Additional Problems for Groups, Lie Groups, Lie Algebras with Applications

by
Willi-Hans Steeb
International School for Scientific Computing
at
University of Johannesburg, South Africa

Igor Tanski
International School for Scientific Computing

Yorick Hardy
Department of Mathematical Sciences
at
University of South Africa, South Africa

updated: February 8, 2017

Preface

The purpose of this manuscript is to supply a collection of additional problems for the book:

Problems and Solutions for Groups, Lie Groups, Lie Algebras and Applications.

World Scientific Publishing, Singapore 2012

ISBN: 13-978-981-4383-90-5

Accompanying problem books for this book are:

Problems and Solutions in Introductory and Advanced Matrix Calculus

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2006
ISBN 981 256 916 2
`http://www.worldscibooks.com/mathematics/6202.html`

Problems and Solutions in Quantum Computing and Quantum Information, third edition

by Willi-Hans Steeb and Yorick Hardy
World Scientific, Singapore, 2006
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6077.html`

The International School for Scientific Computing (ISSC) provides certificate courses for this subject. Please contact the author if you want to do this course or other courses of the ISSC.

e-mail addresses of the authors:

`steebwilli@gmail.com`
`yorickhardy@gmail.com`

Home page of the authors:

`http://issc.uj.ac.za`

Contents

Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\lvert z \rvert$	modulus of the complex number z
	$\lvert x+iy \rvert = (x^2+y^2)^{1/2}, \;\; x, y \in \mathbb{R}$
$f(S)$	image of the set S under the mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
G	group
$Z(G)$	center of the group G
$\mathbb{Z}_n$	cyclic group $\{0, 1, \ldots, n-1\}$
	under addition modulo n
G/N	factor group
D_n	nth dihedral group
S_n	symmetric group on n letters, permutation group
A_n	alternating group on n letters, alternating group
L	Lie algebra
$\mathbf{x}$	column vector in the vector space $\mathbb{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\lVert . \rVert$	norm

$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbb{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbb{R}^3$
S^2	two sphere
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of a matrix A
A^T	transpose of the matrix A
$\overline{A}$	conjugate of the matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of the square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of an $m \times n$ matrix A and an $n \times p$ matrix B
$[A, B] := AB - BA$	commutator of square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator of square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator
$\hat{N}$	Number operator
g	metric tensor field
ϵ	real parameter
$\wedge$	exterior product
d	exterior derivative

The *Pauli spin matrices* are used extensively in the book. They are given by

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In some cases we will also use σ_1, σ_2 and σ_3 to denote σ_x, σ_y and σ_z .

Chapter 1

Groups

Problem 1. Can one find a commutative group with 5 elements such that each element is inverse to itself?

Problem 2. Show that S_3 is *isomorptic* to the *dihedral group* of order 6.

Problem 3. Find all solutions of $z^5 = 1$. Show that the solutions form a group under multiplication. Find all subgroups. Add all the elements of the solution. Discuss.

Problem 4. Let $z \in \mathbb{C}$ and $z \neq 0$.
(i) Do the 2×2 matrices

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$$

form a group under matrix multiplication?
(ii) Do the 3×3 matrices

$$\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & z \\ 0 & 1 & 0 \\ z^{-1} & 0 & 0 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 5. Let $H_1, \ldots, H_n$ be subgroups of the group G. Show that the intersection of subgroups H_j of a group G for $j \in J := \{1, 2, \ldots, n\}$ is again a subgroup of G.

Problem 6. Let G be a group. Given two elements $g_1, g_2 \in G$. One defines the *group commutator* of g_1 and g_2 to be the element

$$g_1^{-1} g_2^{-1} g_1 g_2.$$

Consider the Lie group $SO(1,1,\mathbb{R})$ with

$$g_1(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \qquad g_1(\beta) = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}.$$

Find the group commutator.

Problem 7. Show that the set of the 8 quaternions

$$M := \{ \pm 1, \pm i, \pm j, \pm k \}$$

with the composition

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k$$

form a group, where 1 is the identity element.

Problem 8. (i) Write down all six 3×3 permutation matrices which form a group under matrix multiplication. Find all the eigenvalues of these matrices. Do they form a group under multiplication?
(ii) Find the three normalized eigenvectors for each of the permutation matrices. Then for each of them form a 3×3 unitary matrix by inserting the normalized eigenvectors as column vectors in the matrix. Do these six unitary matrices form a group under matrix multiplication? Obviously for the identity matrix we find the identity matrix again.

Problem 9. Let $\omega = \exp(i2\pi/3)$. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(i) Show that the matrices A, A^2, A^3 form a group under matrix multiplication.
(ii) Show that the matrices B, B^2, B^3 form a group under matrix multiplication.
(iii) Do the matrices $A^j B^k$ $(j,k = 1,2,3)$ form a group under matrix multiplication?
(iv) Do the matrices $A^j \otimes B^k$ $(j,k = 1,2,3)$ form a group under matrix multiplication?

Problem 10. Consider the group S_n. Any element of S_n can be written as a product of disjoint cycles, and two elements of S_n are conjugate if and only if they have the same number of cycles of any given length. Thus, a conjugacy class of S_n can be specified by giving an unordered partition of the number n. We write a partition of n as a tuple $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$, where

$$\sum_{i=1}^{i=n} \lambda_i = n$$

and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. We can also express such a partition pictorially using a Young diagram, which is a series of rows of boxes, in which the i-th row contains i boxes. For example, the partition $(5, 4, 1, 1)$ of 11 can be represented by the following Young diagram.

$$\begin{array}{ccccc} x & x & x & x & x \\ x & x & x & x & \\ x & & & & \\ x & & & & \end{array}$$

A numbering of the Young diagram λ is a labeling of the boxes by the numbers $1, 2, \ldots, n$. For instance, we can number the diagram by

$$\begin{array}{ccc} 2 & 5 & 1 \\ 4 & 3 & \end{array}$$

There are $n!$ such numberings. There is an action of S_n on the set of numberings by permuting the numbers. A tabloid is an equivalence class of numberings of λ, where we consider two numberings T and T' equivalent if the entries of each row of T and T' are the same. For instance, we have

$$\begin{array}{ccc} 2 & 5 & 1 \\ 4 & 3 & \end{array} = \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \end{array}$$

We denote by $\{T\}$ the tabloid corresponding to T . There is an action of S_n on the set of tabloids of shape λ by $\sigma\{T\} = \{\sigma T\}$. We now define M^λ to be the complex vector space with basis the tabloids of shape λ, and we let S_n act by permuting the basis elements.
(i) Consider tabloids of shape $(n-1, 1)$. What is the dimension of M^λ in this case? How S_n acts in this space?
(ii) Consider tabloids of shape $(1, \ldots, 1)$. What is the dimension of M^λ in this case? How S_n acts in this space?

Problem 11. Let G be a group. An *automorphism* of G is an isomorphism sending G onto itself. Show that the set $\mathrm{Aut}(G)$ of automorphisms of G is a group with respect to the operation of composition of automorphisms.

Problem 12. Do the four 4×4 matrices

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

form a group under matrix multiplication? Can the matrices be written as the Kronecker product of two 2×2 permutation matrices?

Problem 13. Let V be a vector space over a field $\mathbb{F}$. Let W be a subspace of V. We define an *equivalence relation* $\sim$ on V by stating that $v_1 \sim v_2$ if $v_1 - v_2 \in W$. The *quotient space* V/W is the set of equivalence classes $[v]$ where $v_1 - v_2 \in W$. Thus we can say that v_1 is equivalent to v_2 modulo W if $v_1 = v_2 + w$ for some $w \in W$. Let

$$V = \mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \; : \; x_1, x_2 \in \mathbb{R} \right\}$$

and

$$W = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \; : \; x_1 \in \mathbb{R} \right\}.$$

(i) Is

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 1 \end{pmatrix} ?$$

(ii) Give the quotient space.

Problem 14. Consider the set

$$G = \{\, (a,b) \in \mathbb{R}^2 : -\infty < a < \infty, \; 0 \le b < 2\pi \,\}.$$

We define the composition

$$(a,b) \bullet (c,d) := (ac, (b+d) \bmod 2\pi).$$

Show that this composition defines a group.

Problem 15. Let $m \in \mathbb{Z}$, $m \ge 2$, and let $G(m)$ be the group of 2×2 matrices of determinant ± 1 and with entries in the ring $\mathbb{Z}/m\mathbb{Z}$, i.e.

$$G(m) := \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \; : \; a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{Z}/m\mathbb{Z}, \; |a_{11}a_{22} - a_{12}a_{21}| = 1 \right\}.$$

Show that the set $G(m)$ forms a finite group under matrix multiplication.

Problem 16. The problem is to formulate the conjugacy criterion: given 2 elements A and B of the group G one can decide whether they are conjugated. According to definition they are conjugated if and only if there exists the third element X of the group G such that

$$B = XAX^{-1}.$$

In this case we say that A and B belong to the same conjugacy class. There is important necessary condition: conjugated elements A and B have the same order (prove this). So we must not further check the conjugacy, when orders are not equal. For the S_n group we have necessary and sufficient condition: conjugated elements A and B have the same cyclic structure (prove this). For the matrix groups we can obtain the conjugacy criterion by pure algebraic means. Give such criterion for the $GL(n, \mathbb{R})$.

Problem 17. Find all the standard *Young tableaux* of the finite group S_4. Derive the inequivalent irreducible representations of S_4 and give their dimensions.

Problem 18. Cosets of a subgroup H of a group G form a group under the induced operation of coset multiplication if and only if $gH = Hg$ for every $g \in G$. We can rewrite $gH = Hg$ as $H = g^{-1}Hg$ for all $g \in G$, where

$$g^{-1}Hg := \{\, g^{-1}hg \,:\, h \in H \,\}.$$

Let G be the group of the 3×3 permutation matrices. Let H be the subgroup given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

These three permutation matrices are all even. Their determinant is equal 1. They form the group A_3. Find $g^{-1}Hg$.

Problem 19. Consider the six 3×3 permutation matrices

$$P_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which form a group under matrix multiplication.
(i) Find the conjugacy classes.
(ii) Find the irreducible representations.
(iii) Write down the character table.

Problem 20. Find the group generated by the 4×4 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Write down the group table. Is the group abelian? Find all the subgroups. Find the eigenvalues of P and show that the form a group under multiplication.

Problem 21. Find the group generated by the 5×5 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

under matrix multiplication. Is the group abelian? Find all the subgroups. Find the eigenvalues of P and show that the form a group under multiplication.

Problem 22. (i) Find the group generated by the two Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

under matrix multiplication.
(ii) Find the group generated by the matrices

$$\sigma_1 \otimes \sigma_1, \qquad \sigma_3 \otimes \sigma_3.$$

Problem 23. Find the group generated by the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

under matrix multiplication. Give the group table. Find the conjugacy classes.

Problem 24. Find the finite group generated by the two permutation matrices

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

under matrix multiplication.

Problem 25. Do the matrices

$$\begin{pmatrix} \cosh(\alpha) & e^{i\phi}\sinh(\alpha) \\ e^{-i\phi}\sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$$

form a group under matrix multiplication?

Problem 26. Do the 4×4 matrices ($\alpha \in \mathbb{R}$)

$$A(\alpha) = \begin{pmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ 0 & \cosh\alpha & \sinh\alpha & 0 \\ 0 & \sinh\alpha & \cosh\alpha & 0 \\ \sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix}$$

form a group under matrix multiplication? Prove or disprove. First find the determinant of the matrix $A(\alpha)$.

Problem 27. Consider a field of characteristic 2 with the elements

$$\mathbb{F} = \{\, 0,\, 1,\, a,\, b \,\}.$$

and the compositions $+$ and $\cdot$ given by

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Do we have a group under the composition $+$? Prove or disprove. Do we have a group under the composition $\cdot$? Prove or disprove.

Problem 28. (i) Let $\alpha \in \mathbb{R}$. Is $x_1^2 + x_2^2$ invariant under the transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}?$$

(ii) Let $\alpha \in \mathbb{R}$. Is $x_1^2 + x_2^2$ invariant under the transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}?$$

Problem 29. The lemniscate of Gerono is given by

$$x^4 - x^2 + y^2 = 0.$$

Find all symmetry operations (i.e. functions $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$) that leaves the lemniscate of Gerono invariant, for example $x \to x$, $y \to -y$. Show that all these operations form a group under function composition. Give the group table. Is the group commutative?

Problem 30. Let I_n be the $n \times n$ unit matrix. Do the four $2n \times 2n$ matrices

$$\begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix}, \quad \begin{pmatrix} I_n & 0_n \\ 0_n & -I_n \end{pmatrix}, \quad \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}, \quad \begin{pmatrix} 0_n & I_n \\ -I_n & 0 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 31. Do the matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U(\phi) = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}.$$

form a group under matrix multiplication?

Problem 32. Consider the Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Do the sixteen 4×4 matrices ($j = 0, 1, 2, 3$)

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_j \end{pmatrix}, \quad \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ -\sigma_j & 0_2 \end{pmatrix}$$

form a group under matrix multiplication?
(ii) Do the sixteen 4×4 matrices ($j = 0, 1, 2, 3$)

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_j \end{pmatrix}, \quad \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ -\sigma_j & 0_2 \end{pmatrix}$$

form a Lie algebra under the commutator?

Problem 33. Suppose that an $n \times n$ group multiplication table is given. Check that this table satisfy group multiplication axioms.
- Describe the verification algorithm.
- Compose the verification program.

Problem 34. For given positive integer number n construct all possible $n \times n$ multiplication tables, which satisfy the group multiplication axioms. Additional restriction, which simplifies the problem: we seek only such tables, which contain in the first row and the first column the identical permutations of the group elements numbers.
- Describe the algorithm.
- Compose the program.

Problem 35. Let positive integer number n be a given group order. How many $n \times n$ multiplication tables satisfy group multiplication axioms?

Problem 36. A Platonic solid is defined as a regular, convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex.
In 3D space there are 5 Platonic solids:
-tetrahedron (4 faces, 6 edges, 4 vertices);
-cube or hexahedron(8 faces, 12 edges, 6 vertices);
-octahedron (6 faces, 12 edges, 8 vertices);
-dodecahedron (20 faces, 30 edges, 12 vertices);
-icosahedron (12 faces, 30 edges, 20 vertices).
For the groups of rotations of first 3 Platonic solids (or regular polyhedra) build the group multiplication tables. Use these tables as test examples for the following problems.

Problem 37. A group cycle graph pictures the cycles of a group. It is particularly useful in visualizing the structure of small groups. A cycle is the set of powers of a given group element g. Different cycles can overlap, or they can have no element in common but the identity. The cycle graph displays each interesting cycle as a polygon. We need to consider only the primitive cycles, those that are not subsets of another cycle. Each of these is generated by some primitive element a. Cycle graphs are generally drawn without a self-loop from the identity element to itself and without any implicit sub-cycles. Let $n \times n$ multiplication table satisfy group multiplication axioms. Build the cycle graph for this group.
- Describe the algorithm.
- Compose the program.

Problem 38. Let $n \times n$ multiplication table satisfy group multiplication axioms. Choose some elements of the group and combine them in all possible ways. We call these elements "generators". The resulting set of elements generally is some subgroup of original group. We try to choose the minimal set of generators in such a way, that the resulting set is original

group itself. One can formulate this problem in terms of geometry of some graph - the Cayley graph of the group. The vertices of the graph are the group elements. The edges from each vertex corresponds to the left product of some generator and the group element. So the graph is unidirected and the number of outgoing edges for each node is the same. It is also equal to the numbers of incoming edges, because generators has unique inverses. When there exists the path to each of group elements starting from the group unity, the set of generators is full. Build the Cayley graph for the group.
- Describe the algorithm.
- Compose the program.

Problem 39. For the groups of rotations of Platonic solids (or regular polyhedra) T and O build the Cayley graphs. Use these tables as test examples for the following problems.

Problem 40. Let us suppose that a Cayley graph is given. Check that this Cayley graph satisfies the group multiplication axioms.
- Describe the algorithm.
- Compose the program.

Problem 41. Suppose that a valid Cayley graph is given. Build the group multiplication table for this group.
- Describe the algorithm.
- Compose the program.

Problem 42. Let $n \times n$ multiplication table satisfy group multiplication axioms. Find all subgroups for this group.

Problem 43. Let an $n \times n$ multiplication table satisfy the group multiplication axioms. Find all normal subgroups for this group.

Problem 44. For the groups of rotational symmetries T and O of Platonic solids (or regular polyhedra) find a set of generators and a set of relations. Use these presentations as test examples for the following problems.

Problem 45. A presentation of a group is a description of the group by 2 sets: set Γ of generators ("group letters") and set R of "group words", called relators. One combines generators in all possible way and simplifies generated group words recursively using identities $r_i = I$ where $r_i \in R$ and I is the group unity. The task is to determine whether after simplification remains finite number of words, if so - to restore the multiplication table of the finite group. The Todd-Coxeter algorithm solves this problem. The algorithm uses 2 notions - chains and links. A chain is a symbol mWn, which represents the equation $mW = n$, where m and n are generators and W is a word. The length of a chain is the length of the word: the number of symbols, counting each generator or its inverse as a single symbol. A chain mWn is a link if it has length 1, i.e. W is a generator or inverse to generator. The links are used to build up a table, called the Link Table:

	A	B	C	...
0	...	...	...	...
1	...	...	...	...
2	...	...	...	...
...	...	...	...	...

Each row of the table corresponds to some numbered group element. We write this number in the first column of the table. Each column of the table corresponds to some generator. We write this generator name in the first row of the table. In the intersections cell we write the result of the right application of the generator to the group element (for example e_3A) in terms of other numbered group elements. The full table contains in all its columns a permutation of row numbers $0, 1, \ldots$. Initially the table is empty and the list of chains is also empty. Adding new numerated element n (the next in sequence, 0 is the group unity element I) we perform following actions:
-add to chain list all chains nr_in (because $r_i = I$);
-fill in n to some empty cell of the Link table. This rule explains genesis of newborn group elements: they denote the right action of some generator on the some existing group element. There are several strategies for this assignment, see discussion below.
Then we try to simplify existing chains using simplification rules:
- if the chain starts from nA and exists the link nAm with the same head, we nA with m in the head of the chain, because the link nAm means $nA = m$.
- if the chain ends by Am and exists link nAm with the same tail, we replace Am with n in the tail of the chain. For if the chain is αAm then this means $\alpha A = m$. The link nAm means $m = nA$, thus $\alpha A = nA$ and we can write $\alpha = n$, q.e.d. When as a result of simplifications series some chain is a link, we exclude it from the list of chains and transfer to the link table. When no further simplifications are possible - we add new numerated element $n + 1$. This step is impossible to trait algorithmically, because all fixed rules of adding new element appears to be insufficient, so it is better to let user do it by hand. The process ends when the Link table is full, this is success. The last stage after that is to restore the multiplication table. Another possible way of process termination is when we try to place the new link in the link table and its appears, that the cell of the table is filled with another value. This must not necessarily mean the error, because it is possible, that we

assigned another number to some existing element. This event signals, that we need to consider once again previous "by hand" steps from this point of view. This problems task is to implement the Todd-Coxeter algorithm.

Problem 46. Rubik square. The Rubik square is a 3×3 square with fixed central (coordinated as $(2,2)$) cell and other numbered cells moving according following rules. Namely, the generators of the Rubik square group are:
- r_1 - the cyclic permutation of the first row;
- r_3 - the cyclic permutation of the third row;
- c_1 - the cyclic permutation of the first column;
- c_3 - the cyclic permutation of the third column.

Thus the rules are slightly similar to the famous "15 puzzle". Our question is inspired by "15 puzzle" question: is it possible to reach an arbitrary position of numbered cells starting from the fixed initial position ?

Problem 47. Liquid square. The Liquid square is a 3×3 square with cell moving according following rules. The generators of the Liquid square group are only those permutations of cells, for which every cell moves to its nearest neighbor place (that is one step to North, West, South or East) or remains in rest. We call this kind of movement "Liquid square" because it provides a discrete model of incompressible fluid motion. Provide the list of generators. Is it possible to reach an arbitrary position of numbered cells starting from the fixed initial position ? .

Problem 48. The roots of equation

$$z^5 - 1 = 0.$$

are: unity ($z_0 = 1$)and powers $z_k = \epsilon^k, k = 1..4$ of $\epsilon = \exp(2\pi i/5)$. Consider the action on this point set of the transformation group

$$T_m(z) = z^m$$

with group multiplication

$$T_m T_n = T_{mn \bmod 5}.$$

Find invariant sets for T_i. Observe, that z_0 and $\{z_1, z_2, z_3, z_4\}$ are invariant sets of all T_i. This corresponds to decomposition

$$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + 1).$$

Observe that only T_4 has smaller invariant sets. Construct sums of roots on these invariant sets and show that these sums satisfy some quadratic equation with integer coefficients. One of this sums is equal to $2\cos(2\pi/5)$. This proves possibility to construct the regular pentagon using the ruler and compass.

Problem 49. The roots of equation

$$z^{17} - 1 = 0.$$

are: unity ($z_0 = 1$)and powers $z_k = \epsilon^k, k = 1..16$ of $\epsilon = \exp(2\pi i/17)$. Consider the action on this point set of the transformation group

$$T_m(z) = z^m$$

with group multiplication

$$T_m T_n = T_{mn \bmod 17}.$$

Find invariant sets for T_i. Observe that z_0 and $\{z_1, \ldots z_{16}\}$ are invariant sets of all T_i. This corresponds to decomposition

$$z^{17} - 1 = (z-1)(z^{16} + z^{15} + \ldots + z^2 + 1).$$

Observe that T_2 (and also T_8, T_9, T_{15}) has 2 smaller invariant sets A_1, A_2, each set contains 8 roots. Show, that sums of elements of these sets satisfy some quadratic equation with integer coefficients, construct this equation. Observe that one of invariant sets of T_4 is a proper subset of A_1 with 4 elements (sets B_1 and B_2). Construct the quadratic equation with integer coefficients for sums of roots on these subsets. Observe that one of invariant sets of T_{16} is a proper subset of B_1 with 2 elements - z_1 and z_{16}. Construct the quadratic equation with integer coefficients for sums of roots on this subset C_1 and its relative to B_1 complement C_2. Finally, this proves possibility to construct the regular 17 - gon (heptadecagon) using the ruler and compass. This is the famous discovery by K. F. Gauss in 1796 at the age of 19. To construct the quadratic equation on each step find the sum of roots, the product and determine coefficients of the equation.

Problem 50. Show that there are only two different fourth order groups. The (commutative) cyclic group and the fourth order inversion group.

Problem 51. Show that there are only two different sixth-order groups. The (commutative) cyclic group and the symmetric group D_3 of a regular triangle.

Problem 52. (i) Find the group generated by the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

under matrix multiplication. Note that $A^2 = B^2 = I_2$ and

$$AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(ii) Find the group generated by the 3×3 matrices

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

under matrix multiplication. Note that $X^2 = Y^3 = I_3$.

Chapter 2

Lie Groups

Problem 1. Consider the compact Lie group $SO(2,\mathbb{R})$ with the element

$$A(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

(i) Let $\oplus$ be the direct sum. Is $A(\alpha)\oplus(1)$ an element of $SO(3,\mathbb{R})$? Is $(1)\oplus A(\alpha)$ an element of $SO(3,\mathbb{R})$?
(ii) Is

$$\begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

an element of $SO(3,\mathbb{R})$?
(iii) Is $(A(\alpha)\oplus(1))((1)\oplus B(\beta))$ an element of $SO(3,\mathbb{R})$, where

$$B(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}\,?$$

Problem 2. Consider the compact Lie groups $SO(3,\mathbb{R})$ and $SO(4,\mathbb{R})$. $SO(4,\mathbb{R})$ has dimension 6 and $SO(3,\mathbb{R})$ has dimension 3. Let A and B be elements of $SO(3,\mathbb{R})$.
(i) Let $\oplus$ be the direct sum. Can any element C of $SO(4,\mathbb{R})$ be written as

$$C = (1)\oplus A + B \oplus (1)\,?$$

Obviously all the elements $A \oplus (1)$ form a sub group of $SO(4,\mathbb{R})$. Analogously all the elements $(1) \oplus (A)$ form a sub group of $SO(4,\mathbb{R})$.
(ii) Find the factor group $SO(4,\mathbb{R})/((1) \oplus SO(3,\mathbb{R}))$.

Problem 3. Let $R \in SO(3)$, $L \in SL(2,\mathbb{R})$, T a 3×2 matrix over $\mathbb{R}$ and $0_{2\times 3}$ the 2×3 unit matrix. We form the 5×5 matrix

$$\begin{pmatrix} R & T \\ 0_{2\times 3} & L \end{pmatrix}.$$

Do these matrices form a group under matrix multiplication?

Problem 4. Show that the Lie group homomorphism (i.e. a differentiable group homomorphism) $\mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$, $t \mapsto (e^{2\pi i t}, e^{2\sqrt{2}\pi i t})$ has a dense image. Use $\mathbb{S}^1 \times \mathbb{S}^1 \cong \mathbb{R}^2/\mathbb{Z}^2$.

Problem 5. Every matrix in the Lie group $SL(2,\mathbb{R})$ can be uniquely written in the form $(a, b, \phi \in \mathbb{R})$

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \exp \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Hence the topological group $SL(2,\mathbb{R})$ is equivalent to $\mathbb{T} \times \mathbb{R}^2$. Write the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

in this form, i.e. find ϕ, a and b.

Problem 6. A volume form on a differentiable manifold is a nowhere-vanishing differential form ω of top degree. This volume form defines a measure on the Borel sets U by

$$\mu(U) = \int_U \omega.$$

For any Lie group a natural volume form may be defined in such a way that it is invariant of the left group action. This volume form is unique up to an arbitrary constant multiplier and the corresponding measure is known as the Haar measure. Let us write the group law for the left multiplication case in the following form:

$$G_c = G_b G_a$$

where
$a_i, i = 1, \dots n$ are parameters of the group element G_a;

$b_i, i = 1, \dots n$ are parameters of the group element G_b;
$c_i, i = 1, \dots n$ are parameters of the group element G_c;

$$c_s = \phi_s(b_1, b_2, \dots b_n; a_1, a_2, \dots, a_n).$$

Let us denote the Jacobian determinant

$$\Delta(G_b, G_a) = \det \left| \frac{\partial c_i}{\partial a_j} \right|$$

Then we have

$$\Delta(E, G_a) = 1$$

$$\Delta(G_b G_a, E) = \Delta(G_b, G_a)\Delta(G_a, E).$$

Let us denote

$$\delta(G_a) = \Delta(G_a, E), \quad \delta(E) = \Delta(E, E) = 1.$$

This implies

$$\Delta(G_b, G_a) = \frac{\delta(G_b G_a)}{\delta(G_a)}.$$

It follows

$$\Delta(G_a^{-1}, G_a) = \frac{1}{\delta(G_a)}.$$

This last quantity we accept as the definition of left invariant volume measure

$$u(G_a) = \Delta(G_a^{-1}, G_a) = \frac{1}{\delta(G_a)} = \frac{1}{\Delta(G_a, E)}.$$

(i) Let $f(G_a)$ be any continuous function. Consider the integral on the whole group space

$$\int_G f(G_a) dG_a := \int_V f(a_i) u(G_a) da_1 da_2 \dots da_n$$

and show that it is left invariant

$$\int_G f(G_a) dG_a = \int_G f(G_b G_a) dG_a$$

or in coordinates

$$\int_V f(a_i) u(G_a) da_1 da_2 \dots da_n = \int_V f(c_i) u(G_a) da_1 da_2 \dots da_n.$$

(ii) Using this method find invariant measures for the following matrix Lie groups:
- the special orthogonal group $SO(2)$

- the group of affine linear transformations of the real line $\mathrm{Aff}(1,\mathbb{R})$
- special linear group $SL(2,\mathbb{R})$.

Problem 7. The *translation group* T_v in $\mathbb{R}^3$ is a three dimensional Lie group with the group multiplication law

$$g_1 g_2 = (a,b,c) \bullet (x,y,z) := (a+x, b+y, c+z), \qquad (x,y,z) \in \mathbb{R}^3.$$

Show that this composition is a valid group law. Find the group unity and the inverse element expression.

Problem 8. Let $z \in \mathbb{C}$. Let A, B be $n \times n$ matrices over $\mathbb{C}$. We say that B is invariant with respect to A if

$$e^{zA} B e^{-zA} = B.$$

Obviously e^{-zA} is the inverse of e^{zA}. Show that, if this condition is satisfied, one has $[A,B] = 0_n$, where 0_n is the $n \times n$ zero matrix. If e^{zA} would be unitary we have $UBU^* = B$.

Problem 9. Let $z \in \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA)B\exp(-zA)$.
(ii) Calculate the commutator $[A,B]$.

Problem 10. Let $z \in \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA)B\exp(-zA)$.
(ii) Calculate the commutator $[A,B]$.

Problem 11. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Find the skew-hermitian matrices Σ_1, Σ_2, Σ_3 such that

$$\sigma_1 = \exp(\Sigma_1), \quad \sigma_2 = \exp(\Sigma_2), \quad \sigma_3 = \exp(\Sigma_3).$$

Find the commutators $[\Sigma_1, \Sigma_2]$, $[\Sigma_2, \Sigma_3]$, $[\Sigma_3, \Sigma_1]$ and compare with the commutators $[\sigma_1, \sigma_2]$, $[\sigma_2, \sigma_3]$, $[\sigma_3, \sigma_1]$.

Problem 12. Show that the surface ∂C of the unit cube

$$C = \{(x_1, x_2, x_3) : 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1\}$$

can be made into a differentiable manifold.

Problem 13. A complex analytic group G is a group which is also a complex analytic manifold, such that the group multiplication $(g_1, g_2) \mapsto g_1 g_2 : G \times G \to G$ and the inversion $g \mapsto g^{-1} : G \to G$ are complex analytic. A vector field V on a complex analytic group G is called left-invariant if it satisfies

$$dL_{g_1}(V_{g_2}) = V_{g_1 g_2})$$

for all $g_1, g_2 \in G$ and L_{g_1} denotes the left translation $g_2 \mapsto g_1 g_2$. Show that any left-invariant vector field V on a complex analytic group G is complex analytic.

Problem 14. The Lie group $SL(2, \mathbb{C})$ consists of the complex matrices with determinant equal to 1. Give a compact and non-compact Lie subgroup of $SL(2, \mathbb{C})$.

Problem 15. Let $z \in \mathbb{C}$. Construct all 2×2 matrices A and B over $\mathbb{C}$ such that

$$\exp(zA) B \exp(-zA) = e^{-z} B.$$

Problem 16. Show that each representation of a compact Lie group G on an n-dimensional real vector space V is equivalent to an orthogonal representation of G on $\mathbb{R}^n$.

Problem 17. Show that the maximal compact subgroup of $SL(3, \mathbb{R})$ is $SO(3, \mathbb{R})$.

Problem 18. Consider the 2×2 matrices over $\mathbb{R}$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

i.e. the matrices are the elements of the Lie group $SL(2, \mathbb{R})$. Consider the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} (a^2 - b^2 - c^2 + d^2)/2 & bd - ac & (a^2 - b^2 + c^2 - d^2)/2 \\ cd - ab & ad + bc & -cd - ab \\ (a^2 + b^2 - c^2 - d^2)/2 & -bd - ac & (a^2 + b^2 + c^2 + d^2)/2 \end{pmatrix}.$$

Does the 2×2 identity matrix maps into the 3×3 identity matrix? Find the map of

$$i\sigma_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Is the trace and determinant preserved?

Problem 19. Consider the Lie group $SU(2)$. Then

$$T_1 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cong S^1$$

is an element of $SU(2)$. Now

$$T_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is also an element of $SU(2)$, but also of $SO(2)$. Are T_1 and T_2 are conjugate in $SU(2)$, i.e. can one find a unitary matrix g in $SU(2)$ such that $gT_1g^{-1} = T_2$?

Problem 20. Show that

$$SU(2) \otimes SU(2) \cong SO(4) \;\Leftrightarrow\; su(2) \otimes I_2 + I_2 \otimes su(2) \cong so(4).$$

Problem 21. Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the *Heisenberg group* with the multiplication law

$$(\mathbf{z}, t) \circ (\zeta, \tau) = (\mathbf{z} + \zeta, t + \tau - \frac{1}{2}\Im(\mathbf{z} \cdot \zeta)$$

Let $n = 1$ and $z = e^{i\phi}$, $\zeta = e^{-i\phi}$, $t = 1$, $\tau = -1$. Find $(z, t) \circ (\zeta, \tau)$.

Chapter 3

Lie Algebras

Problem 1. Let L be a finite-dimensional Lie algebra over a field of characteristic 0 ($\mathbb{R}$, $\mathbb{C}$). Show that the following conditions are equivalent.
(i) L is semisimple
(ii) The Killing form $\kappa(x,y) := \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ is non-degenerate, i.e.

$$\det(\kappa(x,y)) \neq 0.$$

Here ad denotes the adjoint representation.

Problem 2. Let L be a Lie algebra and $R = \mathrm{rad}L \neq L$ its radical. Show that the quotient Lie algebra L/R is semisimple.

Problem 3. Show that if L is a solvable Lie algebra, then the sub Lie algebra $[L,L]$ is nilpotent.

Problem 4. Show that the vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial x},$$

$$x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial}$$

form a Lie algebra under the commutator. They are related to the projective group.

Problem 5. (i) Let L be a non-abelian N-dimensional Lie algebra. Let $N > 2$ and let x be an arbitrary element of L. How do we find another element y of L such that these two elements form Lie subalgebra of L? Propose an algorithm.
(ii) Consider $g\ell(2,\mathbb{C})$ as an example. Take as given vector first basis vector for diagonal matrix entry, then - for nondiagonal matrix entry.

Problem 6. Let L be N-dimensional Lie algebra. Let $N > 2$ and let P be its Lie r-dimensional subalgebra of L. How do we add to P two elements X and Y in such a way that there union forms Lie $r+2$-dimensional subalgebra Q of L? What happens when the number of unknown variables is less then number of equations? Find condition in terms of r and N. Discuss the case when this condition is not valid.

Problem 7. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $[A,B] = 0$.
(i) Can we conclude that $[e^A, e^B] = 0$?
(ii) Can we conclude that $[\sin(A), \cos(B)] = 0$?

Problem 8. Let A, B be 2×2 matrices over $\mathbb{R}$. Assume we know commutator $C = [A,B]$ and the anti-commutator $D = [A,B]_+$. Can we reconstruct A, B from C, D?

Problem 9. Let A, B be $n \times n$ matrices. Consider the map

$$(A,B) \longmapsto A \otimes I_n + I_n \otimes B + A \otimes B\,.$$

Thus

$$(B,A) \longmapsto B \otimes I_n + I_n \otimes A + B \otimes A\,.$$

What is condition on A and B so that the commutator

$$[A \otimes I_n + I_n \otimes B + A \otimes B, B \otimes I_n + I_n \otimes A + B \otimes A]$$

vanishes?

Problem 10. Consider the non-commutative two-dimensional Lie algebra with $[A,B] = A$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that the matrices

$$\{\, A \otimes I_2 + I_2 \otimes A,\;\; B \otimes I_2 + I_2 \otimes B \,\}$$

also form a non-commutative Lie algebra under the commutator. Discuss.

Problem 11. Consider the Lie algebra $s\ell(2,\mathbb{R})$ with

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the commutators $[x,h] = -2x$, $[x,y] = h$, $[y,h] = 2y$. Consider

$$h \mapsto h \otimes I_2 + I_2 \otimes h, \quad x \mapsto x \otimes h + h \otimes x, \quad y \mapsto y \otimes h + h \otimes y.$$

(i) Find the commutators

$$[h\otimes I_2 + I_2 \otimes h, x\otimes h + h \otimes x], \quad [h\otimes I_2 + I_2\otimes h, y\otimes h + h\otimes y], \quad [x\otimes h + h\otimes x, y\otimes h + h\otimes y]$$

and thus show that we have a Lie algebra.
(ii) Is this Lie algebra isomorphic to $s\ell(2,\mathbb{R})$?

Problem 12. (i) Show that the Lie algebras $su(1,1)$ and $s\ell(2,\mathbb{R})$ are isomorphic.
(ii) Show that the Lie algebras $s\ell(2,\mathbb{C})$ and $so(1,3)$ are isomorphic.

Problem 13. The four 2×2 matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis of the Lie algebra $g\ell(2,\mathbb{R})$. Is the Lie algebra semisimple?

Problem 14. Let L be a Lie algebra. Given $H, K \subset L$. We define their bracket by

$$[H,K] := \text{span}\{\, [h,k] \,:\, h \in H,\ k \in K \,\}.$$

The derived Lie algebra of L is the ideal $[L,L]$. Show that the quotient Lie algebra $L/[L,L]$ is an abelian Lie algebra.

Problem 15. Show that the Lie algebra consisting of $n\times n$ matrices over $\mathbb{R}$ (or $\mathbb{C}$) with zeros below the main diagonal is solvable but not nilpotent.

Problem 16. Let L be a Lie algebra. Given $H, K \subset L$. We define their bracket by

$$[H,K] := \text{span}\{\, [h,k] \,:\, h \in H,\ k \in K \,\}.$$

The derived Lie algebra of L is the ideal $[L,L]$. Show that the quotient Lie algebra $L/[L,L]$ is an abelian Lie algebra.

Problem 17. Consider the Lie algebra $A_3 = s\ell(4,\mathbb{C})$. Given the roots, find the structure constants. The Lie algebra A_3 ($= s\ell(4,\mathbb{C})$) has rank 3 and dimension 15. The root system of A_3 consists of 12 four-dimensional vectors. Each vector has one $(+1)$ coordinate, one (-1) coordinate and the rest coordinates are zero. 6 roots are called positive - the number of $(+1)$ coordinate is less then number of (-1) coordinate. The rest are called negative. Check the Jacobi identities for resulting structure constants.

Problem 18. Let L be a finite dimensional Lie algebra. Let $R = \mathrm{rad}L$ be its radical. Then there exists a semisimple Lie subalgebra S of L such that L is the direct sum of its linear subspace R and S, i.e.

$$L = R \oplus S.$$

This is the *Levi decomposition theorem.* Apply it to the Lie algebra $L = g\ell(2,\mathbb{R})$ with the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 19. Consider the semisimple Lie algebra $s\ell(n+1,\mathbb{F})$. Let $E_{i,j}$ $(i,j \in \{1,2,\ldots,n+1\})$ denote the standard basis, i.e. $(n+1)\times(n+1)$ matrices with all entries zero except for the entry in the i-th row and j-th column which is one. We can form a Cartan-Weyl basis with

$$H_j := E_{j,j} - E_{j+1,j+1}, \qquad j \in \{1,2,\ldots,n\}.$$

Show that $E_{i,j}$ are *root vectors* for $i \neq j$, i.e. there exists $\lambda_{H,i,j} \in \mathbb{F}$ such that

$$[H, E_{i,j}] = \lambda_{H,i,j} E_{i,j}$$

for all $H \in \mathrm{span}\{\, H_1, \ldots,\, H_n\}$.

Problem 20. A Lie algebra L is called simple if L is non-abelian and has no proper ideals. Show that for a simple Lie algebra the derived Lie algebra $L^1 = [L,L]$ is equal to L, i.e. $L^1 = L$.

Problem 21. A Lie algebra L is called semisimple if $L \neq 0$ and L has no abelian ideals $\neq 0$. Show that a finite-dimensional Lie algebra L is semisimple if and only if $\mathrm{rad}L = 0$.

Problem 22. Study the Lie algebra $s\ell(2,\mathbb{F})$, where $\mathrm{char}\mathbb{F} = 2$.

Problem 23. Let L be a Lie algebra and $R = \mathrm{rad}L \neq L$ its radical. Show that the quotient Lie algebra L/R is semisimple.

Problem 24. Let L be a finite-dimensional Lie algebra and $R = \mathrm{rad}L$ its radical. Then there exits a semisimple Lie subalgebra K of L such that L is the direct sum of its linear subspaces R and K, i.e.

$$L = R \oplus K.$$

This is *Levi's theorem.* Apply the theorem to the Lie algebra $g\ell(2, \mathbb{R})$.

Problem 25. The semisimple Lie algebra $s\ell(3, \mathbb{R}$ has dimension 8. The standard basis is given by

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_{13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the commutator table.

Problem 26. Let L be a finite-dimensional Lie algebra over a field $\mathbb{F}$. The linear map $K : L \times L \to \mathbb{F}$ defined by

$$K(a, b) := \mathrm{tr}(\mathrm{ad}(a)\mathrm{ad}(b))$$

is called the Cartan-Killing form or Killing form of L, where tr denotes the trace. Show that

$$K([a, b], c) = K(a, [b, c]).$$

Problem 27. Consider the Cartan-Killing form of the finite-dimensional linear Lie algebra $g\ell(n, \mathbb{R})$. Show that

$$K(X, Y) = \mathrm{tr}((\mathrm{ad}X)(\mathrm{ad}Y)) = 2n\mathrm{tr}(XY) - 2(\mathrm{tr}X)(\mathrm{tr}Y).$$

Problem 28. Let $n \in \mathbb{N}$. Consider the linear differential operators

$$J_+ = x^2 \frac{d}{dx} - nx, \qquad J_0 = x\frac{d}{dx} - \frac{1}{2}n, \qquad J_- = \frac{d}{dx}.$$

(i) Show that these linear differential operators obey the simple Lie algebra $s\ell(2,\mathbb{R})$ commutation relations.
(ii) Find J_+x^n, J_0x^n, J_-x^n. Disucss.

Problem 29. Consider the differential operators

$$L_1 = -i\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} - i\sin\alpha\frac{\partial}{\partial\beta} - i\frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial r} + \frac{i}{2}\sin\alpha\cot\beta$$
$$L_2 = -i\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} + i\cos\alpha\frac{\partial}{\partial\beta} - i\frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial r} - \frac{i}{2}\cos\alpha\cot\beta$$
$$L_3 = -i\frac{\partial}{\partial\alpha}.$$

Find the commutators $[L_1, L_2]$, $[L_2, L_3]$, $[L_3, L_1]$.

Problem 30. Let $q \geq 1$ and $\gamma = \ln(q)$. The deformed quantum algebra is generated by the operators J_+, J_- and J_z obeying the commutation relations

$$[J_z, J_+] = J_+, \quad [J_z, J_-] = -J_-, \quad [J_+, J_-] = \frac{\sinh(\gamma J_z)}{\sinh(\gamma/2)}.$$

For $\gamma \to 0$ we obtain the Lie algebra $su(2)$ with $[J_+, J_-] = 2J_z$. Consider the operators

$$J_x = \frac{1}{2}(J_+ + J_-), \qquad J_y = \frac{1}{2i}(J_+ - J_-)$$

and J_z. Find the commutators

$$[J_z, J_x], \quad [J_z, J_y], \quad [J_x, J_y].$$

Problem 31. (i) The (1+1) Poincaré Lie algebra $iso(1,1)$ is generated by one boost generator K and the translation generators along the light-cone P_+ and P_-. The commutators are

$$[K, P_+] = 2P_+, \quad [K, P_-] = -2P_-, \quad [P_+, P_-] = 0.$$

Can one find 2×2 matrices K, P_+, P_- which satisfy these commutation relations?
(ii) Let z be the deformation parameter. The Hopf structure of $U_z iso(1,1)$ is given by the coproduct Δ, co-unit ϵ and antipode γ

$$\Delta(P_+) = I\otimes P_+ + P_+\otimes I, \quad \Delta(P_-) = e^{-zP_+}\otimes P_- + P_-\otimes e^{zP_+}, \quad \Delta(K) = e^{-zP_+}\otimes K + K\otimes e^{zP_+}$$

$$\epsilon(K) = \epsilon(P_+) = \epsilon(P_-) = 0,$$

$$\gamma(K) = -e^{zP_+} K e^{-zP_+}, \quad \gamma(P_+) = -e^{zP_+} P_+ e^{-zP_+}, \quad \gamma(P_-) = -e^{zP_+} P_- e^{-zP_-}.$$

Find the commutators

$$[\Delta(K), \Delta(P_+)], \quad [\Delta(K), \Delta(P_-)], \quad [\Delta(P_+), \Delta(P_-)].$$

Problem 32. Consider the differential operators

$$K_+ = e^{i\phi}\left(-i\cos(\theta)\frac{\partial}{\partial\phi} + \sin(\theta)\frac{\partial}{\partial\theta}\right) + e^{i\phi}\cos(\theta), \qquad K_- = -e^{-i\phi}\left(i\cos(\theta)\frac{\partial}{\partial\phi} + \sin(\theta)\frac{\partial}{\partial\theta}\right)$$

and

$$K_3 = -i\frac{\partial}{\partial\phi} + \frac{1}{2}.$$

Find

$$[K_+, K_-], \qquad [K_3, K_+], \quad [K_3, K_-].$$

Problem 33. Show that the matrices

$$J_+ = \begin{pmatrix} 0 & 1 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & y \\ 0 & -1 & -x \\ 0 & 0 & 0 \end{pmatrix}.$$

form a basis of a Lie algebra. Classify the Lie algebra.

Problem 34. Let f be a smooth function. Find the commutators

$$[x, d^2/dx^2]f(x), \qquad [1/x, d^2/dx^2]f(x).$$

Problem 35. The Lie algebra $SO(6)$ is the real Lie algebra of all 6×6 real antisymmetric matrices. Show that a basis is given by

$$M_{jk} := E_{jk} - E_{kj}, \qquad j, k = 1, \ldots, 6$$

where E_{jk} is the 6×6 matrix with 1 at the entry jk and zero otherwise.

Problem 36. The oscillator Lie algebra h_4 is generated by the basis $\{N, A_+, A_-, M\}$ with the commutators

$$[N, A_+] = A_+, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = M$$

$$[M, N] = 0, \quad [M, A_+] = 0, \quad [M, A_-] = 0.$$

(i) Show that a matrix representation of h_4 is given by

$$R(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad R(A_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R(A_-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad R(M) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the 3×3 matrix

$$\exp(mR(M)) \exp(a_- R(A_-)) \exp(a_+ R(A_+)) \exp(nR(N)).$$

(ii) Show that a representation of h_4 with vector fields is given by

$$V_N = \frac{\partial}{\partial N}, \quad V_{A_+} = e^N \frac{\partial}{\partial A_+}, \quad V_{A_-} = e^{-N} \frac{\partial}{\partial A_-} - A_+ e^{-N} \frac{\partial}{\partial M}, \quad V_M = \frac{\partial}{\partial M}.$$

Problem 37. The Lie algebra $o(2,\mathbb{R})$ of the Lie group $O(2,\mathbb{R})$ is is abelian (one-dimensional). Show that Lie group $O(2,\mathbb{R})$ is not abelian.

Problem 38. Show that $sp(4,\mathbb{R}) \simeq o(3,2)$.

Problem 39. Show that the semi-simple Lie algebras $so(2,1)$ and $su(1,1)$ are isomorphic.

Problem 40. Consider the Lie algebras $o(3,\mathbb{C})$, $o(4,\mathbb{C})$, $o(5,\mathbb{C})$, $o(6,\mathbb{C})$, $s\ell(2,\mathbb{C})$, $s\ell(4,\mathbb{C})$, $sp(4,\mathbb{C})$. Show that

$$o(3,\mathbb{C}) \simeq s\ell(2,\mathbb{C}), \quad o(4,\mathbb{C}) \simeq s\ell(2,\mathbb{C}) \times s\ell(2,\mathbb{C}),$$

$$o(5,\mathbb{C}) \simeq sp(4,\mathbb{C}), \quad o(6,\mathbb{C}) \simeq s\ell(4,\mathbb{C}).$$

Problem 41. Show that the Lie algebra $so(6,\mathbb{C})$ is isomorphic to $s\ell(4,\mathbb{C})$.

Problem 42. Do the three operators

$$S_3 = -x\frac{d}{dx} + \ell, \quad S_- = x, \quad S_+ = -x\frac{d^2}{dx^2} + 2\ell\frac{d}{dx}$$

form a basis of a Lie algebra under the commutator?

Problem 43. Show that the Heisenberg algebra is a non-compact, non-semisimple, but solvable and nilpotent algebra.

=

Problem 44. Show that the vector fields

$$\frac{\partial}{c\partial t}, \quad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}$$

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}, \quad x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}, \quad x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}$$

$$t\frac{\partial}{\partial x_1} + \frac{x_1}{c^2}\frac{\partial}{\partial t}, \quad t\frac{\partial}{\partial x_2} + \frac{x_2}{c^2}\frac{\partial}{\partial t}, \quad t\frac{\partial}{\partial x_3} + \frac{x_3}{c^2}\frac{\partial}{\partial t}$$

form a basis of a Lie algebra under the commutator.

Problem 45. Show that the vector fields

$$\frac{\partial}{c\partial t}, \quad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}$$

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}, \quad x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}, \quad x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}$$

$$t\frac{\partial}{\partial x_1}+\frac{x_1}{c^2}\frac{\partial}{\partial t}-qx_1u\frac{\partial}{\partial u}, \quad t\frac{\partial}{\partial x_2}+\frac{x_2}{c^2}\frac{\partial}{\partial t}-qx_2u\frac{\partial}{\partial u}, \quad t\frac{\partial}{\partial x_3}+\frac{x_3}{c^2}\frac{\partial}{\partial t}-qx_3u\frac{\partial}{\partial u}$$

$$u\frac{\partial}{\partial u}$$

form a basis of a Lie algebra under the commutator, where q is a nonzero constant.

Problem 46. The operators L_n ($n \in \mathbb{Z}$) satisfy the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}I$$

where I is the identity operator. Find $[L_0, L_0]$, $[L_1, L_{-1}]$, $[L_n, L_{-n}]$.

Problem 47. Let $m, n \in \mathbb{Z}$. Let L_n be the generator of the Virasoro algebra with the commutator

$$[L_m, L_n] = (m-n)L_{m+n} + c(m(m^2-1)/12)\delta_{m,-n}.$$

Find $[L_3, L_{-3}]$.

Problem 48. Let $m, n \in \mathbb{Z}$. The *Virasoro algebra* is the infinite-dimensional Lie algebra with basis elements C, L_n satisfing the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m,-n}C, \quad [L_m, C] = 0$$

The operator C commuates with all elements of L_m. It is called the central element of the algebra.
(i) Show that

$$L_m = e^{-mt}\frac{d}{dt}$$

and $C = 0$ is a realization of the algebra.
(ii) Show that

$$L_m = e^{-mt}\frac{\partial}{\partial t} + e^{-mt}\left(m + \frac{1}{2}m(m-1)\alpha e^{-x}\right)\frac{\partial}{\partial x}$$

and $C = 0$ is a realization of the algebra where $\alpha = 0, \pm 1$.

Problem 49. (i) Show that the differential operators

$$D_1 = \frac{1}{2}\left(\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial}{\partial\zeta}\right) - \frac{c^2}{4\zeta} + \zeta\right)$$
$$D_2 = i\left(\zeta\frac{\partial}{\partial\zeta} + \frac{1}{2}\right)$$
$$D_3 = -\frac{1}{2}\left(\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial}{\partial\zeta}\right) - \frac{c^2}{4\zeta} - \zeta\right)$$

form a basis of the Lie algebra $so(2,1)$.
(ii) Find the Casimir operator $D^2 = D_1^2 + D_2^2 + D_3^2$.

Problem 50. Do the vector fields

$$V_1 = i\cos(\gamma)\left(\cot(\theta)\frac{\partial}{\partial\gamma} - \mathrm{cosec}(\theta)\frac{\partial}{\partial\phi}\right) + i\sin(\gamma)\frac{\partial}{\partial\theta}$$
$$V_2 = i\sin(\gamma)\left(\cot(\theta)\frac{\partial}{\partial\gamma} - \mathrm{cosec}(\theta)\frac{\partial}{\partial\phi}\right) + i\cos(\gamma)\frac{\partial}{\partial\theta}$$
$$V_3 = i\frac{\partial}{\partial\gamma}.$$

form a basis of a Lie algebra under the commutator?

Problem 51. Consider the vector space of analytic functions. Find the commutators of

$$\sinh(x), \quad \cosh(x), \quad \frac{d}{dx}.$$

Problem 52. Consider the differential operators

$$S = \frac{1}{2}x\left(-\frac{d^2}{dx^2}+1\right), \quad T = -ix\frac{d}{dx}, \quad U = \frac{1}{2}x\left(-\frac{d^2}{dx^2}-1\right).$$

Find the commutators. Discuss. Find $C = S^2 - T^2 - U^2$ (Casimir operator).

Chapter 4

Applications

Problem 1. Denote by $Vect(S^1)$ the set of smooth vector fields on the unity circle S^1. The Virasoro algebra vir is a vector space $\mathrm{Vect}(S^1) \oplus \mathbb{R}$ equipped with the commutator

$$\left[\left(f(x)\frac{\partial}{\partial x}, a\right), \left(g(x)\frac{\partial}{\partial x}, b\right)\right] := \left((f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x}, \int_{S^1} f'(x)g''(x)dx\right).$$

Show that the commutator is bilinear and antisymmetric.

Problem 2. Consider the group of diffeomorphisms of S^1. This is infinite group with composition as a group law. Its Lie algebra is the set of smooth vector fields on the unity circle S^1 equipped with the commutator

$$\left[f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right] = (f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x}.$$

We denote this set by $\mathrm{Vect}(S^1)$. Let us introduce a positive quadratic form ("energy")

$$H\left(f(x)\frac{\partial}{\partial x}\right) = 1/2\left(\int_{S^1}(f(x))^2dx\right).$$

The nonviscous Burgers equation on the unit circle is the evolutionary equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}.$$

Show that nonviscous Burgers equation is the Euler's equation of geodesic flows on the $\mathrm{Diff}(S^1)$ group (with the right-invariant metric H).

Problem 3. Let $\widetilde{S}U(1,1)$ be the covering Lie group of $SU(1,1)$. A basis of the Lie algebra of $\widetilde{S}U(1,1)$ is given by

$$[J_a, J_b] = i\sum_{c=1}^{3} \epsilon_{abc}\lambda_c J_c, \qquad a,b,c = 1,2,3 \quad \lambda_1 = \lambda_2 = -\lambda_3 = 1$$

with the Casimir operator

$$J_3^2 - J_1^2 - J_2^2 = \mu^2 - \frac{1}{4}.$$

We find that J_2 and $J_1 + J_3$ form the two-dimensional sub Lie algebra

$$[J_2, J_1 + J_3] = i(J_1 + J_3).$$

Show that

$$J_1 + J_3 = \epsilon, \quad J_2 = i\epsilon\frac{d}{d\epsilon}, \quad \int |f(\epsilon)|^2 \frac{d\epsilon}{|\epsilon|} < \infty$$

and

$$J_3 = -\frac{1}{2}\left(\epsilon\frac{d^2}{d\epsilon^2} - \epsilon + \left(\frac{1}{4} - \mu^2\right)\epsilon^{-1}\right)$$

is a representation on $(-\infty, 0]$ and $[0, \infty)$.

Problem 4. Consider the Lie algebra $s\ell(2,\mathbb{R})$ with the basis x, y, h and the commutators

$$[x,h] = -2x, \qquad [x,y] = h, \qquad [h,y] = -2y.$$

Let $t \in \mathbb{R}$.
(i) Find $a(t)$, $b(t)$, $c(t)$ such that

$$\exp(tx)y\exp(-tx) = a(t)x + b(t)y + c(t)h.$$

(ii) Find $a(t)$, $b(t)$, $c(t)$ such that

$$\exp(tx)h\exp(-tx) = a(t)x + b(t)y + c(t)h.$$

(iii) Find $a(t)$, $b(t)$, $c(t)$ such that

$$\exp(ty)h\exp(-ty) = a(t)x + b(t)y + c(t)h.$$

Problem 5. Let $\mathbb{F}$ be a field. In the following we have $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A *Hopf algebra* is a vector space $\mathcal{A}$ endowed with five linear operations

$$\begin{aligned} \mu &: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \\ \eta &: \mathbb{F} \to \mathcal{A} \\ \Delta &: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \\ \epsilon &: \mathcal{A} \to \mathbb{F} \\ \gamma &: \mathcal{A} \to \mathcal{A} \end{aligned}$$

which possess the following properties

$$\begin{aligned}
\mu \cdot (id \otimes \mu) &= \mu \otimes (\mu \otimes id) \\
\mu \cdot (id \otimes \eta) &= id = \mu \cdot (\eta \otimes id) \\
(id \otimes \Delta) \cdot \Delta &= (\Delta \otimes id) \cdot \Delta \\
(\epsilon \otimes id) \cdot \Delta &= id = (id \otimes \epsilon) \cdot \Delta \\
\mu \cdot (id \otimes \gamma) \cdot \Delta &= \eta \cdot \epsilon = \mu \cdot (\gamma \otimes id) \cdot \Delta
\end{aligned}$$

and the co-multiplication Δ and co-unit ϵ are $\mathbb{F}$-algebra morphisms, that is, preserves multiplication. Give an example of a Hopf algebra using a finite group G.

Problem 6. Consider the differential operators

$$J_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \quad J_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad J_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

and

$$J_+ = J_1 + iJ_2, \qquad J_- = J_1 - iJ_2.$$

The commutation relations are

$$[J_1, J_2] = J_3, \qquad [J_2, J_3] = J_1, \qquad [J_3, J_1] = J_2.$$

Express the differential operators in spherical coordinates

$$x_1 = r\cos(\theta)\cos(\phi), \quad x_2 = r\cos(\theta)\sin(\phi), \quad x_3 = r\sin(\theta)$$

where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$.

Problem 7. Let $\partial := \partial/\partial x$. The two pseudodifferential operators

$$L := \partial + \sum_{j=1}^{\infty} u_{j+1}\partial^{-j}, \qquad W := 1 + \sum_{j=1}^{\infty} w_j \partial^{-j}$$

are called the *Lax operator* and *gauge operator*, respectively. With $\partial^{-1}\partial = \partial\partial^{-1} = 1$ the *generalized Leibniz rule* is satisfied

$$\partial^k f \cdot = \sum_{j=0}^{\infty} \binom{k}{j} (\partial^j f)\partial^{k-j}.$$

for any $k \in \mathbb{Z}$ and $L = W\partial W^{-1}$. The *KP hierarchy* is given by the *Lax equations*

$$\partial_n L = [B_n, L]$$

where $\partial_n := \partial/\partial t_n$ and $B_n := L_+^n$ is the differential part of $L^n = L_+^n + L_-^n$ with

$$L_+^n = \sum_{j=0}^{\infty} c_j^n \partial^j, \qquad L_-^n = \sum_{j=-\infty}^{-1} c_j^n \partial^j.$$

Derive the first three equations of the KP hierarchy.

Problem 8. Consider Poincaré's upper half-space version of hyperbolic geometry

$$H^3 := \{ (y_1, y_2, y_3) \in \mathbb{R}^3 \ : \ y_3 > 0 \}$$

endowed with a Riemannian metric tensor field

$$g = \frac{R^2 (dy_1 \otimes dy_1 + dy_2 \otimes dy_2 + dy_3 \otimes dy_3)}{y_3^2}.$$

(i) Find the Killing vector fields of g.
(ii) Now H^3 is isometric to the standard model of three-dimensional hyperbolic geometry, namely to the upper sheet Q^3 of a two-sheeted hyperboloid in $\mathbb{R}^4$

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = R^2, \quad x_0 > 0$$

with the pseudo-Euclidean metric tensor field of $\mathbb{R}^4$ given by

$$\widetilde{g} = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 - dx_0 \otimes dx_0$$

restricted to it. Find the Killing vector fields of $\widetilde{g}$.

Problem 9. Find the symmetries group of H_2O water molecule. Provide the character table.

Problem 10. The *Pascal matrix* of order n is defined as

$$P_n := \left(\frac{(i+j-2)!}{(i-1)!(j-1)!} \right), \qquad i, j = 1, \ldots, n$$

Thus

$$P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

(i) Find the determinant of P_2, P_3, P_4. Find the inverse of P_2, P_3, P_4.
(ii) Find the determinant for P_n. Is P_n an element of the Lie group $SL(n, \mathbb{R})$?

Problem 11. Let U be a 2×2 unitary matrix. What is the condition on U such that $I_2 \otimes U + U \otimes I_2$ is a unitary matrix?

Problem 12. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $\|A\| < 1$, where

$$\|A\| := \sup_{\|\mathbf{x}\|=\mathbf{1}} \|A\mathbf{x}\|.$$

Show that the matrix $B = I_n + A$ is invertible, i.e. $B \in GL(n, \mathbb{R})$. To show that the expansion

$$I_n - A + A^2 - A^3 + \cdots$$

converges apply

$$\begin{aligned} \|A^m - A^{m+1} + A^{m+2} - \cdots \pm A^{m+k-1}\| &\leq \|A^m\| \cdot \|1 + \|A\| + \cdots + \|A\|^{k-1}\| \\ &= \|A\|^m \frac{1 - \|A\|^k}{1 - \|A\|}. \end{aligned}$$

Then calculate $(I_n + A)(I_n - A + A^2 - A^3 + \cdots)$.

Problem 13. Let $b_1^\dagger$, $b_2^\dagger$ be Bose creation operators. Consider the operators

$$K_+ = b_1^\dagger b_2^\dagger, \quad K_- = b_1 b_2, \quad K_3 = \frac{1}{2}(b_1^\dagger b_1 + b_2^\dagger b_2 + I).$$

(i) Calculate the commutators and thus show that we have a basis of the Lie algebra $su(1,1)$.
(ii) Consider the Hamilton operator

$$\hat{H} = \omega(b_1^\dagger b_1 + b_2^\dagger b_2 + I) + \lambda(b_1 b_2 + b_1^\dagger b_2^\dagger).$$

Show that $\hat{H}$ can be expressed using K_+, K_-, K_3.
(iii) Solve the initial problem of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi$$

by setting

$$b_1 = \frac{1}{\sqrt{2\omega}}(\omega q_1 + ip_1), \qquad b_2 = \frac{1}{\sqrt{2\omega}}(\omega q_2 + ip_2)$$

with $p_1 = -i\partial/\partial q_1$, $p_2 = -i\partial/\partial q_2$ and

$$q_1 = \frac{1}{\sqrt{2\omega}}(x_1 + x_2), \qquad q_2 = \frac{1}{\sqrt{2\omega}}(x_1 - x_2).$$

First solve the initial value problem of the linear partial differential equations

$$-iK_+ f(x_1,x_2,t) = \frac{\partial f(x_1,x_2,t)}{\partial t}, \quad -iK_- g(x_1,x_2,t) = \frac{\partial g(x_1,x_2,t)}{\partial t}$$

$$K_3 h(x_1,x_2,t) = \frac{\partial h(x_1,x_2,t)}{\partial t}$$

($\widetilde{f}(x_1,x_2) = f(x_1,x_2,t=0)$ etc) and then use

$$e^{-it\hat{H}} = e^{a(t)C} e^{b(t)B} e^{c(t)A}$$

to solve the initial value problem for the Schrödinger equation, where $A = -iK_+$, $B = -iK_-$, $C = K_3$. The time-dependent functions $a(t)$, $b(t)$, $c(t)$ have to determined from the properties of the Lie algebra $su(1,1)$.

Problem 14. Consider Poincaré's upper half-space version of hyperbolic geometry

$$H^3 := \{(y_1,y_2,y_3) \in \mathbb{R}^3 \ : \ y_3 > 0\}$$

endowed with a Riemannian metric tensor field

$$g = \frac{R^2(dy_1 \otimes dy_1 + dy_2 \otimes dy_2 + dy_3 \otimes dy_3)}{y_3^2}.$$

(i) Find the Killing vector fields of g.
(ii) Now H^3 is isometric to the standard model of three-dimensional hyperbolic geometry, namely to the upper sheet Q^3 of a two-sheeted hyperboloid in $\mathbb{R}^4$

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = R^2, \quad x_0 > 0$$

with the pseudo-Euclidean metric tensor field of $\mathbb{R}^4$ given by

$$\widetilde{g} = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 - dx_0 \otimes dx_0$$

restricted to it. Find the Killing vector fields of $\widetilde{g}$.

Problem 15. Denote by $Vect(S^1)$ the set of smooth vector fields on the unity circle S^1. The Virasoro algebra vir is a vector space $\mathrm{Vect}(S^1) \oplus \mathbb{R}$ equipped with the commutator

$$\left[\left(f(x)\frac{\partial}{\partial x}, a\right), \left(g(x)\frac{\partial}{\partial x}, b\right)\right] := \left((f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x}, \int_{S^1} f'(x)g''(x)dx\right).$$

Show that the commutator is bilinear and antisymmetric.

Problem 16. Consider the semi-simple Lie algebra $so(3,1)$ (also called *Lorentz algebra*). Over the basis $\{ J_i,\, K_i \}_{i=1,2,3}$ the commutators are given by (summation convention)

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, K_j] = \epsilon_{ijk} K_k, \qquad [K_i, K_j] = -\epsilon_{ijk} J_k$$

where $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1$ and $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$. Find the Casimir operators.

Problem 17. Apply the technique described in the problem 20 (text book, chap4) to find a disentangled form of

$$\hat{O}_I(\boldsymbol{\alpha}) = \exp\left(\sum_{k=1}^{3} \alpha_k X_k\right)$$

where X_1, X_2, X_3 obey (Lie algebra $so(3)$)

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

Problem 18. Apply the technique in problem 30 (text book chap4) to find a disentangled form of

$$\hat{O}_I(\boldsymbol{\alpha}) = \exp\left(\alpha_+ K_+ + \alpha_0 K_0 + \alpha_- K_-\right)$$

where K_+, K_0, K_- obey (Lie algebra $su(1,1)$)

$$[K_0, K_+] = K_+, \quad [K_0, K_-] = -K_-, \quad [K_+, K_-] = 2K_0.$$

Problem 19. Consider the matrix

$$A = \begin{pmatrix} \alpha & 0 & 0 & \delta \\ 0 & \beta & \gamma & 0 \\ 0 & \gamma & \beta & 0 \\ \delta & 0 & 0 & \alpha \end{pmatrix}$$

and the Pauli spin matrices σ_x, σ_y, σ_z. Calculate the commutators

$$[A, \sigma_x \otimes \sigma_x], \quad [A, \sigma_y \otimes \sigma_y], \quad [A, \sigma_z \otimes \sigma_z]$$

where $\otimes$ denotes the Kronecker product.

Problem 20. Consider the differential operators

$$J_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \quad J_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad J_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

and

$$J_+ = J_1 + iJ_2, \qquad J_- = J_1 - iJ_2.$$

The commutation relations are

$$[J_1, J_2] = J_3, \qquad [J_2, J_3] = J_1, \qquad [J_3, J_1] = J_2.$$

Express the differential operators in spherical coordinates

$$x_1 = r\cos(\theta)\cos(\phi), \quad x_2 = r\cos(\theta)\sin(\phi), \quad x_3 = r\sin(\theta)$$

where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$.

Problem 21. Find nontrivial subgroups with two elements of the Pauli group $\mathcal{P}_n$ with $n \geq 2$.

Problem 22. The simple Lie algebra $so(2,1)$ consists of the three generators T_3, $T_+ = T_1 + iT_2$, $T_- = T_1 - iT_2$ with the commutation relations

$$[T_+, T_-] = -2T_3, \qquad [T_3, T_+] = T_+, \qquad [T_3, T_-] = -T_-.$$

Thus

$$T_1 = \frac{1}{2}(T_+ + T_-), \qquad T_2 = \frac{1}{2i}(T_+ - T_-).$$

Let $\theta \in \mathbb{R}$. Find

$$e^{-iT_2\theta}T_3e^{iT_2\theta}, \quad e^{-iT_2\theta}T_+e^{iT_2\theta}, \quad e^{-iT_2\theta}T_-e^{iT_2\theta}.$$

Problem 23. (i) Find the group of symmetries of the C_6H_6 Benzene molecule.
(ii) Find a 3×3 matrix representation of the group.

Problem 24. Consider the nine vector fields

$$V_1 = \frac{\partial}{\partial x_1}, \quad V_2 = \frac{\partial}{\partial x_2}, \quad V_3 = \frac{\partial}{\partial x_3}$$

$$V_{12} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}, \quad V_{23} = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}, \quad V_{31} = x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}$$

$$V_{c1} = 2x_1\sum_{j=1}^{3} x_j\frac{\partial}{\partial x_j} - (x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_1}$$

$$V_{c2} = 2x_2\sum_{j=1}^{3} x_j\frac{\partial}{\partial x_j} - (x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_2}$$

$$V_{c3} = 2x_3\sum_{j=1}^{3} x_j\frac{\partial}{\partial x_j} - (x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_3}.$$

Show that the vector fields form a basis of a Lie algebra. Classify the Lie algebra.

Problem 25. (i) Show that $e_6 \supset su(6)$.
(ii) Show that $e_7 \supset su(8)$.
(iii) Show that $g_2 \supset su(3)$.

Problem 26. The infinitesimal generators (vector fields) of Lie algebra $so(4)$ can be written in terms of the variables (x_1, x_2, x_3, x_4) as

$$X_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \quad X_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad X_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2},$$

$$Y_1 = x_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1}, \quad Y_2 = x_2 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_2}, \quad Y_3 = x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}.$$

(i) Find the commutators.
(ii) Show that the Lie algebra is semisimple, but not simple.

Problem 27. Consider the Lie algebra $so(5)$. Find the two Casimir operators. Start from the vector fields

$$V_{jk} = x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}, \qquad j, k = 1, 2, \ldots, 5.$$

Problem 28. Show that the vector fields

$$V_{jk} := x_j \frac{\partial}{\partial x_k}, \qquad j, k = 1, 2, \ldots, n$$

form a Lie algebra under the commutator. Is the Lie algebra semisimple?

Problem 29. Let $f \geq 2$. The *Brauer algebra* $D_f(n)$ is defined algebraically by $2f-2$ generators

$$\{ g_1, g_2, \ldots, g_{f-1}, e_1, e_2, \ldots, e_{f-1} \}$$

with the properties

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\ g_i g_j &= g_j g_i, \quad \text{for } |i-j| \geq 2 \\ e_i g_i &= e_i \\ e_i g_{i-1} e_i &= e_i. \end{aligned}$$

Show that

$$\begin{aligned} e_i e_j &= e_j e_i, \quad \text{for } |i-j| \geq 2 \\ e_i^2 &= n e_i \\ (g_i - 1)^2 (g_i + 1) &= 0. \end{aligned}$$

Problem 30. The *Hecke algebra* $H_n(v)$ of type A_{n-1} is generated by $n-1$ elements $T_1, \ldots, T_{n-1}$ subject to the properties

$$\begin{aligned} T_i T_j &= T_j T_i, \quad \text{for } |j-i| > 1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad \text{for } i < n-1 \\ T_i^2 &= (v-1) T_i + v. \end{aligned}$$

For $v = 0$ this algebra is not semisimple. Find a representation for $H_n(0)$.

Problem 31. Consider the second order ordinary differential equation

$$\frac{d^2u}{dt^2} = f(du/dt, u, t)$$

where f is an analytic function. Taking the time derivative we obtain

$$\frac{d^3u}{dt^3} = \frac{\partial f}{\partial \dot{u}} \frac{d^2u}{dt^2} + \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial t}.$$

Consider the hypersurfaces associated with these two differential equations

$$\begin{aligned} F_1(u, u, u, t) &= u_{tt} - f(u_t, u, t) = 0 \\ F_2(u, u, u, u, t) &= u_{ttt} - \frac{\partial f}{\partial u_t} u_{tt} - \frac{\partial f}{\partial u} u_t - \frac{\partial f}{\partial t} = 0. \end{aligned}$$

(i) Find the condition that

$$V = g(t, u, u_t) \frac{\partial}{\partial u}$$

is a Lie symmetry vector field of the ordinary differential equation. One calculates the prolongation of the vector field V

$$\begin{aligned} \widetilde{V} = V &+ \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial u} + \frac{\partial g}{\partial u_t} u_{tt} \right) \frac{\partial}{\partial u_t} \\ &\left(\frac{\partial^2 g}{\partial t^2} + 2 \frac{\partial^2 g}{\partial u \partial t} u_t + 2 \frac{\partial^2 g}{\partial u_t \partial t} u_{tt} + \frac{\partial^2 g}{\partial u^2} u_t^2 + 2 \frac{\partial^2 g}{\partial u \partial u_t} u_t u_{tt} \right. \\ &\left. + \frac{\partial^2 g}{\partial u_t^2} u_{tt}^2 + \frac{\partial g}{\partial u} u_{tt} + \frac{\partial g}{\partial u_t} u_{ttt} \right). \end{aligned}$$

The invariance condition is then

$$L_{\widetilde{V}} F_1 \hat{=} 0.$$

(ii) Simplify the result if $f(du/dt, u, t) = 0$.
(iii) Simplify the result if $f(du/dt, u, t) = u$.

Problem 32. Show that the vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z},$$

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad (y^2+z^2-x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - 2zx\frac{\partial}{\partial z},$$

$$(z^2+x^2-y^2)\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial x} - 2yz\frac{\partial}{\partial z}, \quad (x^2+y^2-z^2)\frac{\partial}{\partial z} - 2yz\frac{\partial}{\partial y} - 2zx\frac{\partial}{\partial z}$$

form a Lie algebra under commutator. Are these vector fields are Killing vector fields of the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Problem 33. Consider the *sine-Gordon equation*

$$\frac{\partial^2 u}{\partial x \partial t} = \sin(u).$$

(i) Show that the differential equations admits the Lie symmetry vector fields

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}.$$

(ii) Show that the sine-Gordon equation can be derived from the Lagrangian

$$L = \frac{1}{2}u_t u_x - \cos(u).$$

(iii) Show that applying Noether's theorem to the vector field V_3 provides the conserved vector with the components

$$T_1 = -\frac{1}{2}xu_x^2 + t\cos u, \qquad T_2 = \frac{1}{2}tu_t^2 - x\cos u.$$

Problem 34. (i) Show the potential Z-K equation

$$\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0$$

can be derived from the Lagrange function

$$L = -\frac{1}{2}u_x u_t + \frac{1}{6}u_x^3 + \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2.$$

Show that the differential equations admits the Lie symmetry vector fields

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}$$

$$V_3 = 5t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z} - 3u\frac{\partial}{\partial u}$$

$$V_4 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}, \quad V_5 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

$$V_6 = 5t^2\frac{\partial}{\partial t} + (2tx + \frac{3}{2}(y^2+z^2))\frac{\partial}{\partial x} + 6ty\frac{\partial}{\partial y} + 6tz\frac{\partial}{\partial z} - (x^2 + 6tu)\frac{\partial}{\partial u}.$$

Problem 35. Let $c_1, c_2, c_3 \in \mathbb{R}$ with $c_3 \neq 0$. Consider the vector field

$$V = (c_1 + c_2 x + c_3 x^2)\frac{d}{dx}.$$

Calculate $x(t) = \exp(tV)x$. Compare with the solution of the initial value problem of the nonlinear differential equation

$$\frac{dx}{dt} = c_1 + c_2 x + c_3 x^2.$$

Problem 36. Consider the group of diffeomorphisms of S^1. This is infinite group with composition as a group law. Its Lie algebra is the set of smooth vector fields on the unity circle S^1 equipped with the commutator

$$\left[f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right] = (f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x}.$$

We denote this set by $\mathrm{Vect}(S^1)$. Let us introduce a positive quadratic form ("energy")

$$H\left(f(x)\frac{\partial}{\partial x}\right) = 1/2\left(\int_{S^1}(f(x))^2 dx\right).$$

The nonviscous Burgers equation on the unit circle is the evolutionary equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}.$$

Show that nonviscous Burgers equation is the Euler's equation of geodesic flows on the $\mathrm{Diff}(S^1)$ group (with the right-invariant metric H).

Problem 37. (i) Consider the vector space of operators

$$\{ c_{j_1}^\dagger c_{j_2} c_{k_1}^\dagger c_{k_2} \}$$

where $c_j^\dagger$, c_j are Fermi creation and annihilation operators and $j_1, j_2, k_1, k_2 = 1, 2, \ldots, N$. Note that depending on j_1, j_2, k_1, k_2 the operator $c_{j_1}^\dagger c_{j_2} c_{k_1}^\dagger c_{k_2}$ could be the zero operator. Calculate the commutator

$$[c_{j_1}^\dagger c_{j_2} c_{k_1}^\dagger c_{k_2}, c_{m_1}^\dagger c_{m_2} c_{n_1}^\dagger c_{n_2}]$$

and thus show that we have a Lie algebra. Of course the Jacobi identity must also be checked.
(ii) Consider the number operator

$$N = \sum_{j=1}^{N} c_j^\dagger c_j.$$

Calculate the commutator

$$[N, c_{j_1}^\dagger c_{j_2} c_{k_1}^\dagger c_{k_2}].$$

Discuss.

Problem 38. Let $\hat{G} = g\ell(n, \mathbb{C}) \times \mathbb{C}[\lambda, \lambda^{-1}]$ be the loop algebra of a semi-infinite formal Laurent series with coefficients in $g\ell(n, \mathbb{C})$. An element $X(\lambda) \in \hat{G}$ can be expressed as a formal series in the form

$$X(\lambda) = \sum_{i=-\infty}^{m} x_i \otimes \lambda^i, \qquad \text{for all } x_i \in g\ell(, \mathbb{C}).$$

The Lie bracket of $X(\lambda)$ with $Y(\lambda)$ is defined by

$$[X(\lambda), Y(\lambda)] = \sum_{k=-\infty}^{m+l} \sum_{i+j=k} [x_i, y_j] \otimes \lambda^k.$$

Do we have (Jacobi identity)

$$[[X(\lambda), Y(\lambda)], Z(\lambda)] + [[Z(\lambda), X(\lambda)], Y(\lambda)] + [[Y(\lambda), Z(\lambda)], X(\lambda)] = 0?$$

Problem 39. A realization of the Lie algebra $s\ell(2,\mathbb{R})$ using vector fields is

$$V_1 = \frac{d}{dx}, \quad V_2 = u\frac{d}{du}, \quad V_3 = u^2\frac{d}{du}.$$

Consider the *Riccati equation*

$$\frac{du}{dt} = 1 + u + u^2.$$

(i) Find the solution of the initial value problem using the Lie series

$$u(t) = \exp(tV)u|_{u=u_0}$$

where

$$V = V_1 + V_2 + V_3 = (1 + u + u^2)\frac{d}{du}.$$

(ii) Find the functions $f(t)$, $g(t)$, $h(t)$ such that

$$e^{tV} = e^{f(t)d/du}e^{g(t)ud/du}e^{h(t)u^2d/du}.$$

Problem 40. Let $b_1^\dagger$, $b_2^\dagger$ be Bose creation operators. Consider the operators

$$K_+ = b_1^\dagger b_2^\dagger, \quad K_- = b_1 b_2, \quad K_3 = \frac{1}{2}(b_1^\dagger b_1 + b_2^\dagger b_2 + I).$$

(i) Calculate the commutators and thus show that we have a basis of the Lie algebra $su(1,1)$.
(ii) Consider the Hamilton operator

$$\hat{H} = \omega(b_1^\dagger b_1 + b_2^\dagger b_2 + I) + \lambda(b_1 b_2 + b_1^\dagger b_2^\dagger).$$

Show that $\hat{H}$ can be expressed using K_+, K_-, K_3.
(iii) Solve the initial problem of the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

by setting

$$b_1 = \frac{1}{\sqrt{2\omega}}(\omega q_1 + ip_1), \qquad b_2 = \frac{1}{\sqrt{2\omega}}(\omega q_2 + ip_2)$$

with $p_1 = -i\partial/\partial q_1$, $p_2 = -i\partial/\partial q_2$ and

$$q_1 = \frac{1}{\sqrt{2\omega}}(x_1 + x_2), \qquad q_2 = \frac{1}{\sqrt{2\omega}}(x_1 - x_2).$$

First solve the initial value problem of the linear partial differential equations

$$-iK_+ f(x_1,x_2,t) = \frac{\partial f(x_1,x_2,t)}{\partial t}, \quad -iK_- g(x_1,x_2,t) = \frac{\partial g(x_1,x_2,t)}{\partial t}$$

$$K_3 h(x_1,x_2,t) = \frac{\partial h(x_1,x_2,t)}{\partial t}$$

($\widetilde{f}(x_1,x_2) = f(x_1,x_2,t=0)$ etc) and then use

$$e^{-it\hat{H}} = e^{a(t)C}e^{b(t)B}e^{c(t)A}$$

to solve the initial value problem for the Schrödinger equation, where $A = -iK_+$, $B = -iK_-$, $C = K_3$. The time-dependent functions $a(t)$, $b(t)$, $c(t)$ have to determined from the properties of the Lie algebra $su(1,1)$.

Problem 41. Consider the *nonlinear wave equation*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) = 0.$$

Show that this partial differential equation admits the Lie symmetry vector fields

$$\frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial x}, \qquad t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

Calculate the commutators of these vector fields and classify the Lie algebra.

Problem 42. Consider the group of diffeomorphisms of S^1. This is infinite group with composition as a group law. Its Lie algebra is the set of smooth vector fields on the unity circle S^1 equipped with the commutator

$$\left[f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right] = (f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x}.$$

We denote this set by $\mathrm{Vect}(S^1)$. Let us introduce a positive quadratic form ("energy")

$$H\left(f(x)\frac{\partial}{\partial x}\right) = 1/2\left(\int_{S^1}(f(x))^2 dx\right).$$

The nonviscous Burgers equation on the unit circle is the evolutionary equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}.$$

Show that nonviscous Burgers equation is the Euler's equation of geodesic flows on the $\mathrm{Diff}(S^1)$ group (with the right-invariant metric H).

Problem 43. The n-qubit *Pauli group* is defined by

$$\mathcal{P}_n = \{\, I_2,\, \sigma_x,\, \sigma_y,\, \sigma_z \,\}^{\otimes n} \otimes \{\, \pm 1,\, \pm i \,\} \tag{1}$$

where σ_x, σ_y, σ_z are the 2×2 Pauli matrices and I_2 is the 2×2 identity matrix. The dimension of the Hilbert space under consideration is $\dim \mathcal{H} = 2^n$. Thus each element of the Pauli group $\mathcal{P}_n$ is (up to an overall phase ± 1, $\pm i$) a Kronecker product of Pauli matrices and 2×2 identity matrices acting on n qubits. The order of the group is given by 2^{2n+2}.
(i) Find all subgroups of the Pauli group $\mathcal{P}_1$.
(ii) Find all conjugacy classes of the Pauli group $\mathcal{P}_1$.

Problem 44. Consider the system of linear partial differential equations (*Knizhnik-Zamolodchikov equation*)

$$\frac{\partial}{\partial z_i} \Phi = k \sum_{j \neq i}^{n} \frac{P^{(ij)}}{z_i - z_j} \Phi$$

where $\Phi = \Phi(z_1, z_2, \dots, z_n)$ is a function of n complex independent variables, which takes values in the tensor product $V \otimes V \otimes \cdots \otimes V$ of N identical vector spaces V. $P^{(ij)}$ is the permutation of the i-th and j-th factors of tensor product. This is a simple form of the Knizhnik-Zamolodchikov equation (KZ-equation).
(i) Show that the Knizhnik-Zamolodchikov equation is consistent. This means that the differential operators O_i and O_j commute, where

$$O_i := \frac{\partial}{\partial z_i} - k \sum_{j \neq i}^{n} \frac{P^{(ij)}}{z_i - z_j}.$$

Each solution of the KZ equation is fully determined by the value of Φ in some initial point. The set of solutions of the KZ equation is a finite dimensional manifold. We can search solution near a regular point $z_i \neq z_j$, $1 \leq i, j \leq n$ as the $f(z_1, \dots, z_n)$ sum of power series on $z_j \neq 0, 1 \leq j \leq n$ variables. When we start from any regular point (the base point), we can prolong solution of (KZ) along any path using the analytic continuation method. When path returns to the base point, the power series in common case is not the same - it is equal to linear combination of another solutions. The monodromy operator M depends only on the homotopy class of path γ, thus monodromy group gives a representation of the fundamental group.
(ii) Show that when the number of vector spaces is equal to the number of independent variables $N = n$, the set of symmetries of the KZ equation contains the S_n group.

Problem 45. Show that the nonlinear partial differential equation (Boyer-Finley equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \kappa \frac{\partial^2 e^u}{\partial t^2}, \qquad \kappa = \pm 1$$

admits the Lie symmetry vector fields

$$\frac{\partial}{\partial t}, \qquad t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial u}.$$

Problem 46. Consider the quantum algebra $su(2)_q$. Let

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$$

where $q = e^\tau$ is the deformation parameter. Let $j \in \{1/2, 1, 3/2, 2, \dots\}$. The $su(2)_q$ generators are

$$J_3 = \sum_{m=-j}^{j} m|jm\rangle\langle jm|$$

$$J_\pm = \sum_{m=-j}^{j} \sqrt{[j \mp m][j \pm m + 1]}|jm \pm 1\rangle\langle jm|.$$

Find J_+J_-, J_-J_+ and the commutators $[J_3, J_\pm]$, $[J_+, J_-]$.

Problem 47. The q-deformed algebra of $SU_q(2)$ is an associative algebra of the generators J_3, $J_\pm$ ($J_\pm = J_1 \pm iJ_2$) and the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_3]$$

where

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

(i) Let

$$E = e^{zJ_3/2}J_+, \qquad F = e^{-zJ_3/2}J_-.$$

(i) Find the commutators $[J_3, E]$, $[J_3, F]$, $[E, F]$.
(ii) Find the irreducible representation of $su_q(2)$ in a $(2j+1)$-dimensional Hilbert space.
(iii) The coproduct and antipodes have the form

$$\Delta E = \mathbf{1} \otimes E + E \otimes e^{zJ_3}, \quad \Delta F = e^{-zJ_3} \otimes F + F \otimes \mathbf{1}, \quad \Delta J_3 = \mathbf{1} \otimes J_3 + J_3 \otimes \mathbf{1}$$

$$S(E) = -e^{-zJ_3}E, \quad S(F) = -e^{zJ_3}F, \quad S(J_3) = -J_3.$$

Find the commutators

$$[\Delta E, \Delta F], \quad [\Delta E, \Delta J_3], \quad [\Delta F, \Delta J_3].$$

Problem 48. (i) Consider the differential equations

$$\frac{du}{dt} = \cos(u), \qquad \frac{du}{dt} = \sin(u)$$

with the corresponding vector fields

$$V = \cos(u)\frac{d}{du}, \qquad W = \sin(u)\frac{d}{du}.$$

Both system have infinite many fixed points. Calculate the commutator and study the fixed points of this vector field.
(ii) Study the same question for the differential equations

$$\frac{du}{dt} = \cosh(u), \qquad \frac{du}{dt} = \sinh(u).$$

Problem 49. Consider the Hamilton function

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + g\left(\frac{1}{(q_1 - q_2)^2} + \frac{1}{(q_2 - q_3)^2} + \frac{1}{(q_3 - q_1)^2}\right).$$

Write down the Hamilton equations of motion. Find the first integrals and the Lie symmetry vector fields.

Problem 50. Let $g \in L_2(\mathbb{R}^d)$ and

$$S_1 = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \cdots \sum_{m_d=-\infty}^{\infty} g(m_1, m_2, \dots, m_d)$$

$$S_2 = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_d=-\infty}^{\infty} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \cdots \int_{-\infty}^{\infty} dy_d g(y_1, y_2, \dots, y_d) \exp(-2\pi i \sum_{j=1}^{d} n_j y_j).$$

Show that $S_1 = S_2$. This is called the Poisson resummation formula.

Problem 51. The *Kustaanheimo-Stiefel transformation* is defined by the map from $\mathbb{R}^4$ (coordinates u_1, u_2, u_3, u_4) to $\mathbb{R}^3$ (coordinates x_1, x_2, x_3)

$$\begin{aligned}
x_1(u_1, u_2, u_3, u_4) &= 2(u_1u_3 - u_2u_4)\\
x_2(u_1, u_2, u_3, u_4) &= 2(u_1u_4 + u_2u_3)\\
x_3(u_1, u_2, u_3, u_4) &= u_1^2 + u_2^2 - u_3^2 - u_4^2
\end{aligned}$$

together with the constraint

$$u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4 = 0.$$

(i) Show that

$$r^2 = x_1^2 + x_2^2 + x_3^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

(ii) Show that

$$\Delta_3 = \frac{1}{4r}\Delta_4 - \frac{1}{4r^2}V^2$$

where

$$\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_4 = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}$$

and V is the vector field

$$V = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}$$

(iii) Consider the differential one form

$$\alpha = u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4.$$

Find $d\alpha$. Find $L_V\alpha$, where $L_V(.)$ denotes the Lie derivative.
(iv) Let $g(x_1(u_1,u_2,u_3,u_4), x_2(u_1,u_2,u_3,u_4), x_3(u_1,u_2,u_3,u_4))$ be a smooth function. Show that $L_V g = 0$.

Problem 52. Find the Lie symmetries of the partial differential equation

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1.$$

Problem 53. Show that the nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)$$

admits the Lie symmetry vector fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x}, \quad t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

Problem 54. The generators of the Lie algebra $so(2,1)$ satisfy the commutation relations

$$[T_1, T_2] = -iT_1, \quad [T_2, T_3] = -iT_3, \quad [T_1, T_3] = -iT_2.$$

Find the conditions on λ, k, α_0, α_1, α_2 such that

$$T_1(x) = \alpha_2 x^{2-k}\frac{d^2}{dx^2} + \alpha_1 x^{1-k} + \alpha_0 x^{-k}$$
$$T_2(x) = -\frac{i}{k}x\frac{d}{dx} - i\beta$$
$$T_3(x) = \lambda x^k$$

satisfy these commutation relations.

Problem 55. Show that

$$\exp\left(-i\epsilon\frac{\partial}{\partial z}\right) z \exp\left(i\epsilon\frac{\partial}{\partial z}\right) = z - i\epsilon$$
$$\exp\left(-i\epsilon\frac{\partial}{\partial z}\right) \epsilon \exp\left(i\epsilon\frac{\partial}{\partial z}\right) = \epsilon$$
$$\exp\left(-i\epsilon\frac{\partial}{\partial z}\right) \frac{\partial}{\partial z} \exp\left(i\epsilon\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}$$
$$\exp\left(-i\epsilon\frac{\partial}{\partial z}\right) \frac{\partial}{\partial \epsilon} \exp\left(i\epsilon\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial \epsilon} + i\frac{\partial}{\partial z}.$$

Problem 56. Let $m, n \in \mathbb{Z}$. A Kac-Moody algebra is given by the operators P_m, Q_n and the commutation relations

$$[P_m, P_n] = 0$$
$$[P_m, Q_n] = mP_{m+n}$$
$$[Q_m, Q_n] = (m-n)Q_{m+n}$$

Give a representation.

Problem 57. Consider the radial Schrödinger equation for a one lectron atom with nuclear charge Z

$$\left(-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} - \frac{2Z}{r} + \frac{Z^2}{n^2}\right) u_{n_r,\ell}(r) = 0$$

where $n = n_r + \ell + 1$ is the principal quantum number, ℓ is the angular momentum quantum and n_r is a quantum number which denotes the number of radial nodes. We introduce

$$\rho(r) = \frac{Z}{n}r, \qquad \widetilde{u}(\rho(r)) = u(r).$$

Then the radial Schrödinger equation takes the form

$$\left(-\rho\frac{d^2}{d\rho^2}+\frac{\ell(\ell+1)}{\rho}+\rho\right)\widetilde{u}_{n_r,\ell}(\rho)=2n\widetilde{u}_{n_r,\ell}(\rho).$$

Define the operators

$$W_1=\rho, \qquad W_3=-\rho\frac{d^2}{d\rho^2}+\frac{\ell(\ell+1)}{\rho}.$$

(i) We define W_2 as $[W_1,W_3]=2iW_2$. Find W_2.
(ii) We define

$$T_1=\frac{1}{2}(W_3-W_1), \quad T_2=W_2, \quad T_3=\frac{1}{2}(W_3+W_1).$$

Find the commutators and thus show that we have a basis of a Lie algebra.
(iii) Find $T_3\widetilde{u}_{n_r,\ell}(\rho)$. Discuss.
(iv) Give the quadratic Casimir operator.

Problem 58. Consider the Hamilton operator

$$\hat{H}=\frac{1}{2m}\hat{\mathbf{p}}^2-\frac{\alpha}{r}$$

with $r^2=x_1^2+x_2^2+x_3^2$ and the canonical commutation relations

$$[\hat{x}_j,\hat{x}_k]=0, \quad [\hat{p}_j,\hat{p}_k]=0, \quad [\hat{p}_j,\hat{x}_k]=-i\delta_{jk}I.$$

The angular momentum operator $\hat{\mathbf{L}}$ and the *Runge-Lenz vector* $\hat{\mathbf{A}}$ are defined by

$$\hat{\mathbf{L}}:=\hat{\mathbf{x}}\times\hat{\mathbf{p}}, \qquad \hat{\mathbf{A}}:=\frac{1}{2m}(\hat{\mathbf{p}}\times\hat{\mathbf{L}}-\hat{\mathbf{L}}\times\hat{\mathbf{p}})-\alpha\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}=\mathbf{r}/r$. Show that

$$\begin{aligned}
[\hat{\mathbf{L}},\hat{H}]&=0\\
[\hat{\mathbf{A}},\hat{H}]&=0\\
[\hat{L}_j,\hat{L}_k]&=i\epsilon_{jk\ell}\hat{L}_\ell\\
[\hat{L}_j,\hat{A}_k]&=i\epsilon_{jk\ell}\hat{A}_\ell\\
[\hat{A}_j,\hat{A}_k]&=-\frac{2}{m}i\epsilon_{jk\ell}\hat{H}\\
\hat{\mathbf{L}}\cdot\hat{\mathbf{A}}&=0\\
\hat{\mathbf{A}}^2&=\frac{2}{m}(\hat{\mathbf{L}}+1)\hat{H}+\alpha^2.
\end{aligned}$$

Problem 59. Find the commutators of the differential operators

$$T_1 = \frac{1}{2}\left(xy - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)$$
$$T_2 = -\frac{i}{2}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$
$$T_3 = \frac{1}{4}\left(y^2 - \frac{\partial^2}{\partial y^2} - x^2 + \frac{\partial^2}{\partial x^2}\right)$$
$$V = \frac{1}{4}\left(y^2 - \frac{\partial^2}{\partial y^2} + x^2 - \frac{\partial^2}{\partial x^2}\right) - \frac{1}{2}.$$

Problem 60. (i) Let $m \in \mathbb{Z}$. Consider the differential operator

$$L_m = z^{m+1}\frac{\partial}{\partial z}$$

Show that

$$[L_m, L_n] = -(m-n)L_{m+n}.$$

Problem 61. Show that the following differential operators

$$D_- = -\frac{d}{dx}, \qquad D_0 = x\frac{d}{dx} + 1, \qquad D_+ = x^2\frac{d}{dx} + 2x$$

are a realization of the Lie algebra $su(1,1)$.

Problem 62. Show that the differential operators

$$T_0 = j + 1 + z\frac{d}{dz}, \quad T_+ = 2(j+1)z + z^2\frac{d}{dz}, \quad T_- = \frac{d}{dz}$$

satisfy the commutation relations

$$[T_0, T_\pm] = \pm T_\pm, \qquad [T_+, T_-] = -2T_0.$$

Problem 63. Show that the second order differential operators

$$J_+(\lambda) = \frac{1}{4}\left(\left(x - \frac{d}{dx}\right)^2 - \frac{\lambda(\lambda-1)}{x^2}\right)$$
$$J_-(\lambda) = \frac{1}{4}\left(\left(x + \frac{d}{dx}\right)^2 - \frac{\lambda(\lambda-1)}{x^2}\right)$$
$$J_3(\lambda) = \frac{1}{4}\left(-\frac{d^2}{dx^2} + x^2 + \frac{\lambda(\lambda-1)}{x^2}\right)$$

satisfy the commutation relation pf the Lie algebra $su(1,1)$

$$[J_+(\lambda), J_(\lambda)] = -2J_3(\lambda), \quad [J_3(\lambda), J_+(\lambda)] = J_+(\lambda), \quad [J_3(\lambda), J_-(\lambda)] = -J_-(\lambda).$$

Problem 64. Show that the differential operators

$$S = \frac{1}{2}x\left(-\frac{d^2}{dx^2}+1\right), \quad T = -ix\frac{d}{dx}, \quad U = \frac{1}{2}x\left(-\frac{d^2}{dx^2}-1\right)$$

satisfy the commutation relations

$$[S,T] = -iU, \qquad [T,U] = iS, \qquad [U,S] = -iT.$$

Chapter 5

Programming Problems

Problem 1. Let A be an $n \times n$ hermitian matrix. Then $(A + iI_n)^{-1}$ exists and

$$U = (A - iI_n)(A + iI_n)^{-1}$$

is a unitary matrix (so-called *Cayley transform* of A). Note that $+1$ cannot be an eigenvalue of U. Write a SymbolicC++ program that finds the Cayley transform of

$$\sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2. Consider the permutation group S_3. Write a SQL script for the composition of the group elements using the relation table with columns "Op1" for the first operand, "Op2" for the second operand and "Res" for the result of the group operation. Then implement the inverse of each group element using the relation table with 2 columns - first for the group element and second for its inverse. Finally determine the conjugacy classes.

Problem 3. Let $\{ \beta_1, \ldots, \beta_n \}$ be the generators of the *Braid group* B_{n+1} where

$$\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}, \qquad \beta_i^2 = 1, \qquad \beta_i \beta_j = \beta_j \beta_i$$

where $j, i \in \{ 1, \ldots, n \}$ and $|j - i| \neq 1$. Let $\{ \mathbf{e}_1, \ldots, \mathbf{e}_{n+1} \}$ denote the standard basis in $\mathbb{C}^{n+1}$. The action ρ of an element β of the braid group

B_n on a vector $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{C}^n$ is given by

$$\rho(\beta_i)(x_1\mathbf{e}_1 + \ldots + x_{n+1}\mathbf{e}_{n+1})$$
$$= x_1\mathbf{e}_1 + \ldots + x_{i-1}\mathbf{e}_{i-1} + x_{i+1}\mathbf{e}_i + (x_i + 1)\mathbf{e}_{i+1} + x_{i+2}\mathbf{e}_{i+2} + \ldots + x_{n+1}\mathbf{e}_{n+1}.$$

and $\rho(\beta_i\beta_j)\mathbf{x} = \rho(\beta_i)\rho(\beta_j)\mathbf{x}$. We have the property $\rho((\beta_i\beta_{i+1})^3) = \rho((\beta_{i+1}\beta_i)^3) = T_i$ where

$$T_i(\mathbf{x}) = \mathbf{x} + 2(\mathbf{e}_i + \mathbf{e}_{i+1} + \mathbf{e}_{i+2}).$$

Give a C++ implementation of ρ.

Problem 4. When integrating the determining equations system "by hand" one usually solves the problem by parts: finds the subsystem that is easy to integrate and substitute results of integration to the rest of the system. It remains the question: what to do when there is not such subsystem? The general theory of overdetermined PDE provides the answer: the reduction of the system to some standard form. Below we provide description of one of possible simplified versions of Standard Form algorithm.
1. For each equation in the system choose a highest derivative with respect to the standard ordering on derivatives and solve equation for this derivative. This step provides the solved form for each equation of the system

$$D^{\mathbf{a}}u^{\sigma} = f$$

where
- u^{σ} is an independent variable under differentiation;
- $\mathbf{a}$ vector with components $(a_1, a_2, \ldots, a_n)$ which denote quantities of differentiation by corresponding independent variable;
- $D^{\mathbf{a}}u^{\sigma}$ is the leading term of the equation;
- f is some function of derivatives of smaller order than leading term order.
We call the ordering standard, when all derivatives are ordered first by total quantity of derivations and second in lexicographical order. More specifically $D^{\mathbf{a}}u^k \succ D^{\mathbf{b}}u^j$ iff the first non-zero member of the list $[(\sum a_j - \sum b_j), (k-l), (a_1 - b_1), (a_2 - b_2), \ldots, (a_m - b_m)]$ is positive
2. Eliminate every derivative in the right hand side (RHS) of the system which is a (possibly zero order) derivative of some derivative in the LHS of the system obtained in step 1. Do this for every equation in the system until no derivative in the RHS is a derivative of any derivative in the LHS of the system.
3. Take compatibility conditions: for every pair of equations in the system of the form $D^{\mathbf{a}}u^{\sigma} = f$, $D^{\mathbf{b}}u^{\sigma} = f$ append the equation $D^{\mathbf{c}-\mathbf{a}}f = D^{\mathbf{c}-\mathbf{b}}g$ to the system, where $\mathbf{c} = \max(\mathbf{a}, \mathbf{b})$, i.e. $c_i = \max(a_i, b_i)$, $i = 1, .., n$.
4. Repeat steps 1,2,3 until (a) every equation is in solved form with respect to the highest derivative, (b) no explicit substitutions from the LHS of the

system into the RHS of this system are possible, and (c) every compatibility condition $D^{\mathbf{c}-\mathbf{a}}f = D^{\mathbf{c}-\mathbf{b}}g$ is identically satisfied subject to the reductions outlined in 2.

This form of algorithm is proposed in "New Symmetries from old: exploiting Lie algebra structure to determine infinitesimal symmetries of differential equations, Alan Boulton, Master Thesis, The University of British Columbia, 1993".

The full run of the algorithm may be rather time consuming, therefore we propose to perform it step by step. On each step user selects 2 equations, performs for them the step 3, adds the new equation to the system.

The leading term of a new equation may be the total derivative of some existing equation. We propose in this situation the slightly another procedure. Namely, we propose not to add new equation with commutator according to step 3, but to reorganize the new equation: subtract from its RHS the total derivative of the RHS of existing equation, determine the new leading term in the RHS, and only after that add new equation to the system.

The problem is: to implement the steps of simplified Standard Form algorithm as independent procedures.

On the practice the user can after several steps of algorithm return to usual methods of partial integration.

Problem 5. Find standard form for determining equations of the one-dimensional heat equation

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}.$$

Problem 6. Find the standard form for the Cauchy - Riemann equations for a function of 2 complex variables

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad \frac{\partial u}{\partial a} - \frac{\partial v}{\partial b} = 0, \quad \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} = 0.$$

Problem 7. Find the standard form for determining equations for the Rayleigh equation

$$\frac{\partial^2 u}{\partial t^2} + u - \epsilon \left(\frac{\partial u}{\partial t} - \left(\frac{\partial u}{\partial t} \right)^3 \right) = 0.$$

Bibliography

Books

Aldous J. M. and Wilson R. J.
Graphs and Applications: An Introductory Approach, Springer (2000)

Anderson R. L. and Ibragimov N. H.
Lie-Bäcklund Transformations in Applications, SIAM (1979)

Armstrong M. A.
Groups and Symmetry, Springer (1988)

Baumslag B. and Chandler B.
Group Theory, Schaum's Outline Series, McGraw-Hill (1968)

Bäuerle G. G. A. and de Kerf E. A.
Lie Algebras, North-Holland (1990)

Bishop D. M.
Group Theory and Chemistry, Dover Publications (1993)

Bluman G. W. and Cole J. D.
Similarity Methods for Differential Equations, Springer (1974)

Bourbaki N.
Elements of Mathematics: Lie Groups and Lie Algebras, Addison-Wesley (1975)

Bredon G. E.
Introduction to Compact Transformation Groups, Elsevier (1972)

Bluman G. W. and Kumei S.
Symmetries and Differential Equations, Springer (1989)

Bronson R.
Matrix Operations, Schaum's Outlines, McGraw-Hill (1989)

Bump D.
Lie Groups, Springer (2000)

Carter R. W.
Simple Groups of Lie Type, John Wiley (1972)

Chern S. S., Chen W. H. and Lam K. S.
Lectures on Differential Geometry, World Scientific (1999)

Crampin M. and Pirani F. A. E.
Applicable Differential Geometry, Cambridge University Press (1984)

de Souza P. N. and Silva J.-N.
Berkeley Problems in Mathematics, Springer (1998)

Di Franceso P., Mathieu P. and Sénéchal D.
Conformal Field Theory, 2nd edition, Springer (1997)

Dixmier J.
Enveloping Algebras, North-Holland (1974)

Englefield M. J.
Group Theory and the Coulomb Problem, Wiley-Interscience (1972)

Erdmann K. and Wildon M.
Introduction to Lie Algebras, Springer (2006)

Flanders H.
Differential Forms, Academic Press (1963)

Frankel T.
The Geometry of Physics, 2nd edition, Cambridge University Press (2004)

Fuhrmann P. A.
A Polynomial Approach to Linear Algebra, Springer (1996)

Fulton W. and Harris J.
Representation Theory, Springer (1991)

Gallian J. A.
Contemporary Abstract Algebra, sixth edition, Houghton Mifflin (2006)

Gilmore R.
Lie Groups, Lie Algebras and Some of Their Applications, Wiley (1974)

Golub G. H. and Van Loan C. F.
Matrix Computations, third edition, Johns Hopkins University Press (1996)

Göckeler M. and Schücker T.
Differential geometry, gauge theories, and gravity, Cambridge University Press (1987)

Grossman S. I.
Elementary Linear Algebra, third edition, Wadsworth Publishing (1987)

Hall B. C.
Lie Groups, Lie Algebras, and Representations: an elementary introduction, Springer (2003)

Hamermesh M.
Group Theory and its Application to Physical Problems, Addison-Wesley (1962)

Helgason S.
Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators and Spherical Functions, Academic Press (1984)

Helgason S.
Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press (1978)

Horn R. A. and Johnson C. R.
Topics in Matrix Analysis, Cambridge University Press (1999)

Humphreys J. E.
Introduction to Lie Algebras and Representation Theory, Springer (1972)

Ibragimov N. H.
Transformation Groups in Mathematical Physics, Nauka (1983)
(in Russian)

Inui T., Tanabe Y. and Onodera Y.

Group Theory and its Applications in Physics, Springer (1990)

Isham C. J.
Modern Differential Geometry, World Scientific (1989)

Jacobson N.
Lie Algebras, Interscience Publishers (1962)

James G. and Liebeck M.
Representations and Characters of Groups, 2nd edition, Cambridge University Press (2001)

Johnson D. L.
Presentation of Groups, Cambridge University Press (1976)

Jones H. F.
Groups, Representations and Physics, Adam Hilger (1990)

Kac V. G.
Infinite Dimensional Lie Algebras, Cambridge University Press (1985)

Kedlaya K. S., Poonen B. and Vakil R.
The William Lowell Putnam Mathematical Competition 1985–2000, The Mathematical Association of America (2002)

Kirillov A. A.
Elements of the Theory of Representations, Springer, 1976

Knapp A. W.
Lie Groups and Beyond: An Introduction, 2nd edition, Birkhäuser (2002)

Lang S.
Linear Algebra, Addison-Wesley, Reading (1968)

Lang S.
Introduction to Differentiable Manifolds, Wiley-Interscience (1962)

Lee Dong Hoon
The structure of complex Lie groups, Chapman and Hall/CRC (2002)

Lie M. S.
Untersuchungen über unendliche continuirliche Gruppen, Hirzel (1893)

Magnus W., Karrass A and Solitar D.
Combinatorial Group Theory, 2nd edition, Dover (1976)

McCrimmon K.
A Taste of Jordan Algebras, Springer (2004)

Miller W.
Lie Theory and Special Functions, Academic Press (1968)

Miller W.
Symmetry Groups and Their Applications, Academic Press (1972)

Ohtsuki T.
Quantum Invariants, World Scientific Publishing (2002)

Olver P. J.
Applications of Lie Groups to Differential Equations, Graduate Text in Mathematics, vol. 107, Springer (1986)

Ovsyannikov L. V.
Group Analysis of Differential Equations, Academic Press (1982)

Pontryagin L. S.
Topological Groups, Princeton University Press (1958)

Robinson D. J. S.
A course in the theory of groups, Springer (1996)

Schneider H. and Barker G. P.
Matrices and Linear Algebra, Dover Publications (1989)

Scott W. R.
Group Theory, Dover Publications (1987)

Sniatychi J.
Geometric Quantization and Quantum Mechanics, Springer (1980)

Spivak M.
Calculus on Manifolds, Benjamin (1966)

Spivak M.
Differential Geometry, Publish or Perish (1970)

Steeb W.-H. and Hardy Y.
Matrix Calculus and Kronecker Product, 2nd edition, World Scientific Publishing (2011)

Steeb W.-H.
Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra, 2nd edition, World Scientific Publishing (2007)

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, third edition, Volume I: Introductory Level, World Scientific Publishing (2009)

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, third edition, Volume II: Advanced Level, World Scientific Publishing (2009)

Sternberg S.
Lie Algebras (2004)
`http://www.math.harvard.edu/~shlomo/docs/lie_algebras.pdf`

Tung W. K.
Group Theory Physics: Problems and Solutions by Michael Aivazis, World Scientific (1991)

Varadarajan V. S.
Lie Groups, Lie Algebras and Their Representations, Springer (2004)

von Westenholz C.
Differential Forms in Mathematical Physics, revised edition, North-Holland (1981)

Warner F. W.
Foundations of Differentiable Manifolds and Lie Groups, Scott Foresman (1971)

Wawrzynczyk A.
Group Representations and Special Functions, D. Reidel (1984)

Weyl H.
Gruppentheorie und Quantenmechanik, Hirzel (1928)

Wybourne B. G.
Classical Groups for Physicists, John Wiley (1974)

Publications

Agricola I., *Old and New on the Exceptional Group G_2*, Notices of the AMS, **55**, 922 (2008)

Boyer C. P. and Kalnins E. G., *Symmetries of the Hamilton-Jacobi equation*, J. Math. Phys. **18**, 1032 (1977)

Campoamor-Stursberg R., *Casimir operators induced by the Maurer-Cartan equations*, J. Phys. A : Math. Theor. **41**, 365207 (2008)

Campoamor-Stursberg R., *The structure of the invariants of perfect Lie algebras*, J. Phys. A: Math. Gen. 6709–6723 (2003)

Chinea F. J. and Guil F., *Local and non-local conserved currents for an equation related to the nonlinear σ-model*, J. Phys. A : Math. Gen. **15**, 2349 (1982)

DasGupta Ananda, *Disentanglement formulas: An alternative derivation and some applications to squeezed coherent states*, American J. Phys. **64** 1422 (1996)

Delduc F. and Fehér L., *Regular conjugacy classes in the Weyl group and integrable hierarchies*, J. Phys. A : Math. Gen. **28**, 5843 (1995)

Fairlie D. B., Nuyts J. and Zachos C. K., *A Presentation for the Virasoro and Super-Virasoro Algebras*, Comm. Math. Phys. **17**, 595 (1988)

Fernández F. M., *Time Evolution and Lie Algebras*, Phys. Rev. A **40**, 41 (1989)

Flaschka H. *A commutator representation of Painlevé equations*, J. Math. Phys. **21**, 1016-1018 (1980)

Fokas A. S., *Group Theoretical Aspects of Constants of Motion and Separable Solutions in Classical Mechanics*, Journal of Mathematical Analysis and Applications **68**, 347 (1979)

Franco Jos
Lie Algebras Exercises
`http://www.doschivos.com/mate.htm`

Garsia A., Musiker G., Wallach and Xin G., *Invariants, Kronecker Prod-*

ucts and Combinatorics of some Remarkable Diophantine Systems, `arXiv:0810.0060` (2008)

Harrison B. K., *On Methods of Finding Bäcklund Transformations in Systems with more than Two Independent Variables*, Nonlinear Mathematical Physics, **2**, 201 (1995)

Harrison B. K., *The Differential Form Method for Finding Symmetries*, Symmetry, Integrability and Geometry: Methods and Applications **1**, 1 (2005)

Hsu L. and Kamran N., *Symmetries of Second-Order Ordinary Differential Equations and Elie Cartan's Method of Equivalence*, Letters in Mathematical Physics **15**, 91 (1988)

Kalnins E. G. and Miller W., *Lie theory and separation of variables. 11. The EPD equation*, J. Math. Phys. **17**, 369-377 (1976)

Koca Mehmet, *E_8 lattice with icosians and Z_5 symmetry*, J. Phys. A: Math. Gen. **22**, 4125-4134 (1989)

Kyriakopoulos E., *Lie algebra of the hypergeometric functions*, J. Math. Phys. **15**, 172-176 (1974)

Kyriakopoulos E., *Generating functions of the hypergeometric functions*, J. Math. Phys. **15**, 753 (1974)

Legaré M., *Symmetry Reductions of the Lax Pair of the Four-Dimensional Euclidean Self-Dual Yang-Mills Equations*, Nonlinear Mathematical Physics, **3**, 266 (1996)

Luan Pi-Gang, Lee H. C. and Zhang R. B., *Colored solutions of the Yang-Baxter equation from representations of $U_q g\ell(2)$*, J. Math. Phys. **41**, 6529 (2000)

Mahomed F. M. and Leach P. G. L., *Symmetry Lie algebras of nth order ordinary differential equations*, J. Math. Anal. Appl. **151**, 80 (1990)

McGehee R., *Triple Collision in the Collinear Three-Body Problem*, Inventiones Mathematica **27**, 191 (1974)

Morozov O. I., *Structure of Symmetry Groups via Cartan's Method: Survey of Four Approaches*, Symmetry, Integrability and Geometry: Methods and

Applications **1**, 1 (2005)

Nattermann and Doebner H.-D., *Gauge Classification, Lie Symmetries and Integrability of a Family of Nonlinear Schrödinger Equations*, Nonlinear Mathematical Physics **3**, 302 (1996)

Olafsson S., *New non-linear evolution equations related to some superloop algebras*, J. Phys. A : Math. Gen. **22**, 157 (1989)

Patera J., Sharp R. T., Winternitz R. and Zassenhaus H., *Invariants of real low dimension Lie algebras*, **17**, 986 (1976)

Patera J., Sharp R. T., Winternitz R. and Zassenhaus H., *Subgroups of the Poincaré group and their invariants*, **17**, 977 (1976)

Perelomov A. M., *Instanton-like solutions in chiral models*, Physica 4D, 1 (1981)

Planat M. and Jorrand P., *Group theory for quantum gates and quantum coherence*, J. Phys. A: Math. Theor. **41**, 182001 (2008)

Saenkarun S. and Loutsiouk A., *Representations of G_2*, Thai Journal of Mathematics **5**, 309 (2007)

Sakovick S. Yu., *On conservation laws and zero-curvature representations of the Liouville equation*, J. Phys. A : Math. Gen. **27**, L125 (1994)

Sam Steven V., *Solutions to "Introduction to Lie Algebras and Representation Theory"* by James E. Humphreys (2010)

Shadwick W. F., *The KdV prolongation algebra*, J. Math. Phys. **21**, 454 (1980)

Suzuki Masuo, *Decomposition Formulas of Exponential Operators and Lie Exponentials with Some Applications to Quantum Mechanics and Statistical Physics*, J. Math. Phys. **26**, 601 (1985)

Swift J. W., *Hopf bifurcation with the symmetry of the square*, Nonlinearity **1**, 333 (1988)

Wahlquist H. D. and Estabrook F. B., *Prolongation structures of nonlinear evolution equations*, J. Math. Phys. **16**, 1 (1975)

Index