m}$  defined by

$$\langle u, v \rangle_{H^m} := \int_{\Omega} \sum_{|\alpha| \leq m} (D^\alpha u)(D^\alpha v) dx \quad \text{for } u, v \in H^m(\Omega).$$

This inner product generates the Sobolev norm  $\| \cdot \|_{H^m}$  defined by

$$\|u\|_{H^m}^2 = \langle u|u \rangle_{H^m} = \int_{\Omega} \sum_{|\alpha| \leq m} (D^\alpha u)^2 dx.$$

Thus  $H^0(\Omega) = L_2(\Omega)$ . We can write

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle_{L_2(\Omega)}.$$

In other words the Sobolev inner product  $\langle u, v \rangle_{H^m(\Omega)}$  is equal to the sum of the  $L_2(\Omega)$  inner products of  $D^\alpha u$  and  $D^\alpha v$  over all  $\alpha$  such that  $|\alpha| \leq m$ .

(i) Consider the domain  $\Omega = (0, 2)$  and the function

$$u(x) = \begin{cases} x^2 & 0 < x \leq 1 \\ 2x^2 - 2x + 1 & 1 < x < 2. \end{cases}$$

Obviously  $u \in L_2(\Omega)$ . Find the Sobolev space to which  $u$  belongs.

(ii) Find the norm of  $u$ .

**Problem 15.** Let  $c > 0$ . Consider the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dx^2} + c\delta^{(n)}(x)u = Eu$$

where  $\delta^{(n)}$  ( $n = 0, 1, 2, \dots$ ) denotes the  $n$ -th derivative of the delta function. Derive the joining conditions on the wave function  $u$ .

**Problem 16.** Let  $a > 0$ . Show that

$$\sum_{m=-\infty}^{\infty} \exp(i2\pi m(x+q)/a) \equiv a \sum_{k=-\infty}^{\infty} \delta(x+q-ka).$$

**Problem 17.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function and  $x_0$  a root of  $f$ , i.e.  $f(x_0) = 0$ . Show that

$$\delta'(f(x)) = \frac{1}{(f'(x_0))^2} \left( \delta'(x-x_0) + \frac{f''(x_0)}{f'(x_0)} \delta(x-x_0) \right)$$

**Problem 18.** Show that the sum

$$\frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) P_{\ell}(x) P_{\ell}(y)$$

of Legendre polynomials  $P_{\ell}$  is given by the Dirac delta function  $\delta(y-x)$  for  $-1 \leq x \leq +1$  and  $-1 \leq y \leq +1$ .

**Problem 19.** Show that

$$\delta(x-x') = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} (\cos(nx) \cos(nx') + \sin(nx) \sin(nx')).$$

**Miscellaneous****Problem 20.** Let

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

be the Schrödinger equation, where

$$\hat{H} = -\frac{\hbar^2}{2m} \Delta + U(r), \quad \Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

and  $\mathbf{r} = (x_1, x_2, x_3)$ . Let

$$\rho(\mathbf{r}, t) := \bar{\psi}(\mathbf{r}, t) \psi(\mathbf{r}, t)$$

Find  $\mathbf{j}$  such that

$$\operatorname{div} \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

**Problem 21.** A particle is enclosed in a rectangular box with impenetrable walls, inside which it can move freely. The Hilbert space is

$$L_2([0, a] \times [0, b] \times [0, c])$$

where  $a, b, c > 0$ . Find the eigenfunctions and eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions?**Problem 22.** Show that in one-dimensional problems the energy spectrum of the bound states is always non-degenerate.

Hint. Suppose that the opposite is true.

Let  $u_1$  and  $u_2$  be two linearly independent eigenfunctions with the same energy eigenvalues  $E$ .

$$\begin{aligned} \frac{d^2 u_1}{dx^2} + \frac{2m}{\hbar^2} (E - V) u_1 &= 0 \\ \frac{d^2 u_2}{dx^2} + \frac{2m}{\hbar^2} (E - V) u_2 &= 0 \end{aligned}$$

**Problem 23.** Let  $a > 0$  and let  $f_a : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f_a(x) = \begin{cases} x/a^2 + 1/a & \text{for } -a \leq x \leq 0 \\ -x/a^2 + 1/a & \text{for } 0 \leq x \leq a \end{cases}$$

The function  $f_a$  generates regular functional. Find the derivative of  $f_a$  in the sense of generalized functions.

**Problem 24.** Consider a one-dimensional lattice (chain) with lattice constant  $a$ . Let  $k$  be the sum over the first Brillouin zone we have

$$\frac{1}{N} \sum_{k \in 1.BZ} F(\epsilon(k)) \rightarrow \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} F(\epsilon(k)) dk = G$$

where

$$\epsilon(k) = \epsilon_0 - 2\epsilon_1 \cos(ka).$$

Using the identity

$$\int_{-\infty}^{\infty} \delta(E - \epsilon(k)) F(E) dE \equiv F(\epsilon(k))$$

we can write

$$G = \frac{a}{2\pi} \int_{-\infty}^{\infty} F(E) \left( \int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k)) dk \right) dE.$$

Calculate

$$g(E) = \int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k)) dk$$

where  $g(E)$  is called the density of states.

**Problem 25.** Let  $\epsilon > 0$ . Consider the Schrödinger eigenvalue equation

$$\left( -\frac{d^2}{dx^2} + 2\epsilon\delta(x) \right) u(x, \epsilon) = E(\epsilon)u(x, \epsilon)$$

with the boundary conditions  $u(\pm 1, \epsilon) = 0$ . Here  $\epsilon$  is the coupling constant and determines the penetrability of the potential barrier. Find the eigenfunctions and the eigenvalues.

**Problem 26.** Show that in the sense of generalized functions

$$\begin{aligned} \delta(x) &= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} e^{-|x|/\epsilon} \\ \delta(x) &= \frac{1}{\pi} \lim_{\epsilon \rightarrow \infty} \epsilon \frac{\sin^2(\epsilon x)}{(\epsilon x)^2} \\ \delta(x) &= \frac{1}{4} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( 1 + \frac{|x|}{\epsilon} \right) e^{-|x|/\epsilon}. \end{aligned}$$

**Problem 27.** Give two interpretations of the series of derivatives of  $\delta$  functions

$$f(k) = 2\pi \sum_{n=0}^{\infty} c_n (-1)^n \delta^{(n)}(k). \quad (1)$$



**Problem 28.** Let  $a > 0$ . Show that

$$\delta(x^2 - a^2) = \frac{1}{2a}(\delta(x - a) + \delta(x + a)).$$

**Problem 29.** (i) Show that the Fourier transform in the sense of generalized function of the *Dirac comb*

$$\sum_{n \in \mathbb{Z}} \delta(x - n)$$

is again a Dirac comb.

(ii) Find the Fourier transform in the sense of generalized functions of

$$1 + \sqrt{2\pi}\delta(x).$$

**Problem 30.** (i) Consider the nonlinear differential equation

$$3u \frac{du}{dx} = 2 \frac{du}{dx} \frac{d^2u}{dx^2} + u \frac{d^3u}{dx^3}.$$

Show that  $u(x) = e^{-|x|}$  is a solution in the sense of generalized function.

(ii) Consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + 3u \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^3 u}{\partial x^3}.$$

Show that  $u(x, t) = c \exp(-|x - ct|)$  (*peakon*) is a solution in the sense of generalized functions.

**Problem 31.** Let  $f$  be a differentiable function with a simple zero at  $x = a$  such that  $f(x = a) = 0$  and  $df(x = a)/dx \neq 0$ . Let  $g$  be a differentiable function with a simple zero at  $x = b \neq a$  such that  $g(x = b) = 0$  and  $dg(x = b)/dx \neq 0$ . Show that

$$\delta(f(x)g(x)) = \frac{1}{|f'(a)g(a)|} \delta(x - a) + \frac{1}{|f(b)g'(b)|} \delta(x - b)$$

where  $'$  denotes differentiation.

**Problem 32.** Consider the non-relativistic hydrogen atom, where  $a_0$  is the Bohr radius and  $a = a_0/Z$ . The Schrödinger-Coulomb Green function  $G(\mathbf{r}_1, \mathbf{r}_2; E)$  corresponding to the energy variable  $E$  is the solution of the partial differential equation

$$\left( -\frac{\hbar^2}{2m} \nabla_1^2 - \frac{\hbar^2}{amr_1} - E \right) G(\mathbf{r}_1, \mathbf{r}_2; E) = \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

with the appropriate boundary conditions. Show that expanding  $G$  in terms of spherical harmonics  $Y_{\ell m}$

$$G(\mathbf{r}_1, \mathbf{r}_2; E) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell}(r_1, r_2; E) Y_{\ell m}(\theta_1, \phi_1) Y_{\ell m}^*(\theta_2, \phi_2)$$

we find for the radial part  $g_{\ell}$  of the Schrödinger-Coulomb Green function

$$\left( \frac{1}{r_1^2} \frac{d}{dr_1} \left( r_1^2 \frac{d}{dr_1} \right) - \frac{\ell(\ell+1)}{r_1^2} + \frac{2}{ar_1} - \frac{1}{\nu^2 a^2} \right) g_{\ell}(r_1, r_2; \nu) = -\frac{2m}{\hbar^2} \frac{\delta(r_1 - r_2)}{r_1 r_2}$$

where  $\nu^2 a^2 := -\hbar^2/(2mE)$ .

Hint. Utilize the identity

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\delta(r_1 - r_2)}{r_1 r_2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta_1, \phi_1) Y_{\ell m}^*(\theta_2, \phi_2)$$

**Problem 33.** Show that

$$\delta\left(t - \frac{x}{c}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t(t-x/c)} d\omega.$$

**Problem 34.** Let  $\epsilon > 0$ . Show that

$$f_{\epsilon}(x - a) = \frac{1}{\sqrt{\pi\epsilon}} \exp\left(-\frac{(x - a)^2}{\epsilon}\right)$$

tends to  $\delta(x - a)$  in the sense of generalized function if  $\epsilon \rightarrow 0_+$ .

**Problem 35.** Let  $J_0$  be the Bessel functions. Show that

$$\delta(x)\delta(y) = \frac{1}{2\pi} \int_0^{\infty} k J_0(k\sqrt{x^2 + y^2}) dk$$

in the sense of generalized functions.

**Problem 36.** Let  $\alpha \in [0, 1)$ . Show that

$$\int_0^{\infty} x^{\alpha-1} P\left(\frac{1}{1-x^2}\right) = \frac{\pi}{2} \cot(\pi\alpha/2).$$

**Problem 37.** Show that

$$\delta(\mathbf{x} - \mathbf{x}') = \lim_{\alpha \rightarrow \beta} \left( \frac{2\pi}{\beta - \alpha} \right)^{3/2} \exp\left(-\frac{\alpha\beta}{2(\beta - \alpha)} (\mathbf{x} - \mathbf{x}')^2\right).$$

**Problem 38.** Let  $p \in [0, 1]$  and

$$\rho(x) = \frac{1}{2} p e^{-|x|} + (1-p)\delta(x).$$

Then  $\rho(x) \geq 0$ . Show that in the sense of generalized function

$$\int_{\mathbb{R}} \rho(x) dx = 1.$$

**Problem 39.** The two-dimensional Dirac comb function is defined by

$$C(x_1, x_2) := \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \delta(x_1 - m) \delta(x_2 - n).$$

Find the Fourier transform of  $C$  in the sense of generalized functions.

**Problem 40.** Let  $T > 0$ . Consider the sequence of functions

$$f_n(t) = \frac{1}{n!} \frac{n}{T} \left( \frac{nt}{T} \right)^n \exp(-nt/T)$$

where  $n = 1, 2, \dots$ . Find  $f_n(t)$  for  $n \rightarrow \infty$  in the sense of generalized functions. Find the Laplace transform of  $f_n(t)$ .

**Problem 41.** What charge distribution  $\rho(r)$  does the spherical symmetric potential

$$V(r) = \frac{e^{-\mu r}}{r}$$

give? For  $r \neq 0$  Poisson's equation in spherical coordinates is given by

$$\Delta V(\mathbf{r}) = \frac{1}{r} \frac{d^2}{dr^2} (rV(\mathbf{r})) + R(\theta, \phi)V(\mathbf{r}) = -4\pi\rho(\mathbf{r})$$

where  $R(\theta, \phi)$  is the differential operator depending on the angles  $\theta, \phi$ .

**Problem 42.** Let  $a_1, a_2, \dots, a_n \neq 0$ . Show that

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 d\epsilon_1 \cdots \int_0^1 d\epsilon_n \frac{\delta(1 - \sum_{j=1}^n \epsilon_j)}{(\sum_{j=1}^n \epsilon_j a_j)^n}.$$

**Problem 43.** Prove the identity in the sense of generalized function

$$f(y) \frac{\partial}{\partial y} \delta(x-y) \equiv -f(x) \frac{\partial}{\partial x} \delta(x-y) + \frac{df(y)}{dy} \delta(x-y).$$

**Problem 44.** Let  $\phi \in S(\mathbb{R})$ . Show that in the sense of generalized functions

$$\lim_{\zeta \rightarrow \infty} \frac{\sin(\zeta x)}{x} \phi(x) dx = \pi \phi(0).$$

**Problem 45.** Show that

$$u(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x - x_0)^2}{4Dt}\right)$$

satisfies the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

with the initial condition

$$u(x, 0) = \delta(x - x_0).$$

**Problem 46.** Let  $A, B$  be  $n \times n$  matrices. Show that

$$\begin{aligned} e^{A+B} &= \int_0^\infty d\alpha_1 e^{\alpha_1 A} \delta(1 - \alpha_1) + \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 e^{\alpha_1 A} B e^{\alpha_2 A} \delta(1 - \alpha_1 - \alpha_2) \\ &\quad + \int_0^\infty \int_0^\infty \int_0^\infty e^{\alpha_1 A} B e^{\alpha_2 A} B e^{\alpha_3 A} \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) + \cdots \end{aligned}$$

**Problem 47.** Consider the Hilbert space  $L_2([-1, 1])$ . The *Legendre polynomials* are given by

$$P_0 = 1, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

where  $n = 1, 2, \dots$ . They satisfy

$$\int_{-1}^{+1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{m,n}, \quad n, m = 0, 1, 2, \dots$$

Let

$$\delta(x) = \sum_{j'=0}^{\infty} d_{j'} P_{j'}(x).$$

Find the expansion coefficients  $d_{j'}$ .

## 8.2 Supplementary Problems

**Problem 1.** Show that the 2-dimensional complex  $\delta$ -function can be written as ( $\alpha \in \mathbb{C}$ )

$$\delta^{(2)}(z) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\alpha \exp(\alpha^* z - z^* \alpha) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\alpha \exp(i(\alpha^* z + z^* \alpha)).$$

**Problem 2.** Show that

$$\delta(x - x') = \frac{1}{\pi} \left( 1 + 2 \sum_{k=1}^{\infty} \cos(kx) \cos(kx') \right).$$

**Problem 3.** Consider the Hilbert space  $L_2([0, \infty))$ . The *Laguerre polynomials* are defined as

$$L_n(x) = e^x \frac{d^n}{dx^n} (e^{-x} x^n), \quad n = 0, 1, 2, \dots$$

For the Hilbert space  $L_2([0, \infty))$  we have the basis

$$B = \{ e^{-x/2} L_n(x) : n = 0, 1, 2, \dots \}.$$

Let  $a \in \mathbb{R}$ . Show that

$$\delta(x - a) = e^{-(x+a)/2} \sum_{k=0}^{\infty} L_k(x) L_k(a).$$

**Problem 4.** Show that

$$\delta(\cos(\theta_1) - \cos(\theta_2)) \delta(\phi_1 - \phi_2) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1, \phi_1) Y_{\ell,m}^*(\theta_2, \phi_2).$$

**Problem 5.** Let  $H_k$  ( $k = 0, 1, 2, \dots$ ) be the Hermite polynomials. Show that

$$\delta(x - a) = \frac{e^{-(x^2+a^2)/2}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{H_k(x) H_k(a)}{2^k k!}$$

**Problem 6.** Show that

$$\frac{\partial}{\partial \bar{z}} \left( \frac{1}{z} \right) = \pi \delta(z)$$

where

$$\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

**Problem 7.** Let  $a \in \mathbb{R}$ . Show that

$$\delta(x - a) = \int_{-\infty}^{+\infty} Ai(s - x) Ai(s - a) ds$$

where  $Ai$  is the Airy function.

**Problem 8.** Let  $x \in \mathbb{R}$  and  $\alpha > 0$ . Show that

$$\sum_{n=-\infty}^{\infty} \delta(x - n\alpha) \equiv \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \exp(2\pi i n x / \alpha).$$

**Problem 9.** Find the first and second derivative of the function

$$f(x) := \begin{cases} x \exp(-cx^2) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

in the sense of generalized function. Note that the function is continuous.

**Problem 10.** Show that

$$\int_0^{\infty} \cos(\omega t) dt = \pi \delta(\omega).$$

**Problem 11.** Let  $r^2 = x_1^2 + x_2^2$  and  $r'^2 = x_1'^2 + x_2'^2$ .

(i) Show that

$$\delta(x_1, x_2) = \frac{1}{2\pi r} \delta(r)$$

(ii) Show that

$$\delta(x_1 - x_1', x_2 - x_2') = \frac{1}{r} \delta(r - r') \delta(\theta - \theta'), \quad r' > 0$$

where we applied polar coordinates.

**Problem 12.** Let  $r^2 = x_1^2 + x_2^2 + x_3^2$ . Show that

$$\delta(x_1 - x_1', x_2 - x_2', x_3 - x_3') = \frac{1}{r \sin(\theta)} \delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi')$$

where we applied spherical coordinates.

**Problem 13.** Consider the linear partial differential equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0.$$

Show that  $u(x, t) = f(x - ct)$  is a solution for any one-dimensional generalized function.

**Problem 14.** Let  $p > 0$ . Show that

$$\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{i(m+1/2)\phi} \cos(p|m+1/2|x) = \frac{1}{2p} (\delta(x - \phi/p) + \delta(x + \phi/p)).$$

**Problem 15.** Let  $a > 0$ . Consider

$$f_a(x) = \begin{cases} 0 & \text{for } x < -a \\ \exp(-\exp(x/(x^2 - a^2))) & \text{for } -a \leq x \leq +a \\ 1 & \text{for } a < x \end{cases}$$

Is the function analytic? Is the function an element of  $C^\infty(\mathbb{R})$ ? Find

$$\delta_a(x) = \frac{d}{dx} f_a(x).$$

Then consider  $a \rightarrow 0$ .

**Problem 16.** The *Morlet wavelet* consists of a plane wave modulated by a Gaussian, i.e.

$$\psi(\eta) = \frac{1}{\pi^{1/4}} e^{i\omega\eta} e^{-\eta^2/2}$$

where  $\omega$  is the dimensionless frequency. Show that if  $\omega = 6$  the admissibility condition is satisfied.

**Problem 17.** Let

$$f_0(x) = \exp(-x^2/2).$$

We define the mother wavelets  $f_n$  as

$$f_n(x) = -\frac{d}{dx} f_{n-1}(x), \quad n = 1, 2, \dots$$

Show that the family of  $f_n$ 's obey the Hermite recursion relation

$$f_n(x) = x f_{n-1}(x) - (n-1) f_{n-2}(x), \quad n = 2, 3, \dots$$

**Problem 18.** Show that

$$H(x-a) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \exp(iu(\tau-x)).$$

**Problem 19.** Show that (distributional identity on  $L_1(\mathbb{R})$ )

$$\frac{1}{\pi^2} \int_{\mathbb{R}} \frac{1}{(t-x)(s-x)} dx = \delta(t-s)$$

where the integral is evaluated in the principal value sense.

**Problem 20.** Let  $c > 0$ . Show that an integral representation of the delta function is given by

$$\delta(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tx} dt$$

where the path of the  $t$ -integration can be closed to the right or left.

**Problem 21.** Show that in the sense of generalized functions

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(ikx) dx, \quad \theta(x) = \int_0^{\infty} \delta(\lambda-x) d\lambda.$$





# Bibliography

Davies E. B.  
*Linear Operators and their Spectra*  
Cambridge studies in advanced mathematics, Cambridge University Press,  
2007

Dunford E. B. and Schwartz J. T.  
*Linear Operators. Part 1: General Theory*  
Interscience, New York (1966)

Golub G. H. and Van Loan C. F.  
*Matrix Computations*, Third Edition,  
Johns Hopkins University Press (1996)

Jones D. S.  
*The Theory of Generalized Functions*, Cambridge University Press (1982)

Kato T.  
*Perturbation Theory of Linear Operators*  
Springer, New York (1966)

Keener J. P.  
*Principles of Applied Mathematics: Transformation and Approximation.*  
Perseus Books, 2nd revised edition (2000)

Miller W.  
*Symmetry Groups and Their Applications*  
Academic Press, New York (1972)

Reddy B. Daya  
*Introductory Functional Analysis*,  
Springer, New York (1990)

Riesz F. and Nagy B. Sz., *Functional Analysis*, Frederick Unger Pub., New

York, 1965

Schwartz L.

*Théorie des distributions*, Hermann, 2 vols. (1966)

Steeb W.-H.

*Matrix Calculus and Kronecker Product with Applications and C++ Programs*

World Scientific Publishing, Singapore (1997)

Steeb W.-H.

*Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra*

World Scientific Publishing, Singapore (1996)

Steeb W.-H.

*Hilbert Spaces, Wavelets, Generalized Functions and Quantum Mechanics*

Kluwer Academic Publishers, Dordrecht (1998)

Steeb W.-H.

*Problems and Solutions in Theoretical and Mathematical Physics,*

Third Edition, Volume I: Introductory Level

World Scientific Publishing, Singapore (2009)

Steeb W.-H.

*Problems and Solutions in Theoretical and Mathematical Physics,*

Third Edition, Volume II: Advanced Level

World Scientific Publishing, Singapore (2009)

Steeb W.-H., Hardy Y., Hardy A. and Stoop R.

*Problems and Solutions in Scientific Computing with C++ and Java Simulations*

World Scientific Publishing, Singapore (2004)

Titchmarsh E. C. *Eigenfunction Expansions*, Part One, second edition Oxford University Press, Oxford, 1962

Vladimirov V. S.

*Equations of Mathematical Physics*, Marcel Dekker, New York (1971)

Weidmann J.

*Linear Operators in Hilbert Spaces*

Springer-Verlag, New York (1980)

Yosida K.  
Functional Analysis, Fifth Edition  
Springer Verlag (1978)

Zuily C.  
Problems in Distributions and Partial Differential Equations  
North-Holland, Amsterdam (1988)

# Index

- Admissibility condition, 64
- Airy function, 51
- Analysing wavelet, 64
- Anosov map, 45
- Antilinear operator, 8
  
- Bell basis, 11, 14
- Bernstein polynomials, 35, 38
- Bessel function, 44
- Bessel inequality, 9
  
- Cauchy sequence, 7
- Chebyshev polynomial, 51
- Chebyshev polynomials, 32
- Comb, 8
- Continuous wavelet transform, 64
- Convolution theorem, 59
  
- Dilations, 64
- Dirac comb, 81
- Dispersion relation, 51
  
- Fock space, 47, 48
- Frame, 3
- Fredholm alternative, 3
  
- Gaussian function, 60
- Gaussian measure, 47
- Gram-Schmidt orthogonalization process, 11
- Gram-Schmidt technique, 16
  
- Haar scaling function, 64
- Hardy space, 6
- Harper operator, 55
- Hat function, 58
  
- Hilbert transform, 66
  
- Inverse wavelet transform, 64
  
- Kernel, 3
- Kernel function, 61
  
- Laguerre polynomials, 43, 85
- laguerre polynomials, 34
- Legendre polynomials, 25, 31, 55, 84
- Legendre polynomials, 32
- Lie group, 48
  
- Momentum operator, 72
- Morlet wavelet, 87
- Mutually unbiased basis, 19
  
- Null space, 3
  
- Paley-Wiener space, 39
- Parseval's relation, 44
- Peakon, 81
- Poisson equation, 83
- Position operator, 42
- projection theorem, 4
  
- Range, 3
- Reproducing kernel, 48
- Reproducing kernel Hilbert space, 39
- Resolvent, 56
  
- Schauder basis, 37
- Schwarz inequality, 2
- Schwarz-Cauchy inequality, 2
- Shannon wavelet, 67
- Shifted Legendre polynomials, 28

Sinc function, 39  
Sobolev space, 77  
Spectral theorem, 12  
Spherical harmonics, 74  
Symplectic tomogram, 75  
  
Translations, 64  
Triangle function, 37  
Triangle inequality, 2  
  
Volterra integral equation, 29