Problems and Solutions in Differential Geometry and Applications

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Preface

The purpose of this book is to supply a collection of problems in differential geometry.

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Contents

Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidian space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
M	manifold
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\|z\|$	modulus of complex number z
	$(\|x+iy\| = (x^2+y^2)^{1/2},\ \ x, y \in \mathbf{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbf{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\\| . \\|$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbf{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbf{R}^3$
S_n	symmetric group
A_n	alternating group
D_n	n-th dihedral group

A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\text{rank}(A)$	rank of matrix A
A^T	transpose of matrix A
$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
V	vector field of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$\otimes$	tensor product
$\wedge$	exterior product, Grassmann product, wedge product
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

Chapter 1

Curves, Surfaces and Manifolds

Problem 1. Consider the compact differentiable manifold

$$S^2 := \{ (x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 + x_3^2 = 1 \}.$$

An element $\eta \in S^2$ can be written as

$$\eta = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$$

where $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$. The *stereographic projection* is a map

$$\Pi \,:\, S^2 \setminus \{ (0, 0, -1) \} \to \mathbb{R}^2$$

given by

$$x_1(\theta, \phi) = \frac{2\sin(\theta)\cos(\phi)}{1 + \cos(\theta)}, \qquad x_2(\theta, \phi) = \frac{2\sin(\theta)\sin(\phi)}{1 + \cos(\theta)}.$$

(i) Let $\theta = 0$ and ϕ arbitrary. Find x_1, x_2. Give a geometric interpretation.
(ii) Find the inverse of the map, i.e., find

$$\Pi^{-1} \,:\, \mathbb{R}^2 \to S^2 \setminus \{ (0, 0, -1) \}.$$

Problem 2. The parameter representation for the *torus* is given by

$$x_1(u_1, u_2) = (R + r\cos(u_1))\cos(u_2)$$
$$x_2(u_1, u_2) = (R + r\cos(u_1))\sin(u_2)$$
$$x_3(u_1, u_2) = r\sin(u_1)$$

where $u_1 \in [0, 2\pi]$ and $u_2 \in [0, 2\pi]$ and $R > r$. Let

$$\mathbf{t}_1(u_1, u_2) := \begin{pmatrix} \partial x_1/\partial u_1 \\ \partial x_2/\partial u_1 \\ \partial x_3/\partial u_1 \end{pmatrix}, \qquad \mathbf{t}_2(u_1, u_2) := \begin{pmatrix} \partial x_1/\partial u_2 \\ \partial x_2/\partial u_2 \\ \partial x_3/\partial u_2 \end{pmatrix}.$$

The surface element of the torus is given by

$$do = \sqrt{g} du_1 du_2$$

where

$$g = g_{11}g_{22} - g_{12}g_{21}$$

and

$$g_{jk}(u_1, u_2) := \mathbf{t}_j(u_1, u_2) \cdot \mathbf{t}_k(u_1, u_2)$$

with $\cdot$ denoting the scalar product. Calculate the surface area of the torus.

Problem 3. Let $x, y \in \mathbb{R}$. Consider the map

$$\xi(x, y) = \frac{x}{1 + x^2 + y^2}, \qquad \eta(x, y) = \frac{y}{1 + x^2 + y^2}, \qquad \zeta(x, y) = \frac{x^2 + y^2}{1 + x^2 + y^2}.$$

Calculate

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2.$$

Discuss. Find $\xi(0, 0)$, $\eta(0, 0)$, $\zeta(0, 0)$ and $\xi(1, 1)$, $\eta(1, 1)$, $\zeta(1, 1)$.

Problem 4. Consider the two-dimensional unit sphere

$$S^2 := \{ \mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \}.$$

Show that S^2 is an orientable two-dimensional differentiable manifold. Use the following orientation-preserving atlas

$$U_1 = \{ \mathbf{x} \in S^2 : x_3 > 0 \}, \qquad U_2 = \{ \mathbf{x} \in S^2 : x_3 < 0 \},$$

$$U_3 = \{ \mathbf{x} \in S^2 : x_2 > 0 \}, \qquad U_4 = \{ \mathbf{x} \in S^2 : x_2 < 0 \},$$

$$U_5 = \{ \mathbf{x} \in S^2 : x_1 > 0 \}, \qquad U_6 = \{ \mathbf{x} \in S^2 : x_1 < 0 \}.$$

Problem 5. $\mathbb{C}^n$ is an n-dimensional complex manifold. The complex projective space $\mathbf{P}^n(\mathbb{C})$ which is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$, that is

$$\mathbf{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{ \mathbf{0} \}) / \sim$$

for the equivalence relation

$$(u_0, u_1, \ldots, u_n) \sim (v_0, v_1, \ldots, v_n) \Leftrightarrow \exists\, \lambda \in \mathbb{C}^* \ : \ \lambda u_j = v_j \ \forall\ 0 \le j \le n$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Show that $\mathbf{P}^1(\mathbb{C})$ is a one-dimensional complex manifold.

Problem 6. Let

$$S^n := \{ (x_1, x_2, \ldots, x_{n+1}) \ : \ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 \}.$$

(i) Show that S^3 can be considered as a subset of $\mathbb{C}^2$ ($\mathbb{C}^2 \cong \mathbb{R}^4$)

$$S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 1 \}.$$

(ii) The *Hopf map* $\pi : S^3 \to S^2$ is defined by

$$\pi(z_1, z_2) := (\overline{z}_1 z_2 + \overline{z}_2 z_1, -i\overline{z}_1 z_2 + i\overline{z}_2 z_1, |z_1|^2 - |z_2|^2).$$

Find the parametrization of S^3, i.e. find $z_1(\theta, \phi)$, $z_2(\theta, \phi)$ and thus show that indeed π maps S^3 onto S^2.
(iii) Show that $\pi(z_1, z_2) = \pi(z_1', z_2')$ if and only if $z_j' = e^{i\alpha} z_j$ ($j = 1, 2$) and $\alpha \in \mathbb{R}$.

Problem 7. The n-dimensional complex projective space $\mathbb{C}\mathbf{P}^n$ is the set of all complex lines on $\mathbb{C}^{n+1}$ passing through the origin. Let f be the map that takes nonzero vectors in $\mathbb{C}^2$ to vectors in $\mathbb{R}^3$ by

$$f(z_1, z_2) = \left(\frac{z_1\overline{z}_2 + \overline{z}_1 z_2}{z_1\overline{z}_1 + \overline{z}_2 z_2}, \frac{z_1\overline{z}_2 - \overline{z}_1 z_2}{i(z_1\overline{z}_1 + \overline{z}_2 z_2)}, \frac{z_1\overline{z}_1 - \overline{z}_2 z_2}{z_1\overline{z}_1 + \overline{z}_2 z_2} \right)$$

The map f defines a bijection between $\mathbb{C}\mathbf{P}^1$ and the unit sphere in $\mathbb{R}^3$. Consider the normalized vectors in $\mathbb{C}^2$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -i \end{pmatrix}.$$

Apply f to these vectors in $\mathbb{C}^2$.

Problem 8. The *stereographic projection* is the map $\phi : S^2 \setminus N \to \mathbb{C}$ defined by

$$\phi(x, y, z) = \frac{x}{1 - z} + i\frac{y}{1 - z}.$$

Show that the inverse of the stereographic projection takes a complex number $u + iv$ ($u, v \in \mathbb{R}$)

$$\left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right)$$

to the unit sphere.

Problem 9. Show that the projective space $\mathbf{P}^n(\mathbb{C})$ is a compact manifold.

Problem 10. Consider the solid torus $M = S^1 \times D^2$, where D^2 is the unit disk in $\mathbb{R}^2$. On it we define coordinates (φ, x, y) such that $\varphi \in S^1$ and $(x, y) \in D^2$, that is, $x^2 + y^2 \leq 1$. Using these coordinates we define the map

$$f : M \to M, \qquad f(\varphi, x, y) = \left(2\varphi, \frac{1}{10}x + \frac{1}{2}\cos(\varphi), \frac{1}{10}y + \frac{1}{2}\sin(\varphi)\right).$$

(i) Show that this map is well-defined, that is, $f(M) \subset M$.
(ii) Show that f is injective.

Problem 11. Show that a parameter representation of the *hyperboloid*

$$x_1^2 - x_2^2 - x_3^2 = 1$$

is given by

$$x_1(t) = \cosh(t), \quad x_2(t) = \sinh(t)\cos(\theta), \quad x_3(t) = \sinh(t)\sin(\theta)$$

where $0 \leq t < \infty$ and $0 \leq \theta \leq 2\pi$.

Problem 12. Consider the upper sheet of the *hyperboloid*

$$H^2 := \{\, \mathbf{v} \in \mathbb{R}^3 \,:\, \mathbf{v}^2 = v_0^2 - v_1^2 - v_2^2 = 1\,,\, v_0 > 0 \,\}.$$

Find a parametrization for $\mathbf{v}$.

Problem 13. Find the stereographic projection of the two-dimensional sphere

$$S^2 := \{\, \mathbf{v} \in \mathbb{R}^3 \,:\, \mathbf{v}^2 = v_0^2 + v_1^2 + v_2^2 = 1 \,\}.$$

Problem 14. Consider the curve

$$\alpha(t) = \begin{pmatrix} t \\ \cosh(t) \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Show that the *curvature* is given by

$$\kappa(t) = \frac{1}{\cosh^2(t)}.$$

Problem 15. Consider the *unit ball*

$$S^2 := \{ (x, y, z) \in \mathbb{R}^3 \, : \, x^2 + y^2 + z^2 = 1 \}.$$

Let $\alpha(t) = (x(t), y(t), z(t))$ be a parametrized differentiable curve on S^2. Show that the vector $(x(t), y(t), z(t))$ (t fixed) is normal to the sphere at the point $(x(t), y(t), z(t))$.

Problem 16. A generic *superquadric surface* can be defined as a closed surface in $\mathbb{R}^3$

$$\mathbf{r}(\eta, \omega) \equiv \begin{pmatrix} x(\eta, \omega) \\ y(\eta, \omega) \\ z(\eta, \omega) \end{pmatrix} = \begin{pmatrix} a_1 \cos^{\epsilon_1}(\eta) \cos^{\epsilon_2}(\omega) \\ a_2 \cos^{\epsilon_1}(\eta) \sin^{\epsilon_2}(\omega) \\ a_3 \sin^{\epsilon_1}(\eta) \end{pmatrix}, \quad -\pi/2 \le \eta \le \pi/2, \;\; -\pi \le \omega < \pi.$$

There are five parameters ϵ_1, ϵ_2, a_1, a_2, a_3. Here ϵ_1 and ϵ_2 are the deformation parameters that control the shape with $\epsilon_1, \epsilon_2 \in (0, 2)$. The parameter a_1, a_2, a_3 define the size in x, y and z direction. Find the implicit representation.

Problem 17. Let

$$\begin{aligned} x_1(z, \bar{z}) &= \operatorname{sech}\left(\frac{z + \bar{z}}{2}\right) \cosh\left(\frac{z - \bar{z}}{2}\right) \\ x_2(z, \bar{z}) &= i\operatorname{sech}\left(\frac{z + \bar{z}}{2}\right) \sinh\left(\frac{z - \bar{z}}{2}\right) \\ x_3(z, \bar{z}) &= -\tanh\left(\frac{z + \bar{z}}{2}\right). \end{aligned}$$

Find $x_1^2 + x_2^2 + x_3^2$. Note that

$$\operatorname{sech}(z) := \frac{2}{e^z + e^{-z}}.$$

Problem 18. Let $\mathbf{d} = (d_0, d_1, \ldots, d_n)$ be an $(n+1)$-tuple of integers $d_j > 1$. We define

$$V(\mathbf{d}) := \{ \mathbf{z} = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \, : \, f(\mathbf{z}) := z_0^{d_0} + z_1^{d_1} + \cdots + z_n^{d_n} = 0 \}.$$

Let $\mathbb{S}^{2n+1}$ denote the unit sphere in $\mathbb{C}^{n+1}$, i.e.

$$z_0 \bar{z}_0 + z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n = 2.$$

We define

$$\Sigma(\mathbf{d}) := V(\mathbf{d}) \cap \mathbb{S}^{2n+1}.$$

Show that $\Sigma(\mathbf{d})$ is a smooth manifold of dimension $2n-1$. The manifolds $\Sigma(\mathbf{d})$ are called *Brieskorn manifolds*.

Problem 19. Let $w \in \mathbb{C}$. Consider the *stereographic projection*

$$r(w) = \left(\frac{2\Re(w)}{|w|^2+1}, \frac{2\Im(w)}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1} \right).$$

(i) Let $w = 1$. Find $r(w)$.
(ii) Let $w = i$. Find $r(w)$.
(iii) Let $w = e^{i\phi}$. Find $r(w)$.
(iv) Let $w = 1/2$. Find $r(w)$.

Problem 20. (i) Consider the rational curve in the plane

$$y^2 = x^2 + x^3.$$

Find the parameter representation $x(t)$, $y(t)$.
(ii) Consider the rational curve in the plane

$$x^2 + y^2 = 1.$$

Find the parameter representation $x(t)$, $y(t)$.

Problem 21. Let $a > 0$. Consider the transformation Minkowski coordinates $(t, z$ and *Rindler coordinates* (ζ, η)

$$t(\zeta,\eta) = \frac{1}{a}\exp(a\zeta)\sinh(a\eta), \qquad z(\zeta,\eta) = \frac{1}{a}\exp(a\zeta)\cosh(a\eta).$$

Find the inverse transformation.

Problem 22. Show that the *helicoid*

$$\mathbf{x}(u,v) = (a\sinh(v)\cos(u), a\sinh(v)\sin(u), au)$$

is a *minimal surface*.

Problem 23. Let A be a symmetric $n \times n$ matrix over $\mathbb{R}$. Let $0 \neq b \in \mathbb{R}$. Show that the surface

$$M = \{\, \mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{x}^T A \mathbf{x} = b \,\}$$

is an $(n-1)$ dimensional submanifold of the manifold $\mathbb{R}^n$.

Problem 24. Let C be the topological space given by the boundary of

$$[0,1]^n := [0,1] \times \cdots \times [0,1].$$

This means C is the surface of the n-dimensional unit cube. Show that C can be endowed with the structure of a differential manifold.

Problem 25. Find the Gaussian curvature for the *torus* given by the parametrization

$$\mathbf{x}(u,v) = ((a + r\cos(u))\cos(v), (a + r\cos(u))\sin(v), r\sin(u))$$

where $0 < u < 2\pi$ and $0 < v < 2\pi$.

Problem 26. The *Möbius band* can be parametrized as

$$\mathbf{x}(u,v) = ((2 - v\sin(u/2))\sin(u), (2 - v\sin(u/2))\cos(u), v\cos(u/2))\,.$$

Show that the *Gaussian curvature* is given by

$$K(u,v) = \frac{1}{(v^2/4 + (2 - v\sin(u/2))^2)^2}.$$

Problem 27. Given the surface in $\mathbb{R}^3$

$$f(t,\theta) = \left(\left(1 + t\sin\frac{\theta}{2}\right)\cos(\theta), \left(1 + t\cos\frac{\theta}{2}\right)\sin(\theta), t\sin(\frac{\theta}{2})\right)$$

where

$$t \in \left(-\frac{1}{2}, \frac{1}{2}\right) \qquad \theta \in \mathbb{R}.$$

(i) Build three models of this surface using paper, glue and a scissors. Color the first model with the South African flag. For the second model keep t fixed (say $t = 0$) and cut the second model along the θ parameter. For the third model keep θ fixed (say $\theta = 0$) and cut the model along the t parameter. Submit all three models.
(ii) Describe the curves with respect to t for θ fixed. Describe the curve with respect to θ for t fixed.
(iii) The map given above can also be written in the form

$$x(t,\theta) = \left(1 + t\sin\frac{\theta}{2}\right)\cos(\theta)$$

$$y(t,\theta) = \left(1 + t\cos\frac{\theta}{2}\right)\sin(\theta)$$

$$z(t,\theta) = t\sin(\frac{\theta}{2}).$$

For fixed t the curve

$$(x(\theta), y(\theta), z(\theta))$$

can be considered as a solution of a differential equation. Find this differential equation. Then t plays the role of a bifurcation parameter.

Problem 28. Let M be a differentiable manifold. Suppose that $f : M \to M$ is a diffeomorphism with $N_m(f) < \infty$, $m = 1, 2, \ldots$. Here $N_m(f)$ is the number of fixed points of the m-th iterate of f, i.e. $f^{(m)}$. One defines the *zeta function* of f as the formal power series

$$\zeta_f(t) := \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m(f) t^m\right).$$

(i) Show that $\zeta_f(t)$ is an invariant of the topological conjugacy class of f.
(ii) Find $N_m(f)$ for the map $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = \sinh(x)$.

Problem 29. Consider the curve given by

$$x_1(t) = \cos(t)(2\cos(t) - 1)$$
$$x_2(t) = \sin(t)(2\cos(t) - 1)$$

where $t \in [0, 2\pi]$. Draw the curve with GNUPLOT. Find the longest distance between two points on the curve.

Problem 30. (i) Consider the transformation in $\mathbb{R}^3$

$$x_0(a,\theta_1) = \cosh(a)$$
$$x_1(a,\theta_1) = \sinh(a)\sin(\theta_1)$$
$$x_2(a,\theta_1) = \sinh(a)\cos(\theta_1)$$

where $a \geq 0$ and $0 \leq \theta_1 < 2\pi$. Find

$$x_0^2 - x_1^2 - x_2^2.$$

(ii) Consider the transformation in $\mathbb{R}^4$

$$x_0(a,\theta_1,\theta_2) = \cosh(a)$$
$$x_1(a,\theta_1,\theta_2) = \sinh(a)\sin(\theta_2)\sin(\theta_1)$$
$$x_2(a,\theta_1,\theta_2) = \sinh(a)\sin(\theta_2)\cos(\theta_1)$$
$$x_3(a,\theta_1,\theta_2) = \sinh(a)\cos(\theta_2)$$

where $a \geq 0$, $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_2 \leq \pi$. Find

$$x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Extend the transformation to $\mathbb{R}^n$.

Problem 31. A fixed charge Q is located on the z-axis with coordinates $\mathbf{r}_a = (0, 0, d/2)$, where d is interfocal distance of the *prolate spheroidal coordinates*

$$x(\eta, \xi, \phi) = \frac{1}{2} d((1-\eta^2)(\xi^2 - 1))^{1/2} \cos(\phi)$$
$$y(\eta, \xi, \phi) = \frac{1}{2} d((1-\eta^2)(\xi^2 - 1))^{1/2} \sin(\phi)$$
$$z(\eta, \xi, \phi) = \frac{1}{2} d\eta\xi$$

where $-1 \leq \eta \leq +1$, $1 \leq \xi \leq \infty$, $0 \leq \phi \leq 2\pi$. Express the *Coulomb potential*

$$V = \frac{Q}{|\mathbf{r} - \mathbf{r}_a|}$$

in prolate spheroidal coordinates.

Problem 32. Let $\alpha, \theta, \phi, \omega \in \mathbb{R}$. Consider the vector in $\mathbb{R}^5$

$$\mathbf{x}(\alpha, \theta, \phi, \omega) = \begin{pmatrix} \cosh(\alpha)\sin(\theta)\cos(\phi) \\ \cosh(\alpha)\sin(\theta)\sin(\phi) \\ \cosh(\alpha)\cos(\theta) \\ \sinh(\alpha)\cos(\omega) \\ \sinh(\alpha)\sin(\omega) \end{pmatrix}.$$

Find

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2.$$

This vector plays a role for the Lie group $SO(3,2)$. The invariant measure is

$$\cosh^2(\alpha)\sinh(\alpha)\sin(\theta) d\alpha d\theta d\phi d\omega.$$

Problem 33. Show that the surface ∂C of the unit cube

$$C = \{ (x_1, x_2, x_3) \, : \, 0 \leq x_1 \leq 1, \; 0 \leq x_2 \leq 1, \; 0 \leq x_3 \leq 1 \}$$

can be made into a differentiable manifold.

Problem 34. The equation of the *monkey saddle* surface in $\mathbb{R}^3$ is given by

$$x_3 = x_1(x_1^2 - 3x_2^2)$$

with the parameter representation

$$x_1(u_1,u_2) = u_1, \qquad x_2(u_1,u_2) = u_2, \quad x_3(u_1,u_2) = u_1^3 - 3u_1u_2^2.$$

Find the mean and Gaussian curvature.
Let

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Find g restricted to the monkey saddle surface. Find the curvature scalar.

Problem 35. Let $a > 0$ and consider the surface

$$\begin{aligned} x_1(u_1,u_2) &= a\frac{1-u_2^2}{1+u_2^2}\cos(u_1) \\ x_2(u_1,u_2) &= a\frac{1-u_2^2}{1+u_2^2}\sin(u_1) \\ x_3(u_1,u_2) &= \frac{2au_2}{1+u_2^2}. \end{aligned}$$

Find $x_1^2 + x_2^2 + x_3^2$.

Problem 36. Show that an *open disc*

$$D^2 := \{ (x_1,x_2) \in \mathbb{R}^2 \, : \, x_1^2 + x_2^2 < 1 \}$$

is homeomorphic to $\mathbb{R}^2$.

Problem 37. Let $r > 0$. The *Klein bagel* is a specific immersion of the Klein bottle manifold into three dimensions with the parameter representation

$$\begin{aligned} x_1(u_1,u_2) &= (r + \cos(u_1/2)\sin(u_2) - \sin(u_1/2)\sin(2u_2))\cos(u_1) \\ x_2(u_1,u_2) &= (r + \cos(u_1/2)\sin(u_2) - \sin(u_1/2)\sin(2u_2))\sin(u_1) \\ x_3(u_1,u_2) &= \sin(u_1/2)\sin(u_2) + \cos(u_1/2)\sin(2u_2) \end{aligned}$$

where $0 \leq u_1 < 2\pi$ and $0 \leq u_2 < 2\pi$. Find the mean curvature and Gaussian curvature.

Problem 38. Consider the circle

$$S^1 := \{(x_1,x_2) \in \mathbb{R}^2 \, : \, x_1^2 + x_2^2 = 1 \}$$

and the square

$$I^2 = \{ (x_1,x_2) \in \mathbb{R} \, : \, (|x_1| = 1, |x_2| \leq 1), (|x_1| \leq 1, |x_2| = 1) \}.$$

Find a homeomorphism.

Problem 39. The transformation between the orthogonal ellipsoidal coordinates (ρ, μ, ν) and the Cartesian coordinates (x_1, x_2, x_3) is

$$x_1^2 = \frac{\rho^2\mu^2\nu^2}{h^2k^2}$$
$$x_2^2 = \frac{(\rho^2-\mu^2)(\mu^2-h^2)(h^2-\nu^2)}{h^2(k^2-h^2)}$$
$$x_3^2 = \frac{(\rho^2-k^2)(k^2-\mu^2)(k^2-\nu^2)}{k^2(k^2-h^2)}$$

where $k^2 = a_1^2 - a_3^2$, $h^2 = a_1^2 - a_2^2$ and $a_1 > a_2 > a_3$ denote the three semi-axes of the ellipsoid. The three surfaces in $\mathbb{R}^3$, $\rho =$ constant, $(k \le \rho \le \infty)$, $\mu =$ constant, $(h \le \mu \le k)$ and $\nu =$ constant, $(0 \le \nu \le h$, represent ellipsoids and hyperboloids of one and two sheets, respectively. Find the inverse transformation.

Problem 40. Let $x_1, x_2, x_3 \in \mathbb{R}$ and

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

Let $w \in \mathbb{C}$ with

$$w = \frac{x_1 + ix_2}{1 + x_3}.$$

Find x_1, x_2, x_3 as functions of w and w^*.

Problem 41. (i) Let M be a manifold and $f : M \to M$, $g : M \to M$. Assume that f is invertible. Then we say that the map f is a *symmetry* of the map g if

$$f \circ g \circ f^{-1} = g.$$

Let $M = \mathbb{R}$ and $f(x) = \sinh(x)$. Find all g such that $f \circ g \circ f^{-1} = g$.
(ii) Let f and g be invertible maps. We say that g has a reversing symmetry f if

$$f \circ g \circ f^{-1} = g^{-1}.$$

Let $M = \mathbb{R}$ and $f(x) = \sinh(x)$. Find all g that satisfy this equation.

Problem 42. Consider the map $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x) = (2\cos(x - \pi/2), \sin(2(x - \pi/2))).$$

Show that $(\mathbf{f}, \mathbb{R})$ is an immersed submanifold of the manifold $\mathbb{R}^2$, but not an embedded submanifold.

Problem 43. Use GNU-plot to plot the curve

$$x_1(t) = \cos(3t), \quad x_2(t) = \sin(5t)$$

in the (x_1, x_2)-plane with $t \in [0, 2\pi]$.

Problem 44. A special set of coordinates on S^n called spheroconical (or elliptic spherical) coordinates are defined as follows: For a given set of real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}$ and nonzero x_1, ..., x_{n+1} the coordinates λ_j $(j = 1, \ldots, n)$ are the solutions of the equation

$$\sum_{j=1}^{n+1} \frac{x_j^2}{\lambda - \alpha_j}.$$

Find the solutions for $n = 2$.

Problem 45. Given the surface in $\mathbb{R}^3$

$$f(t,\theta) = \left(\left(1 + t\sin\frac{\theta}{2}\right)\cos(\theta), \left(1 + t\cos\frac{\theta}{2}\right)\sin(\theta), t\sin\frac{\theta}{2}\right)$$

where $t \in (-1/2, 1/2)$ and $\theta \in \mathbb{R}$.
(i) Build three models of this using paper, glue and a scissor. Color the first model with the South African flag. For the second model keep t fixed (say $t = 0$) and cut the second model along the θ parameter. For the third model keep θ fixed (say $\theta = 0$) and cut the model along the t parameter. Submit all three models.
(ii) Describe the curves with respect to t for θ fixed. Describe the curves with respect to θ for t fixed.
(iii) The map given above can also be written in the form

$$x(t,\theta) = \left(1 + t\sin\frac{\theta}{2}\right)\cos(\theta)$$
$$y(t,\theta) = \left(1 + t\cos\frac{\theta}{2}\right)\sin(\theta)$$
$$z(t,\theta) = t\sin(\frac{\theta}{2}).$$

For fixed t the curve $(x(\theta), y(\theta), z(\theta))$ can be considered as a solution of an system of first order differential equations. Find this system, where t plays the role of a bifurcation parameter.

Problem 46. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $n \geq 2$. Let $r \in \mathbb{N}$ or ∞, I be a non-empty interval of real numbers and t in I. A vector-valued function

$$\gamma : I \to \mathbb{R}^n$$

of class C^r (this means that γ is r times continuously differentiable) is called a parametric curve of class C^r of the curve γ. t is called the parameter of the curve γ. The parameter t may represent time and the curve $\gamma(t)$ as the trajectory of a moving particle in space. If I is a closed interval $[a,b]$, then $\gamma(a)$ the starting point and $\gamma(b)$ is the endpoint of the curve γ. If $\gamma : (a,b) \to \mathbb{R}^n$ is injective, we call the curve simple. If γ is a parametric curve which can be locally described as a power series, we call the curve analytic or of class C^ω. A C^k-curve

$$\gamma : [a,b] \to \mathbb{R}^n$$

is called regular of order m if for any t in interval I

$$\{d\gamma(t)/dt, d^2\gamma(t)/dt^2, ..., d^m\gamma(t)/dt^m\} \quad m \le k$$

are linearly independent in the vector space $\mathbb{R}^n$. A *Frenet frame* is a moving reference frame of n orthonormal vectors $\mathbf{e}_j(t)$ ($j = 1, \dots, n$) which are used to describe a curve locally at each point (t). Using the Frenet frame we can describe local properties (e.g. curvature, torsion) in terms of a local reference system than using a global one like the Euclidean coordinates. Given a C^{n+1}-curve in $\mathbb{R}^n$ which is regular of order n the Frenet frame for the curve is the set of orthonormal vectors

$$\mathbf{e}_1(t), \dots, \mathbf{e}_n(t)$$

called Frenet vectors. They are constructed from the derivatives of (t) using the GramSchmidt orthogonalization algorithm with

$$\mathbf{e}_1(t) = \frac{d\gamma(t)/dt}{\|d\gamma(t)/dt\|}, \quad \mathbf{e}_j(t) = \frac{\overline{\mathbf{e}_j}(t)}{\|\overline{\mathbf{e}_j}(t)\|}, \quad j = 2, \dots, n$$

where

$$\overline{\mathbf{e}_j}(t) = \gamma^{(j)}(t) - \sum_{i=1}^{j-1} \langle \gamma^{(j)}(t), \mathbf{e}_i(t) \rangle \, \mathbf{e}_i(t)$$

where $\gamma^{(j)}$ denotes the j derivative with respect to t and $\langle \, , \, \rangle$ denotes the scalar product in the Euclidean space $\mathbb{R}^n$. The Frenet frame is invariant under reparametrization and are therefore differential geometric properties of the curve. Find the Frenet frame for the curve ($t \in \mathbb{R}$)

$$\gamma(t) = \begin{pmatrix} \cos(t) \\ t \\ \sin(t) \end{pmatrix}.$$

Problem 47. Show that the Lemniscate of Gerono $x_1^4 = x_1^2 - x_2^2$ can be parametrized by

$$(x_1(t), x_2(t)) = (\sin(t), \sin(t)\cos(t))$$

where $0 \leq t \leq \pi$.

Problem 48. Study the curve

$$x_1(t) = \cos\left(c_0 t + \frac{c_1}{\omega}\sin(\omega t)\right)$$
$$x_2(t) = -\sin\left(c_0 t + \frac{c_1}{\omega}\sin(\omega t)\right)$$

in the plane with $c_0, c_1, \omega > 0$, where c_0, c_1, ω have the dimension of a frequency and t is the time.

Problem 49. The *Hammer projection* is an equal-area cartographic projections that maps the entire surface of a sphere to the interior of an ellipse of semiaxis $\sqrt{8}$ and $\sqrt{2}$. The Hammer projection is given by the transformation between (θ, ϕ) and (x_1, x_2)

$$x_1(\theta,\phi) = \frac{\sqrt{8}\sin(\theta)\sin(\phi/2)}{\sqrt{1+\sin(\theta)\cos(\phi/2)}}, \quad x_2(\theta,\phi) = \frac{\sqrt{2}\cos(\theta)}{\sqrt{1+\sin(\theta)\cos(\phi/2)}}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.
(i) Show that $x_1^2/8 + x_2^2/2 < 1$.
(ii) Find $\theta(x_1, x_2)$ and $\phi(x_1, x_2)$.

Problem 50. Consider the surface in $\mathbb{R}^3$

$$x_1^2 + x_2^2 - x_3^2 = 1.$$

Show that parametrization of this surface is given by

$$x_1(u_1,u_2) = \cosh(u_1)\cos(u_2), \quad x_2(u_1,u_2) = \cosh(u_1)\sin(u_2), \quad x_3(u_1,u_2) = \sinh(u_1)$$

where $-1 \leq \leq 1$ and $-\pi \leq u_2 \leq \pi$.

Problem 51. (i) Let $R > 0$. Study the manifold

$$\frac{x_1^2}{R^2 e^{-\epsilon}} + \frac{x_2^2}{R^2 e^{-\epsilon}} + \frac{x_3^2}{R^2 e^{2\epsilon}} = 1$$

where ϵ is a deformation parameter.
(ii) Show that the volume V of the spheroid is given by $V = (4\pi/3)R^3$.

Problem 52. Plot the graph

$$r(\theta) = 1 + 2\cos(2\theta).$$

Problem 53. Let $a > 0$. Consider

$$x_1(u,v) = a\frac{1-v^2}{1+v^2}\cos(u), \quad x_2(u,v) = a\frac{1-v^2}{1+v^2}\sin(u), \quad x_3(u,v) = a\frac{2v}{1+v^2}.$$

(i) Show that

$$x_1^2(u,v) + x_2^2(u,v) + x_3^2(u,v) = a^2.$$

(ii) Calculate

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}$$

where $\times$ denotes the vector product. Discuss.

Problem 54. Show that the *helicoid*

$$\mathbf{x}(u,v) = (a\sinh(v)\cos(u), a\sinh(v)\sin(u), au)$$

is a minimal surface.

Problem 55. The *Enneper surface* is given by

$$x_1(u_1,u_2) = 3u_2 - 3u_1^2 u_2 + u_2^3, \quad x_2(u_1,u_2) = 3u_1 - 3u_1 u_2^2 + u_1^3, \quad x_3(u_1,u_2) = -6u_1u_2.$$

Show that the affine invariants are given by

$$F(u_1,u_2) = k(1+u_1^2+u_2^2), \quad A(u_1,u_2) = 2ku_2, \quad B(u_1,u_2) = 2ku_1$$

where $k = 3\sqrt{6}$.

Problem 56. (i) Show that the map $\mathbf{f} : (\pi/4, 7\pi/4) \to \mathbb{R}^2$

$$\mathbf{f}(\theta) = \begin{pmatrix} \sin(\theta)\cos(2\theta) \\ \cos(\theta)\cos(2\theta) \end{pmatrix}$$

is an injective immersion.
(ii) Show that the image of $\mathbf{f}$ is an injectively immersed submanifold.

Problem 57. Let $t \in (0,1)$. *Minimal Thomson surfaces* are given by

$$\begin{aligned} x_1(u_1,u_2) &= -(1-t^2)^{-1/2}(tu_2 + \cos(u_1)\sinh(u_2)) \\ x_2(u_1,u_2) &= (1-t^2)^{-1/2}(u_1 + t\sin(u_1)\cosh(u_2)) \\ x_3(u_1,u_2) &= \sin(u_1)\sinh(u_2). \end{aligned}$$

Show that the corresponding affine invariants are

$$\begin{aligned} F(u_1,u_2) &= (1-t^2)^{-1/2}(\cosh(u_2) + t\cos(u_1)) \\ A(u_1,u_2) &= (1-t^2)^{-1/2}\sinh(u_2) \\ B(u_1,u_2) &= -t(1-t^2)^{-1/2}\sin(u_1). \end{aligned}$$

Problem 58. Let n be a positive integer. Consider the manifold

$$C_n := \left\{ (x,y) \in \mathbb{R}^2 \, : \, \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\}.$$

We have a circle in the plane with radius $1/n$ and centre $(1/n, 0)$. Find a area of the circle.

Problem 59. Describe the set

$$S = \{ (x,y) \in \mathbb{R} \, : \, sin(y)\cosh(x) = 1 \}.$$

Then study the complex numbers given by $z = x + iy$ with $x, y \in S$.

Problem 60. Consider the two manifolds

$$x_1^2 + x_2^2 = 1, \qquad y_1^2 + y_2^2 = 1.$$

Show that

$$|x_1 y_1 + x_2 y_2| \leq 1.$$

Hint. Set

$$x_1(t) = \cos(t), \quad x_2(t) = \sin(t), \quad y_1(t) = \cos(\tau), \quad y_2(t) = \sin(\tau).$$

Problem 61. Consider the two-dimensional Euclidean space and the metric tensor field in polar coordinates

$$g = dr \otimes dr + r^2 d\theta \otimes d\theta.$$

Let $u \in \mathbb{R}$ and $R > 0$. Consider the transformation

$$(r, \theta) \mapsto (e^{u/R}, \theta).$$

Find the metric tensor field.

Problem 62. Consider the analytic function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = 3x_1^2 + 4x_2 + x_3$$

and the smooth surface in $\mathbb{R}^3$

$$S = \{ (x_1, x_2, x_3) \, : \, f(x_1, x_2, x_3) = -2 \}.$$

(i) Show that $\mathbf{p} = (1, 1, -9) \in \mathbb{R}^3$ satisfies $f(x_1, x_2, x_3) = -2$,

(ii) Find the normal vector $\mathbf{n}$ at $\mathbf{p}$.
(iii) Let

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Calculate $\mathbf{v}^T(\nabla f)_{\mathbf{p}}$. Find the conditions on v_1, v_2, v_3 such that $\mathbf{v}^T(\nabla f)_{\mathbf{p}} = 0$ and

$$T_{\mathbf{p}} = \{\, \mathbf{v} \,:\, \mathbf{v}^T(\nabla f)_{\mathbf{p}} = 0 \,\}.$$

Problem 63. Consider the *space cardioid*

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} (1-\cos(t))\cos(t) \\ (1-cos(t))\sin(t) \\ \sin(t) \end{pmatrix}.$$

Find the curvature and torsion.

Problem 64. Let

$$\mathbb{L}^3 = SU(2) \setminus SL(2,\mathbb{C})$$

be the homogeneous space of second order unimodular hermitian positive definite matrices. This is model of the classical Lobachevsky space. Let $g_{jk} \in \mathbb{C}$ with $j,k = 1,2$. We define

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad g_{11}g_{22} - g_{12}g_{21} = 1.$$

Now any $x \in \mathbb{L}^3$ can be written as

$$x = g^*g = \begin{pmatrix} g_{11}\bar{g}_{11} + g_{21}\bar{g}_{21} & \bar{g}_{11}g_{12} + \bar{g}_{21}g_{22} \\ g_{11}\bar{g}_{12} + g_{21}\bar{g}_{22} & g_{12}\bar{g}_{12} + g_{22}\bar{g}_{22} \end{pmatrix}.$$

Find $\det(x)$.

Problem 65. Let $\alpha \in \mathbb{R}$. Consider the 2×2 matrix

$$F(\alpha) = \begin{pmatrix} f_{11}(\alpha) & f_{12}(\alpha) \\ f_{21}(\alpha) & f_{22}(\alpha) \end{pmatrix}$$

with $f_{jk} : \mathbb{R} \to \mathbb{R}$ be analytic functions. Let

$$X := \left.\frac{dF(\alpha)}{d\alpha}\right|_{\alpha=0} = \left.\begin{pmatrix} df_{11}(\alpha)/d\alpha & df_{12}(\alpha)/d\alpha \\ df_{21}(\alpha)/d\alpha & df_{22}(\alpha)/d\alpha \end{pmatrix}\right|_{\alpha=0}$$

Find the conditions on the functions f_{jk} such that

$$\exp(\alpha X) = F(\alpha).$$

Apply the *Cayley-Hamilton theorem.* Set $f'_{jk}(0) = df_{jk}(\alpha)/d\alpha|_{\alpha=0}$ and

$$tr := f'_{11}(0) + f'_{22}(0), \qquad det := f'_{11}(0)f'_{22}(0) - f'_{12}(0)f'_{21}(0).$$

Problem 66. Consider the differential equation

$$\left(\frac{dy}{dx}\right)^3 + x\frac{dy}{dx} - y = 0$$

with the solution $y(x) = Cx + C^3$. The singular solution is given by $4x^3 + 27y^2 = 0$ as can be seen as follows. Differentiation of $4x^3 + 27y^2 = 0$ yields $ydy/dx + (2/9)x^2 = 0$. Inserting this equation into the differential equation provides

$$-\frac{8x^6}{9^2} - 2x^3y^2 - 9y^4 = 0$$

which is satisfied with $y^2 = -4x^3/27$. Draw the curve $F(x,y) = 4x^3 + 27y^2$. Find the equation of the tangent at $x_0 = -1$, $y_0 = 2/(3\sqrt{3})$.

Problem 67. A four-dimensional *torus* $S^3 \times S^1$ can be defined as

$$(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} - a)^2 + w^2 = 1$$

where $a > 1$ is the constant radius of S^3.
(i) Show that the four-dimensional torus can be parametrized as

$$\begin{aligned} x_1(\psi,\rho,\phi_1,\phi_2) &= (a + \cos(\psi))\rho\cos(\phi_1) \\ x_2(\psi,\rho,\phi_1,\phi_2) &= (a + \cos(\psi))\rho\sin(\phi_1) \\ x_3(\psi,\rho,\phi_1,\phi_2) &= (a + \cos(\psi))\sqrt{1-\rho^2}\cos(\phi_2) \\ x_4(\psi,\rho,\phi_1,\phi_2) &= (a + \cos(\psi))\sqrt{1-\rho^2}\sin(\phi_2) \\ w(\psi,\rho,\phi_1,\phi_2) &= \sin(\psi) \end{aligned}$$

where $\phi_1 \in [0, 2\pi]$, $\phi_2 \in [0, 2\pi]$, $\psi \in [0, 2\pi]$, $\rho \in [0, 1]$.
(ii) Find the metric tensor field $g_{S^3 \times S^1}$ starting of with

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 + dx_4 \otimes dx_4 + dw \otimes dw.$$

Chapter 2

Vector Fields and Lie Series

Problem 1. Consider the vector fields

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

defined on $\mathbb{R}^2$.
(i) Do the vector fields V, W form a basis of a Lie algebra? If so, what type of Lie algebra do we have.
(ii) Express the two vector fields in polar coordinates $x(r,\theta) = r\cos(\theta)$, $y(r,\theta) = r\sin(\theta)$.
(iii) Calculate the commutator of the two vector fields expressed in polar coordinates. Compare with the result of (i).

Problem 2. Consider the vector fields

$$V_1 = \frac{d}{dx}, \qquad V_2 = x\frac{d}{dx}, \qquad V_3 = x^2\frac{d}{dx}.$$

(i) Show that the vector fields form a basis of a Lie algebra under the commutator.
(ii) Find the adjoint representation of this Lie algebra.
(iii) Find the Killing form.
(iv) Find the Casimir operator.

Problem 3. Consider the vector fields

$$V_1 = \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta\sin\psi\frac{\partial}{\partial\psi}$$
$$V_2 = -\sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta\cos\psi\frac{\partial}{\partial\psi}$$
$$V_3 = \frac{\partial}{\partial\psi}.$$

Calculate the commutators and show that V_1, V_2, V_3 form a basis of a Lie algebra.

Problem 4. Let $X_1, X_2, \ldots, X_r$ be the basis of a Lie algebra with the commutator

$$[X_i, X_j] = \sum_{k=1}^{r} C_{ij}^k X_k$$

where the C_{ij}^k are the structure constants. The structure constants satisfy (third fundamental theorem)

$$C_{ij}^k = -C_{ji}^k$$
$$\sum_{m=1}^{r}\left(C_{ij}^m C_{mk}^\ell + C_{jk}^m C_{mi}^\ell + C_{ki}^m C_{mj}^\ell\right) = 0\,.$$

We replace the X_i's by c-number differential operators (vector fields)

$$X_i \mapsto V_i = \sum_{\ell=1}^{r}\sum_{k=1}^{r} x_k C_{i\ell}^k \frac{\partial}{\partial x_\ell}, \qquad i = 1, 2, \ldots, r.$$

Let

$$V_j = \sum_{n=1}^{r}\sum_{m=1}^{r} x_m C_{jn}^m \frac{\partial}{\partial x_n}.$$

Show that

$$[V_i, V_j] = \sum_{k=1}^{n} C_{ij}^k V_k$$

where

$$V_k = \sum_{n=1}^{r}\sum_{m=1}^{r} x_m C_{kn}^m \frac{\partial}{\partial x_n}.$$

Problem 5. Consider the vector fields (differential operators)

$$E = x\frac{\partial}{\partial y}, \quad F = y\frac{\partial}{\partial x}, \quad H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Show that these vector fields form a basis of a Lie algebra, i.e. calculate the commutators. Consider the basis for $n \in \mathbb{Z}$

$$\{ x^j y^k \, : \, j, k \in \mathbb{Z}, \, j + k = n \}.$$

Find $E(x^j y^k)$, $F(x^j y^k)$, $H(x^j y^k)$.

Problem 6. Show that the sets of vector fields

$$\left\{ \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}, \quad x^2 \frac{\partial}{\partial x} \right\}$$

$$\left\{ \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad x^2 \frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u} \right\}$$

$$\left\{ \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad x^2 \frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u} \right\}$$

form each a basis of the Lie algebra $s\ell(2, \mathbb{C})$ under the commutator.

Problem 7. Consider the Lie algebra $o(3, 2)$. Show that the vector fields form a basis of this Lie algebra

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x}, \quad V_3 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \frac{1}{4}x^2\frac{\partial}{\partial u}$$

$$V_4 = \frac{\partial}{\partial x}, \quad V_5 = t\frac{\partial}{\partial x} + \frac{1}{2}x\frac{\partial}{\partial u}, \quad V_6 = \frac{\partial}{\partial u}$$

$$V_7 = \frac{1}{2}x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad V_8 = \frac{1}{2}xt\frac{\partial}{\partial t} + (tu + \frac{1}{4}x^2)\frac{\partial}{\partial x} + \frac{1}{2}xu\frac{\partial}{\partial u}$$

$$V_9 = \frac{1}{4}x^2\frac{\partial}{\partial t} + ux\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}, \quad V_{10} = \frac{1}{2}x\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}.$$

Show that the vector fields $V_1, \dots, V_7$ form a Lie subalgebra.

Problem 8. Let $X_1, X_2, \dots, X_r$ be the basis of a Lie algebra with the commutator

$$[X_i, X_j] = \sum_{k=1}^{r} C_{ij}^k X_k$$

where the C_{ij}^k are the structure constants. The structure constants satisfy (third fundamental theorem)

$$C_{ij}^k = -C_{ji}^k$$

$$\sum_{m=1}^{r} \left(C_{ij}^m C_{mk}^\ell + C_{jk}^m C_{mi}^\ell + C_{ki}^m C_{mj}^\ell \right) = 0\,.$$

We replace the X_i's by c-number differential operators (linear vector fields)

$$X_i \mapsto V_i = \sum_{\ell=1}^{r} \sum_{k=1}^{r} x_k C_{i\ell}^k \frac{\partial}{\partial x_\ell}, \qquad i = 1, 2, \dots, r.$$

which preserve the commutators.
Consider the Lie algebra with $r = 3$ and the generators X_1, X_2, X_3 and the commutators

$$[X_1, X_3] = X_1, \qquad [X_2, X_3] = -X_2.$$

All other commutators are 0. The Lie algebra is solvable. Find the corresponding linear vectors fields. Find the smooth functions f such that

$$V_j f(\mathbf{x}) = 0 \qquad \text{for all } j = 1, 2, 3.$$

Problem 9. Let V, W be two smooth vector fields

$$V = f_1 \frac{\partial}{\partial u_1} + f_2 \frac{\partial}{\partial u_2} + f_3 \frac{\partial}{\partial u_3}$$
$$W = g_1 \frac{\partial}{\partial u_1} + g_2 \frac{\partial}{\partial u_2} + g_3 \frac{\partial}{\partial u_3}$$

defined on $\mathbb{R}^3$. Let $d\mathbf{u}/dt = \mathbf{f}(\mathbf{u})$ and $d\mathbf{u}/dt = \mathbf{g}(\mathbf{u})$ be the corresponding autonomous system of first order differential equations. The fixed points of V are defined by the solutions of the equations $f_j(u_1^*, u_2^*, u_3^*) = 0$ ($j = 1, 2, 3$) and the fixed points of W are defined as the solutions of the equations $g_j(u_1^*, u_2^*, u_3^*) = 0$ ($j = 1, 2, 3$). What can be said about the fixed points of $[V, W]$?

Problem 10. Consider the nonlinear differential equations

$$\frac{du}{dt} = u^2 - u, \qquad \frac{du}{dt} = -\sin(u).$$

with the corresponding vector fields

$$V = (u^2 - u)\frac{d}{du}, \qquad W = -\sin(u)\frac{d}{du}.$$

(i) Show that both differential equations admit the fixed point $u^* = 0$.
(ii) Consider the vector field given by the commutator of the two vector fields V and W, i.e. $[V, W]$. Show that the corresponding differential equation of this vector field also admits the fixed point $u^* = 0$.

Problem 11. Let $z \in \mathbb{C}$. Consider the vector field

$$L_n := z^{n+1}\frac{d}{dz}, \qquad n \in \mathbb{Z}$$

Calculate the commutator $[L_m, L_n]$.

Problem 12. Consider the vector fields

$$\frac{\partial}{\partial u_{jk}}, \quad u_{jm}\frac{\partial}{\partial u_{jk}}, \quad u_{\ell k}\frac{\partial}{\partial u_{jk}}, \quad u_{jm}u_{\ell k}\frac{\partial}{\partial u_{jk}}$$

where $j = 1, 2, \ldots, p$; $k = 1, 2, \ldots, n$; $m = 1, 2, \ldots, n$; $\ell = 1, 2, \ldots, p$. Find the commutators. Do the vector fields form a basis of a Lie algebra. Discuss.

Problem 13. Consider the vector fields

$$V_1 = \frac{\partial}{\partial r}, \quad V_2 = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad V_3 = \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}.$$

Find the commutators

$$[V_1, V_2], \quad [V_2, V_3], \quad [V_3, V_1].$$

Problem 14. Show that the differential operators (vector fields)

$$\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad (xy - z)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}$$

generate a finite-dimensional Lie algebra.

Problem 15. Consider smooth vector fields in $\mathbb{R}^3$

$$V = V_1(\mathbf{x})\frac{\partial}{\partial x_1} + V_2(\mathbf{x})\frac{\partial}{\partial x_2} + V_3(\mathbf{x})\frac{\partial}{\partial x_3}$$
$$W = W_1(\mathbf{x})\frac{\partial}{\partial x_1} + W_2(\mathbf{x})\frac{\partial}{\partial x_2} + W_3(\mathbf{x})\frac{\partial}{\partial x_3}.$$

Now

$$\text{curl}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3} \\ \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1} \\ \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2} \end{pmatrix}, \qquad \text{curl}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial W_3}{\partial x_2} - \frac{\partial W_2}{\partial x_3} \\ \frac{\partial W_1}{\partial x_3} - \frac{\partial W_3}{\partial x_1} \\ \frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2} \end{pmatrix}.$$

We consider now the smooth vector fields

$$V_c = (\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3})\frac{\partial}{\partial x_1} + (\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1})\frac{\partial}{\partial x_2} + (\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2})\frac{\partial}{\partial x_3}$$

$$W_c = (\frac{\partial W_3}{\partial x_2} - \frac{\partial W_2}{\partial x_3})\frac{\partial}{\partial x_1} + (\frac{\partial W_1}{\partial x_3} - \frac{\partial W_3}{\partial x_1})\frac{\partial}{\partial x_2} + (\frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2})\frac{\partial}{\partial x_3}.$$

Note that if

$$\alpha = V_1(\mathbf{x})dx_1 + V_2(\mathbf{x})dx_2 + V_3(\mathbf{x})dx_3$$

then

$$d\alpha = \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right) dx_2 \wedge dx_3\,.$$

(i) Calculate the commutator $[V_c, W_c]$. Assume that $[V, W] = 0$. Can we conclude $[V_c, W_c] = 0$?
(ii) Assume that $[V, W] = R$. Can we conclude that $[V_c, W_c] = R_c$?

Problem 16. Consider the first order ordinary differential equation

$$\frac{du}{dt} = u + 1$$

with the corresponding vector field

$$V = (u+1)\frac{d}{du}.$$

Calculate the map

$$u \mapsto \exp(tV)u.$$

Solve the inital value problem of the differential equation and compare.

Problem 17. Consider the vector fields

$$V_1 = (u_2 + u_1u_3)\frac{\partial}{\partial u_1} + (-u_1 + u_2u_3)\frac{\partial}{\partial u_2} + (1 + u_3^2)\frac{\partial}{\partial u_3}$$
$$V_2 = (1 + u_1^2)\frac{\partial}{\partial u_1} + (u_1u_2 + u_3)\frac{\partial}{\partial u_2} + (-u_2 + u_1u_3)\frac{\partial}{\partial u_3}$$
$$V_3 = (u_1u_2 - u_3)\frac{\partial}{\partial u_1} + (1 + u_2^2)\frac{\partial}{\partial u_2} + (u_1 + u_2u_3)\frac{\partial}{\partial u_3}.$$

Find the commutators $[V_1, V_2]$, $[V_2, V_3]$, $[V_3, V_1]$ and thus show that we have a basis if the Lie algebra $so(3, \mathbb{R})$.

Problem 18. Let $\{\,,\,\}$ denote the *Poisson bracket*. Consider the functions

$$S_1 = \frac{1}{4}(x_1^2 + p_1^2 - x_2^2 - p_2^2), \quad S_2 = \frac{1}{2}(p_1p_2 + x_1x_2, \quad S_3 = \frac{1}{2}(x_1p_2 - x_2p_1).$$

Calculate $\{S_1, S_2\}$, $\{S_2, S_3\}$, $\{S_3, S_1\}$ so thus estabilish that we have a basis of a Lie algebra. Classify the Lie algebra.

Problem 19. Consider the vector fields in $\mathbb{R}^2$

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad V_3 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}.$$

Find the fixed points of the corresponding autonomous systems of first order differential equations. Study their stability.

Problem 20. Consider in $\mathbb{R}^3$ the vector fields

$$V_{12} = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}, \quad V_{23} = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}, \quad V_{31} = x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}$$

with the commutators

$$[V_{12}, V_{23}] = V_{31}, \quad [V_{23}, V_{31}] = V_{12}, \quad [V_{31}, V_{12}] = V_{23}.$$

Thus we have a basis of the simple Lie algebra $so(3, \mathbb{R})$.
(i) Find the curl of these vector fields.
(ii) Let

$$\omega = dx_1 \wedge dx_2 \wedge dx_3$$

be the volume form in $\mathbb{R}^3$. Find the differential two-forms

$$V_{12}\rfloor\omega, \quad V_{23}\rfloor\omega, \quad V_{31}\rfloor\omega.$$

(iii) Let * be the Hodge star operator. Find the one forms

$$*(V_{12}\rfloor\omega), \quad *(V_{23}\rfloor\omega), \quad *(V_{31}\rfloor\omega).$$

Problem 21. The *Kustaanheimo-Stiefel transformation* is defined by the map from $\mathbb{R}^4$ (coordinates u_1, u_2, u_3, u_4) to $\mathbb{R}^3$ (coordinates x_1, x_2, x_3)

$$x_1(u_1, u_2, u_3, u_4) = 2(u_1u_3 - u_2u_4)$$
$$x_2(u_1, u_2, u_3, u_4) = 2(u_1u_4 + u_2u_3)$$
$$x_3(u_1, u_2, u_3, u_4) = u_1^2 + u_2^2 - u_3^2 - u_4^2$$

together with the constraint

$$u_2du_1 - u_1du_2 - u_4du_3 + u_3du_4 = 0.$$

(i) Show that

$$r^2 = x_1^2 + x_2^2 + x_3^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

(ii) Show that

$$\Delta_3 = \frac{1}{4r}\Delta_4 - \frac{1}{4r^2}V^2$$

where

$$\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_4 = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}$$

and V is the vector field

$$V = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} - u_4 \frac{\partial}{\partial u_3} + u_3 \frac{\partial}{\partial u_4}.$$

(iii) Consider the differential one form

$$\alpha = u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4.$$

Find $d\alpha$. Find $L_V \alpha$, where $L_V(.)$ denotes the Lie derivative.
(iv) Let $g(x_1(u_1, u_2, u_3, u_4), x_2(u_1, u_2, u_3, u_4), x_3(u_1, u_2, u_3, u_4))$ be a smooth function. Show that $L_V g = 0$.

Problem 22. Give four different representations of the simple Lie algebra $s\ell(2, \mathbb{R})$ using vector fields V_1, V_2, V_3 which have to satisfy

$$[V_1, V_2] = V_1, \quad [V_2, V_3] = V_3, \quad [V_1, V_3] = 2V_2.$$

Problem 23. (i) Let $n \geq 1$. The *Heisenberg group* $\mathbb{H}^n$ can be considered as $\mathbb{C} \times \mathbb{R}$ endowed with a polynomial group law $\cdot : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$. Its Lie algebra identifies with the tangent space $T_0\mathbb{H}^n$ at the identity $0 \in \mathbb{H}^N$. Consider the tangent bundle $T\mathbb{H}^n$, where

$$X_j(p) := \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t}, \quad Y_j(p) := \frac{\partial}{\partial y_j} - \frac{x_j}{2}\frac{\partial}{\partial t}, \quad T(p) := \frac{\partial}{\partial t}$$

and $p \in \mathbb{H}^n$. Find the commutators of the vector fields

$$[X_j, Y_k], \quad [X_j, T], \quad [Y_j, T].$$

(ii) Consider the differential one-form

$$\alpha := dt + \frac{1}{2}\sum_{j=1}^{n}(y_j dx_j - x_j dy_j)$$

which is the contact form of $\mathbb{H}^n$. Find the Lie derivatives

$$L_{X_j}\alpha, \quad L_{Y_j}\alpha, \quad L_T\alpha.$$

Find $d\alpha$ and the Lie derivatives

$$L_{X_j}d\alpha, \quad L_{Y_j}d\alpha, \quad L_T d\alpha.$$

Problem 24. Consider the smooth vector fields in $\mathbb{R}^n$

$$V = \sum_{j,k=1}^{n} a_{jk} x_j \frac{\partial}{\partial x_k}, \qquad W = \sum_{j,k,\ell=1}^{n} c_{jk\ell} x_j x_k \frac{\partial}{\partial x_\ell}$$

where $a_{jk}, c_{jk\ell} \in \mathbb{R}$. Find the conditions on a_{jk} and $c_{jk\ell}$ such that $[V, W] = 0$.

Problem 25. Find two smooth vector fields V and W in $\mathbb{R}^n$ such that

$$[[W, V], V] = 0 \quad \text{but} \quad [W, V] \neq 0.$$

Find two $n \times n$ matrices A and B such that

$$[[B, A], A] = 0 \quad \text{but} \quad [B, A] \neq 0.$$

Problem 26. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Calculate

$$\exp\left(i\pi\alpha \frac{d}{d\alpha}\right) \alpha, \quad \exp\left(i\pi\alpha \frac{d}{d\alpha}\right) \alpha^2, \quad \exp\left(i\pi\alpha \frac{d}{d\alpha}\right) f(\alpha).$$

Problem 27. Do the vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t}, \quad t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}$$

form a basis of a Lie algebra under the commutator?

Problem 28. Give a vector field V in $\mathbb{R}^3$ such that

$$V \times \text{curl} V \neq \mathbf{0}.$$

Give a vector field V in $\mathbb{R}^3$ such that

$$V \times \text{curl} V = \mathbf{0}.$$

Problem 29. Let $f_1 : \mathbb{R}^2 \to \mathbb{R}$, $f_2 : \mathbb{R}^2 \to \mathbb{R}$ be analytic function. Consider the analytic vector fields

$$V = f_1(x_1, x_2)\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \qquad W = \frac{\partial}{\partial x_1} + f_2(x_1, x_2)\frac{\partial}{\partial x_2}$$

in $\mathbb{R}^2$.

(i) Find the conditions on f_1 and f_2 such that $[V, W] = 0$.
(ii) Find the conditions on f_1 and f_2 such that $[V, W] = V + W$.

Problem 30. Show that the vector fields

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad V_3 = (y^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}$$

form a basis for the Lie algebra $s\ell(2, \mathbb{R})$. Solve the initial value problem for the autonomous system

$$\frac{dx}{dt} = y^2 - x^2, \qquad \frac{dy}{dt} = -2xy.$$

Problem 31. Consider the vector fields

$$V_1 = \frac{\partial}{\partial x_0}, \quad V_2 = x_0\frac{\partial}{\partial x_0} + \frac{2}{3}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)$$

$$V_3 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}, \quad V_4 = x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}, \quad V_5 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}.$$

Find the commutators and thus show we have a basis of a Lie algebra.

Problem 32. Let $\xi, \eta > 0$. Consider the transformation to three-dimensional parabolic coordinates

$$x_1(\xi, \eta, \phi) = \xi\eta\cos(\phi), \quad x_2(\xi, \eta, \phi) = \xi\eta\sin(\phi), \quad x_3(\xi, \eta, \phi) = \frac{1}{2}(\eta^2 - \xi^2).$$

Let

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Show that under this transformation

$$g = (\eta^2 + \xi^2)d\eta \otimes d\eta + (\eta^2 + \xi^2)d\xi \otimes d\xi + \eta^2\xi^2 d\phi \otimes d\phi.$$

Problem 33. Consider the Darboux-Halphen system

$$\frac{dx_1}{dt} = x_2x_3 - x_1x_2 - x_3x_1, \quad \frac{dx_2}{dt} = x_3x_1 - x_1x_2 - x_2x_3, \quad \frac{dx_3}{dt} = x_1x_2 - x_3x_1 - x_2x_3$$

with the corresponding vector field

$$V = (x_2x_3 - x_1x_2 - x_3x_1)\frac{\partial}{\partial x_1} + (x_3x_1 - x_1x_2 - x_2x_3)\frac{\partial}{\partial x_2} + (x_1x_2 - x_3x_1 - x_2x_3)\frac{\partial}{\partial x_3}.$$

(i) Is the autonomous system of differential equations invariant under the transformation ($\alpha\delta - \beta\gamma \neq 0$)

$$(t, x_j) \mapsto \left(\frac{\alpha t + \beta}{\gamma t + \delta}, 2\gamma \frac{\gamma t + \delta}{\alpha\delta - \gamma\beta} + \frac{(\gamma t + \delta)^2}{\alpha\delta - \gamma\beta} x_j \right)$$

with $j = 1, 2, 3$.
(ii) Consider the vector fields

$$U = 2(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}), \qquad W = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

Find the commutators $[U, V]$, $U, W]$, $[V, W]$. Do we have basis of a Lie algebra? Discuss.

Problem 34. (i) Consider the vector field $V_1(x_1, x_2) = x_2 \frac{\partial}{\partial x_1}$ with the corresponding autonmous system of differential equations

$$\frac{dx_1}{d\tau} = x_2, \quad \frac{dx_2}{d\tau} = 0.$$

Find the solution of the initial value problem. Discuss.
(ii) Consider the vector field $V_3(x_1, x_2) = \frac{x_1^2}{2} \frac{\partial}{\partial x_2}$ with the corresponding autonmous system of differential equations

$$\frac{dx_1}{d\tau} = 0, \quad \frac{dx_2}{d\tau} = x_1^2.$$

Find the solution of the intial value problem. Discuss.
(iii) Find the vector field $V_3 = [V_1, V_2]$, where [,] denotes the commutator. Write down the corresponding autonmous system of differential equations and solve the initial value problem. Discuss.
(iv) Find the vector field $V_4 = V_1 + V_2$ and write down the corresponding autonmous system of differential equations and solve the initial value problem. Discuss.

Problem 35. (i) Find the Lie algebra generated by

$$V_1 = x_2 \frac{\partial}{\partial x_2}, \quad V_2 = -x_2 \frac{\partial}{\partial x_1}.$$

(ii) Let c be a constant. Find the Lie algebra generated by

$$V_1 = x_2 \frac{\partial}{\partial x_2} + c x_3 \frac{\partial}{\partial x_3}, \quad V_2 = -x_2 \frac{\partial}{\partial x_1}, \quad V_3 = -c x_3 \frac{\partial}{\partial x_1}.$$

Problem 36. Consider the autonomous system of first order ordinary differential equations

$$\frac{du_1}{dt} = -u_1 u_3, \quad \frac{du_2}{dt} = u_2 u_3, \quad \frac{du_3}{dt} = u_1^2 - u_2^2$$

with the vector field

$$V = -u_1 u_3 \frac{\partial}{\partial u_1} + u_2 u_3 \frac{\partial}{\partial u_2} + (u_1^2 - u_2^2) \frac{\partial}{\partial u_3}.$$

Show that

$$I_1 = \frac{1}{2}(u_1 + u_2 + u_3), \quad I_2 = u_1 u_2$$

are first integrals.

Problem 37. Let $\alpha \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Calculate

$$\cosh\left(\alpha \frac{d}{dx}\right) f(x), \qquad \sinh\left(\alpha \frac{d}{dx}\right) f(x).$$

Chapter 3

Metric Tensor Fields

Problem 1. Let $a > b > 0$ and define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi).$$

The function f is a parametrized torus T^2 in $\mathbb{R}^3$. Consider the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

(i) Calculate $g|_{T^2}$.
(ii) Calculate the Christoffel symbols Γ^m_{ab} from $g|_{T^2}$.
(iii) Calculate the curvature.
(iv) Give the differential equations for the geodesics.

Problem 2. The two-dimensional *de Sitter space* $\mathbb{V}$ with the topology $\mathbb{R} \times \mathbb{S}$ may be visualized as a one-sheet hyperboloid $\mathbb{H}_{r_0}$ embedded in 3-dimensional Minkowski space $\mathbb{M}$, i.e.

$$\mathbb{H}_{r_0} = \{ (y^0, y^1, y^2) \in \mathbb{M} \, : \, (y^2)^2 + (y^1)^2 - (y^0)^2 = r_0^2, \;\; r_0 > 0 \, \}$$

where r_0 is the parameter of the one-sheet hyperboloid $\mathbb{H}_{r_0}$. The induced metric, $g_{\mu\nu}$ $(\mu, \nu = 0, 1)$, on $\mathbb{H}_{r_0}$ is the de Sitter metric.
(i) Show that we can parametrize (parameters ρ and θ) the hyperboloid as follows

$$y^0(\rho, \theta) = -\frac{r_0\cos(\rho/r_0)}{\sin(\rho/r_0)}, \qquad y^1(\rho, \theta) = \frac{r_0\cos(\theta/r_0)}{\sin(\rho/r_0)}, \qquad y^2(\rho, \theta) = \frac{r_0\sin(\theta/r_0))}{\sin(\rho/r_0)}$$

where $0 < \rho < \pi r_0$ and $0 \le \theta < 2\pi r_0$.

(ii) Using this parametrization find the metric tensor field induced on $\mathbb{H}_{r_0}$.

Problem 3. Consider the metric tensor field

$$g = -dZ \otimes dZ - dT \otimes dT + dW \otimes dW.$$

Consider the parametrization

$$\begin{aligned} Z(z,t) &= \cosh(\epsilon z)\cos(\epsilon t) \\ T(z,t) &= \cosh(\epsilon z)\sin(\epsilon t) \\ W(z,t) &= \sinh(\epsilon z). \end{aligned}$$

(i) Find $Z^2 + T^2 - W^2$.
(ii) Express g using this parametrization.

Problem 4. The *anti-de Sitter space* is defined as the surface

$$X^2 + Y^2 + Z^2 - U^2 - V^2 = -1$$

embedded in a five-dimensional flat space with the metric tensor field

$$g = dX \otimes dX + dY \otimes dY + dZ \otimes dZ - dU \otimes dU - dV \otimes dV.$$

This is a solution of Einstein's equations with the cosmological constant $\Lambda = -3$. Its intrinsic curvature is constant and negative. Find the metric tensor field in terms of the intrinsic coordinates (ρ, θ, ϕ, t) where

$$\begin{aligned} X(\rho,\theta,\phi,t) &= \frac{2\rho}{1-\rho^2}\sin\theta\cos\phi \\ Y(\rho,\theta,\phi,t) &= \frac{2\rho}{1-\rho^2}\sin\theta\sin\phi \\ Z(\rho,\theta,\phi,t) &= \frac{2\rho}{1-\rho^2}\cos\theta \\ U(\rho,\theta,\phi,t) &= \frac{1+\rho^2}{1-\rho^2}\cos t \\ V(\rho,\theta,\phi,t) &= \frac{1+\rho^2}{1-\rho^2}\sin t \end{aligned}$$

where $0 \le \rho < 1$, $0 \le \phi < 2\pi$, $0 \le \theta < \pi$, $-\pi \le t < \pi$.

Problem 5. Consider the *Poincaré upper half-plane*

$$H_+^2 := \{\,(x,y) \in \mathbb{R}^2 \,:\, y > 0\,\}$$

with metric tensor field

$$g = \frac{1}{y}dx \otimes \frac{1}{y}dx + \frac{1}{y}dy \otimes \frac{1}{y}dy$$

which is conformal with the standard inner product. Find the curvature forms.

Problem 6. Consider the manifold M of the upper space $x_2 > 0$ of $\mathbb{R}^2$ endowed with the metric tensor field

$$g = \frac{dx_1 \otimes dx_1 + dx_2 \otimes dx_2}{x_2^2}.$$

Show that the metric tensor field admits the symmetry $(x_1, x_2) \to (-x_1, x_2)$ and the transformation ($z = x_1 + ix_2$)

$$z \to z' = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1$$

preserve the metric tensor field. Find the Gaussian curvature of g.

Problem 7. Consider the manifold M of the upper space $x_n > 0$ of $\mathbb{R}^n$ endowed with the metric tensor field

$$g = \frac{dx_1 \otimes dx_1 + \cdots + dx_n \otimes dx_n}{x_n^2}.$$

Find the Gaussian curvature.

Problem 8. The *Klein bagel* (figure 8 immersion) is a specific immersion of the Klein bootle manifold into three dimensions. The figure 8 immersion has the parametrization

$$x(u,v) = (r + \cos(u/2)\sin(v) - \sin(u/2)\sin(2v))\cos(u)$$
$$y(u,v) = (r + \cos(u/2)\sin(v) - \sin(u/2)\sin(2v))\sin(u)$$
$$z(u,v) = \sin(u/2)\sin(v) + \cos(u/2)\sin(2v)$$

where r is a positive constant and $0 \le u < 2\pi$, $0 \le v < 2\pi$. Find the Riemann curvature of the Klein bagel.

Problem 9. Consider the compact differentiable manifold S^3

$$S^3 := \{ (x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}$$

and the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 + dx_4 \otimes dx_4.$$

(i) Express g using the following parametrization

$$x_1(\alpha,\beta,\theta)=\cos(\alpha)\cos(\theta)$$
$$x_2(\alpha,\beta,\theta)=\sin(\alpha)\cos(\theta)$$
$$x_3(\alpha,\beta,\theta)=\cos(\beta)\sin(\theta)$$
$$x_4(\alpha,\beta,\theta)=\sin(\beta)\sin(\theta)$$

where $0\le\theta\le\pi/2$, $0\le\alpha,\beta\le 2\pi$.
(ii) Now S^3 is the manifold of the compact Lie group $SU(2)$. Thus we can define the vector fields (angular momentum operators)

$$L_1=\frac{1}{2}\cos(\alpha+\beta)(\tan\theta\frac{\partial}{\partial\alpha}-\cot\theta\frac{\partial}{\partial\beta})-\sin(\alpha+\beta)\frac{\partial}{\partial\theta}$$
$$L_2=\frac{1}{2}\sin(\alpha+\beta)(\tan\theta\frac{\partial}{\partial\alpha}-\cot\theta\frac{\partial}{\partial\beta})+\cos(\alpha+\beta)\frac{\partial}{\partial\theta}$$
$$L_3=-\left(\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\right).$$

Find the commutation relation $[L_j,L_k]$ for $j,k=1,2,3$.
(iii) Find the dual basis of L_1, L_2, L_3.

Problem 10. Consider the metric tensor field

$$g=dx_1\otimes dx_1+dx_2\otimes dx_2+dx_3\otimes dx_3.$$

The parabolic set of unit-less coordinates (u,v,θ) is defined by a transformation of Cartesian coordinates ($0\le u\le\infty, 0\le v\le\infty$ and $0\le\theta\le 2\pi$)

$$x_1(u,v,\theta)=auv\cos\theta,\quad x_2(u,v,\theta)=auv\sin\theta,\quad x_3(u,v,\theta)=\frac{1}{2}a(u^2-v^2).$$

Express g using this parabolic coordinates.

Problem 11. Consider the metric tensor field

$$g=cdt_0\otimes cdt_0-dx_0\otimes dx_0-dy_0\otimes dy_0-dz_0\otimes dz_0$$

and the transformation

$$t_0=t$$
$$x_0=r\cos(\phi+\omega t)$$
$$y_0=r\sin(\phi+\omega t)$$
$$z_0=z.$$

Express g in the new coordinates t, r, z, ϕ.

Problem 12. Consider the upper half-plane $\{(x_1, x_2) : x_2 > 0\}$ endowed with the metric tensor field

$$g = \frac{1}{x_2^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2)$$

defines a two-dimensional Riemann manifold.
(i) Show that the Gaussian curvature is given by $R = -1$.
(ii) Find the surface element dS and the Laplace operater Δ.
(iii) Consider the conformal mapping from the upper half-plane $\{z = x_1 + ix_2 : x_2 > 0\}$ to the unit disk $\{w = re^{i\theta} : r \leq 1\}$

$$w(z) = \frac{iz + 1}{z + i}.$$

Express g in r and θ.

Problem 13. (i) Consider the metric tensor field

$$g(u_1, u_2) = du_1 \otimes du_1 + e^{2u_1} du_2 \otimes du_2, \quad -\infty < u_1, u_2 < +\infty.$$

Show that Gaussian curvature $K(u_1, u_2)$ has the value -1.
(ii) Consider the transformation

$$x_1(u_1, u_2) = u_2, \qquad x_2(u_1, u_2) = e^{-u_2}.$$

Express g using the coordinates x_1, x_2.
(iii) Consider the transformation

$$x_1(\rho, \phi) = x_{10} + \rho\cos(\phi), \qquad x_2(\rho, \phi) = \rho\sin(\phi)$$

where x_{10} is a constant. Express g in ρ and ϕ

Problem 14. Let $N \geq 2$ and $a > 0$. An N-dimensional Riemann manifold of constant negative Gaussian curvature $K = -1/a^2$ is described by the metric tensor field

$$g = dr \otimes dr + a^2 \sinh\left(\frac{r}{a}\right) d\sigma_{N-1} \otimes d\sigma_{N-1}$$

where $r \in [0, \infty)$ measures the distance to the origin and $d\sigma_{N-1} \otimes d\sigma_{N-1}$ denotes the metric tensor field of the unit sphere S_{N-1}.
(i) Show that volume element dV is covariantly defined as

$$dV_N = \left(a \sinh\left(\frac{r}{a}\right)\right)^{N-1} dr d\Omega_{N-1}$$

where $d\Omega_{N-1}$ is the surface element of the unit-sphere S_{N-1}.
(ii) Show that the radial part Δ_r of the Laplace operator for the metric tensor field given above is

$$\Delta_r = \frac{1}{(\sinh(r/a))^{N-1}} \frac{\partial}{\partial r} \left((\sinh(r/a))^{N-1} \frac{\partial}{\partial r} \right).$$

Problem 15. The Poincaré upper half-plane is defined as

$$\mathbf{H} := \{ \zeta = x + iy \, : \, x \in \mathbb{R}, \, y > 0 \}$$

together with the metric tensor field

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

(i) Show that under the Cayley transfom

$$\zeta = \frac{-iz + i}{z + 1}, \qquad z = x_1 + ix_2 = \frac{-\zeta + i}{\zeta + i}$$

the Poincaré upper half-plane is mapped onto the Poincaré disc with metric

$$g_{jk} = \frac{2}{1 - r^2} \mathrm{diag}(1, r^2), \quad r^2 = x_1^2 + x_2^2.$$

(ii) Show that under the transformation

$$\eta = X + iY = -\ln(-i\zeta) = 2\tan^{-1}(z)$$

the Poincaré upper half-plane is mapped onto the hyperbolic strip with metric

$$g_{jk} = \frac{1}{\cos^2(Y)} \delta_{jk}.$$

Problem 16. Let $R > 0$ and fixed. The *oblate spheroidal coordinates* are given by

$$x_1(\eta, \xi, \phi) = R\sqrt{(1 - \eta^2)(\xi^2 + 1)} \cos(\phi)$$
$$x_2(\eta, \xi, \phi) = R\sqrt{(1 - \eta^2)(\xi^2 + 1)} \sin(\phi)$$
$$x_3(\eta, \xi, \phi) = R\eta\xi$$

where $-1 \leq \eta \leq 1$, $0 \leq \xi < \infty$, $0 \leq \phi \leq 2\pi$ with the x_3 axis as the axis of revolution. Express the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3$$

in oblate spheroidal coordinates.

Problem 17. Consider the manifold $\mathbb{R}^3$. Let $a, b, c > 0$ and $a \neq b$, $a \neq c$, $b \neq c$. The *sphero-conical coordinates* s_2, s_3 are defined to be the roots of the quadratic equation

$$\frac{x_1^2}{a+s} + \frac{x_2^2}{b+s} + \frac{x_3^2}{c+s} = 0.$$

The first sphero-conical coordinate s_1 is given as the sum of the squares

$$s_1 = x_1^2 + x_2^2 + x_3^2.$$

The formula that expresses the Cartesian coordinates x_1, x_2, x_3 through s_1, s_2, s_3 are

$$\begin{aligned} x_1^2 &= \frac{s_1(a+s_2)(a+s_3)}{(a-b)(a-c)} \\ x_2^2 &= \frac{s_1(b+s_2)(b+s_3)}{(b-a)(b-c)} \\ x_3^2 &= \frac{s_1(c+s_2)(c+s_3)}{(c-a)(c-b)}. \end{aligned}$$

Given the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Express this metric tensor field using sphero-conical coordinates.

Problem 18. Consider the metric tensor field

$$g = -dT \otimes dT + dX \otimes dX + dY \otimes dY + dZ \otimes dZ$$

and the invertible coordinates transformation ($b > 0$)

$$\begin{aligned} T(t,x,y,z) &= \frac{1}{b}(e^{bz}\cosh(bt) - 1) \\ X(t,x,y,z) &= x \\ Y(t,x,y,z) &= y \\ Z(t,x,y,z) &= \frac{1}{b}e^{bz}\sinh(bt). \end{aligned}$$

Express the metric tensor field in the new coordinates. Given the inverse transformation.

Problem 19. Consider the metric tensor field

$$g = dT \otimes dT - dX \otimes dX$$

where $0 < X < \infty$ and $-\infty < T < \infty$. Show that under the transformation

$$T(r, \eta) = r \sinh(\eta), \qquad X(r, \eta) = r \cosh(\eta)$$

($0 < r < \infty, -\infty < \eta < \infty$) the metric tensor field takes the form (Rindler chart)

$$g = r^2 d\eta \otimes d\eta - dr \otimes dr.$$

Problem 20. Consider the metric tensor field

$$g = -e^{2\Phi(z)} dt \otimes dt + dz \otimes dz.$$

The proper acceleration of a test particle at rest with respect to this metric tensor field is given by $\partial\Phi/\partial z$. Hence if the gravitational potential has the form $\Phi(z) = az$ ($a > 0$) then all the test particles at rest have the same acceleration of magnitude a in the positive z-direction. Show that the metric tensor takes the form

$$g = -(a\rho)^2 dt \otimes dt + \frac{1}{a\rho^2} d\rho \otimes d\rho$$

under the transformation

$$\rho(z) = \frac{1}{a} e^{az}.$$

Problem 21. Show that the Killing vector fields of the metric tensor field

$$g = a(t) dx \otimes dx + b(t) e^{2x} (dy \otimes dy + dz \otimes dz) + dt \otimes dt$$

are given by

$$V_1 = \frac{\partial}{\partial y}, \quad V_2 = \frac{\partial}{\partial z}$$

$$V_3 = -\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad V_4 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

Problem 22. Show that the de Sitter space is an exact solution of the vacuum *Einstein equation* with a positive cosmological constant Λ

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

Problem 23. The cosmological constant Λ is a dimensionful parameter with unit of $1/(length)^2$. Show that the metric tensor field

$$g = -cd\tau \otimes cd\tau + e^{2c\tau/a} d\chi \otimes d\chi + a^2(d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi)$$

where $a > 0$ has the dimension of a length. Show that this metric tensor field satisfies the vacuum *Einstein equation* with a positive cosmological constant Λ

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

where $a = 1/\sqrt{(}\Lambda)$.

Problem 24. The metric tensor field g of a weak, plane,elliptically polarized gravitional wave propagating in the x-direction can be written as

$$g = cdt \otimes cdt - dx \otimes dx - (1 - h_{22}(x,t)) dy \otimes dy - (1 + h_{22}(x,t)) dz \otimes dz$$
$$+ h_{23}(x,t) dy \otimes dz + h_{23}(x,t) dz \otimes dy$$

where

$$h_{22}(x,t) = h \sin(k(ct - x) + \phi), \qquad h_{23}(x,t) = \widetilde{h} \sin(k(ct - x) + \widetilde{\phi})$$

with k the wave vector, h, $\widetilde{h}$ the amplitudes and ϕ, $\widetilde{\phi}$ the initial phase. They completely determine the state of the polarization of the gravitional wave. Show that in terms of the retarded and advanced coordinates

$$u(x,t) = \frac{1}{2}(ct - x), \qquad v(x,t) = \frac{1}{2}(ct + x)$$

the coordinates y, z and v can be omitted.

Problem 25. Consider the Poincaré metric tensor field

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

Find the geodesic equations and solve them.

Problem 26. Consider the metric tensor field

$$g = cdt \otimes cdt - dx \otimes dx$$

Express the metric in the coordinates u, v with

$$ct = a \sinh(u) \cosh(v), \quad x = a \cosh(u) \sinh(v)$$

with $a > 0$ and dimension *meter*.

Problem 27. Consider the metric tensor field

$$g = -dT \otimes dT + dX \otimes dX + dY \otimes dY + dZ \otimes dZ$$

and the invertible coordinates transformation ($b > 0$)

$$T(t,x,y,z) = \frac{1}{b}(e^{bz}\cosh(bt) - 1), \quad X(t,x,y,z) = x,$$

$$Y(t,x,y,z) = y, \quad Z(t,x,y,z) = \frac{1}{b}e^{bz}\sinh(bt).$$

Express the metric tensor field in the new coordinates. Given the inverse transformation.

Problem 28. Show that the metric tensor field

$$g = c^2(1 - 2a/r)dt \otimes dt - dr \otimes dr - r^2 d\phi \otimes d\phi - dz \otimes dz$$

is not a solution of Einstein's equation.

Problem 29. Consider the Euclidean space $\mathbb{R}^3$ with there metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Let $c_1, c_2, c_3 > 0$. The *hyperboloid*

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} - \frac{x_3^2}{c_3^2} = 1$$

can be written in parameter form as

$$\begin{aligned} x_1(\theta,\phi) &= c_1 \cos(\theta)\sec(\phi) \\ x_2(\theta,\phi) &= c_2 \sin(\theta)\sec(\phi) \\ x_3(\theta,\phi) &= c_3 \tan(\phi) \end{aligned}$$

where $\sec(\phi) = 1/\cos(\phi)$. Find the metric tensor field for the hyperbolid.

Problem 30. Consider the *K"ahler potential*

$$K = \frac{1}{2}\ln\left(1 + \sum_{\ell=1}^{n} z_\ell \bar{z}_\ell\right).$$

Let

$$g_{j\bar{k}} = g_{\bar{k}j} = \frac{\partial^2 K}{\partial z_j \partial \bar{z}_k}.$$

Find the metric tensor field.

Problem 31. Consider the metric tensor field

$$g = dx_0 \otimes dx_0 - dx_1 \otimes dx_1 - dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3) \otimes (x_1 dx_1 + x_2 dx_2 + x_3 dx_3)}{R^2 - (x_1^2 + x_2^2 + x_3^2)}$$

where R is a positive constant and $x_0 = ct$. Apply the transfomation

$$\begin{aligned} x_1(r,\alpha,\beta,u) &= R\sin(r/R)\sin(\alpha)\cos(\beta) \\ x_2(r,\alpha,\beta,u) &= R\sin(r/R)\sin(\alpha)\sin(\beta) \\ x_3(r,\alpha,\beta,u) &= R\sin(r/R)\cos(\alpha) \\ x_0(r,\alpha,\beta,u) &= u + r. \end{aligned}$$

Chapter 4

Differential Forms and Applications

We denote by $\wedge$ the *exterior product*. It is also called the wedge product or Grassmann product. The exterior product is associative. We denote by d the *exterior derivative*. The exterior derivative d is linear.

Problem 1. Let f, g be two smooth functions defined on $\mathbb{R}^2$. Find the differential two-form $df \wedge dg$.

Problem 2. Consider the analytic functions $f_1 : \mathbb{R}^2 \to \mathbb{R}$, $f_2 : \mathbb{R}^2 \to \mathbb{R}$

$$f_1(x_1, x_2) = x_1 + x_2, \qquad f_2(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

(i) Find df_1 and df_2. Then calculate $df_1 \wedge df_2$.
(ii) Solve the system of equations

$$df_1 \wedge df_2 = 0, \qquad x_1^2 + x_2^2 - 1 = 0.$$

Problem 3. Consider the complex number $z = re^{i\phi}$. Calculate

$$\frac{dz \wedge d\bar{z}}{z}.$$

Problem 4. (i) Consider the differential one form

$$\alpha = x_1 dx_2 - x_2 dx_1$$

on $\mathbb{R}^2$. Show that α is invariant under the transformation

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Show that $\omega = dx_1 \wedge dx_2$ is invariant under this transformation.
(ii) Let α be the $(n-1)$ differential form on $\mathbb{R}^n$ given by

$$\alpha = \sum_{j=1}^{n} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

where $\widehat{\ }$ indicates omission. Show that α is invariant under the orthogonal group of $\mathbb{R}^n$. Show that $\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is invariant under the orthogonal group.

Problem 5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth planar mapping with constant Jacobian determinant $J = 1$, written as

$$Q = Q(p, q), \qquad P = P(p, q).$$

For coordinates in $\mathbb{R}^2$ the (area) differential two-form is given as

$$\omega = dp \wedge dq.$$

(i) Find $f^*\omega$.
(ii) Show that $pdq - f^*(pdq) = dF$ for some smooth function $F : \mathbb{R}^2 \to \mathbb{R}$.

Problem 6. Consider the differential one-form in $\mathbb{R}^3$

$$\alpha = x_1 dx_2 + x_2 dx_3 + x_3 dx_1.$$

Find $\alpha \wedge d\alpha$. Find the solutions of the equation $\alpha \wedge d\alpha = 0$.

Problem 7. Consider the differential one-form in $\mathbb{R}^3$

$$\alpha = dx_3 - x_2 dx_1 - dx_2.$$

Show that $\alpha \wedge d\alpha \neq 0$.

Problem 8. Let $j, k, \ell \in \{1, 2, \ldots, n\}$. Consider the differential one-forms

$$\alpha_{jk} := \frac{dz_j - dz_k}{z_j - z_k}.$$

Calculate

$$\alpha_{jk} \wedge \alpha_{k\ell} + \alpha_{k\ell} \wedge \alpha_{\ell j} + \alpha_{\ell j} \wedge \alpha_{jk}.$$

Problem 9. Consider all 2×2 matrices with $UU^* = I_2$, $\det U = 1$ i.e., $U \in SU(2)$. Then U can be written as

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad a, b \in \mathbb{C}$$

with the constraint $aa^* + bb^* = 1$. Let

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Show that

$$(z_1')(z_1')^* + (z_2')(z_2')^* = z_1 z_1^* + z_2 z_2^*.$$

(ii) Consider

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Show that $dz_1' \wedge dz_2' = dz_1 \wedge dz_2$.

Problem 10. A transformation $(\mathbf{q}, \mathbf{p}) \to (\mathbf{Q}, \mathbf{P})$ is called *symplectic* if it preserves the differential two-form

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

Consider the Hamilton function

$$H(\mathbf{q}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2\mu} - \frac{\mu M}{|\mathbf{q}|}, \qquad \mathbf{p} := \mu \frac{d\mathbf{q}}{dt}$$

where μ and M are positive constants and $\mathbf{p} = (p_1, p_2)^T$, $\mathbf{q} = (q_1, q_2)^T$. The phase space is $\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2$. The parameter μ is the reduced mass $m_1 m_2 / M$. The symplectic two-form is

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

Show that ω is invariant under the transformation

$$\mathbf{f} : ((r, \phi), (R, \Phi)) \to (q_1, q_2, p_1, p_2)$$

with

$$\begin{aligned} q_1(r,\phi,R,\Phi) &= r\cos\phi \\ q_2(r,\phi,R,\Phi) &= r\sin\phi \\ p_1(r,\phi,R,\Phi) &= R\cos\phi - \frac{\Phi}{r}\sin\phi \\ p_2(r,\phi,R,\Phi) &= R\sin\phi + \frac{\Phi}{r}\cos\phi. \end{aligned}$$

Find the Hamilton function in this new symplectic variables.

Problem 11. Consider the differential one-form

$$\alpha = \frac{i}{4}\sum_{j=0}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j).$$

Let $z_j = x_j + iy_j$. Find α. Find $d\alpha$.

Problem 12. Consider the vector space $\mathbb{R}^3$ and the smooth vector field

$$V = V_1(\mathbf{x})\frac{\partial}{\partial x_1} + V_2(\mathbf{x})\frac{\partial}{\partial x_2} + V_3(\mathbf{x})\frac{\partial}{\partial x_3}.$$

Given the differential two forms

$$\omega_1 = x_1 dx_2 \wedge dx_3, \quad \omega_2 = x_2 dx_3 \wedge dx_1, \quad \omega_3 = x_3 dx_1 \wedge dx_2.$$

Find the conditions on V_1, V_2, V_3 such that the following three conditions are satisfied

$$\begin{aligned} L_V\omega_1 &\equiv V\rfloor d\omega_1 + d(V\rfloor\omega_1) = 0 \\ L_V\omega_2 &\equiv V\rfloor d\omega_2 + d(V\rfloor\omega_2) = 0 \\ L_V\omega_3 &\equiv V\rfloor d\omega_3 + d(V\rfloor\omega_3) = 0. \end{aligned}$$

Then solve the initial value problem of the autonomous system of first order differential equations corresponding to the vector field V.

Problem 13. Let $z = x + iy$ $(x, y \in \mathbb{R})$. Find $dz \otimes d\bar{z}$ and $dz \wedge d\bar{z}$.

Problem 14. Consider the vector space $\mathbb{R}^3$. Find a differential one-form α such that $d\alpha \neq 0$ but $\alpha \wedge d\alpha = 0$.

Problem 15. In vector analysis in $\mathbb{R}^3$ we have the identity

$$\vec{\nabla}(\vec{A}\times\vec{B}) \equiv \vec{B}\mathrm{curl}\vec{A} - \vec{A}\mathrm{curl}\vec{B}.$$

Express this identity using differential forms, the exterior derivative and the exterior product.

Problem 16. Consider the differential $n+1$ form

$$\alpha = df\wedge\omega + dt\wedge df\wedge\sum_{j=1}^{n}(-1)^{j+1}V_j(\mathbf{x},t)dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n + (divV)f dt\wedge\omega$$

where the circumflex indicates omission and $\omega = dx_1 \wedge \cdots \wedge dx_n$. Here $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function of $\mathbf{x}$, t and V is a smooth vector field.
(i) Show that the sectioned form

$$\widetilde{\alpha} = df(\mathbf{x},t)\wedge\omega + dt\wedge df(\mathbf{x},t)\wedge\left(\sum_{j=1}^{n}(-1)^{j+1}V_j(\mathbf{x},t)dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\right)$$
$$+(divV(\mathbf{x},t))f(\mathbf{x},t)dt\wedge\omega$$

where we distinguish between the independent variables $x_1, \ldots, x_n$, t and the dependent variable f leads using the requirement that $\widetilde{\alpha} = 0$ to the generalized Liouville equation.
(ii) Show that the differential form α is closed, i.e. $d\alpha = 0$.

Problem 17. Let $M = \mathbb{R}^n$ and $\mathbf{p} \in \mathbb{R}^n$. Let $T_{\mathbf{p}}(\mathbb{R}^n)$ be the tangent space at $\mathbf{p}$. A differential one-form at $\mathbf{p}$ is a linear map ϕ from $T_{\mathbf{p}}(\mathbb{R}^n)$ into $\mathbb{R}$. This map satisfies the following properties

$$\phi(V_{\mathbf{p}}) \in \mathbb{R}, \qquad \text{for all } V_{\mathbf{p}} \in \mathbb{R}^n$$
$$\phi(aV_{\mathbf{p}} + bW_{\mathbf{p}}) = a\phi(V_{\mathbf{p}}) + b\phi(W_{\mathbf{p}}) \qquad \text{for all } a,b \in \mathbb{R},\ V_{\mathbf{p}}, W_{\mathbf{p}} \in T_{\mathbf{p}}(\mathbb{R}^n).$$

A differential one-form is a smooth choice of a linear map ϕ defined above for each point $\mathbf{p}$ in the vector space $\mathbb{R}^n$. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued $C^\infty(\mathbb{R}^n)$ function. One defines the df of the function f as the differential one-form such that

$$df(V) = V(f)$$

for every smooth vector field V in $\mathbb{R}^n$. Thus at any point $\mathbf{p}$, the differential df of a smooth function f is an operator that assigns to a tangent vector $V_{\mathbf{p}}$ the directional derivative of the function f in the direction of this vector, i.e.

$$df(V)(\mathbf{p}) = V_{\mathbf{p}}(f) = \nabla f(\mathbf{p}) \cdot V(\mathbf{p}).$$

If we apply the differential of the coordinate functions x_j ($j = 1, \ldots, n$) we obtain

$$dx_j\left(\frac{\partial}{\partial x_k}\right) \equiv \frac{\partial}{\partial x_j}\rfloor dx_k = \frac{\partial x_j}{\partial x_k} = \delta_{jk}.$$

(i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x_1, x_2) = x_1^2 + x_2^2$$

and

$$V = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}.$$

Find $df(V)$.

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x_1, x_2) = x_1^2 + x_2^2$$

and

$$V = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}.$$

Find $df(V)$.

Problem 18. Consider the manifold $M = \mathbb{R}^4$ and the differential two-form

$$\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

Let

$$\alpha = (a^2 + p_1^2)dq_1 \wedge dp_2 - p_1 p_2 (dq_1 \wedge dp_1 - dq_2 \wedge dp_2) - (b^2 + p_2^2)dq_2 \wedge dp_1$$

where a and b are constants. Find $d\alpha$. Can $d\alpha$ be written in the form $d\alpha = \beta \wedge \Omega$, where β is a differential one-form?

Problem 19. A necessary and sufficient condition for the Pfaffian system of equations

$$\omega_j = 0, \quad j = 1, \ldots, r$$

to be completely integrable is

$$d\omega_j \equiv 0 \mod (\omega_1, \ldots, \omega_r), \quad j = 1, \ldots, r$$

Let

$$\omega \equiv P_1(\mathbf{x})dx_1 + P_2(\mathbf{x})dx_2 + P_3(\mathbf{x})dx_3 = 0 \tag{1}$$

be a total differential equation in $\mathbb{R}^3$, where P_1, P_2, P_3 are analytic functions on $\mathbb{R}^3$. Complete integrabilty of ω means that in every sufficiently small neighbourhood there exists a smooth function f such that

$$f(x_1, x_2, x_3) = \text{const}$$

is a first integral of (1). A necessary and sufficient condition for (1) to be completely integrable is

$$d\omega \wedge \omega = 0.$$

Problem 20. Consider the differential one-form in space-time

$$\alpha = a_1(\mathbf{x})dx_1 + a_2(\mathbf{x})dx_2 + a_3(\mathbf{x})dx_3 + a_4(\mathbf{x})dx_4$$

with $\mathbf{x} = (x_1, x_2.x_3, x_4)$ $(x_4 = ct)$.
(i) Find the conditions on the a_j's such that $d\alpha = 0$.
(ii) Find the conditions on the a_j's such that $d\alpha \neq 0$ and $\alpha \wedge d\alpha = 0$.
(iii) Find the conditions on the a_j's such that $\alpha \wedge d\alpha \neq 0$ and $d\alpha \wedge d\alpha = 0$.
(iv) Find the conditions on the a_j's such that $d\alpha \wedge d\alpha \neq 0$.
(v) Consider the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 - dx_4 \otimes dx_4.$$

Find the condition such that $d(*\alpha) = 0$, where $*$ denotes the Hodge star operator.

Problem 21. Let $z = x + iy$, $x, y \in \mathbb{R}$. Calculate

$$-idz \wedge d\bar{z}.$$

Problem 22. Consider the manifold $M = \mathbb{R}^2$ and the metric tensor field $g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2$. Let

$$\omega = \omega_1(\mathbf{x})dx_1 + \omega_2(\mathbf{x})dx_2$$

be a differential one-form in $M$with $\omega_1, \omega_2 \in C^\infty(\mathbb{R}^2)$. Show that ω can be written as

$$\omega = d\alpha + \delta\beta + \gamma$$

where α is a $C^\infty(\mathbb{R}^2)$ function, β is a two-form given by $\beta = b(\mathbf{x})dx_1 \wedge dx_2$ $(b(\mathbf{x}) \in C^\infty(\mathbb{R}^2))$ and $\gamma = \gamma_1(\mathbf{x})dx_1 + \gamma_2(\mathbf{x})dx_2$ is a harmonic one-form, i.e. $(d\delta + \delta d)\gamma = 0$. We define

$$\delta\beta := (-1) * d * \beta.$$

Problem 23. Given a Lagrange function L. Show that the *Cartan form* for a Lagrange function is given by

$$\alpha = L(\mathbf{x}, \mathbf{v}, t)dt + \sum_{j=1}^{n} \left(\frac{\partial L}{\partial v_j}(dx_j - v_j dt) \right). \tag{1}$$

Let

$$H = \sum_{j=1}^{n} v_j \frac{\partial L}{\partial v_j} - L, \qquad p_j = \frac{\partial L}{\partial v_j}. \tag{2}$$

Find the Cartan form for the Hamilton function.

Problem 24. Let x_1, x_2, ..., x_n be the independent variables. Let $u_1(\mathbf{x})$, $u_2(\mathbf{x})$, ... $u_m(\mathbf{x})$ be the dependent variables. There are $m \times n$ derivatives $\partial u_j(\mathbf{x})/\partial x_i$. We introduce the coordinates

$$(x_i, u_j, u_{ji}) \equiv (x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, y_m, u_{11}, u_{12}, \ldots, u_{mn}).$$

Consider the n-differential form (called the *Cartan form*) can be written as

$$\Theta = \left(L - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial \mathcal{L}}{\partial u_{j,i}} u_{j,i} \right) \Omega + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial L}{\partial u_{j,i}} du_j \wedge \left(\frac{\partial}{\partial x_i} \rfloor \Omega \right)$$

where

$$\Omega := dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

Let

$$H := \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial L}{\partial u_{j,i}} u_{j,i} \right) - L, \qquad p_i^j := \frac{\partial L}{\partial u_{j,i}}.$$

Show that we find the Cartan form for the Hamilton

$$\begin{aligned} \Theta := & -H dx_1 \wedge dx_2 \ldots dx_{n-1} \wedge dx_n \\ & + \sum_{i=1}^{n} \sum_{j=1}^{m} p_i^j dx_u^j \wedge dx_1 \wedge \ldots dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \ldots \wedge dx_n \end{aligned}$$

where the hat indicates that this term is omitted.

Problem 25. Consider the differential 2-form

$$\beta = \frac{4 dz \wedge d\bar{z}}{(1 + |z|^2)^2}$$

and the linear fractional transformations

$$z = \frac{aw + b}{cw + d}, \qquad ad - bc = 1.$$

What is the conditions on a, b, c, d such that β is invariant under the transformation?

Problem 26. Consider the two-dimensional sphere

$$S_1^2 + S_2^2 + S_3^2 = S^2$$

where $S > 0$ is the radius of the sphere. Consider the symplectic structure on this sphere with the symplectic differential two form

$$\omega := -\frac{1}{2S^2} \sum_{jk\ell=1}^{3} \epsilon_{jk\ell} S_j dS_k \wedge dS_\ell$$

($\epsilon_{123} = 1$) and the Hamilton vector fields

$$V_{S_j} := \sum_{k\ell=1}^{3} \epsilon_{jk\ell} S_k \frac{\partial}{\partial S_\ell}.$$

The Poisson bracket is defined by

$$[S_j, S_k]_{\text{PB}} := -V_{S_j} S_k.$$

(i) Calculate $[S_j, S_k]_{\text{PB}}$.
(ii) Calculate $V_{S_j} \rfloor \omega$.
(iii) Calculate $V_{S_j} \rfloor V_{S_k} \rfloor \omega$.
(iv) Calculate the Lie derivative $L_{V_{S_j}} \omega$.

Problem 27. Consider the system of partial differential equations (continuity and Euler equation of hydrodynamics in one space dimension)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + 2c\frac{\partial c}{\partial x} = \frac{\partial H}{\partial x}, \qquad \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + \frac{1}{2}c\frac{\partial u}{\partial x} = 0$$

where u and c are the velocities of the fluid and of the disturbance with respect to the fluid, respectively. H the depth is a given function of x. Show that the partial differential equations can be written in the forms $d\alpha = 0$ and $d\omega = 0$, where α and β are differntial one-forms. Owing to $d\alpha = 0$ and $d\omega = 0$ one can find locally (Poincaré lemma) zero-forms (functions) (also called potentials) such that

$$\alpha = d\Phi, \qquad \beta = d\Psi.$$

Problem 28. Let $a > 0$. *Toroidal coordinates* are given by

$$x_1(\mu,\theta,\phi) = \frac{a\sinh\mu\cos\phi}{\cosh\mu - \cos\theta}, \quad x_2(\mu,\theta,\phi) = \frac{a\sinh\mu\sin\phi}{\cosh\mu - \cos\theta}, \quad x_3(\mu,\theta,\phi) = \frac{a\sin\theta}{\cosh\mu - \cos\theta}$$

where

$$0 < \mu < \infty, \quad -\pi < \theta < \pi, \quad 0 < \phi < 2\pi.$$

Express the volume element $dx_1 \wedge dx_2 \wedge dx_3$ using toroidal coordinates.

Problem 29. Let α, β be smooth differential one-forms. The linear operator $d_\alpha(.)$ is defined by

$$d_\alpha(\beta) := d\beta + \alpha \wedge \beta.$$

Let

$$\alpha = x_1 dx_2 + x_2 dx_3 + x_3 dx_1, \qquad \beta = x_1 x_2 dx_3.$$

Find $d_\alpha(\beta)$. Solve $d_\alpha(\beta) = 0$.

Problem 30. Consider the differential two-form in $\mathbb{R}^3$

$$\alpha = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$$

and the vector field

$$V = x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_1}.$$

Find

$$V \rfloor \alpha, \quad V \rfloor d\alpha, \quad L_V \alpha, \quad L_V d\alpha.$$

Problem 31. Let $n \geq 2$ and ω be the volume form in $\mathbb{R}^n$

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

(i) Find the condition on the smooth vector V in $\mathbb{R}^n$ such

$$V \rfloor \omega = 0.$$

(ii) Let V, W be two smooth vector fields in $\mathbb{R}^n$. Find the conditions on V, W such that

$$W \rfloor (V \rfloor \omega) = 0.$$

Problem 32. Consider the manifold $\mathbb{R}^n$. Calculate

$$\frac{\partial}{\partial x_j} \rfloor (dx_k \wedge dx_\ell)$$

where $j, k, \ell = 1, \ldots, n$.

Problem 33. Consider the vector fields

$$V_{12} = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \quad V_{23} = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \quad V_{31} = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}$$

in $\mathbb{R}^3$ and the volume form $\omega = dx_1 \wedge dx_2 \wedge dx_3$.
(i) Find the commutators

$$[V_{12}, V_{23}], \quad [V_{23}, V_{31}], \quad [V_{31}, V_{12}].$$

Discuss.
(ii) Find

$$V_{12} \rfloor \omega, \quad V_{23} \rfloor \omega, \quad V_{31} \rfloor \omega.$$

(iii) Let $*$ be the Hodge star operator in $\mathbb{R}^3$ with metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Find $*(V_{12} \rfloor \omega)$, $*(V_{23} \rfloor \omega)$, $*(V_{31} \rfloor \omega)$.
(iv) Find

$$d(*(V_{12} \rfloor \omega)), \quad d(*(V_{23} \rfloor \omega)), \quad d(*(V_{31} \rfloor \omega)).$$

Problem 34. (i) Let V, W be smooth vector fields in $\mathbb{R}^n$ $(n \geq 2)$ and α, β be differential one-forms. Calculate

$$L_{[V,W]}(\alpha \wedge \beta).$$

(ii) Assume that $L_V \alpha = 0$ and $L_W \beta = 0$. Simplify the result from (i).
(iii) Assume that $L_V \alpha = f\alpha$ and $L_W \beta = g\beta$, where f, g are smooth functions. Simplify the result from (i).
(iv) Let $L_V \alpha = \beta$ and $L_W \beta = \alpha$. Simplify the result from (i).

Problem 35. A symplectic structure on a $2n$-dimensional manifold M is a closed non-degenerate differential two-form ω such that $d\omega = 0$ and ω^n does not vanish. Every symplectic form is locally diffeomorphic to the standard differential form

$$\omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

on $\mathbb{R}^{2n}$. Consider the vector field

$$V = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_{2n} \frac{\partial}{\partial x_{2n}}$$

in $\mathbb{R}^{2n}$. Find $V \rfloor \omega_0$ and $L_V \omega_0$.

Problem 36. Let $a > b > 0$. Consider the transformation

$$x_1(\theta, \phi) = (a + b\cos\phi)\cos\theta, \qquad x_2(\theta, \phi) = (a + b\cos\phi)\sin\theta.$$

Find $dx_1 \wedge dx_2$ and $dx_1 \otimes dx_1 + dx_2 \otimes dx_2$.

Problem 37. Consider the differential one-form

$$\alpha = (2xy - x^2)dx + (x + y^2)dy.$$

(i) Calculate $d\alpha$.
(ii) Calculate

$$\oint \alpha$$

with the closed path $C_1 - C_2$ starting from $(0,0)$ moving along via the curve $C_1 : y = x^2$ to $(1,1)$ and back to $(0,0)$ via the curve $C_2 : y = \sqrt{x}$. Let D be the (convex) domain enclosed by the two curves C_1 and C_2.
(iii) Calculate the double integral

$$\int\int_D d\alpha$$

where D is the domain given in (i), i.e. $C_1 - C_2$ is the boundary of D. Thus verify the theorem of Gauss-Green.

Problem 38. Consider the differential one form in the plane

$$\alpha = x_2^2 dx_1 + x_1^2 dx_2$$

Calculate the integral

$$\oint_C \alpha$$

where C is the closed curve which the boundary of a triangle with vertices $(0,0)$, $(1,1)$, $(1,0)$ and counterclockwise orientation. Apply Green's theorem

$$\oint_C f(x_1, x_2)dx_1 + g(x_1, x_2)dx_2 = \int\int_D \left(\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) dx_1 dx_2.$$

Problem 39. (i) The *lemniscate of Gerono* is described by the equation

$$x^4 = x^2 - y^2.$$

Show that a parametrization is given by

$$x(t) = \sin(t), \qquad y(t) = \sin(t)\cos(t)$$

with $t \in [0, \pi]$.
(ii) Consider the differential one-form

$$\alpha = xdy$$

in the plane $\mathbb{R}^2$. Let $x(t) = \sin(t)$, $y(t) = \sin(t)\cos(t)$. Find $\alpha(t)$.
(iii) Calculate

$$-\int_0^\pi x(t)dy(t).$$

Disucss.

Problem 40. (i) Consider the smooth differential one form

$$\alpha = f_1(x_1,x_2,x_3)dx_1 + f_2(x_1,x_2,x_3)dx_2 + f_3(x_1,x_2,x_3)dx_3$$

in $\mathbb{R}^3$. Find the differential equations from

$$\alpha \wedge d\alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha = 0.$$

(i) Consider the smooth differential one form

$$\alpha = f_1(x_1,x_2,x_3,x_4)dx_1 + f_2(x_1,x_2,x_3,x_4)dx_2 + f_3(x_1,x_2,x_3,x_4)dx_3 + f_4(x_1,x_2,x_3,x_4)dx_4$$

in $\mathbb{R}^4$. Find the differential equations from

$$\alpha \wedge d\alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha = 0.$$

Problem 41. Consider the differentiable manifold

$$S^3 = \{\, (x_1,x_2,x_3,x_4) \,:\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \,\}.$$

(i) Show that the matrix

$$U(x_1,x_2,x_3,x_4) = -i \begin{pmatrix} x_3 + ix_4 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 + ix_3 \end{pmatrix}$$

is unitary. Show that the matrix is an element of $SU(2)$.
(ii) Consider the parameters (θ,ψ,ϕ) with $0 \le \theta < \pi$, $0 \le \psi < 4\pi$, $0 \le \phi < 2\pi$. Show that

$$x_1(\theta,\psi,\phi) + ix_2(\theta,\psi,\phi) = \cos(\theta/2)e^{i(\psi+\phi)/2}$$
$$x_3(\theta,\psi,\phi) + ix_4(\theta,\psi,\phi) = \sin(\theta/2)e^{i(\psi-\phi)/2}$$

is a parametrization. Thus the matrix given in (i) takes the form

$$-i \begin{pmatrix} \sin(\theta/2)e^{i(\psi-\phi)/2} & \cos(\theta/2)e^{-i(\psi+\phi)/2} \\ \cos(\theta/2)e^{i(\psi+\phi)/2} & -\sin(\theta/2)e^{-i(\psi-\phi)/2} \end{pmatrix}.$$

(iii) Let $(\xi_1, \xi_2, \xi_3) = (\theta, \psi, \phi)$ with $0 \le \theta < \pi$, $0 \le \psi < 4\pi$, $0 \le \phi < 2\pi$. Show that

$$\frac{1}{24\pi^2}\int_0^{\pi} d\theta \int_0^{4\pi} d\psi \int_0^{2\pi} d\phi \sum_{j,k,\ell=1}^{3} \epsilon_{jk\ell} \mathrm{tr}(U^{-1}\frac{\partial U}{\partial \xi_j} U^{-1}\frac{\partial U}{\partial \xi_k} U^{-1}\frac{\partial U}{\partial \xi_\ell}) = 1$$

where $\epsilon_{123} = \epsilon_{321} = \epsilon_{132} = +1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$ and 0 otherwise.
(iv) Consider the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 + dx_4 \otimes dx_4.$$

Using the parametrization show that

$$g_{S^3} = \frac{1}{4}(d\theta \otimes d\theta + d\psi \otimes d\psi + d\phi \otimes d\phi + \cos(\theta) d\psi \otimes d\phi + \cos(\theta) d\phi \otimes d\psi).$$

(v) Consider the differential one forms e_1, e_2, e_3 defined by

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -x_4 & -x_3 & x_2 & x_1 \\ x_3 & -x_4 & -x_1 & x_2 \\ -x_2 & x_1 & -x_4 & x_3 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{pmatrix}.$$

Show that

$$g_{S^3} = de_1 \otimes de_1 + de_2 \otimes de_2 + de_3 \otimes de_3.$$

(vi) Show that

$$de_j = \sum_{k,\ell=1}^{3} \epsilon_{jk\ell} e_k \wedge e_\ell$$

i.e. $de_1 = 2e_2 \wedge e_3$, $de_2 = 2e_3 \wedge e_1$, $de_3 = 2e_1 \wedge e_2$.

Problem 42. Let V, W be two smooth vector fields defined on $\mathbb{R}^3$. We write

$$V = V_1(\mathbf{x})\frac{\partial}{\partial x_1} + V_2(\mathbf{x})\frac{\partial}{\partial x_2} + V_3(\mathbf{x})\frac{\partial}{\partial x_3}$$
$$W = W_1(\mathbf{x})\frac{\partial}{\partial x_1} + W_2(\mathbf{x})\frac{\partial}{\partial x_2} + W_3(\mathbf{x})\frac{\partial}{\partial x_3}.$$

Let

$$\omega = dx_1 wegdedx_2 \wedge dx_3$$

be the volume form in $\mathbb{R}^3$. Then $L_V\omega = (\div(V))\omega$, where $L_V(.)$ denotes the Lie derivative and $\div$ denotes the diveregence of the vector field. Find the divergence of the vector field given by the commutator $[V, W]$. Apply it to the vector fields asscociated with the autonomous systems of first order differential equations

$$\frac{dx_1}{dt} = x_2 - x_1, \quad \frac{dx_2}{dt} = x_1x_1x_1x_1x_1, \quad \frac{dx_3}{dt} = x_1x_2 - bx_3$$

and

$$\frac{dx_1}{dt} = x_1x_1, \quad \frac{dx_2}{dt} = x_1x_1x_1x_1, \quad \frac{dx_3}{dt} = x_1x_1x_1x_1.$$

The first system is the *Lorenz model* and the second system is *Chen's model*.

Problem 43. Let A be a differential one-form in space-time with the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 - dx_4 \otimes dx_4$$

with $x_4 = ct$. Let $F = dA$. Find $F \wedge *F$, where $*$ is the Hodge star operator.

Problem 44. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a two-dimensional analytic map.
(i) Find the condition on $\mathbf{f}$ such that $dx_1 \wedge dx_2$ is invariant, i.e. $\mathbf{f}$ should be area preserving.
(ii) Find the condition on $\mathbf{f}$ such that $x_1dx_1 + x_2dx_2$ is invariant.
(iii) Find the condition on $\mathbf{f}$ such that $x_1dx_1 - x_2dx_2$ is invariant.
(iv) Find the condition on $\mathbf{f}$ such that $x_1dx_2 + x_2dx_1$ is invariant.
(v) Find the condition on $\mathbf{f}$ such that $x_1dx_2 - x_2dx_1$ is invariant.

Problem 45. Consider the smooth one-form in $\mathbb{R}^3$

$$\alpha = f_1(\mathbf{x})dx_1 + f_2(\mathbf{x})dx_2 + f_3(\mathbf{x})dx_3, \quad \beta = g_1(\mathbf{x})dx_1 + g_2(\mathbf{x})dx_2 + g_3(\mathbf{x})dx_3.$$

Find the differential equation from the condition

$$d(\alpha \wedge \beta) = 0$$

and provide solution of it.

Problem 46. Let $c > 0$. Consider the elliptical coordinates

$$x_1(\alpha, \beta) = c\cosh(\alpha)\cos(\beta), \quad x_2(\alpha, \beta) = c\sinh(\alpha)\sin(\beta).$$

Find the differential two-form $\omega = d_1 \wedge dx_2$ in this coordinate system.

Problem 47. Let θ, ϕ, ψ be the Euler angles and consider the differential one-forms

$$\begin{aligned}
\sigma_1 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \\
\sigma_2 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi \\
\sigma_3 &= d\psi + \cos\theta d\phi.
\end{aligned}$$

Find

$$\sigma_1 \wedge sigma_2 + \sigma_2 \wedge \sigma_3 + \sigma_3 \wedge \sigma_1, \quad \sigma_1 \wedge \sigma_2 \wedge \sigma_3.$$

Problem 48. Let **B** be a vector field in $\mathbb{R}^3$. Calculate

$$(\nabla \times B) \times B.$$

Formulate the problem with differential forms.

Problem 49. Let $if(z)$ be a C^∞ function on a closed disc $B \subset \mathbb{C}$. Show that the differential equation

$$\bar{\partial}_z = if(z)$$

has a C^∞ solution $w(z)$ in the interior of B with

$$w(z) = \frac{1}{2\pi} \int_B \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

Problem 50. Let

$$\alpha = dx_1 + x_1 dx_2 + x_1 x_2 dx_3.$$

Find $\alpha \wedge d\alpha$.

Problem 51. Let $z = re^{i\phi}$. Find $dz \wedge d\bar{z}$.

Problem 52. Let $z_1, z_2 \in \mathbb{C}$. Consider the differential one-form

$$\omega = \frac{1}{2\pi i} \frac{dz_1 - dz_2}{z_1 - z_2}.$$

Find $d\omega$ and $\omega \wedge \omega$.

Problem 53. Let $z \in \mathbb{C}$ and $z = x + iy$ with $x, y \in \mathbb{R}$. Find

$$\alpha = z^* dz - z dz^*.$$

Problem 54. Consider the manifold $M = \mathbb{R}^2$ and the differential one form

$$\alpha = \frac{1}{2}(xdy - ydx).$$

(i) Find the differential two form $d\omega$.

(ii) Consider the domains in $\mathbb{R}^2$

$$D = \{ (x,y) \, : \, x^2+y^2 \leq 1 \}, \qquad \partial D = \{ (x,y) \, : \, x^2+y^2 = 1 \}$$

i.e. ∂D is the boundaary of D. Show that (Stokes theorem)

$$\int_D d\alpha = \int_{\partial D} \alpha.$$

Apply polar coordinates, i.e. $x(r,\phi) = r\cos(\phi)$, $y(r,\phi) = r\sin(\phi)$.

Problem 55. Let $M = \mathbb{R}^2$ and $\alpha = x_1 dx_2 - x_2 dx_1$. Then $d\alpha = 2dx_1 \wedge dx_2$. Now let $M = \mathbb{R}^2 \setminus \{(0,0)\}$. Consider the differential one form

$$\beta = \frac{1}{x_1^2+x_2^2}(x_1 dx_2 - x_2 dx_1)$$

(i) Find $d\beta$.
(ii) Show that

$$d(\arctan(y/x)) = \frac{1}{x_1^2+x_2^2}(x_1 dx_2 - x_2 dx_1).$$

Problem 56. Let $M = \mathbb{R}^2$. Consider the differential one-form

$$\alpha = (2x_1^3 + 3x_2)dx_1 + (3x_1 + x_2 - 1)dx_2.$$

(i) Find $d\alpha$.
(ii) Can one find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $df = \alpha$.

Problem 57. Consider the differential one-form

$$\alpha = xdy - ydx$$

in $M = \mathbb{R}^2$.
(i) Find $d\alpha$.
(ii) Let $c \in \mathbb{R}$. Show that $y - cx = 0$ satisfies $\alpha = 0$.

Problem 58. (i) Consider the differential one-forms in $\mathbb{R}^4$

$$\alpha_1 = -x_1 dx_0 + x_0 dx_1 - x_3 dx_2 + x_2 dx_3$$
$$\alpha_2 = -x_2 dx_0 + x_3 dx_1 + x_0 dx_2 - x_1 dx_3$$
$$\alpha_3 = -x_3 dx_0 - x_2 dx_1 + x_1 dx_2 + x_0 dx_3.$$

Find $d\alpha_1$, $d\alpha_2$, $d\alpha_3$ and $\alpha_2 \wedge \alpha_3$, $\alpha_3 \wedge \alpha_1$, $\alpha_1 \wedge \alpha_2$ and thus show that $d\alpha_1 = 2\alpha_2 \wedge \alpha_3$, $d\alpha_2 = 2\alpha_3 \wedge \alpha_1$, $d\alpha_3 = 2\alpha_1 \wedge \alpha_2$.
(ii) Consider the vector fields in $\mathbb{R}^4$

$$V_1 = -x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3}$$
$$V_2 = -x_2 \frac{\partial}{\partial x_0} + x_3 \frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}$$
$$V_3 = -x_3 \frac{\partial}{\partial x_0} - x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_0 \frac{\partial}{\partial x_3}.$$

Find the commutators $[V_1, V_2]$, $[V_2, V_3]$, $[V_3, V_1]$.
(iii) Find the interior product (contraction)

$$V_1 \rfloor \alpha_1, \quad V_2 \rfloor \alpha_2, \quad V_3 \rfloor \alpha_3.$$

Problem 59. Consider the manifold $M = \mathbb{R}^2$, the differential two form $\omega = dx \wedge dy$ and the smooth vector field

$$V = V_1(x,y)\frac{\partial}{\partial x} + V_2(x,y)\frac{\partial}{\partial y}.$$

Find the condition on a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$V \rfloor \omega = df.$$

Problem 60. Consider the analytic function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

and the analytic function $g : \mathbb{R}^3 \to \mathbb{R}$ given by

$$g(x_1, x_2, x_3) = x_1 x_2 x_3.$$

Find df, dg and then $df \wedge dg$. Solve $df \wedge dg = 0$.

Problem 61. Let $R > 0$. The anti-de Sitter metric tensor field g is given

$$g = -\omega_t \otimes \omega_t + \omega_r \otimes \omega_r + \omega_\theta \otimes \omega_\theta + \omega_\phi \otimes \omega_\phi$$

with the spherical orthonormal coframe (differential one forms)

$$\omega_t = e^{\Theta(r)} dt, \quad \omega_r = e^{-\Theta(r)} dr, \quad \omega_\theta = r d\theta, \quad \omega_\phi = r \sin(\theta) d\phi$$

with $e^{2\Theta(r)} = 1 + (r/R)^2$ and r, θ, ϕ are spherical coordinates. Show that the Riemannian curvature two-form

$$\Omega_{\alpha,\beta} = -\frac{1}{R^2}\omega_\beta \wedge \omega_\alpha, \qquad \alpha, \beta \in \{ t, r, \theta, \phi \}$$

is that of a constant negative curvature space with radius of curvature R.

Problem 62. Let $k \in \mathbb{R}$ and $k \neq 0$. Consider the three differential one-forms

$$\omega_1 = e^{-kx_1} dx_2, \quad \omega_2 = dx_3, \omega_3 = dx_1.$$

(i) Find $d\omega_1$, $d\omega_2$, $d\omega_3$.
(ii) Find $\omega_1 \wedge \omega_1$, $\omega_2 \wedge \omega_2$, $\omega_3 \wedge \omega_3$.
(iii) Find $\omega_1 \wedge \omega_2$, $\omega_2 \omega_1$, $\omega_2 \wedge \omega_3$, $\omega_3 \wedge \omega_2$, $\omega_3 \wedge \omega_1$, $\omega_1 \wedge \omega_3$.
(iv) Find the expansion coefficients $C^j_{k,\ell}$ $(j, k, \ell = 1, 2, 3)$ such that

$$d\omega_j = \frac{1}{2} \sum_{k,\ell=1}^{3} C^j_{k,\ell} \omega_k \wedge \omega_\ell.$$

(v) Consider the vector fields

$$V_1 = e^{kx_1} \frac{\partial}{\partial x_2}, \quad V_2 = \frac{\partial}{\partial x_3}, \quad V_3 = \frac{\partial}{\partial x_1}.$$

Find the commutators $[V_1, V_2]$, $[V_2, V_3]$, $[V_3, V_1]$.

Problem 63. Consider the differential two-form in $\mathbb{R}^4$

$$\begin{aligned}\beta = &a_{12}(\mathbf{x}) dx_1 \wedge dx_2 + a_{13}(\mathbf{x}) dx_1 \wedge dx_3 + a_{14}(\mathbf{x}) dx_1 \wedge dx_4 \\ &+ a_{23}(\mathbf{x}) dx_2 \wedge dx_3 + a_{24}(\mathbf{x}) dx_2 \wedge dx_4 + a_{34}(\mathbf{x}) dx_3 \wedge dx_4\end{aligned}$$

where $a_{jk} : \mathbb{R}^4 \to \mathbb{R}$ are smooth functions. Find $d\beta$ and the conditions from $d\beta = 0$.

Problem 64. Let $f_1(x_1, x_2) = x_1 + x_2$ and $f_2(x_1, x_2) = x_1^2 + x_2^2$. Solve the system of equations

$$df_1 \wedge df_2 = 0, \qquad x_1^2 + x_2^2 = 1.$$

Problem 65. Consider the differential two forms in $\mathbb{R}^3$

$$\begin{aligned}\beta_1 &= x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2 \\ \beta_2 &= \frac{1}{1 + x_1^2 + x_2^2 + x_3^2} \beta_1.\end{aligned}$$

Find $d\beta_1$ and $d\beta_2$.

Problem 66. Consider the differential one forms in $\mathbb{R}^n$

$$\alpha_1 = \sum_{j=1}^{n} x_j dx_j$$

$$\alpha_2 = x_2 dx_1 + x_3 dx_2 + \cdots + x_n dx_{n-1} + x_1 dx_n.$$

(i) Find the two forms $d\alpha_1$ and $d\alpha_2$.
(ii) Find $\alpha_1 \wedge \alpha_2$ and then $d(\alpha_1 \wedge d\alpha_2)$.

Problem 67. Consider the differential two forms $dx_1 \wedge dx_2$ in $\mathbb{R}^2$ and the transformation

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Find $dx_1' \wedge dx_2'$.

Problem 68. Let β be the differential two form in $\mathbb{R}^3$

$$\beta = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2.$$

Find $d\beta$.

Problem 69. Consider the differential two-form $dx_1 \wedge dx_2$ in $\mathbb{R}^2$ and the transformation

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Find $dx_1' \wedge dx_2'$.

Problem 70. (i) Find smooth maps $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{f}^*(dx_1 \wedge dx_2) = dx_2 \wedge dx_3, \quad \mathbf{f}^*(dx_2 \wedge dx_3) = dx_1 \wedge dx_2, \quad \mathbf{f}^*(dx_3 \wedge dx_1) = dx_1 \wedge dx_2.$$

(ii) Find smooth vector fields V in $\mathbb{R}^3$ such that

$$L_V(dx_1 \wedge dx_2) = dx_2 \wedge dx_3, \quad L_V(dx_2 \wedge dx_3) = dx_1 \wedge dx_2, \quad L_V(dx_3 \wedge dx_1) = dx_1 \wedge dx_2.$$

Problem 71. Consider the smooth map $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$

$$f_1(x_1, x_2, x_3) = x_1 x_2 - x_3, \quad f_2(x_1, x_2, x_3) = x_1, \quad f_3(x_1, x_2, x_3) = x_2.$$

(i) Show that the map is invertible and find the inverse.
(ii) Find

$$\mathbf{f}^*(dx_1 \wedge dx_2), \quad \mathbf{f}^*(dx_2 \wedge dx_3), \quad \mathbf{f}^*(dx_3 \wedge dx_1).$$

Disuss.
(iii) Consider the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Find $\mathbf{f}^*(g)$. Discuss.

Problem 72. Consider the differential one-forms in $\mathbb{R}^3$

$$\alpha_1 = \frac{dx_3 - x_1 dx_2 + x_2 dx_1}{1 + x_1^2 + x_2^2 + x_3^2}$$
$$\alpha_2 = \frac{dx_1 - x_2 dx_3 + x_3 dx_2}{1 + x_1^2 + x_2^2 + x_3^2}$$
$$\alpha_3 = \frac{dx_2 + x_1 dx_3 - x_3 dx_1}{1 + x_1^2 + x_2^2 + x_3^2}.$$

Find the dual basis of the vector fields V_1, V_2, V_3.

Problem 73. (i) Find smooth maps $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{f}^*(dx_1 \wedge dx_2) = dx_2 \wedge dx_3, \quad \mathbf{f}^*(dx_2 \wedge dx_3) = dx_1 \wedge dx_2, \quad \mathbf{f}^*(dx_3 \wedge dx_1) = dx_1 \wedge dx_2.$$

(ii) Find smooth vector fields V in $\mathbb{R}^3$ such that

$$L_V(dx_1 \wedge dx_2) = dx_2 \wedge dx_3, \quad L_V(dx_2 \wedge dx_3) = dx_1 \wedge dx_2, \quad L_V(dx_3 \wedge dx_1) = dx_1 \wedge dx_2.$$

Problem 74. (i) Consider the smooth differential one-form in $\mathbb{R}^3$

$$\alpha = -e^{x_1} x_3 dx_1 + \sin(x_3) dx_2 + (x_2 \cos(x_3) - e^{x_1}) dx_3.$$

Find $d\alpha$. Can one find a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $df = \alpha$.
(ii) Consider the smooth differential one-form in $\mathbb{R}^3$

$$\alpha = (3x_1 x_3 + 2x_2) dx_1 + x_1 dx_2 + x_1^2 dx_3.$$

Find $d\alpha$. Can one find a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $df = \alpha$. Discuss.
(iii) Consider the smooth differential one-form in $\mathbb{R}^3$

$$\alpha = x_2 dx_1 + dx_2 + dx_3.$$

Find $d\alpha$. Can one find a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $df = \alpha$. Discuss. Consider the differential one-form $\widetilde{\alpha} = x_1 \alpha$.

Problem 75. Let a_1, a_2, a_3 be real constants. Consider the differential one-form

$$\alpha = (a_2 \cos(x_2) + a_3 \sin(x_3))dx_1 + (a_1 \sin(x_1) + a_3 \cos(x_3))dx_2 + (a_1 \cos(x_1) + a_2 \sin(x_2))dx_3.$$

Find $d\alpha$ and solve the equation $d\alpha = 0$ and the equation $d\alpha = \alpha \wedge \alpha$.

Chapter 5

Lie Derivative and Applications

Problem 1. Let V be a smooth vector field defined on $\mathbb{R}^n$

$$V = \sum_{i=1}^{n} V_i(\mathbf{x}) \frac{\partial}{\partial x_i}.$$

Let T be a $(1,1)$ smooth tensor field defined on $\mathbb{R}^n$

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} \otimes dx_j \, .$$

Let $L_V T$ be the Lie derivative of T with respect to the vector field V. Show that if $L_V T = 0$ then

$$L_V \operatorname{tr}(a(\mathbf{x})) = 0$$

where $a(\mathbf{x})$ is the $n \times n$ matrix $(a_{ij}(\mathbf{x}))$ and tr denotes the trace.

Problem 2. Let V, W be vector fields. Let f, g be C^∞ functions and α be a differential form. Assume that

$$L_V \alpha = f\alpha, \qquad L_W \alpha = g\alpha.$$

Show that

$$L_{[V,W]}\alpha = (L_V f - L_W g)\alpha. \tag{1}$$

Problem 3. Let f and V be smooth function and smooth vector field in $\mathbb{R}^n$. Find

$$V \rfloor df.$$

Problem 4. Let V_j $(j = 1, \ldots, n)$ be smooth vector fields and α a smooth differential one-form. Assume that

$$L_{V_j}\alpha = d\phi_j, \qquad j = 1, 2, \ldots, n$$

where ϕ_j are smooth functions.
(i) Find

$$L_{[V_j, V_k]}\alpha.$$

(ii) Assume that the vector fields V_j $(j = 1, \ldots, n)$ form basis of a Lie algebra, i.e.

$$[V_j, V_k] = \sum_{\ell=1}^{n} c_{jk}^{\ell} V_\ell$$

where c_{jk}^{ℓ} are the structure constants. Find the conditions on the functions ϕ_j.

Problem 5. Find the first integrals of the autonomous system of ordinary first order differential equations

$$\frac{dx_1}{dt} = x_1x_2 + x_1x_3$$
$$\frac{dx_2}{dt} = x_2x_3 - x_1x_2$$
$$\frac{dx_3}{dt} = -x_1x_3 - x_2x_3.$$

Problem 6. (i) Consider the smoth vector fields

$$X = X_1(x_1, x_2)\frac{\partial}{\partial x_1} + X_2(x_1, x_2)\frac{\partial}{\partial x_2}$$

and the two differential form

$$\omega = dx_1 \wedge dx_2.$$

Find the equation

$$d(X \rfloor \omega) = 0$$

where $\rfloor$ denotes the contraction (inner product). One also writes

$$d(\omega(X)) = 0.$$

Calculate the Lie derivative $L_X\omega$.
(ii) Consider the smoth vector fields

$$X = \sum_{j=1}^{4} X_j(\mathbf{x})\frac{\partial}{\partial x_j}$$

and the differential two form

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4.$$

Find the equation

$$d(X\rfloor\omega) = 0$$

where $\rfloor$ denotes the contraction (inner product). One also writes $d(\omega(X)) = 0$. Calculate $L_X\omega$.

Problem 7. Let α be a smooth differential one-form and V be a smooth vector field. Assume that

$$L_V\alpha = f\alpha$$

where f is a smooth function. Define the function F as

$$F := V\rfloor\alpha$$

where $\rfloor$ denotes the contraction. Show that

$$dF = f\alpha - V\rfloor d\alpha.$$

Problem 8. Let V, W be two smooth vector fields defined on $\mathbb{R}^3$. We write

$$V = V_1(\mathbf{x})\frac{\partial}{\partial x_1} + V_2(\mathbf{x})\frac{\partial}{\partial x_2} + V_3(\mathbf{x})\frac{\partial}{\partial x_3}$$
$$W = W_1(\mathbf{x})\frac{\partial}{\partial x_1} + W_2(\mathbf{x})\frac{\partial}{\partial x_2} + W_3(\mathbf{x})\frac{\partial}{\partial x_3}.$$

Let

$$\omega = dx_1 \wedge dx_2 \wedge dx_3$$

be the volume form in $\mathbb{R}^3$. Then $L_V\omega = (\text{div}(V))\omega$, where $L_V(.)$ denotes the Lie derivative and div$V$ denotes the diveregence of the vector field V. Find the divergence of the vector field given by the commutator $[V, W]$. Apply it to the vector fields asscociated with the autonomous systems of first order differential equations

$$\frac{dx_1}{dt} = \sigma(x_2 - x_1), \quad \frac{dx_2}{dt} = \alpha x_1 - x_2 - x_1x_3, \quad \frac{dx_3}{dt} = -\beta x_3 + x_1x_2$$

and

$$\frac{dx_1}{dt} = a(x_2 - x_1), \quad \frac{dx_2}{dt} = (c-a)x_1 + cx_2 - x_1x_3, \quad \frac{dx_3}{dt} = -bx_3 + x_1x_2 .$$

The first system is the *Lorenz model* and the second system is *Chen's model*.

Problem 9. Consider the smooth vector field

$$V = \sum_{j=1}^{n} V_j(\mathbf{u})\frac{\partial}{\partial u_j}$$

defined on $\mathbb{R}^n$. Consider the smooth differential one-form

$$\alpha = \sum_{k=1}^{n} f_k(\mathbf{u})du_k .$$

Find the Lie derivative $L_V\alpha$. What is the condition such that $L_V\alpha = 0$.

Problem 10. Consider the smooth vector fields V and W defined on $\mathbb{R}^n$. Let f and g be smooth functions. Assume that

$$L_V f = 0, \qquad L_W g = 0 .$$

Find

$$L_{[V,W]}(f+g), \qquad L_{[V,W]}(fg) .$$

Problem 11. Let V, W be two smooth vector fields defined on $\mathbb{R}^n$. Let f, g be smooth function defined on $\mathbb{R}^n$. Assume that

$$Vf = 0, \qquad Wg = 0$$

i.e. f, g are first integrals of the dynamical system given by the vector fields V and W.
(i) Calculate

$$[V,W](fg), \qquad [V,W](gf)$$

where [,] denotes the commutator.
(ii) Calculate

$$[V,W](f(g))$$

where $f(g)$ denotes function composition.

Problem 12. Consider the manifold $M = \mathbb{R}^2$. Let V be a smooth vector field in M. Let (x,y) be the local coordinate system. Assume that

$$L_V dx = dy, \qquad L_V dy = dx$$

where $L_V(.)$ denotes the Lie derivative. Find

$$L_V(dx \wedge dy).$$

Problem 13. Let V, W be two smooth vector fields

$$V = \sum_{j=1}^{n} V_j(\mathbf{x}) \frac{\partial}{\partial x_j}, \qquad W = \sum_{j=1}^{n} W_j(\mathbf{x}) \frac{\partial}{\partial x_j}$$

defined on $\mathbb{R}^n$. Assume that

$$[V, W] = f(\mathbf{x}) W.$$

Let

$$\Omega = dx_1 \wedge \cdots \wedge dx_n$$

be the volume form and $\alpha := W \rfloor \Omega$. Find the Lie derivative

$$L_V \alpha.$$

Discuss.

Problem 14. Let $M = \mathbb{R}^2$ and let x, y denote the Euclidean coordinates on $\mathbb{R}^2$. Consider the differential one-form

$$\alpha = \frac{1}{2}(xdy - ydx).$$

Consider the vector field defined on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$

$$V = \frac{1}{x^2 + y^2} \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right).$$

Find

$$V \rfloor d\alpha$$

and the Lie derivative $L_V \alpha$.

Problem 15. Consider the two smooth vector fields in $\mathbb{R}^2$

$$V = V_1(\mathbf{x}) \frac{\partial}{\partial x_1} + V_2(\mathbf{x}) \frac{\partial}{\partial x_2}, \qquad W = W_1(\mathbf{x}) \frac{\partial}{\partial x_1} + W_2(\mathbf{x}) \frac{\partial}{\partial x_2}.$$

Assume that $[W, V] = 0$. Find the Lie derivatives

$$L_V(V_1 W_2 - V_2 W_1), \qquad L_W(V_1 W_2 - V_2 W_1).$$

Discuss.

Problem 16. Consider the smooth manifold $M = \mathbb{R}^3$ with coordintes (x, p, z) and the differential one form

$$\alpha = dz - pdx.$$

(i) Show that $\alpha \wedge d\alpha \neq 0$. Consider the vector fields

$$V = \frac{\partial}{\partial p}, \qquad W = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z}.$$

Find

$$V \rfloor \alpha, \qquad W \rfloor \alpha\,.$$

(ii) Consider the smooth manifold $M = \mathbb{R}^5$ with coordinates (x_1, x_2, p_1, p_2, z) and the differential one-form

$$\alpha = dz - \sum_{j=1}^{2} p_j dz_j.$$

Show that $\alpha \wedge d\alpha \wedge \neq 0$. Consider the vector fields

$$V_1 = \frac{\partial}{\partial p_1}, \quad V_2 = \frac{\partial}{\partial p_2}, \quad W_1 = \frac{\partial}{\partial x_1} + p_1\frac{\partial}{\partial z}, \quad W_2 = \frac{\partial}{\partial x_2} + p_2\frac{\partial}{\partial z}.$$

Find

$$V_1 \rfloor \alpha, \quad V_2 \rfloor \alpha, \quad W_1 \rfloor \alpha, \quad W_2 \rfloor \alpha.$$

Problem 17. Let V be a smooth vector field in $\mathbb{R}^3$. Find the condition on V such that

$$L_V(x_1 dx_2 + x_2 dx_3 + x_3 dx_1) = 0.$$

Problem 18. Let $M = \mathbb{R}^2$. Consider

$$V = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}, \qquad \omega = dx_1 \wedge dx_2.$$

Calculate the Lie derivative $L_V\omega$.

Problem 19. Let $du_1/dt = V_1(\mathbf{u})$, $\ldots$, $du_n/dt = V_n(\mathbf{u})$ be an autonomous system of ordinary differential equations, where $V_j(\mathbf{u}) \in C^\infty(\mathbb{R}^n)$ for all $j = 1, \ldots, n$. A function $\phi \in C^\infty(\mathbb{R}^n)$ is called *conformal invariant* with respect to the vector field

$$V = V_1(\mathbf{u})\frac{\partial}{\partial u_1} + \cdots + V_n(\mathbf{u})\frac{\partial}{\partial u_n}$$

if

$$L_V\phi = \rho\phi$$

where $\rho \in C^\infty(\mathbb{R}^n)$. Let $n = 2$ and consider the vector fields

$$V = u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1}, \qquad W = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}.$$

Show that $\phi(\mathbf{u}) = u_1^2 + u_2^2$ is conformal invariant under V and W. Find the commutator $[V, W]$.

Problem 20. Consider the mainfold $\mathbb{R}^2$ and the smooth vector field

$$V = V_1(x_1, x_2)\frac{\partial}{\partial x_1} + V_2(x_1, x_2)\frac{\partial}{\partial x_2}.$$

Find V_1, V_2 such that

$$L_V(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) = 0$$

$$L_V\left(\frac{\partial}{\partial x_1} \otimes dx_1 + \frac{\partial}{\partial x_2} \otimes dx_2\right) = 0$$

$$L_V\left(\frac{\partial}{\partial x_1} \otimes \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \otimes \frac{\partial}{\partial x_2}\right) = 0.$$

Problem 21. Let M be differentiable manifold and $\phi : M \to \mathbb{R}$ be a smooth function. Let α be a smooth diferential one form defined on M. Show that if V is a vector field defined on M such that $d\phi = V\rfloor d\alpha$, then

$$L_V\alpha = d(V\rfloor\alpha + \phi).$$

Problem 22. Consider the manifold $M = \mathbb{R}^n$ and the volume form

$$\Omega = dx_1 \wedge \cdots \wedge dx_n.$$

Consider the analytic vector field

$$V = \sum_{j=1}^{n} V_j(\mathbf{x})\frac{\partial}{\partial x_j}.$$

(i) Find $\omega = V\rfloor\Omega$.
(ii) Find $L_V\Omega$.

Problem 23. Consider the autonomous system of first order ordinary differential equations

$$\frac{du_j}{dt} = V_j(\mathbf{u}), \qquad j = 1, 2, \ldots, n$$

where the V_j's are polynomials. The corresponding vector field is

$$V = \sum_{j=1}^{n} V_j \frac{\partial}{\partial x_j}.$$

Let f be an analytic function. The Lie derivative of f is

$$L_V f = \sum_{j=1}^{n} V_j \frac{\partial f}{\partial x_j}.$$

A *Darboux polynomial* is a polynomial g such that there is another polynomial p satisfying

$$L_V g = pg.$$

The couple is called a *Darboux element*. If m is the greatest of $\deg V_j$ ($j = 1, \ldots, n$), then $\deg p \leq m - 1$. All the irreducible factors of a Darboux polynomial are Darboux. The search for Darboux polynomials can be restricted to irreducible g. If the autonomous system of first order differential equations is homogeneous of degree m, i.e. all V_j are homgeneous of degree m, then p is homogeneous of degree $m-1$ and all homgeneous components of g are Darboux. The search can be restricted to homgeneous g.
(i) Show that the product of two Darboux polynomials is a Darboux polynomial.
(ii) Consider the Lotka-Volterra model for three species

$$\begin{aligned}
\frac{du_1}{dt} &= u_1(c_3 u_2 + u_3) \\
\frac{du_2}{dt} &= u_2(c_1 u_3 + u_1) \\
\frac{du_3}{dt} &= u_3(c_2 u_1 + u_2)
\end{aligned}$$

where c_1, c_2, c_3 are real parameters. Find the determining equation for the Darboux element.

Problem 24. Consider a smooth vector field in $\mathbb{R}^3$

$$V = V_1(\mathbf{x})\frac{\partial}{\partial x_1} + V_2(\mathbf{x})\frac{\partial}{\partial x_2} + V_3(\mathbf{x})\frac{\partial}{\partial x_3}$$

and the differential two-form

$$\beta = dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + dx_3 \wedge dx_1.$$

Find $V \rfloor \alpha$ and $d(V \rfloor \alpha)$. Thus find $L_V \alpha$. Find solutions of the partial differential equations given by $L_V \alpha = 0$.

Problem 25. Consider the unit ball

$$x^2 + y^2 + z^2 = 1.$$

and the vector field

$$\begin{aligned} V = & (a_0 + a_1 x + a_2 y + a_3 z + x(e_1 x + e_2 y + e_3 z))\frac{\partial}{\partial x} \\ & +(b_0 + b_1 x + b_2 y + b_3 z + y(e_1 x + e_2 y + e_3 z))\frac{\partial}{\partial y} \\ & +(c_0 + c_1 x + c_2 y + c_3 z + z(e_1 x + e_2 y + e_3 z))\frac{\partial}{\partial z}. \end{aligned}$$

Find the coefficients from the conditions

$$L_V(x^2 + y^2 + z^2) = 0, \qquad x^2 + y^2 + z^2 = 1.$$

Problem 26. Some quantities in physics owing to the transformation laws have to be considered as currents instead of differential forms. Let M be an orientable n-dimensional differentiable manifold of class C^∞. We denote by $\Phi_k(M)$ the set of all differential forms of degree k with compact support. Let $\phi \in \Phi_k(M)$ and let α be an exterior differential form of degree $n-k$ with locally integrable coefficients. Then, as an example of a current, we have

$$T_\alpha(\phi) \equiv \alpha(\phi) := \int_M \alpha \wedge \phi.$$

Define the Lie derivative for this current.

Problem 27. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth Hamilton function with the corresponding vector field

$$V_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} \right).$$

Let

$$W = \sum_{j=1}^{n} \left(f_j(\mathbf{p}, \mathbf{q})\frac{\partial}{\partial q_j} + g_j(\mathbf{p}, \mathbf{q})\frac{\partial}{\partial p_j} \right)$$

be another smooth vector field. Assume that

$$[V_H, W] = \lambda W \tag{1}$$

where λ is a smooth function of $\mathbf{p}$ and $\mathbf{q}$. Let

$$\omega = dq_1 \wedge \cdots \wedge dq_n \wedge dp_1 \wedge \cdots \wedge dp_n$$

be the standard volume differential form. Let

$$\alpha = W \rfloor \omega.$$

Show that (1) can be written as

$$L_{V_H}\alpha = \lambda\alpha.$$

Problem 28. Consider the vector field V associated with the Lorenz model

$$\frac{du_1}{dt} = \sigma(u_2 - u_1)$$
$$\frac{du_2}{dt} = -u_1u_3 + ru_1 - u_2$$
$$\frac{du_3}{dt} = u_1u_2 - bu_3.$$

Let

$$\alpha = u_1du_2 + u_2du_3 + u_3du_1.$$

Calculate the Lie derivative

$$L_V\alpha$$

Discuss.

Problem 29. Consider the metric tensor field

$$g = -cdt \otimes cdt + dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\phi \otimes d\phi$$

and the vector field

$$V = \frac{1}{\sqrt{1 - \omega^2 r^2 \sin^2\theta / c^2}} \left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \phi} \right)$$

where c is the speed of light and ω a fixed frequency. Find the Lie derivative $L_V g$.

Problem 30. Consider the $2n+1$ dimensional anti-de Sitter space AdS_{2n+1}. This is a hypersurface in the vector space $\mathbb{R}^{2n+2}$ defined by the equation $R(\mathbf{x}) = -1$, where

$$R(\mathbf{x}) = -(x_0)^2 - (x_1)^2 + (x_2)^2 + \cdots + (x_{2n+1})^2.$$

One introduces the even coordinates $\mathbf{p}$ and odd coordinates $\mathbf{q}$. Then we can write

$$R(\mathbf{p}, \mathbf{q}) = -p_1^2 - q_1^2 + p_2^2 + q_2^2 + \cdots + p_{n+1}^2 + q_{n+1}^2.$$

We consider $\mathbb{R}^{2n+2}$ as a symplectic manifold with the canonical symplectic differential form

$$\omega = \sum_{k=1}^{n+1} dp_k \wedge dq_k.$$

Let

$$\alpha = \frac{1}{2} \sum_{k=1}^{n+1} (p_k dq_k - q_k dp_k).$$

Consider the vector field V in $\mathbb{R}^{2n+2}$ given by

$$V = \frac{1}{2} \sum_{k=1}^{n+1} \left(p_k \frac{\partial}{\partial p_k} + q_k \frac{\partial}{\partial q_k} \right).$$

Find the Lie derivative $L_V R$ and $V \rfloor \omega$.

Problem 31. Consider the Lotka-Volterra equation

$$\frac{du_1}{dt} = (a - bu_2)u_1, \qquad \frac{du_2}{dt} = (cu_1 - d)u_2$$

where a, b, c, d are constants and $u_1 > 0$ and $u_2 > 0$. The corresponding vector field V is

$$V = (a - bu_2)u_1 \frac{\partial}{\partial u_1} + (cu_1 - d)u_2 \frac{\partial}{\partial u_2}.$$

Let

$$\omega = f(u_1, u_2) du_1 \wedge du_2$$

where f is a smooth nonzero function. Find a smooth function H (Hamilton function) such that

$$\omega \rfloor V = dH.$$

Note that from this condition since $ddH = 0$ we obtain

$$d(\omega \rfloor V) = 0.$$

Problem 32. Let I, f be analytic functions of u_1, u_2. Consider the autonomous system of differential equations

$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = \begin{pmatrix} 0 & f(\mathbf{u}) \\ -f(\mathbf{u}) & 0 \end{pmatrix} \begin{pmatrix} \partial I/\partial u_1 \\ \partial I/\partial u_2 \end{pmatrix}.$$

Show that I is a first integral of this autonomous system of differential equations.

Problem 33. Consider the smooth vector field

$$V = V_1(\mathbf{u})\frac{\partial}{\partial u_1} + V_2(\mathbf{u})\frac{\partial}{\partial u_2}$$

in $\mathbb{R}^2$. Let $f_1(u_1)$, $f_2(u_2)$ be smooth functions.
(i) Calculate the Lie derivative

$$L_V\left(f_1(u_1)du_1 \otimes \frac{\partial}{\partial u_1} + f_2(u_2)du_2 \otimes \frac{\partial}{\partial u_2}\right).$$

Find the condition arising from setting the Lie derivative equal to 0.
(ii) Calculate the Lie derivative

$$L_V\left(f_1(u_1)du_1 \otimes du_1 + f_2(u_2)du_2 \otimes du_2\right).$$

Find the conditions arising from setting the Lie derivative equal to 0. Compare the conditions to the conditions from (i).

Problem 34. Let V, W be smooth vector fields defined in $\mathbb{R}^n$. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions. Consider now the pairs (V, f), (W, g). One defines a commutator of such pairs as

$$[(V, f), (W, g)] := ([V, W], L_V g - L_W f).$$

Let

$$V = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2}, \qquad W = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_1}$$

and $f(u_1, u_2) = g(u_1, u_2) = u_1^2 + u_2^2$. Calculate the commutator.

Problem 35. Consider the two differential form in $\mathbb{R}^3$

$$\beta = x_3 dx_1 \wedge dx_2 + x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1.$$

Find $d\beta$. Find the Lie dervivative $L_V\beta$. Find the condition on the vector field V such that $L_V\beta = 0$.

Problem 36. Let V be the vector field for the Lorenz model

$$V = \sigma(-u_1 + u_2)\frac{\partial}{\partial u_1} + (-u_1u_3 + ru_1 - u_2)\frac{\partial}{\partial u_2} + (u_1u_2 - bu_3)\frac{\partial}{\partial u_3}.$$

Find the Lie derivative $L_V(du_1 \wedge du_2)$, $L_V(du_2 \wedge du_3)$, $L_V(du_3 \wedge du_1)$. Discuss.

Problem 37. (i) Consider the tensor fields in $\mathbb{R}^2$

$$T_1 = \sum_{j,k=1}^{2} t_{jk}(\mathbf{x})dx_j \otimes dx_k, \quad T_2 = \sum_{j,k=1}^{2} t_{jk}(\mathbf{x})dx_j \otimes \frac{\partial}{\partial x_k}, \quad T_3 = \sum_{j,k=1}^{2} t_{jk}(\mathbf{x})\frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_k}.$$

Find the condition on the vector field

$$V = \sum_{\ell=1}^{2} V_\ell(\mathbf{x})\frac{\partial}{\partial x_\ell}$$

such that

$$L_V T_1 = 0, \quad L_V T_2 = 0, \quad L_V T_3 = 0.$$

(ii) Simplify for the case $t_{jk}(\mathbf{x}) = 1$ for all $j, k = 1, 2$.

Problem 38. Let $n \geq 2$. Consider the smooth vector field in $\mathbb{R}^n$

$$V = \sum_{j=1}^{n} V_j(\mathbf{x})\frac{\partial}{\partial x_j}.$$

Find the Lie derivative of the tensor fields

$$\frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_k}, \quad dx_j \otimes \frac{\partial}{\partial x_k}, \quad dx_j \otimes dx_k$$

with $j, k = 1, \ldots, n$. Set the Lie derivative to 0 and study the partial differential equations of V_j.

Problem 39. V, W be smooth vector fields in $\mathbb{R}^3$. Let

$$L_V(dx_1 \wedge dx_2 \wedge dx_3) = (\mathrm{div}(V))dx_1 \wedge dx_2 \wedge dx_3, \qquad L_W(dx_1 \wedge dx_2 \wedge dx_3) = (\mathrm{div}(W))dx_1 \wedge dx_2 \wedge dx_3.$$

Calculate

$$L_{[V,W]}dx_1 \wedge dx_2 \wedge dx_3.$$

Problem 40. Let V be a smooth vector field in $\mathbb{R}^2$. Assume that

$$L_V(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) = 0, \quad L_V\left(\frac{\partial}{\partial x_1} \otimes \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \otimes \frac{\partial}{\partial x_2}\right) = 0.$$

Can we conclude that

$$L_V\left(dx_1 \otimes \frac{\partial}{\partial x_1} + dx_2 \otimes \frac{\partial}{\partial x_2}\right) = 0\,?$$

Problem 41. The Heisenberg group H is a non-commutative Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined by

$$(x,y,z)\bullet(x',y',z') := (x+x', y+y', z+z'-x'y+xy').$$

(i) Find the idenity element. Find the inverse element.
(ii) Consider the metric tensor

$$g = -dx\otimes dx + dy\otimes dy + x^2 dy\otimes dy + xdy\otimes dz + xdz\otimes dy + dz\otimes dz$$

and the vector fields

$$V_1 = \frac{\partial}{\partial z}, \quad V_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad V_3 = \frac{\partial}{\partial x}.$$

Show that the vector fields form a basis of a Lie algebra. Classify the Lie algebra. Calculate the Lie derivatives

$$L_{V_1}g, \qquad L_{V_2}g, \qquad L_{V_3}g.$$

Discuss.

Problem 42. Consider the $2n+1$ smooth vector fields

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

$(j = 1,\dots,n)$ and the differential one form

$$\theta = dt + \frac{1}{2}\sum_{j=1}^{n}(y_j dx_j - x_j dy_j).$$

(i) Find the commutators

$$[X_j, Y_j], \quad [X_j, T], \quad [Y_j, T].$$

(ii) Find

$$\exp(\alpha X_j)x_j, \quad \exp(\beta Y_j)y_j, \quad \exp(\gamma T)t\,.$$

(iii) Find the Lie derivatives

$$L_{X_j}\theta, \quad L_{Y_j}\theta, \quad L_T\theta.$$

Problem 43. Let V, W be vector fields and α be a differential form. Find the Lie derivative

$$L_V(W \otimes \alpha).$$

Problem 44. Consider the manifold $M = \mathbb{R}^2$. Let V be a smooth vector field in M. Let (x, y) be the local coordinate system. Assume that

$$L_V dx = dy, \qquad L_V dy = dx.$$

Find

$$L_V(dx \wedge dy).$$

Problem 45. Consider the metric tensor field

$$g = dt \otimes dt - dv \otimes dv - kx dt \otimes dy - kx dy \otimes dt + (k^2x^2 - e^{kv}) dy \otimes dy - e^{-kv} dx \otimes dx$$

the differential two-form

$$F = \frac{1}{\sqrt{2}} k e^{ikv} (dv \wedge dt + kx dy \wedge dv + i dx \wedge dy)$$

and the vector fields

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = ky\frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \quad V_4 = \frac{\partial}{\partial v} + \frac{1}{2}kx\frac{\partial}{\partial x} - \frac{1}{2}ky\frac{\partial}{\partial y}.$$

Show that

$$L_{V_1} g = L_{V_2} g = L_{V_3} g = L_{V_4} g = 0.$$

and

$$L_{V_1} F = L_{V_2} F = L_{V_3} F = L_{V_4} F = 0.$$

Chapter 6

Killing Vector Fields and Lie Algebras

Let g be a metric tensor field and V be a vector field. Then V is called a Killing vector field if

$$L_V g = 0$$

i.e. the Lie derivative of g with respect to V vanishes. The Killing vector fields provide a basis of a Lie algebra.

Problem 1. Consider the two-dimensional Euclidean space with the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2.$$

Find the Killing vector fields, i.e. the analytic vector fields V such that

$$L_V g = 0$$

where L_V denotes the Lie derivative. Show that the set of Killing vector fields form a Lie algebra under the commutator.

Problem 2. Consider the metric tensor field

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy), \qquad -\infty < x < \infty, \quad 0 < y < \infty.$$

Find the Killing vector fields.

Problem 3. A standard model of the complex hyperbolic space is the complex unit ball

$$B^n := \{\, \mathbf{z} \in \mathbb{C} \,:\, |\mathbf{z}| < 1 \,\}$$

with the *Bergman metric*

$$g = \sum_{j,k=1}^{n} g_{j,k}(\mathbf{z}) dz_j \otimes d\bar{z}_k$$

where

$$g_{j,k} = \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_k} \ln(1 - |\mathbf{z}|^2)\,.$$

Find the Killing vector fields of g.

Problem 4. Consider the metric tensor field

$$g = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi.$$

Show that g admits the Killing vector fields

$$V_1 = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}$$

$$V_2 = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}$$

$$V_3 = \frac{\partial}{\partial\phi}.$$

Is the Lie algebra given by the vector fields semisimple?

Problem 5. A *de Sitter universe* may be represented by the hypersurface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = R^2$$

where R is a real constant. This hypersurface is embedded in a five dimensional flat space whose metric tensor field is

$$g = dx_0 \otimes dx_0 - dx_1 \otimes dx_1 - dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - dx_4 \otimes dx_4.$$

Find the Killing vector fields V of g, i.e. the solutions of $L_V g = 0$.

Problem 6. For the Poincaré upper half plane

$$H = \{\, z = x_1 + ix_2 \,:\, y > 0 \,\}$$

the metric tensor field is given by

$$g = \frac{1}{x_2^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2).$$

Find the Killing vector fields for g, i.e.

$$V = V_1(x_1, x_2)\frac{\partial}{\partial x_1} + V_2(x_1, x_2)\frac{\partial}{\partial x_2}$$

where $L_V g = 0$.

Problem 7. Consider the metric tensor field

$$g = dt \otimes dt - e^{P_1(t)} dx \otimes dx - e^{P_2(t)} dy \otimes dy - e^{P_3(t)} dz \otimes dz$$

where P_j ($j = 1, 2, 3$) are smooth functions of t. Find the Killing vector fields.

Chapter 7

Lie-Algebra Valued Differential Forms

Problem 1. Let A be an $n \times n$ matrix. Assume that the entries are analytic functions of x. Assume that A is invertible for all x. Let d be the exterior derivative. We have the identity

$$d(\det(A)) \equiv \det(A)\mathrm{tr}(A^{-1}dA).$$

Let

$$A = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}.$$

Calculate the left-hand side and right hand side of the identity.

Problem 2. Let

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Obviously, $R \in SO(2)$. Calculate R^{-1}, dR, $R^{-1}dR$ and $dR(R^{-1})$, where $R^{-1}dR$ is the left-invariant matrix differential one-form and $dR(R^{-1})$ is the right-invariant matrix differential one-form.

Problem 3. Let G be a Lie group with Lie algebra L. A differential form ω on G is called *left invariant* if

$$f(x)^*\omega = \omega \tag{1}$$

for all $x \in G$, $f(x)$ denoting the left translation $g \to xg$ on G. Let $X_1, \ldots,$ X_n be a basis of L and $\omega_1, \ldots \omega_n$ the one-forms on G determined by

$$\omega_i(\widetilde{X}_j) = \delta_{ij} \tag{2}$$

where $\widetilde{X}_i$ are the corresponding left invariant vector fields on G and δ_{ij} is the Kronecker delta. Show that

$$d\omega_i = -\frac{1}{2} \sum_{j,k=1}^{n} c_{jk}^i \omega_j \wedge \omega_k, \qquad i = 1, 2, \ldots, n \tag{3}$$

where the *structural constants* c_{jk}^i are given by

$$[X_j, X_k] = \sum_{i=1}^{n} c_{jk}^i X_i. \tag{4}$$

System (3) is known as the *Maurer-Cartan equations.*

Problem 4. Let G be a Lie group whose Lie algebra is L. L is identified with the left invariant vector fields on G. Now suppose that $X_1, \ldots, X_n$ is a basis of L and that $\omega_1, \ldots \omega_n$ is a dual basis of left invariant one-forms. There is a natural Lie algebra valued one-form $\widetilde{\omega}$ on G which can be written as

$$\widetilde{\omega} := \sum_{i=1}^{n} \omega_i \otimes X_i \tag{1}$$

where

$$(X_i, \omega_j) = \delta_{ij}. \tag{2}$$

Show that

$$d\widetilde{\omega} + \frac{1}{2}[\widetilde{\omega}, \widetilde{\omega}] = 0 \tag{3}$$

where

$$[\widetilde{\omega}, \widetilde{\omega}] := \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega_i \wedge \omega_j) \otimes [X_i, X_j]. \tag{4}$$

Obviously, (3) are the Maurer-Cartan equations.

Problem 5. Consider the Lie algebra

$$G := \left\{ \begin{pmatrix} e^\alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha \in \mathbb{R}\,,\ \beta \in \mathbb{R} \right\}. \tag{1}$$

Let

$$X := \begin{pmatrix} e^\alpha & \beta \\ 0 & 1 \end{pmatrix} \tag{2}$$

and

$$\Omega := X^{-1}dX. \tag{3}$$

Show that

$$d\Omega + \Omega \wedge \Omega = 0. \tag{4}$$

Problem 6. Let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{1}$$

where

$$x_{11}x_{22} - x_{12}x_{21} = 1$$

so that X is a general element of the Lie group $SL(2,\mathbb{R})$. Then $X^{-1}dX$, considered as a matrix of one-forms, takes its value in the Lie algebra $sl(2,\mathbb{R})$, the Lie algebra of $SL(2,\mathbb{R})$. If

$$X^{-1}dX = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & -\omega_1 \end{pmatrix} \tag{2}$$

then $\{\,\omega^j\,\}$ are the left-invariant forms of $SL(2,\mathbb{R})$.
(i) Show that there is a (local) $SL(2,\mathbb{R})$-valued function A on $\mathbb{R}^2$ such that

$$A^{-1}dA = \begin{pmatrix} \Theta^1 & \Theta^2 \\ \Theta^3 & -\Theta^1 \end{pmatrix} = \Theta. \tag{3}$$

Write Θ for this $sl(2,\mathbb{R})$-valued one-form on $\mathbb{R}^2$.
(ii) Show that then $dG = G\Theta$ and that each row (r,s) of the matrix G satisfies

$$dr = r\theta_1 + s\theta_3, \qquad ds = r\theta_2 - s\theta_1. \tag{4}$$

Note that *Maurer-Cartan equations* for the forms $\{\theta_1,\theta_2,\theta_3\}$ may be written

$$d\Theta + \Theta \wedge \Theta = 0. \tag{5}$$

(iii) Show that any element of $SL(2,\mathbb{R})$ can be expressed uniquely as the product of an upper triangular matrix and a rotation matrix (the *Iwasawa decomposition*). Define an upper-triangular-matrix-valued function T and a rotation-matrix-valued function R on $\mathbb{R}^2$ by $A = TR^{-1}$. Thus show that

$$T^{-1}dT = R^{-1}dR + R^{-1}\Theta R.$$

Problem 7. Let

$$SL(2,\mathbb{R}) := \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\} \tag{1}$$

be the group of all (2×2)-real unimodular matrices. Its right-invariant Maurer-Cartan form is

$$\omega = dXX^{-1} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \tag{2}$$

where

$$\omega_{11} + \omega_{22} = 0. \tag{3}$$

Show that ω satisfies the *structure equation* of $SL(2,\mathbb{R})$, (also called the Maurer-Cartan equation)

$$d\omega = \omega \wedge \omega$$

or, written explicitly,

$$d\omega_{11} = \omega_{12} \wedge \omega_{21}, \qquad d\omega_{12} = 2\omega_{11} \wedge \omega_{12}, \qquad d\omega_{21} = 2\omega_{21} \wedge \omega_{11}.$$

Problem 8. Let

$$SL(2,\mathbb{R}) := \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1 \right\} \tag{1}$$

be the group of all (2×2)-real unimodular matrices. Its right-invariant Maurer-Cartan form is

$$\omega = dXX^{-1} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \tag{2}$$

where

$$\omega_{11} + \omega_{22} = 0. \tag{3}$$

Then ω satisfies (see previous problem) the *structure equation* of $SL(2,\mathbb{R})$, (also called the Maurer-Cartan equation)

$$d\omega = \omega \wedge \omega$$

or, written explicitly,

$$d\omega_{11} = \omega_{12} \wedge \omega_{21}, \qquad d\omega_{12} = 2\omega_{11} \wedge \omega_{12}, \qquad d\omega_{21} = 2\omega_{21} \wedge \omega_{11}. \tag{4}$$

(ii) Let U be a neighbourhood in the (x,t)-plane and consider the smooth mapping

$$f : U \to SL(2,\mathbb{R}). \tag{5}$$

The pull-backs of the Maurer-Cartan forms can be written

$$\omega_{11} = \eta(x,t)dx + A(x,t)dt, \quad \omega_{12} = q(x,t)dx + B(x,t)dt, \quad \omega_{21} = r(x,t)dx + C(x,t)dt \tag{6}$$

where the coefficients are functions of x, t. Show that

$$-\frac{\partial \eta}{\partial t} + \frac{\partial A}{\partial x} - qC + rB = 0 \tag{7a}$$

$$-\frac{\partial q}{\partial t} + \frac{\partial B}{\partial x} - 2\eta B + 2qA = 0 \tag{7b}$$

$$-\frac{\partial r}{\partial t} + \frac{\partial C}{\partial x} - 2rA + 2\eta C = 0. \tag{7c}$$

(ii) Consider the special case that $r = +1$ and η is a real parameter independent of x, t. Writing

$$q = u(x,t), \tag{8}$$

show that

$$A(x,t) = \eta C(x,t) + \frac{1}{2}\frac{\partial C}{\partial x}, \qquad B(x,t) = u(x,t)C(x,t) - \eta(x,t)\frac{\partial C}{\partial x} - \frac{1}{2}\frac{\partial^2 C}{\partial x^2}. \tag{9}$$

Show that substitution into the second equation of (7) gives

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}C + 2u\frac{\partial C}{\partial x} = 2\eta^2\frac{\partial C}{\partial x} - \frac{1}{2}\frac{\partial^3 C}{\partial x^3}. \tag{10}$$

(iii) Let

$$C = \eta^2 - \frac{1}{2}u \tag{11}$$

Show that (10) becomes

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} - \frac{3}{2}u\frac{\partial u}{\partial x}, \tag{12}$$

which is the well-known *Korteweg-de Vries equation*.

Problem 9. We consider the case where $M = \mathbb{R}^2$ and $L = sl(2,\mathbb{R})$. In local coordinates (x,t) a Lie algebra-valued one-differential-form is given by

$$\widetilde{\alpha} = \sum_{i=1}^{3} \alpha_i \otimes T_i \tag{1}$$

where

$$\alpha_i := a_i(x,t)dx + A_i(x,t)dt \tag{2}$$

and $\{T_1, T_2, T_3\}$ is a basis of the semi-simple Lie algebra $s\ell(2,\mathbb{R})$. A convenient choice is

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

(i) Show that the condition that the covariant derivative vanishes

$$D_{\widetilde{\alpha}}\widetilde{\alpha} = 0 \tag{4}$$

leads to the following systems of partial differential equations of first order

$$-\frac{\partial a_1}{\partial t} + \frac{\partial A_1}{\partial x} + a_2 A_3 - a_3 A_2 = 0 \tag{5a}$$

$$-\frac{\partial a_2}{\partial t} + \frac{\partial A_2}{\partial x} + 2(a_1 A_2 - a_2 A_1) = 0 \tag{5b}$$

$$\frac{\partial a_3}{\partial t} + \frac{\partial A_3}{\partial x} - 2(a_1 A_3 - a_3 A_1) = 0. \tag{5c}$$

(ii) Show that the sine Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0 \tag{6}$$

can be represented as follows

$$a_2 = -\frac{1}{4}(\cos u + 1) \qquad A_1 = \frac{1}{4}(\cos u - 1) \tag{7a}$$

$$a_2 = \frac{1}{4}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} - \sin u\right), \qquad A_2 = \frac{1}{4}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \sin u\right) \tag{7b}$$

$$a_3 = -\frac{1}{4}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \sin u\right) \qquad A_4 = -\frac{1}{4}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} - \sin u\right) \tag{7c}$$

(iii) Prove the following. Let

$$a_1 = f_1(u), \qquad A_1 = f_2(u) \tag{8a}$$

$$a_2 = c_1\frac{\partial u}{\partial x} + c_2\frac{\partial u}{\partial t} + f_3(u), \qquad A_2 = c_3\frac{\partial u}{\partial x} + c_4\frac{\partial u}{\partial t} + f_4(u) \tag{8b}$$

$$a_3 = c_5\frac{\partial u}{\partial x} + c_6\frac{\partial u}{\partial t} + f_5(u), \qquad A_3 = c_7\frac{\partial u}{\partial x} + c_8\frac{\partial u}{\partial t} + f_6(u) \tag{8c}$$

where $f_1, \ldots, f_6$ are smooth functions and $c_1, \ldots, c_8 \in \mathbb{R}$. Then the Lie algebra-valued differential from $\widetilde{\alpha}$ satisfies the condition (4) if

$$c_1 = c_2 = c_3 = c_4, \qquad c_5 = c_6 = c_7 = c_8 \tag{9a}$$

$$f_4 = -f_3, \qquad f_6 = -f_5 \tag{9b}$$

$$f_5 = cf_3 \qquad (c \in \{+1, -1\}) \tag{9c}$$

$$-c_1\frac{\partial^2 u}{\partial t^2} + c_1\frac{\partial^2 u}{\partial x^2} + 2f_3(-f_1 - f_2) = 0 \tag{9d}$$

and for $c = 1$

$$\frac{df_1}{dt} = -4c_1 f_3, \qquad \frac{df_2}{dt} = 4c_1 f_3, \qquad \frac{d^2 f_3}{dt^2} = -16c_1^2 f_3 \qquad (9e)$$

where $c_1 = -c_5$.

For $c = -1$

$$\frac{df_1}{dt} = 4c_1 f_3, \qquad \frac{df_2}{dt} = -4c_1 f_3, \qquad \frac{d^2 f_3}{dt^2} = 16c_1^2 f_3 \qquad (9f)$$

where $c_1 = c_5$.

(iv) Show that the solutions to these differential equations lead to the non-linear wave equations

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = C_1 \cosh u + C_2 \sinh u \qquad (10)$$

and

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = C_1 \sin u + C_2 \cos u \qquad (11)$$

(C_1, $C_2 \in \mathbb{R}$) can be written as the covariant exterior derivative of a Lie algebra-valued one-form, where the underlying Lie algebra is $sl(2, \mathbb{R})$.

Problem 10. Let (M, g) be a Riemann manifold with $\dim(M) = m$. Let s be an orthonormal local frame on U with dual coframe σ and let ∇ be the Levi-Civita covariant derivative. Then we have

(1) $g|_U = \sum_{i=1}^{m} \sigma^i \otimes \sigma^i$

(2) $\nabla s = s.\omega, \;\; \omega^i_j = -\omega^j_i, \;$ so $\; \omega \in \Omega^1(U, so(m))$

(3) $d\sigma + \omega \wedge \sigma = 0, \;\; d\sigma^i + \sum_{k=1}^{m} \omega^i_k \wedge \sigma^k = 0$

(4) $Rs = s.\Omega, \;\; \Omega = d\omega + \omega \wedge \omega \in \Omega^2(U, so(m)), \;\; \Omega^i_j = d\omega^i_j + \sum_{k=1}^{m} \omega^i_k \wedge \omega^k_j$

(5) $\Omega \wedge \sigma = 0, \;\; \sum_{k=1}^{m} \Omega^i_k \wedge \sigma^k = 0,$ first Bianchi identity

(6) $d\Omega + \omega \wedge \Omega - \Omega \wedge \omega \equiv d\Omega + [\omega, \Omega]_\wedge = 0,$ second Bianchi identity

If (M, g) is a pseudo Riemann manifold,

$$\eta_{ij} = g(s_i, s_j) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$$

the standard inner product matrix of the same signature (p, q) $(p+q = m)$, then we have instead

$$(1')\ g = \sum_{i=1}^{m} \eta_{ii} \sigma^i \otimes \sigma^i$$

$$(2')\ \eta_{jj} \omega_i^j = -\eta_{ii} \omega_j^i \ \text{ thus } \omega = (\omega_i^j) \in \Omega^1(U, so(p, q))$$

$$(3')\ \eta_{jj} \Omega_i^j = -\eta_{ii} \Omega_j^i \ \text{ thus } \Omega = (\Omega_i^j) \in \Omega^2(U, so(p, q)).$$

Consider the manifold $S^2 \subset \mathbb{R}^3$. Calculate the quantities given above. Consider the parametrization (leaving out one longitude)

$$f : (0, 2\pi) \times (-\pi, \pi) \to \mathbb{R}^3, \qquad f(\phi, \theta) = \begin{pmatrix} \cos(\phi)\cos(\theta) \\ \sin(\phi)\cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Problem 11. Show that the Korteweg-de Vries and nonlinear Schrödinger equations are reductions of the self-dual Yang-Mills equations. We work on $\mathbb{R}^4$ with coordinates $x^a = (x, y, u, t)$ and metric tensor field

$$g = dx \otimes dx - dy \otimes dy + du \otimes dt - dt \otimes du$$

of signature (2,2) and a totally skew orientation tensor $\epsilon_{abcd} = \epsilon_{[abcd]}$. We consider a Yang-Mills connection $D_a := \partial_a - A_a$ where the A_a where the A_a are, for the moment, elements of the Lie algebra of $SL(2, \mathbb{C})$. The A_a are defined up to gauge transformations

$$A_a \to h A_a h^{-1} - (\partial_a h) h^{-1}$$

where $h(x_a) \in SL(2, \mathbb{C})$. The connection is said to be self-dual when (summation convention)

$$\frac{1}{2} \epsilon_{ab}^{cd} [D_c, D_d] = [D_a, D_b]. \tag{3}$$

Problem 12. With the notation given above the *self-dual Yang-Mills equations* are given by

$$* D_{\widetilde{\alpha}} \widetilde{\alpha} = D_\alpha \alpha \tag{1}$$

Find the components of the self-dual Yang Mills equation.

Problem 13. Consider the non-compact Lie group $SU(1,1)$ and the compact Lie group $U(1)$. Let $z \in \mathbb{C}$ and $|z| < 1$. Consider the coset space $SU(1,1)/U(1)$ with the element ($\alpha \in \mathbb{R}$)

$$U(z,\alpha) = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & -z \\ -\bar{z} & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.$$

Consequently the coset space $SU(1,1)/U(1)$ can be viewed as an open unit disc in the complex plane. Consider the Cartan differential one-forms forms

$$\mu = i\frac{\bar{z}dz - zd\bar{z}}{1-|z|^2}, \quad \omega_+ = \frac{idz}{1-|z|^2}, \quad \omega_- = -\frac{idz}{1-|z|^2}.$$

Show that (Cartan-Mauer equations)

$$d\mu = 2i\omega_- \wedge \omega_+, \quad d\omega_+ = i\mu \wedge \omega_+, \quad d\omega_- = -i\mu \wedge \omega_-.$$

Show that

$$\omega_+ \wedge \omega_- = \frac{1}{(1-|z|^2)^2} dz \wedge d\bar{z}.$$

Chapter 8

Lie Symmetries and Differential Equations

Problem 1. Show that the second order ordinary linear differential equation

$$\frac{d^2u}{dt^2} = 0$$

admits the eight Lie symmetries

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial u}, \quad t\frac{\partial}{\partial t}, \quad t\frac{\partial}{\partial u}$$

$$u\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial t}, \quad ut\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u}, \quad ut\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t}.$$

Find the commutators. Classify the Lie algebra.

Problem 2. Show that the third order ordinary linear differential equation

$$\frac{d^3u}{dt^3} = 0$$

admits the seven Lie symmetries

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial u}, \quad t\frac{\partial}{\partial t}, \quad t\frac{\partial}{\partial u}$$

$$t^2\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial u}, \quad ut\frac{\partial}{\partial u} + \frac{1}{2}t^2\frac{\partial}{\partial t}.$$

Find the commutators.

Problem 3. Consider the nonlinear partial differential equation

$$\frac{\partial^3 u}{\partial x^3} + u\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

where c is a constant. Show that the partial differential equation admits the Lie symmetry vector fields

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x},$$

$$V_3 = 3t\frac{\partial}{\partial t} + (x + 2ct)\frac{\partial}{\partial x}, \quad V_4 = t\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Problem 4. Consider the stationary incompressible *Prandtl boundary layer equation*

$$\frac{\partial^3 u}{\partial \eta^3} = \frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{\partial u}{\partial \xi}\frac{\partial^2 u}{\partial \eta \partial \xi}.$$

Using the classical Lie method we obtain the similarity reduction

$$u(\xi, \eta) = \xi^\beta y(x), \qquad x = \eta\xi^{\beta-1} + f(\xi)$$

where f is an arbitrary differentiable function of ξ. Find the ordinary differential equation for y.

Problem 5. Show that the *Chazy equation*

$$\frac{d^3 y}{dx^3} = 2y\frac{d^2 y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2$$

admits the vector fields

$$\frac{\partial}{\partial x}, \qquad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad x^2\frac{\partial}{\partial x} - (2xy + 6)\frac{\partial}{\partial y}$$

as symmetry vector fields. Show that the first two symmetry vector fields can be used to reduce the Chazy equation to a first order equation.

Problem 6. Show that the *Laplace equation*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u = 0$$

admits the Lie symmetries

$$P_x = \frac{\partial}{\partial x}, \quad P_y = \frac{\partial}{\partial y}, \quad P_z = \frac{\partial}{\partial z}$$

$$M_{yx} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad M_{xz} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad M_{zy} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

$$D = -\left(\frac{1}{2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

$$K_x = -2xD - r^2\frac{\partial}{\partial x}, \quad K_y = -2yD - r^2\frac{\partial}{\partial y}, \quad K_z = -2xD - r^2\frac{\partial}{\partial z}$$

where $r^2 := x^2 + y^2 + z^2$.

Problem 7. Consider the nonlinear one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right) = 0$$

where $n = 1, 2, \ldots$. An equivalent set of differential forms is given by

$$\alpha = du - u_t dt - u_x dx$$
$$\beta = (u_t - nu^{n-1}u_x^2)dx \wedge dt - u^n du_x \wedge dt$$

with the coordinates t, x, u, u_t, u_x The exterior derivative of α is given

$$d\alpha = -du_t \wedge dt - du_x \wedge dx.$$

Consider the vector field

$$V = V_t\frac{\partial}{\partial t} + V_x\frac{\partial}{\partial x} + V_u\frac{\partial}{\partial u} + V_{u_t}\frac{\partial}{\partial u_t} + V_{u_x}\frac{\partial}{\partial u_x}.$$

Then the symmetry vector fields of the partial differential equation are determined by

$$L_V\alpha = g\alpha$$
$$L_V\beta = h\beta + w\alpha + rd\alpha$$

where $L_V(.)$ denotes the Lie derivative, g, h, r are smooth functions depending on t, x, u, u_t, u_x and w is a differential one-form also depending on t, x, u, u_t, u_x. Find the symmetry vector fields from these two conditions. Note that we have

$$L_V(d\alpha) = d(L_V\alpha) = d(g\alpha) = (dg) \wedge \alpha + gd\alpha.$$

Problem 8. The *Harry Dym equation* is given by

$$\frac{\partial u}{\partial t} - u^3 \frac{\partial^3 u}{\partial x^3} = 0.$$

Show that it admits the Lie symmetry vector fields

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}$$

$$V_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad V_4 = -3t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad V_5 = x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}.$$

Is the Lie algebra spanned by these generators semi-simple?

Problem 9. Given the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial t} = f(u)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. Find the condition that

$$V = a(x,t,u)\frac{\partial}{\partial x} + b(x,t,u)\frac{\partial}{\partial t} + c(x,t,u)\frac{\partial}{\partial u}$$

is a symmetry vector field of the partial differential equation. Start with the corresponding vertical vector field

$$V_v = (-a(x,t,u)u_x - b(x,t,u)u_t + c(x,t,u))\frac{\partial}{\partial u}$$

and calculate first the prolongation. Utilize the differential consequencies which follow from the partial differential equations

$$u_{xt} - f(u) = 0, \quad u_{xxt} - \frac{df}{du}u_x = 0, \quad u_{xtt} - \frac{df}{du}u_t = 0.$$

Problem 10. Consider the n-dimensional smooth manifold $M = \mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$ and an arbitrary smooth first order differential equaion on M

$$F(x_1, \ldots, x_n, \partial u/\partial x_1, \ldots, \partial u/\partial x_n, u) = 0.$$

Find the symmetry vector fields (sometimes called the infinitesimal symmetries) of this first order partial differential equation. Consider the cotangenet bundle $T^*(M)$ over the manifold M with coordinates

$$(x_1, \ldots, x_n, p_1, \ldots, p_n)$$

and construct the product manifold $T^*(M)\times\mathbb{R}$. Then $T^*(M)$ has a canonical differential one-form

$$\sum_{j=1}^{n} p_j dx_j$$

which provides the contact differential one-form

$$\alpha = du - \sum_{j=1}^{n} p_j dx_j$$

on $T^*(M)\times\mathbb{R}$. The solutions of the partial differential equation are surfaces in $T^*(M)\times\mathbb{R}$

$$F(x_1,\dots,x_n,p_1,\dots,p_n,u)=0$$

which annul the differential one-form α. We construct the closed ideal I defined by

$$\begin{aligned}
&F(x_1,\dots,x_n,p_1,\dots,p_n,u)\\
&\alpha = du - \sum_{j=1}^{n} p_j dx_j\\
&dF = \sum_{j=1}^{n}\left(\frac{\partial F}{\partial x_j}dx_j + \frac{\partial F}{\partial p_j}dp_j\right) + \frac{\partial F}{\partial u}du\\
&d\alpha = \sum_{j=1}^{n} dx_j \wedge dp_j .
\end{aligned}$$

The surfaces in $T^*(M)\times\mathbb{R}$ which annul I will be the solutions of the first order partial differential equation. Let

$$V(x_1,\dots,x_n,p_1,\dots,p_n,u) = \sum_{j=1}^{n} V_{x_j}\frac{\partial}{\partial x_j} + \sum_{j=1}^{n} V_{p_j}\frac{\partial}{\partial p_j} + V_u\frac{\partial}{\partial u}$$

be a smooth vector field. Let L_V denote the Lie derivative. Then the conditions for V to be a symmetry vector field are

$$\begin{aligned}
&L_V F = gF\\
&L_V\alpha = \lambda\alpha + \eta dF + \left(\sum_{j=1}^{n}(A_j dx_j + B_j dp_j)\right)F.
\end{aligned}$$

Here λ, η, A_j, B_j are smooth functions of $x_1,\dots,x_n,p_1,\dots,p_n$ and u on $T^*(\mathbb{R}^n)\times\mathbb{R}$, where g, A_j, B_j must be nonsingular in a neighbourhood of $F=0$. Find V.

Chapter 9

Integration

Problem 1. Let $\alpha(t) = (x(t), y(t))$ be a positive oriented simple closed curve, i.e. $x(b) = x(a)$, $y(b) = y(a)$. Show that

$$A = -\int_a^b y(t)x'(t)dt = \int_a^b x(t)y'(t)dt = \frac{1}{2}\int_a^b (x(t)y'(t) - y(t)x'(t))dt\,.$$

Problem 2. Any $SU(2)$ matrix A can be written as $(x_0, x_1, x_2, x_3 \in \mathbb{R})$

$$A = \begin{pmatrix} x_0 - ix_3 & -ix_1 - x_2 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix}, \qquad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \tag{1}$$

i.e., $\det A = 1$. Using *Euler angles* α, β, γ the matrix can also be written as

$$A = \begin{pmatrix} \cos(\beta/2)e^{i(\alpha+\gamma)/2} & -\sin(\beta/2)e^{i(\alpha-\gamma)/2} \\ \sin(\beta/2)e^{-i(\alpha-\gamma)/2} & \cos(\beta/2)e^{-i(\alpha+\gamma)/2} \end{pmatrix}. \tag{2}$$

(i) Show that the invariant measure dg of $SU(2)$ can be written as

$$dg = \frac{1}{\pi^2}\delta(x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1)dx_0dx_1dx_2dx_3$$

where δ is the Dirac delta function.
(ii) Show that dg is normalized, i.e.

$$\int dg = 1\,.$$

(iii) Using (1) and (2) find $x_1(\alpha,\beta,\gamma)$, $x_2(\alpha,\beta,\gamma)$, $x_3(\alpha,\beta,\gamma)$. Find the Jacobian determinant.
(iv) Using the results from (iii) show that the invariant measure can be written as

$$\frac{1}{16\pi^2}\sin\beta d\alpha d\beta d\gamma\,.$$

Problem 3. Let M be a smooth, compact, and oriented n-manifold. Let $f : M \to \mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\}$ be a smooth map. The *Kronecker characteristic* is given by the following integral

$$K(f) := (\mathrm{vol}S^n)^{-1}\int_M \|f(\mathbf{x})\|^{-(n+1)} \det\left(f(\mathbf{x}), \frac{\partial f}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right) d\mathbf{x}$$

where $(\mathbf{x} = (x_1, x_2, \ldots, x_n))$ are local coordinates of M and $d\mathbf{x} = dx_1 dx_2 \cdots dx_n$. Express this integral in terms of differential forms.

Problem 4. Let C be the unit circle centered at the orign $(0,0)$. Calculate

$$\frac{1}{2\pi}\oint_C \frac{PdQ - QdP}{P^2 + Q^2}$$

where $P(x,y) = -y$, $Q(x,y) = x$.

Problem 5. Let $S^n \subset \mathbb{R}^{n+1}$ be given by

$$S^n := \{\,(x_1, \ldots, x_{n+1}) \,:\, x_1^2 + \cdots + x_{n+1}^2 = 1\,\}\,.$$

Show that the invariant normalized n-differential form on S^n is given by

$$\omega = \frac{1}{2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)\frac{dx_1 \wedge \cdots \wedge dx_n}{|x_{n+1}|}$$

where Γ denotes the gamma function.

Problem 6. A volume differential form on a manifold M of dimension n is an n-form ω such that $\omega(p) \neq 0$ at each point $p \in M$. Consider $M = \mathbb{R}^3$ (or an open set here) with coordiante system $(x_1, x_{,}x_3)$ with respect to the usual right-handed orthonormal frame. Then the volume differential form is defined as

$$\omega = dx_1 \wedge dx_2 \wedge dx_3$$

and hence any differential three-form can be written as

$$\eta = f(x_1, x_2, x_3) dx_1 \wedge dx_2 \wedge dx_3$$

for some function f. The integral of η is (if it exists)

$$\int_{\mathbb{R}^3} \eta = \int_{\mathbb{R}^3} f(x_1, x_2, x_3) dx_1 dx_2 dx_3 \,.$$

(i) Express ω in terms of spherical coordinates (r, θ, ϕ) with $r \geq 0$, $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$

$$x_1(r,\theta,\phi) = r \sin\theta \cos\phi, \quad x_2(r,\theta,\phi) = r \sin\theta \sin\phi, \quad x_3(r,\theta,\phi) = r\cos\theta \,.$$

(ii) Express ω in terms of *prolate spherical coordinates* (ξ, η, ϕ) $(a > 0)$

$$x_1(\xi,\eta,\phi) = a \sinh\xi \sin\eta \cos\phi$$
$$x_2(\xi,\eta,\phi) = a \sinh\xi \sin\eta \sin\phi$$
$$x_3(\xi,\eta,\phi) = a \cosh\xi \cos\eta \,.$$

Problem 7. Consider the differential 1-form

$$\alpha = \frac{x_2 dx_1 - x_1 dx_2}{x_1^2 + x_2^2}$$

defined on

$$U = \mathbb{R}^2 \setminus \{(0,0)\} \,.$$

(i) Calculate $d\alpha$.
(ii) Calculate

$$\oint \alpha$$

using polar coordinates.

Chapter 10

Lie Groups and Lie Algebras

Problem 1. Let R_{ij} denote the generators of an $SO(n)$ rotation in the $x_i - x_j$ plane of the n-dimensional Euclidean space. Give an n-dimensional matrix representation of these generators and use it to derive the Lie algebra $so(n)$ of the compact Lie group $SO(n)$.

Problem 2. The Lie group $SL(2,\mathbb{C})$ consists all 2×2 matrices over $\mathbb{C}$ with determinant equal to 1. The group is not compact. The maximal compact subgroup of $SL(2,\mathbb{C})$ is $SU(2)$. Give a 2×2 matrix A which is an element of $SL(2,\mathbb{C})$, but not an element of $SU(2)$.

Problem 3. Consider the Lie group $G = O(2,1)$ and its Lie algebra $o(2,1) = \{K_1, K_2, L_3\}$, where K_1, K_2 are Lorentz boosts and L_3 and infinitesimal rotation. The maximal subalgebras of $o(2,1)$ are represented by $\{K_1, K_2 + L_3\}$ and $\{L_3\}$, nonmaximal subalgebras by $\{K_1\}$ and $\{K_2 + L_3\}$. The two-dimensional subalgebra corresponds to the projective group of a real line. The one-dimensional subalgebras correspond to the groups $O(2)$, $O(1,1)$ and the translations $T(1)$, respectively. Find the $o(2,1)$ infinitesimal generators.

Problem 4. The group generator of the compact Lie group $SU(2)$ can be written as

$$J_1 = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1}\right), \quad J_2 = \frac{i}{2}\left(z_2\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2}\right), \quad J_3 = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right).$$

(i) Find

$$J_+ = J_1 + iJ_2, \qquad J_- = J_1 - iJ_2 .$$

(ii) Let $j = 0, 1, 2, \ldots$ and $m = -j, -j+1, \ldots, 0, \ldots, j$. We define

$$e^j_m(z_1, z_2) = \frac{1}{\sqrt{(j+m)!(j-m)!}} z_1^{j+m} z_2^{j-m} .$$

Find

$$J_+ e^j_m(z_1, z_2), \quad J_- e^j_m(z_1, z_2), \quad J_3 e^j_m(z_1, z_2)$$

(iii) Let

$$J^2 = J_1^2 + J_2^2 + J_3^2 \equiv \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 .$$

Find

$$J^2 e^j_m(z_1, z_2) .$$

Problem 5. Show that the operators

$$L_+ = \overline{z}z, \qquad L_- = -\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}$$

$$L_3 = -\frac{1}{2}\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}} + 1\right), \qquad L_0 = -\frac{1}{2}\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}} + 1\right).$$

form a basis for the Lie algebra $su(1,1)$ under the commutator.

Problem 6. Consider the semi-simple Lie algebra $s\ell(3,\mathbb{R})$. The dimension of $s\ell(3,\mathbb{R})$ is 8. Show that the 8 differential operators

$$J_3^1 = y^2\frac{\partial}{\partial y} + xy\frac{\partial}{\partial x} - ny, \qquad J_2^1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - nx,$$

$$J_3^2 = -y\frac{\partial}{\partial x}, \quad J_1^2 = -\frac{\partial}{\partial x}, \quad J_1^3 = -\frac{\partial}{\partial y}, \quad J_2^3 = -x\frac{\partial}{\partial y},$$

$$J_d = y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x} - n, \quad \widetilde{J}_d = 2y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} - n$$

where $x, y \in \mathbb{R}$ and n is a real number. Find all the Lie subalgebras.

Chapter 11

Miscellaneous

Problem 1. Show that the *Burgers equation*

$$\frac{\partial u}{\partial t} = (1+u)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}$$

can be derived from the metric tensor field

$$g = \left(\frac{u^2}{4} + \eta^2\right) dx \otimes dx + \left(\frac{\eta^2 u}{2} + \frac{u}{4}\left(\frac{u^2}{2} + \frac{\partial u}{\partial x}\right)\right) dx \otimes dt$$
$$+ \left(\frac{\eta^2 u}{2} + \frac{u}{4}\left(\frac{u^2}{2} + \frac{\partial u}{\partial x}\right)\right) dt \otimes dx + \left(\left(\frac{u^2}{4} + \frac{1}{2}\frac{\partial u}{\partial x}\right)^2 + \frac{\eta^2}{4}u\right) dt \otimes dt$$

by setting the curvature R of g equal to 1. Here η is a real parameter.

Problem 2. Two systems of nonlinear differential equations that are integrable by the inverse scattering method are said to be *gauge equivalent* if the corresponding flat connections U_j, V_j, $j = 1, 2$, are defined in the same fibre bundle and obtained from each other by a λ-independent gauge transformation, i.e. if

$$U_1 = gU_2g^{-1} + \frac{\partial g}{\partial x}g^{-1}, \qquad V_1 = gV_2g^{-1} + \frac{\partial g}{\partial t}g^{-1} \tag{1}$$

where $g(x,t) \in GL(n, \mathbb{R})$. We have

$$\frac{\partial U_1}{\partial t} - \frac{\partial V_1}{\partial x} + [U_1, V_1] = 0. \tag{2}$$

Show that
$$\frac{\partial U_2}{\partial t} - \frac{\partial V_2}{\partial x} + [U_2, V_2] = 0. \tag{3}$$

Problem 3. Consider the *nonlinear Schrödinger equation* in one space dimension
$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2\psi = 0 \tag{1}$$
and the *Heisenberg ferromagnet equation* in one space dimension
$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2}, \qquad \mathbf{S}^2 = 1 \tag{2}$$
where $\mathbf{S} = (S_1, S_2, S_3)^T$. Both equations are integrable by the inverse scattering method. Both arise as the consistency condition of a system of linear differential equations
$$\frac{\partial \Phi}{\partial t} = U(x,t,\lambda)\Phi, \qquad \frac{\partial \phi}{\partial x} = V(x,t,\lambda)\Phi \tag{3}$$
where λ is a complex parameter. The consistency conditions have the form
$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0 \tag{4}$$
(i) Show that $\phi_1 = g\phi_2$.
(ii) Show that (1) and (2) are gauge equivalent.

Problem 4. The study of certain questions in the theory of $SU(2)$ gauge fields reduced to the construction of exact solutions of the following nonlinear system of partial differential equations
$$u\left(\frac{\partial^2 u}{\partial y \partial \bar{y}} + \frac{\partial^2 u}{\partial z \partial \bar{z}}\right) - \frac{\partial u}{\partial y}\frac{\partial u}{\partial \bar{y}} - \frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial v}{\partial z}\frac{\partial \bar{v}}{\partial \bar{z}} = 0.$$
$$u\left(\frac{\partial^2 v}{\partial y \partial \bar{y}} + \frac{\partial^2 v}{\partial z \partial \bar{z}}\right) - 2\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial \bar{y}} + \frac{\partial v}{\partial z}\frac{\partial u}{\partial \bar{z}}\right) = 0$$
$$u\left(\frac{\partial^2 \bar{v}}{\partial \bar{y} \partial y} + \frac{\partial^2 \bar{v}}{\partial \bar{z} \partial z}\right) - 2\left(\frac{\partial \bar{v}}{\partial \bar{y}}\frac{\partial u}{\partial y} + \frac{\partial \bar{v}}{\partial \bar{z}}\frac{\partial u}{\partial z}\right) = 0, \tag{1}$$
where u is a real function and v and $\bar{v}$ are complex unknown functions of the real variables $x_1, \ldots, x_4$. The quantities y and z are complex variables expressed in terms of $x_1, \ldots, x_4$ by the formulas
$$\sqrt{2}y := x_1 + ix_2, \qquad \sqrt{2}z := x_3 - ix_4 \tag{2}$$

and the bar over letters indicates the operation of complex conjugations.
(i) Show that a class of exact solutions of the system (1) can be constructed, namely solutions for the linear system

$$\frac{\partial v}{\partial y} - \frac{\partial u}{\partial \bar{z}} = 0, \qquad \frac{\partial v}{\partial z} - \frac{\partial u}{\partial \bar{y}} = 0 \tag{3}$$

where we assume that u, v, and $\bar{v}$ are functions of the variables

$$r := (2y\bar{y})^{1/2} = (x_1^2 + x_2^2)^{1/2} \tag{4}$$

and x_3, i.e., for the stationary, axially symmetric case. (ii) Show that a class of exact solutions of (1) can be given, where

$$u = u(w), \qquad v = v(w), \qquad \bar{v} = \bar{v}(w) \tag{5}$$

where w is a solution of the Laplace equation in complex notation

$$\frac{\partial^2 u}{\partial y \partial \bar{y}} + \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0, \tag{6}$$

and u, v and $\bar{v}$ satisfy

$$u\frac{d^2 u}{dw^2} - \left(\frac{du}{dw}\right)^2 + \frac{dv}{dw}\frac{d\bar{v}}{dw} = 0, \qquad u\frac{d^2 v}{dw^2} - 2\frac{dv}{dw}\frac{du}{dw} = 0. \tag{7}$$

Hint. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{8}$$

Problem 5. The spherically symmetric SU(2) Yang-Mills equations can be written as

$$\frac{\partial \varphi_1}{\partial t} - \frac{\partial \varphi_2}{\partial r} = -A_0\varphi_2 - A_1\varphi_1 \tag{1a}$$

$$\frac{\partial \varphi_2}{\partial t} + \frac{\partial \varphi_1}{\partial r} = -A_1\varphi_2 + A_0\varphi_1 \tag{1b}$$

$$r^2\left(\frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial r}\right) = 1 - (\varphi_1^2 + \varphi_2^2) \tag{1c}$$

where r is the spatial radius-vector and t is the time. To find partial solutions of these equations, two methods can be used. The first method is the inverse scattering theory technique, where the $[L, A]$-pair is found, and the second method is based on Bäcklund transformations.
(ii) Show that system (1) can be reduced to the classical Liouville equation, and its general solution can be obtained for any gauge condition.

Problem 6. We consider the *Georgi-Glashow model* with gauge group $SU(2)$ broken down to $U(1)$ by *Higgs triplets*. The Lagrangian of the model is

$$\mathcal{L} := -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi) \tag{1}$$

where

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu \tag{2}$$

$$D_\mu \phi_a := \partial_\mu \phi_a + g\epsilon_{abc} A^b_\mu \phi_c \tag{3}$$

and

$$V(\phi) := -\frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2. \tag{4}$$

(i) Show that the equations of motion are

$$D_\nu F^{\mu\nu a} = -g\epsilon_{abc}(D^\mu \phi_b)\phi_c, \qquad D_\mu D^\mu \phi_a = (m^2 - \lambda\phi^2)\phi_a. \tag{5}$$

(ii) Show that the vacuum expectation value of the scalar field and Higgs boson mass are

$$\langle \phi^2 \rangle = F^2 = \frac{m^2}{\lambda} \tag{6}$$

and

$$M_H = \sqrt{2\lambda} F,$$

respectively. Mass of the gauge boson is $M_w = gF$.
(iii) Using the time-dependent t' Hooft-Polyakov ansatz

$$A^a_0(r,t) = 0, \qquad A^a_i(r,t) = -\epsilon_{ain} r_n \frac{1 - K(r,t)}{r^2}, \qquad \phi_a(r,t) = \frac{1}{g} r_a \frac{H(r,t)}{r^2} \tag{7}$$

where $r_n = x_n$ and r is the radial variable. Show that the equations of motion (5) can be written as

$$r^2 \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2}\right) K = (K^2 + H^2 - 1) \tag{8a}$$

$$r^2 \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2}\right) H = H\left(2K^2 - m^2 r^2 + \frac{\lambda H^2}{g^2}\right). \tag{8b}$$

(iv) Show that with

$$\beta := \frac{\lambda}{g^2} = \frac{M_H^2}{2M_w^2}$$

and introducing the variables $\xi := M_w r$ and $\tau := M_w t$, system (8) becomes

$$\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \tau^2}\right) K = \frac{K(K^2 + H^2 - 1)}{\xi^2} \tag{10a}$$

$$\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \tau^2}\right) H = \frac{H(2K^2 + \beta(H^2 - \xi^2))}{\xi^2}. \qquad (10b)$$

(v) The total energy of the system E is given by

$$C(\beta) = \frac{g^2 E}{4\pi M_w} =$$

$$\int_0^\infty \left(K_\tau^2 + \frac{H_\tau^2}{2} + K_\xi^2 + \frac{1}{2}\left(\frac{\partial H}{\partial \xi} - \frac{H}{\xi}\right)^2 + \frac{1}{2\xi^2}(K^2-1)^2 + \frac{K^2 H^2}{\xi^2} + \frac{\beta}{4\xi^2}(H^2 - \xi^2)^2 \right) d\xi. \qquad (10)$$

As time-independent version of the ansatz (3) gives the 't Hooft-Polyakov monopole solution with winding number 1. Show that for finiteness of energy the field variables should satisfy the following conditions

$$H \to 0, \qquad K \to 1 \quad \text{as} \quad \xi \to 0 \qquad (11)$$

and

$$H \to \xi, \qquad K \to 0 \quad \text{as} \quad \xi \to \infty. \qquad (12)$$

The 't Hooft-Polyakov monopole is more realistic than the Wu-Yang monopole; it is non-singular and has finite energy.
(vi) Show that in the limit $\beta \to 0$, known as the Prasad-Somerfeld limit, we have the static solutions,

$$K(\xi) = \frac{\xi}{\sinh \xi}, \qquad H(\xi) = \xi \coth \xi - 1. \qquad (13)$$

Problem 7. Consider the Lorenz model

$$\frac{dx}{dt} = -\sigma x + \sigma y = V_1(x,y,z)$$
$$\frac{dy}{dt} = -xz + rx - y = V_2(x,y,z)$$
$$\frac{dz}{dt} = xy - bz = V_3(x,y,z)$$

with the vector field

$$V = V_1(x,y,z)\frac{\partial}{\partial x} + V_2(x,y,z)\frac{\partial}{\partial y} + V_3(x,y,z)\frac{\partial}{\partial z}$$

(i) Find curl$V$.
(ii) Show that curl(curl$V$) $= \mathbf{0}$.

(iii) Since curl(curl(V)) $= \mathbf{0}$ we can find a smooth function ϕ such that

$$\text{curl}V = \text{grad}(\phi)\,.$$

Find ϕ.

Problem 8. Consider the linear operators L and M defined by

$$L\psi(x,t,\lambda) := \left(i\frac{\partial}{\partial x} + U(x,t,\lambda)\right)\psi(x,t,\lambda)$$

$$M\psi(x,t,\lambda) := \left(i\frac{\partial}{\partial t} + V(x,t,\lambda)\right)\psi(x,t,\lambda)\,.$$

Find the condition on L and M such that $[L, M] = 0$, where $[\,,\,]$ denotes the commutator. The potentials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are typicaly chosen as elements of some semisimple Lie algebra.

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Index