Problems
in
Quantum Computing

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Preface

The purpose of this book is to supply a collection of problems in quantum computing.

Prescribed books for problems.

1) Problems and Solutions in Quantum Computing and Quantum Information (third edition)

by Willi-Hans Steeb and Yorick Hardy
World Scientific, Singapore, 2011
ISBN-13 978-981-4366-32-8
`http://www.worldscibooks.com/physics/8249.html`

2) Classical and Quantum Computing with C++ and Java Simulations

by Yorick Hardy and Willi-Hans Steeb
Birkhauser Verlag, Boston, 2002
ISBN 376-436-610-0

3) Matrix Calculus and Kronecker Product

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2010
ISBN 978-981-4335-31-7
`http://www.worldscibooks.com/mathematics/8030.html`

4) Problems and Solutions in Introductory and Advanced Matrix Calculus

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2006
ISBN 981 256 916 2
`http://www.worldscibooks.com/mathematics/6202.html`

5) Continous Symmetries, Lie Algebras, Differential Equations and Computer Algebra, second edition

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2007
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6515.html`

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Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\lvert z\rvert$	modulus of complex number z
	$\lvert x+iy\rvert = (x^2+y^2)^{1/2},\ \ x, y \in \mathbb{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbb{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\lVert . \rVert$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbb{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbb{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of matrix A
A^T	transpose of matrix A

$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

The Pauli spin matrices are used extensively in the book. They are given by

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In some cases we will also use σ_1, σ_2 and σ_3 to denote σ_x, σ_y and σ_z .

Chapter 1

Qubits

Problem 1. Let $|0\rangle$, $|1\rangle$ be the standard basis in the Hilbert space $\mathbb{C}^2$, i.e.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $(0 \le \theta < \frac{\pi}{4})$

$$|\Psi_+(\theta)\rangle := \cos(\theta)|0\rangle + \sin(\theta)|1\rangle, \qquad |\Psi_-(\theta)\rangle := \cos(\theta)|0\rangle - \sin(\theta)|1\rangle.$$

(i) Find the scalar product $\langle \Psi_-(\theta)|\Psi_+(\theta)\rangle$. Discuss.
(ii) Consider the states

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

and the projection operators (projection matrices)

$$\Pi_+ := |+\rangle\langle +|, \qquad \Pi_- := |-\rangle\langle -|.$$

Find

$$\langle \Psi_+(\theta)|\Pi_+|\Psi_+(\theta)\rangle,\ \langle \Psi_+(\theta)|\Pi_-|\Psi_+(\theta)\rangle,\ \langle \Psi_-(\theta)|\Pi_+|\Psi_-(\theta)\rangle,\ \langle \Psi_-(\theta)|\Pi_-|\Psi_-(\theta)\rangle$$

and the 2×2 matrices $\Pi_+ + \Pi_-$ and $\Pi_+\Pi_-$. Discuss.

Problem 2. (i) Consider the normalized vector in the Hilbert space $\mathbb{C}^3$

$$\mathbf{n} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}.$$

Show that the vector is normalized.
(ii) Calculate the 2×2 matrix

$$U(\theta, \phi) = \mathbf{n} \cdot \boldsymbol{\sigma} \equiv n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$$

where σ_1, σ_2, σ_3 are the Pauli spin matrices.
(iii) Is the matrix $U(\theta, \phi)$ unitary? Find the trace and the determinant. Is the matrix $U(\theta, \phi)$ hermitian?
(iv) Find the eigenvalues and normalized eigenvectors of $U(\theta, \phi)$.

Problem 3. Consider the states

$$\psi_1(\phi) = \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}, \qquad \psi_2(\phi) = \begin{pmatrix} -\sin(\phi/2) \\ \cos(\phi/2) \end{pmatrix}.$$

in the Hilbert space $\mathbb{C}^2$.
(i) Show that these states can be generated from the standard basis using the exponential function and the Pauli matrix σ_2, i.e. calculate

$$\exp\left(-i\frac{\phi}{2}\sigma_2\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \exp\left(-i\frac{\phi}{2}\sigma_2\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(ii) Find the states after the transformation $\phi \to \phi + 2\pi$.

Problem 4. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and I_2 the 2×2 identity matrix. Find the eigenvalues and normalized eigenvectors of the Hamilton operator

$$\hat{H} = \varepsilon_0 I_2 + \hbar\omega\sigma_3 + \Delta_1\sigma_1 + \Delta_2\sigma_2$$

where $\varepsilon_0 > 0$. Are the normalized eigenvectors orthonormal to each other?

Problem 5. Let $\hat{H}$ be a 2×2 hermitian matrix. Consider the normalized state

$$|\psi\rangle = \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

in the Hilbert space $\mathbb{C}^2$. Assume that

$$\langle\psi|\hat{H}|\psi\rangle = \hbar\omega\cos(\phi)\sin(2\theta), \qquad \langle\psi|\hat{H}^2|\psi\rangle = \hbar^2\omega^2.$$

Reconstruct the hermitian matrix $\hat{H}$ from these three assumptions. Note that

$$\cos(\theta)\sin(\theta) \equiv \frac{1}{2}\sin(2\theta), \quad e^{i\phi} = \cos(\phi)+i\sin(\phi), \quad e^{-i\phi} = \cos(\phi)-i\sin(\phi).$$

Problem 6. (i) Consider the symmetric matrix over $\mathbb{R}$

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$$

and the state

$$|\psi\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Calculate the *variance*

$$V_H(|\psi\rangle) = \langle\psi|H^2|\psi\rangle - (\langle\psi|H|\psi\rangle)^2.$$

(ii) Consider the Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the state

$$|\psi\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Calculate the *variance*

$$V_H(\psi) := \langle\psi|H^2|\psi\rangle - (\langle\psi|H|\psi\rangle)^2$$

and discuss the dependence on θ.

Problem 7. Let $\hat{A}$ and $\hat{B}$ be $n \times n$ hermitian matrices. Let $|\psi\rangle$ be a normalized state in the Hilbert space $\mathbb{C}^n$. Then we have the inequality

$$(\Delta\hat{A})(\Delta\hat{B}) \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$$

where

$$\Delta\hat{A} := \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}, \qquad \Delta\hat{B} := \sqrt{\langle\hat{B}^2\rangle - \langle\hat{B}\rangle^2}$$

and

$$\langle\hat{A}\rangle := \langle\psi|\hat{A}|\psi\rangle, \qquad \langle\hat{B}\rangle := \langle\psi|\hat{B}|\psi\rangle.$$

Consider the hermitian spin-$\frac{1}{2}$ matrices

$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad s_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad s_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\hat{A} = s_1$ and $\hat{B} = s_2$. Find states $|\psi\rangle$ such that

$$(\Delta\hat{A})(\Delta\hat{B}) = \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$$

i.e. the inequality given above should be an equality.

Problem 8. Given the two normalized states

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find a unitary matrix U such that $|\psi\rangle = U|\phi\rangle$. Give the eigenvalues of U.

Problem 9. Let

$$A = \sum_{k=0}^{3} a_k \sigma_k, \qquad B = \sum_{\ell=0}^{3} b_\ell \sigma_\ell$$

where $\sigma_0 = I_2$ and $a_k, b_\ell \in \mathbb{R}$ with $a_3 \neq 0$ and $b_1 = a_1 b_3 / a_3$, $b_2 = a_2 b_3 / a_3$. Calculate the commutator $[A, B]$.

Problem 10. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Show that

$$\cos(\omega t)\sigma_1 - \sin(\omega t)\sigma_2 = e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_-, \quad e^{\pm\omega t}\sigma_\pm e^{i\omega t\sigma_3/2} = e^{i\omega t\sigma_3/2}\sigma_\pm$$

where $\sigma_\pm := (\sigma_1 \pm i\sigma_2)/2$.

Problem 11. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Can one find a 2×2 invertible matrix K with $K = K^{-1}$ and

$$K\sigma_1 K = \sigma_1, \qquad K\sigma_2 K = -\sigma_2, \qquad K\sigma_3 K = \sigma_3\,?$$

Problem 12. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and $\alpha \in \mathbb{R}$.
(i) Calculate the 2×2 matrices

$$\exp(-i\alpha\sigma_1/2), \qquad \exp(-i\alpha\sigma_2/2), \qquad \exp(-i\alpha\sigma_3/2).$$

Are the matrices unitary?
(ii) Let

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find the state $\exp(-i\alpha\sigma_1/2)|\psi\rangle$ and calculate

$$\langle\psi|\exp(-i\alpha\sigma_1/2)|\psi\rangle \quad \text{and} \quad |\langle\psi|\exp(-i\alpha\sigma_1/2)|\psi\rangle|^2.$$

Problem 13. Let $|0\rangle$, $|1\rangle$ be an orthonormal basis in a two-dimensional Hilbert space. Consider the Hamilton operator

$$\hat{H} = -\frac{1}{2}\hbar\omega(e^{-i\phi}|1\rangle\langle 0| + e^{i\phi}|0\rangle\langle 1|).$$

Find $\exp(-i\hat{H}t/\hbar)$.

Problem 14. Consider the Hamilton operator

$$\hat{H}(\lambda) = \begin{pmatrix} \hbar\omega & \lambda a_{12} \\ \lambda a_{12} & -\hbar\omega \end{pmatrix}$$

where $a_{12} \in \mathbb{R}$. Let I_2 be the 2×2 identity matrix and E a real parameter. Solve the system of equations

$$\det(\hat{H}(\lambda) - I_2 E) = 0$$
$$\frac{d}{dE}\det(\hat{H}(\lambda) - I_2 E) = 0$$

with respect to E and λ.

Problem 15. Consider the Hamilton operator

$$\hat{H} = \begin{pmatrix} \hbar\omega & \Delta \\ \Delta & -\hbar\omega \end{pmatrix}.$$

Consider the unitary matrix

$$U = \begin{pmatrix} \cos(\phi) & -e^{-i\theta}\sin(\phi) \\ e^{i\theta}\sin(\phi) & \cos(\phi) \end{pmatrix}.$$

Can one find ϕ, θ such that $U^*\hat{H}U$ is a diagonal matrix?

Problem 16. Consider the Pauli spin matrices σ_1, σ_2 and σ_3. Can one find an $\alpha \in \mathbb{R}$ such that

$$\exp(i\alpha\sigma_3)\sigma_1\exp(-i\alpha\sigma_3) = \sigma_2\,?$$

Problem 17. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Find the conditions on $\alpha_1, \alpha_2, \alpha_3$ such that

$$U = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

is a unitary matrix.

Problem 18. Consider the map $\mathbf{f} : \mathbb{C}^2 \to \mathbb{R}^3$ defined by

$$\mathbf{f} \;:\; \begin{pmatrix} \cos(\theta) \\ e^{i\phi}\sin(\theta) \end{pmatrix} \mapsto \begin{pmatrix} \sin(2\theta)\cos(\phi) \\ \sin(2\theta)\sin(\phi) \\ \cos(2\theta) \end{pmatrix}.$$

Are the vectors in $\mathbb{C}^2$ and $\mathbb{R}^3$ normalized? Consider the four normalized vectors in $\mathbb{C}^2$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Find the vectors in $\mathbb{R}^3$.

Problem 19. Let σ_1, σ_2 and σ_3 be the Pauli spin matrices. Calculate

$$U(\alpha, \beta, \gamma) = e^{-i\alpha\sigma_3/2} e^{-i\beta\sigma_2/2} e^{-i\gamma\sigma_3/2}$$

where α, β, γ are the Euler angles with the range $0 \leq \alpha < 2\pi$, $0 \leq \beta \leq \pi$ and $0 \leq \gamma < 2\pi$.

Problem 20. Let $\hat{H}_0$ and $\hat{H}_1$ be a pair of real symmetric $n \times n$ matrices, where $\hat{H}_0$ is a diagonal matrix. Let

$$\hat{H}(\epsilon) := \hat{H}_0 + \epsilon \hat{H}_1. \tag{1}$$

When ϵ is real, $\hat{H}(\epsilon)$ is diagonalizable with eigenvalues $E_1(\epsilon)$, $\ldots$, $E_n(\epsilon)$. The eigenvalues are given by the characteristic polynomial

$$P(E, \epsilon) := \det(\hat{H}(\epsilon) - EI_n) = 0 \tag{2}$$

where I_n is the $n \times n$ unit matrix. When ϵ is complex, the eigenvalues may be viewed as the n values of a single function $E(\epsilon)$ of ϵ, analytic on a Riemann surface with N sheets joined at branch point singularities in the complex plane. The *exceptional points* in the complex ϵ plane are defined by the solution of (2) together with

$$\frac{d}{dE} \det(\hat{H}(\epsilon) - EI_n) = 0. \tag{3}$$

(i) Consider the two-level system

$$\hat{H}(\epsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the exceptional points of $\hat{H}(\epsilon)$.
(ii) Let ϵ_1 and ϵ_2 be the two exceptional points. Find the eigenvalues and eigenvectors of the matrices $\hat{H}(\epsilon_1)$ and $\hat{H}(\epsilon_2)$. Discuss.

Problem 21. Study the eigenvalue problem for the matrix

$$\sigma_3 + e^{i\phi}\sigma_1$$

for $\phi \in [0, \pi/2]$.

Problem 22. (i) Let $\phi \in \mathbb{R}$. Is the matrix

$$A = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

hermitian, unitary?
(ii) Find the rank of the matrix.
(iii) Find the eigenvalues and eigenvectors of A.
(iv) Let I_2 be the 2×2 unit matrix. Find the eigenvalues of $A \otimes I_2$.

Problem 23. Let $|\phi_1\rangle$, $|\phi_2\rangle$ be two normalized vectors in the Hilbert space $\mathbb{R}^2$. Assume that

$$\langle \phi_1 | \phi_2 \rangle = \frac{1}{2}.$$

Give a geometric interpretation of this equation.

Problem 24. Consider the vectors

$$|\psi_1\rangle = \begin{pmatrix} i^i \\ e^{i\pi} \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} i^i \\ \sin(i) \end{pmatrix}$$

in the Hilbert space $\mathbb{C}^2$. Are the vectors normalized? If not normalize the vectors.

Problem 25. Let $\mathcal{H}$ be an arbitrary Hilbert space. Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary normalized states in $\mathcal{H}$. Find all the solutions of the equation

$$\langle \phi | \psi \rangle \langle \psi | \phi \rangle = i.$$

Problem 26. What is the condition on ϕ_1, ϕ_2, ϕ_3 (all real) such that

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\phi_1} \\ e^{i\phi_2} & e^{i\phi_3} \end{pmatrix}$$

is a unitary matrix?

Problem 27. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Find unitary matrices U_1, U_2, U_3, U_4 such that

$$U_1^{-1} A U_1 = B, \quad U_2^{-1} B U_2 = C, \quad U_3^{-1} C U_3 = D, \quad U_4^{-1} D U_4 = A.$$

Problem 28. (i) Consider the normalized state

$$|\psi\rangle = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) \end{pmatrix}.$$

in the Hilbert space $\mathbb{C}^2$. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Calculate

$$n_j := \langle\psi|\sigma_j|\psi\rangle, \qquad j = 1, 2, 3$$

Is the vector

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

in $\mathbb{R}^3$ normalized?
(ii) Consider the Hamilton operator

$$\hat{H}(t) = -\frac{\mu\hbar}{2}\mathbf{B}(t)\cdot\sigma \equiv -\frac{\mu\hbar}{2}(B_1(t)\sigma_1 + B_2(t)\sigma_2 + B_3(t)\sigma_3)$$

where $\mathbf{B}(t)$ is a time-dependent homogeneous magnetic field. Show that the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

can be written as

$$\frac{d}{dt}\mathbf{n}(t) = -\mu\mathbf{B}(t)\times\mathbf{n}$$

where $\times$ denotes the vector product.

Problem 29. Find the square roots of the Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e. find the matrices R_j such that $R_j^2 = \sigma_j$ $(j = 0, 1, 2, 3)$.

Problem 30. Consider a d-dimensional Hilbert space with two orthonormal bases

$$|b_{11}\rangle, \quad |b_{12}\rangle, \quad \dots \quad |b_{1d}\rangle \in \mathcal{B}_1$$
$$|b_{21}\rangle, \quad |b_{22}\rangle, \quad \dots \quad |b_{2d}\rangle \in \mathcal{B}_2.$$

The two bases are said to be *mutually unbiased bases* if

$$|\langle b_{2j}|b_{1k}\rangle| = \frac{1}{\sqrt{d}}$$

for all $j, k = 1, \ldots, d$ and $\langle \, | \, \rangle$ denotes the scalar product in the Hilbert space. Consider the Hilbert space $M_2(\mathbb{C})$ of 2×2 matrices over $\mathbb{C}$, where the scalar product is defined as

$$\langle A|B\rangle = \mathrm{tr}(AB^*), \qquad A, B \in M_2(\mathbb{C})$$

Thus $d = \dim(M_2(\mathbb{C})) = 4$. The standard basis in this Hilbert space is given by

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let U_H be the Hadamard matrix

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad U_H^* = U_H.$$

(i) Show that the matrices $\widetilde{E}_{jk}$ $(j, k = 1, 2)$

$$\widetilde{E}_{jk} = U_H E_{jk} U_H^*, \quad j, k = 1, 2$$

and the standard basis form mutually unbiased bases.
(ii) Apply the vec-operator to the matrices E_{jk} and $\widetilde{E}_{jk}$ $(j, k = 1, 2)$ to find mutually unbiased bases in the Hilbert space $\mathbb{C}^4$.

Problem 31. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Find all 2×2 matrices A such that

$$[\sigma_1, A] = [\sigma_2, A] = [\sigma_3, A] = 0_2.$$

Problem 32. Let $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathbb{R}$. The 2×2 matrix $U = (\mathbf{v}_1 \; \mathbf{v}_2)$ contains the two column vectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} \\ e^{i\phi_2} \end{pmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_3} \\ e^{i\phi_4} \end{pmatrix}.$$

Find the conditions on $\phi_1, \phi_2, \phi_3, \phi_4$ such that

$$\langle \mathbf{v}_1 | \mathbf{v}_2 \rangle = 0.$$

Is the matrix U unitary if this condition is satisfied.

Problem 33. (i) Find the norms of the vectors in the Hilbert space $\mathbb{C}^2$

$$|\psi\rangle = \begin{pmatrix} e^{i\alpha} \\ e^{-i\alpha} \end{pmatrix}, \qquad |\phi\rangle = \begin{pmatrix} \sin(i) \\ \cos(i) \end{pmatrix}$$

where $\alpha \in \mathbb{R}$.
(ii) Normalize the vectors $|\psi\rangle$ and $|\phi\rangle$.
(iii) After normalizing the vectors calculate the probability

$$p(\alpha) = |\langle \psi(\alpha)|\phi\rangle|^2.$$

Discuss p as a function of α.

Problem 34. Let $\alpha \in \mathbb{R}$. Consider the vector in $\mathbb{C}^2$

$$\mathbf{v} = \begin{pmatrix} \cosh(\alpha) \\ \sinh(\alpha) \end{pmatrix}.$$

Normalize the vector and then study the cases $\alpha \to +\infty$ and $\alpha \to -\infty$. Can one find a non-zero (column) vector $\mathbf{u}$ in $\mathbb{C}^2$ such that

$$\mathbf{u}^* \mathbf{v} = 0\,?$$

Problem 35. Is the state

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix}$$

normalized? Find a normalized vector which is orthogonal to this vector. If so calculate the density matrix $\rho = |\psi\rangle\langle\psi|$.

Problem 36. Let σ_1, σ_2, σ_3 be the Pauli spin matrices.
(i) Consider the normalized state $|\psi\rangle$ in the Hilbert space $\mathbb{C}^2$

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

and the Pauli spin matrices σ_1, σ_2, σ_3. Find

$$\langle\psi|\sigma_1|\psi\rangle, \quad \langle\psi|\sigma_2|\psi\rangle, \quad \langle\psi|\sigma_3|\psi\rangle.$$

(ii) Consider the normalized state in $\mathbb{C}^2$

$$\psi = \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad \Rightarrow \quad \psi^* = (\, e^{-i\phi}\cos(\theta) \quad \sin(\theta)\,)\,.$$

Find the vector $\mathbf{v} = (v_1\ v_2\ v_3)^T$ in $\mathbb{R}^3$ with

$$v_1 = \psi^*\sigma_1\psi, \quad v_2 = \psi^*\sigma_2\psi, \quad v_3 = \psi^*\sigma_3\psi.$$

Is the vector $\mathbf{v}$ normalized?

Problem 37. Let $\phi, \theta \in \mathbb{R}$ and

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\sin(\theta)}e^{i\phi/2} \\ -\sqrt{1-\sin(\theta)}e^{-i\phi/2} \end{pmatrix}$$

be the eigenvector of a 2×2 matrix with eigenvalue $\lambda_1 = +1$ and

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1-\sin(\theta)}e^{i\phi/2} \\ \sqrt{1+\sin(\theta)}e^{-i\phi/2} \end{pmatrix}$$

the second eigenvector with eigenvalue $\lambda_2 = -1$. Find $\mathbf{v}_1^* \mathbf{v}_1^*$, $\mathbf{v}_2^* \mathbf{v}_2$ and $\mathbf{v}_1^* \mathbf{v}_2$. Discuss. Find

$$\lambda_1 \mathbf{v}_1 \mathbf{v}_1^* + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^*.$$

Discuss.

Chapter 2

Kronecker and Tensor Product

Problem 1. Let $\epsilon := e^{2\pi i/3} \equiv (-1+i\sqrt{3})/2$. Consider the eight states in $\mathbb{C}^8$

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$|\psi_4\rangle = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$|\psi_5\rangle = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \bar{\epsilon} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$|\psi_6\rangle = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \bar{\epsilon} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$|\psi_7\rangle = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \bar{\epsilon} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$|\psi_8\rangle = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \bar{\epsilon} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

(i) Calculate the scalar product $\langle \psi_j | \psi_k \rangle$ for $j,k = 1,2,\ldots,8$.
(ii) Which of the vectors are entangled?

Problem 2. (i) Can we find 2×2 matrices A and B with $\det A = a_{11}a_{22} - a_{12}a_{21} \neq 0$, $\det B = b_{11}b_{22} - b_{12}b_{21} \neq 0$ (i.e. we assume that A and B are invertible) such that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = (A \otimes B)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} ?$$

On the left-hand side we have the Bell state $|\Phi^+\rangle$ and on the right-hand side we have the Bell state $|\Psi^+\rangle$. Since A and B are invertible we find that $A \otimes B$ is also invertible with $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
(ii) Can we also find 2×2 matrices A, B such that $\det(A) = \det(B) = 1$, i.e. $A, B \in SL(2, \mathbb{R})$?

Problem 3. Can we find 2×2 matrices A, B, C with $\det(A) = 1$, $\det(B) = 1$ and $\det(C) = 1$ such that

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (A \otimes B \otimes C)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ?$$

On the left-hand side we have the *W-state* and on the right-hand side we have the *GHZ-state*.

Problem 4. Let A, B be $n \times n$ hermitian matrices over $\mathbb{C}$. Let K be an $n \times n$ hermitian matrix over $\mathbb{C}$ and $H = \hbar\omega K$ be a Hamilton operator, where $\hbar$ is the Planck constant and ω the frequency. The Heisenberg equation of motion for the operators A and B are given by

$$i\hbar\frac{dA}{dt} = [A, H](t), \qquad i\hbar\frac{dB}{dt} = [B, H](t).$$

The solutions can be given as

$$A(t) = e^{it\hat{H}/\hbar}Ae^{-it\hat{H}/\hbar}, \qquad B(t) = e^{it\hat{H}/\hbar}Be^{-it\hat{H}/\hbar}.$$

(i) Find the time evolution of $A \otimes B$, $B \otimes A$, $A \otimes A$ and $B \otimes B$.
(ii) Assume that $[A, H] = 2i\hbar\omega B$ and $[B, H] = -2i\hbar\omega A$. Simplify the Heisenberg equation of motion with these conditions.

Problem 5. Let A be an $m \times m$ hermitian matrix and let B be an $n \times n$ hermitian matrix. Then $A \otimes B$, $A \otimes I_n$, $I_m \otimes B$ are also hermitian matrices, where I_m is the $m \times m$ identity matrix. Let ϵ_1, ϵ_2 and ϵ_3 be real parameters. Consider the Hamilton operator

$$H = \hbar\omega(\epsilon_1 A \otimes B + \epsilon_2 A \otimes I_n + \epsilon_3 I_m \otimes B).$$

The *partition function* $Z(\beta)$ is given by $Z(\beta) = \mathrm{tr}(\exp(-\beta H))$, where H is the (hermitian) Hamilton operator and tr denotes the trace. From the partition function we obtain the Helmholtz free energy, entropy and specific heat.
(i) Calculate $Z(\beta)$ for the Hamilton operator given above.
(ii) Consider the special case that $n = m = 2$ and A, B are any of the Pauli spin matrices σ_1, σ_2, σ_3.

Problem 6. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Is

$$\mathrm{tr}(e^A \otimes e^B) = \mathrm{tr}(e^{A \otimes B})\,?$$

Prove or disprove.

Problem 7. (i) Let A, B be $n \times n$ matrices and I_n be the $n \times n$ identity matrix. Show that

$$(A \otimes I_n)(I_n \otimes B)e^{A \otimes I_n + I_n \otimes B} = (Ae^A) \otimes (Be^B).$$

(ii) Let λ be an eigenvalue of A and μ be an eigenvalue of B. Provide an eigenvalue of $(Ae^A) \otimes (Be^B)$.

Problem 8. (i) Let A be an invertible $n \times n$ matrix. Find the inverse matrix of

$$(A^{-1} \otimes I_n)(I_n \otimes A).$$

(ii) Let B be an invertible $n \times n$ matrix. Calculate

$$(A^{-1} \otimes I_n)(I_n \otimes A)(B^{-1} \otimes I_n)(I_n \otimes B).$$

Problem 9. The two-qubit Pauli group $\mathcal{P}_2$ can be generated as

$$\mathcal{P}_2 = \langle \sigma_1 \otimes \sigma_1, \sigma_3 \otimes \sigma_3, \sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_3, \sigma_3 \otimes \sigma_1 \rangle.$$

It is of order 64. Generate the element $i(I_2 \otimes I_2)$.

Problem 10. Consider the hermitian matrices of the three dipole operators

$$L_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the hermitian matrices of five quadrupole operators

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad Q_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplying these eight hermitian matrices by i we obtain a basis for the semi-simple Lie algebra $su(3)$. Consider the Hamilton operator

$$\hat{H} = \kappa_0 Q_0 \otimes Q_0 + \kappa_1 (V_1 \otimes V_1 + V_2 \otimes V_2) + \kappa_2 (U_1 \otimes U_1 + U_2 \otimes U_2).$$

Find the eigenvalues and eigenvectors of $\hat{H}$.

Problem 11. Consider the Pauli spin matrices σ_3, σ_1, σ_2. The eigenvalues are given by $+1$ and -1 with the corresponding normalized eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Consider the three 4×4 matrices

$$\sigma_1 \otimes \sigma_1, \qquad \sigma_2 \otimes \sigma_2, \qquad \sigma_3 \otimes \sigma_3.$$

(i) Find the eigenvalues.
(ii) Show that the eigenvectors can be given as product states (unentangled states), but also as entangled states (i.e. they cannot be written as product states). Explain.

Problem 12. (i) Consider the two 4×4 matrices $\sigma_1 \otimes \sigma_3$, $\sigma_3 \otimes \sigma_1$. Find the eigenvalues.
(ii) Show that the eigenvectors can be given as product states (unentangled states), but also as entangled states (i.e. they cannot be written as product states). Explain.

Problem 13. Consider the Pauli spin matrix σ_2. Find the eigenvalues and eigenvectors for σ_2 and $\sigma_2 \otimes \sigma_2$. For $\sigma_2 \otimes \sigma_2$ show that we find two sets of entangled states for the eigenvectors and set of unentangled eigenvectors (product states).

Problem 14. Find the eigenvalues and eigenvectors of the Hamilton operator

$$\hat{H} = \hbar\omega_1\sigma_3 \otimes I_2 + \hbar\omega_2 I_2 \otimes \sigma_3 + \epsilon\sigma_2 \otimes \sigma_2.$$

Problem 15. (i) Find the eigenvalues and eigenvectors of

$$\sigma_1 \otimes \sigma_2 \otimes \sigma_3.$$

Can one find entangled eigenvectors?
(ii) Find the eigenvalues and eigenvectors of the Hamilton operator

$$\hat{H} = \epsilon_1(\sigma_1 \otimes I_2 \otimes I_2) + \epsilon_2(I_2 \otimes \sigma_2 \otimes I_2) + \epsilon_3(I_2 \otimes I_2 \otimes \sigma_3) + \gamma(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \gamma \in \mathbb{R}$.

Problem 16. (i) Let U_1, U_2 be unitary 2×2 matrices and Π_1, Π_2 be 2×2 projection matrices with $\Pi_1\Pi_2 = 0$ and $\Pi_1 + \Pi_2 = I_2$. Show that

$$U_1 \otimes \Pi_1 + U_2 \otimes \Pi_2$$

is unitary.
(ii) Let $U_1 = \sigma_1$, $U_2 = \sigma_3$ and

$$\Pi_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Pi_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Find the normalized state

$$(U_1 \otimes \Pi_1 + U_2 \otimes \Pi_2)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Show that this state is entangled, i.e. it can not be written as a product state.

Problem 17. Consider the $n \times n$ unitary matrices $U_1, \ldots, U_n$ and the $n \times n$ projection matrices $\Pi_1, \ldots, \Pi_n$ such that $\Pi_j\Pi_k = \delta_{jk}I_n$ and $\Pi_1 + \cdots + \Pi_n = I_n$. Show that the $n^2 \times n^2$ matrix

$$\sum_{j=1}^{n}(U_j \otimes \Pi_j)$$

is unitary.

Problem 18. (i) Let A be an $n \times n$ matrix over $\mathbb{C}$ and Π be an $m \times m$ projection matrix. Let $z \in \mathbb{C}$. Calculate

$$\exp(z(A \otimes \Pi)).$$

(ii) Let A_1, A_2 be $n \times n$ matrices over $\mathbb{C}$. Let Π_1, Π_2 be $m \times m$ projection matrices with $\Pi_1 \Pi_2 = 0$. Calculate

$$\exp(z(A_1 \otimes \Pi_1 + A_2 \otimes \Pi_2)).$$

(iii) Use the result from (ii) to find the unitary matrix

$$U(t) = \exp(-i\hat{H}t/\hbar)$$

where $\hat{H} = \hbar\omega(A_1 \otimes \Pi_1 + A_2 \otimes \Pi_2)$ and we assume that A_1 and A_2 are hermitian matrices.
(iv) Apply the result of (iii) to

$$A_1 = \sigma_1, \quad \Pi_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \sigma_3, \quad \Pi_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Problem 19. Every 4×4 unitary matrix U can be written as

$$U = (U_1 \otimes U_2)\exp(i(\alpha\sigma_1 \otimes \sigma_1 + \beta\sigma_2 \otimes \sigma_2 + \gamma\sigma_3 \otimes \sigma_3))(U_3 \otimes U_4)$$

where $U_j \in U(2)$ $(j = 1, 2, 3, 4)$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Calculate

$$\exp(i(\alpha\sigma_1 \otimes \sigma_1 + \beta\sigma_2 \otimes \sigma_2 + \gamma\sigma_3 \otimes \sigma_3)).$$

Problem 20. Consider the Hilbert space $\mathbb{C}^{16}$ and the normalized state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle)$$

where

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Give a computer algebra implementation that calculates the 256 expectation values

$$T_{jk\ell m} = \langle\psi|\sigma_j \otimes \sigma_k \otimes \sigma_\ell \otimes \sigma_n|\psi\rangle, \quad j, k, \ell, m = 0, 1, 2, 3$$

where σ_0, σ_1, σ_2, σ_3 are the Pauli spin matrices with $\sigma_0 = I_2$ (2×2) identity matrix.

Problem 21. Consider the unitary matrices

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad C = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1-i & -1-i \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Find $(B \otimes C)(R(I_2 \otimes A)R)(H \otimes H)$.

Problem 22. Consider the spin matrix for spin-$\frac{1}{2}$

$$s_1 = \frac{1}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with the eigenvalues $1/2$ and $-1/2$ and the corresponding normalized eigenvectors

$$\mathbf{e}_{1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_{-1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Do the four vectors

$$\frac{1}{\sqrt{2}}(\mathbf{e}_{1/2} \otimes \mathbf{e}_{1/2} + \mathbf{e}_{-1/2} \otimes \mathbf{e}_{-1/2}), \quad \frac{1}{\sqrt{2}}(\mathbf{e}_{1/2} \otimes \mathbf{e}_{1/2} - \mathbf{e}_{-1/2} \otimes \mathbf{e}_{-1/2}),$$

$$\frac{1}{\sqrt{2}}(\mathbf{e}_{1/2} \otimes \mathbf{e}_{-1/2} + \mathbf{e}_{-1/2} \otimes \mathbf{e}_{-1/2}), \quad \frac{1}{\sqrt{2}}(\mathbf{e}_{1/2} \otimes \mathbf{e}_{-1/2} - \mathbf{e}_{-1/2} \otimes \mathbf{e}_{-1/2}),$$

form a basis in $\mathbb{C}^4$? Prove or disprove.

Problem 23. Let $N \geq 1$. Consider the Hilbert space $\mathbb{C}^{2^N}$. The $(N+1)$ *Dicke states* are defined by

$$\left| \frac{N}{2}, \ell - \frac{N}{2} \right\rangle := \frac{1}{\sqrt{^N C_\ell}}(\underbrace{|0\rangle \otimes \cdots \otimes |0\rangle}_{\ell} \otimes \underbrace{|1\rangle \otimes \cdots \otimes |1\rangle}_{N-\ell} + \text{permutations})$$

where $\ell = 0, 1, \ldots, N$ and $^N C_\ell = N!/(\ell!(N-\ell)!)$. Write down the Dicke states for $N = 2$ and $N = 3$. Which of the states are entangled?

Problem 24. Consider the 2×2 permutation matrices

$$P_1 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Show that

$$\Pi_1 = \frac{1}{2}(P_1 + P_2), \qquad \Pi_2 = \frac{1}{2}(P_1 - P_2)$$

are projection matrices. Find $\Pi_1 \Pi_2$. Discuss.
(ii) Show that

$$\Pi_1 = \frac{1}{2}(P_1 \otimes P_1 + P_2 \otimes P_2), \qquad \Pi_2 = \frac{1}{2}(P_1 \otimes P_1 - P_2 \otimes P_2)$$

are projection matrices. Find $\Pi_1 \Pi_2$. Discuss.

Problem 25. Consider the six 3×3 permutation matrices

$$P_1 = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with the signatures of the permutation $P_1 \to +1$, $P_2 \to -1$, $P_3 \to -1$, $P_4 \to +1$, $P_5 \to +1$, $P_6 \to -1$.
(i) Is

$$\Pi_1 = \frac{1}{6}(P_1 + P_2 + P_3 + P_4 + P_5 + P_6)$$

a projection matrix?
(ii) Is

$$\Pi_2 = \frac{1}{6}(P_1 - P_2 - P_3 + P_4 + P_5 - P_6)$$

a projection matrix? Calculate $\Pi_1 \Pi_2$. Discuss.
(iii) Is

$$\Pi_1 = \frac{1}{6}(P_1 \otimes P_1 + P_2 \otimes P_2 + P_3 \otimes P_3 + P_4 \otimes P_4 + P_5 \otimes P_5 + P_6 \otimes P_6)$$

a projection matrix?
(iv) Is

$$\Pi_2 = \frac{1}{6}(P_1 \otimes P_1 - P_2 \otimes P_2 - P_3 \otimes P_3 + P_4 \otimes P_4 + P_5 \otimes P_5 - P_6 \otimes P_6)$$

a projection matrix? Find $\Pi_1\Pi_2$. Discuss.

Problem 26. Consider the Hilbert space $\mathbb{C}^9$ and the three normalized states

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\\0\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$$

$$|\psi_{23}\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\0\\1\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$$

$$|\psi_{31}\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}0\\0\\1\end{pmatrix}\right).$$

(i) Are the states entangled?
(ii) Find the density matrices.
(iii) Form a mixed state from the three density matrices.

Problem 27. Consider the two Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^d$ and the product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. A state $|\psi\rangle \in \mathcal{H}$ is called maximally entangled if

$$\mathrm{tr}_{\mathcal{H}_1}(|\psi\rangle\langle\psi|) = \mathrm{tr}_{\mathcal{H}_2}(|\psi\rangle\langle\psi|) = \frac{1}{d}.$$

Apply this definition to the Bell states in $\mathcal{H} = \mathbb{C}^4$, i.e. $d = 2$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \quad |\psi_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.$$

Problem 28. (i) Let

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

be the standard basis in $\mathbb{C}^2$. Calculate the 4×4 matrix

$$P := \sum_{j=1}^{2}\sum_{k=1}^{2} |j\rangle\langle k| \otimes |k\rangle\langle j|.$$

What type of matrix is this?
(ii) Calculate P^2. Discuss.
(iii) Let

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

be the *Hadamard basis* in $\mathbb{C}^2$. Calculate the 4×4 matrix

$$Q := \sum_{j=1}^{2}\sum_{k=1}^{2} |j\rangle\langle k| \otimes |k\rangle\langle j|.$$

What type of matrix is this?
(iv) Calculate Q^2. Discuss.

Problem 29. Can the normalized state

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

in the Hilbert space $\mathbb{C}^8$ be written as a product state of three normalized vectors in $\mathbb{C}^2$?

Problem 30. (i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Find the commutators and anticommutators

$$[\sigma_1, \sigma_2], \quad [\sigma_2, \sigma_3], \quad [\sigma_3, \sigma_1], \quad [\sigma_1, \sigma_2]_+, \quad [\sigma_2, \sigma_3]_+, \quad [\sigma_3, \sigma_1]_+.$$

(ii) Consider the 4×4 matrices $\sigma_1 \otimes \sigma_2$, $\sigma_2 \otimes \sigma_3$, $\sigma_3 \otimes \sigma_1$. Find the commutators and anticommutators

$$[\sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_3], \quad [\sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1], \quad [\sigma_3 \otimes \sigma_1, \sigma_1 \otimes \sigma_2]$$

$$[\sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_3]_+, \quad [\sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1]_+, \quad [\sigma_3 \otimes \sigma_1, \sigma_1 \otimes \sigma_2]_+$$

(iii) Consider the 8×8 matrices $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$, $\sigma_3 \otimes \sigma_1 \otimes \sigma_2$, $\sigma_2 \otimes \sigma_3 \otimes \sigma_1$. Find the commutators and anticommutators

$$[\sigma_1 \otimes \sigma_2 \otimes \sigma_3, \sigma_3 \otimes \sigma_1 \otimes \sigma_2], \quad [\sigma_3 \otimes \sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_3 \otimes \sigma_1], \quad [\sigma_2 \otimes \sigma_3 \otimes \sigma_1, \sigma_1 \otimes \sigma_2 \otimes \sigma_3]$$

$$[\sigma_1 \otimes \sigma_2 \otimes \sigma_3, \sigma_3 \otimes \sigma_1 \otimes \sigma_2]_+, \quad [\sigma_3 \otimes \sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_3 \otimes \sigma_1]_+, \quad [\sigma_2 \otimes \sigma_3 \otimes \sigma_1, \sigma_1 \otimes \sigma_2 \otimes \sigma_3]_+.$$

Problem 31. Let σ_0, σ_1, σ_2, σ_3 be the Pauli spin matrices, where $\sigma_0 = I_2$ is the 2×2 unit matrix. Is

$$P = \frac{1}{2}\sum_{j=0}^{3}(\sigma_j \otimes \sigma_j)$$

a permutation matrix?

Problem 32. (i) Let σ_0, σ_1, σ_2, σ_3 be the Pauli spin matrices, where $\sigma_0 = I_2$ is the 2×2 unit matrix. Let

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

be a vector in $\mathbb{R}^3$ with $\|\mathbf{v}\| \leq 1$. Show that

$$\rho_{\mathbf{v}} = \frac{1}{2}(\sigma_0 + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3)$$

is a density matrix.
(ii) Is

$$\rho = \frac{1}{4}(\sigma_0 \otimes \sigma_0 + \sum_{j=1}^{3} v_j\sigma_j \otimes \sigma_j)$$

a density matrix?
(iii) Is

$$\rho = \frac{1}{2^3}(\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + \sum_{j=1}^{3} v_j\sigma_j \otimes \sigma_j \otimes \sigma_j)$$

a density matrix? Extend the result to n Kronecker products.

Problem 33. Consider the invertible matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Can the matrix be written as the Kronecker product of two 2×2 matrices?

Problem 34. Are the two state in $\mathbb{C}^9$

$$|\psi_1\rangle = \frac{1}{\sqrt{6}}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$|\psi_2\rangle = -\frac{1}{\sqrt{3}}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

orthogonal to each other?

Problem 35. Let σ_1, σ_3 be the Pauli spin matrices. Find the 4×4 permutation matrix P such that

$$P(\sigma_1 \otimes \sigma_3)P^{-1} = \sigma_3 \otimes \sigma_1.$$

Problem 36. Consider the two normalized states

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\alpha}\cos(\beta) \\ \sin(\beta) \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\delta}\cos(\gamma) \\ \sin(\gamma) \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in [0, 2\pi)$. Find

$$\max_{\alpha,\beta,\gamma,\delta} |\langle\psi|\phi\rangle|^2.$$

Problem 37. Let

$$|0\rangle, \quad |1\rangle, \quad \ldots \quad, |n\rangle$$

be an orthonormal basis in $\mathbb{C}^{n+1}$. Are the states

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2n}} \sum_{j=1}^{n} |j\rangle \otimes |j\rangle$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle - \frac{1}{\sqrt{2n}} \sum_{j=1}^{n} |j\rangle \otimes |j\rangle$$

normalized? Are the state orthogonal to each other? Is

$$\rho = (|\psi_0\rangle\langle\psi_0|) \otimes (|\psi_1\rangle\langle\psi_1|)$$

a density matrix?

Problem 38. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Show that the 4×4 matrix

$$R = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \right) (\sigma_1 \otimes \sigma_2) \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \right)$$

can be written as direct sum of two 2×2 matrices. Discuss.

Problem 39. Let σ_2 be the second Pauli spin matrix. Then

$$\sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Find the normalized state ($\gamma \in \mathbb{R}$)

$$e^{i\gamma\sigma_2\otimes\sigma_2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \equiv e^{i\gamma\sigma_2\otimes\sigma_2}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right).$$

Is the state entangled? Discuss.

Problem 40. Let $\omega = \exp(i\pi)$. Consider the 4×4 matrices

$$S_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\otimes I_2, \qquad S_2 = I_2 \otimes \begin{pmatrix}1 & 0\\0 & \omega\end{pmatrix}$$

(i) Show that $S_1^2 = I_4$, $S_2^2 = I_4$ and $S_1S_2S_1^{-1}S_2^{-1} = \omega I_4$.
(ii) Find the commutator $[S_1, S_2]$.

Problem 41. Let $\gamma, \alpha_j, \beta_j \in \mathbb{R}$. Can any vector in $\mathbb{C}^4$ be written as

$$(U_1(\alpha)\otimes U_2(\beta))\begin{pmatrix}\cos(\gamma/2)\\0\\0\\\sin(\gamma/2)\end{pmatrix}?$$

Here $U_1(\alpha)$, $U_2(\beta)$ are the unitary matrices

$$U_1(\alpha) = \exp(i\alpha_3\sigma_3/2)\exp(i\alpha_1\sigma_2/2)\exp(i\alpha_2\sigma_3/2)$$

$$U_2(\beta) = \exp(i\beta_3\sigma_3/2)\exp(i\beta_1\sigma_2/2)\exp(i\beta_2\sigma_3/2)$$

and σ_1, σ_2, σ_3 are the Pauli spin matrices.

Chapter 3

Matrix Properties

Problem 1. The vectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $\mathbb{C}^3$. Find the unitary matrices U_{12}, U_{23}, U_{31} such that

$$U_{12}\mathbf{v}_1 = \mathbf{v}_2, \quad U_{23}\mathbf{v}_2 = \mathbf{v}_3, \quad U_{31}\mathbf{v}_3 = \mathbf{v}_1.$$

Then calculate $U_{31}U_{23}U_{12}$ and the matrix

$$V = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^* + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^* + \lambda_3 \mathbf{v}_3 \mathbf{v}_3^*$$

where the complex numbers λ_1, λ_2, λ_3 satisfy $\lambda_1\overline{\lambda}_1 = 1$, $\lambda_2\overline{\lambda}_2 = 1$, $\lambda_3\overline{\lambda}_3 = 1$. Is the matrix unitary?

Problem 2. Can the unitary matrix (permutation matrix)

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

be written as the Kronecker product of two 2×2 matrices, i.e. $U = A \otimes B$?

Problem 3. Let A, B, C be $n \times n$ matrices. Let I_n be the $n \times n$ identity matrix.
(i) What can be said about the eigenvalues and eigenvectors of

$$A \otimes I_n \otimes I_n + I_n \otimes B \otimes I_n + I_n \otimes I_n \otimes C$$

if we know the eigenvalues and eigenvectors of A, B, C?
(ii) Is

$$e^{A \otimes I_n \otimes I_n + I_n \otimes B \otimes I_n + I_n \otimes I_n \otimes C} = e^A \otimes e^B \otimes e^C \, ?$$

Problem 4. Let σ_1, σ_3 be the Pauli spin matrices. Calculate ($\theta \in \mathbb{R}$)

$$R(\theta) = \exp(-i(\theta/2)(\sigma_1 + \sigma_3)/\sqrt{2})$$

Is the matrix $R(\theta)$ unitary?

Problem 5. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Does the set of 4×4 matrices

$$\{ I_2 \otimes I_2, \quad \sigma_1 \otimes \sigma_1, \quad -\sigma_2 \otimes \sigma_2, \quad \sigma_3 \otimes \sigma_3 \}$$

form a group under matrix multiplication?

Problem 6. The spin matrices for spin-$\frac{3}{2}$ particles are given by

$$J_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$$

$$J_3 = \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

(i) Show that the matrices are hermitian.
(ii) Find the eigenvalues and eigenvectors of these matrices.
(iii) Calculate the commutation relations, i.e. $[J_1, J_2]$, $[J_2, J_3]$, $[J_3, J_1]$.
(iv) Are the matrices unitary?

Problem 7. Two orthonormal bases in an n-dimensional complex Hilbert space

$$\{\, |\mathbf{u}_j\rangle \,:\, j = 1, 2, \ldots, n \,\}, \qquad \{\, |\mathbf{v}_j\rangle \,:\, j = 1, 2, \ldots, n \,\}$$

are called *mutually unbiased* if inner products (scalar products) between all possible pairs of vectors taken from distinct bases have the same magnitude $1/\sqrt{n}$, i.e.

$$|\langle \mathbf{u}_j | \mathbf{v}_k \rangle| = \frac{1}{\sqrt{n}} \qquad \text{for all} \quad j, k \in \{\, 1, 2, \ldots, n \,\}.$$

(i) Find such bases for the Hilbert space $\mathbb{C}^2$. Start of with the standard basis

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(ii) Find such bases for the Hilbert space $\mathbb{C}^3$. Start of with the standard basis

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(iii) Find such bases for the Hilbert space $\mathbb{C}^4$ using the result from $\mathbb{C}^2$ and the Kronecker product.

Problem 8. (i) Let A, B be $n \times n$ matrices over $\mathbb{C}$ such that $A^2 = I_n$ and $B^2 = I_n$. Furthermore assume that

$$[A, B]_+ \equiv AB + BA = 0_n$$

i.e. the anticommutator vanishes. Let $\alpha, \beta \in \mathbb{C}$. Calculate $e^{\alpha A + \beta B}$ using

$$e^{\alpha A + \beta B} = \sum_{j=0}^{\infty} \frac{(\alpha A + \beta B)^j}{j!}.$$

(ii) Consider the case that $n = 2$ and

$$\alpha = -i\omega t, \qquad A = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\beta = -i\Delta t/\hbar, \qquad B = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iii) Consider the case that $n = 8$ and

$$\alpha = -i\omega t, \qquad A = \sigma_3 \otimes \sigma_3 \otimes \sigma_3$$

$$\beta = -i\Delta t/\hbar, \qquad B = \sigma_1 \otimes \sigma_1 \otimes \sigma_1.$$

Problem 9. Let A, B be $n \times n$ matrices with $A^2 = I_n$ and $B^2 = I_n$. Assume that the commutator of A and B vanishes, i.e.

$$[A, B] = AB - BA = 0_n.$$

Let $a, b \in \mathbb{C}$. Calculate

$$e^{aA+bB}.$$

(ii) Let $a = -i\omega t$, $b = -i\Delta t/\hbar$ (Δ real) and

$$A = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \quad B = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1$$

with n (even) factors of the Kronecker products. Then the conditions given above are satisfied. Simplify the result from (i) with this assumption.

Problem 10. Let A, B be $n \times n$ matrices with $A^2 = I_n$ and $B^2 = I_n$. Assume that the anticommutator of A and B vanishes, i.e.

$$[A, B]_+ = AB + BA = 0_n.$$

(i) Let $a, b \in \mathbb{C}$. Calculate

$$e^{aA+bB}.$$

(ii) Let $a = -i\omega t$, $b = -i\Delta t/\hbar$ (Δ real) and

$$A = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \quad B = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1$$

with n (odd) factors of the Kronecker products. Then the conditions given above are satisfied. Simplify the result from (i) with this assumption.

Problem 11. Consider the Hilbert space $\mathbb{C}^3$ and the standard basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consider the unitary matrices

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where $\omega = e^{2\pi i/3}$.
(i) Calculate the state $R|j\rangle$, $T|j\rangle$, where $j = 0, 1, 2$.

(ii) Find the commutator $[R, T]$.
(iii) Consider the normalized state

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes 2\rangle).$$

Calculate the state $(R \otimes T)|\psi\rangle$ and discuss.

Problem 12. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Show that

$$[\sigma_m \otimes \sigma_n, \sigma_k \otimes I_2] \equiv [\sigma_m, \sigma_k] \otimes \sigma_n$$
$$[\sigma_m \otimes \sigma_n, I_2 \otimes \sigma_k] \equiv \sigma_m \otimes [\sigma_n, \sigma_k]$$

where $k, m, n \in \{1, 2, 3\}$.

Problem 13. Given two arbitrary normalized states $|\psi\rangle$ and $|\phi\rangle$ in $\mathbb{C}^2$. Find a 2×2 unitary matrix U such that $|\psi\rangle = U|\phi\rangle$, i.e. U must be expressed in terms of the compenents of the states $|\psi\rangle$ and $\phi\rangle$.

Problem 14. Consider the Hamilton operator in $\mathbb{C}^4$

$$\hat{H} = -t(|00\rangle\langle 11| + |11\rangle\langle 00|) + v(|00\rangle\langle 00| + |11\rangle\langle 11|.$$

The kinetic parameter is $t \geq 0$ and v is the potential parameter. Find the eigenvalues and eigenvectors of $\hat{H}$. Keep $t = 1$ fixed and disucss the dependence of the eigenvalues of $\hat{H}$ as a function of v.

Problem 15. Let H be an $n \times n$ hermitian matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues with the pairwise orthogonal normalized eigenvectors $\mathbf{v}_1$, $\ldots$, $\mathbf{v}_n$. Then we can write

$$H = \sum_{\ell=1}^{n} \lambda_\ell \mathbf{v}_\ell \mathbf{v}_\ell^*.$$

Let

$$P = I_n - \mathbf{v}_j \mathbf{v}_j^* - \mathbf{v}_k \mathbf{v}_k^* + \mathbf{v}_j \mathbf{v}_k^* + \mathbf{v}_k \mathbf{v}_j^*, \qquad j \neq k.$$

(i) What is the condition on the eigenvalues of H such that $PHP^* = H$.
(ii) Find P^2.

Problem 16. Let B be an $n \times n$ matrix with $B^2 = I_n$. Show that

$$\exp\left(-\frac{1}{2} i\pi(B - I_n)\right) \equiv B.$$

Problem 17. Consider the vector

$$|\psi\rangle = \begin{pmatrix} \sin(\phi_1)\sin(\phi_2)\sin(\theta) \\ \sin(\phi_1)\sin(\phi_2)\cos(\theta) \\ \sin(\phi_1)\cos(\phi_2) \\ \cos(\phi_1) \end{pmatrix}$$

in the Hilbert space $\mathbb{R}^4$ with $\phi_1, \phi_2, \theta \in \mathbb{R}$. Find the norm of this vector. For which values of ϕ_1, ϕ_2, θ is the norm a mimimum? What is the use of this vector?

Problem 18. Let R be a nonsingular $n \times n$ matrix. Let A and B be $n \times n$ matrices. Assume that $R^{-1}AR$ and $R^{-1}BR$ are diagonal matrices. Calculate the commutator $[A, B]$.

Problem 19. Let A, B be two $n \times n$ matrices. Assume that

$$\mathrm{tr}(A) = 0, \qquad \mathrm{tr}(B) = 0.$$

Can we conclude that $\mathrm{tr}(AB) = 0$? Prove or disprove.

Problem 20. We know that any $n \times n$ hermitian matrix has only real eigenvalues. Assume that a given $n \times n$ matrix has only real eigenvalues. Can we conclude that the matrix is hermitian? Prove or disprove.

Problem 21. Consider the Hilbert space $\mathbb{C}^n$. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$, $\mathbf{e}_n$ be the standard basis in $\mathbb{C}^n$, S_n be the symmetric group of order $n!$ and U_σ be the unitary matrix on $\otimes^n \mathbb{C}^n$ such that

$$U_\sigma(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n) := \mathbf{e}_{\sigma(1)} \otimes \cdots \otimes \mathbf{e}_{\sigma(n)}$$

where $\sigma \in S_n$. We define the matrix ("antisymmetrization operator") in the Hilbert space $\otimes^n \mathbb{C}^n$ by

$$\Pi_n := \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) U_\sigma$$

where sgn is the signature of the permutation $\sigma \in S_n$. The matrices Π_n are projection matrices. Find Π_2 and Π_3.

Problem 22. (i) The four-dimensional face-centered hypercubic lattice plays a central role in simulating three-dimensional hydrodynamics on a cellular automata machine. Consider the four-dimensional face-centered hypercubic lattice in connection with entanglement. This lattice is generated from the four basis vectors

$$(\pm 1, \pm 1, 0, 0). \tag{1}$$

Permuting the components of these four vectors in $\mathbb{R}^4$ we find 20 additional vectors. Show that the 24 vectors can be classified as follows. In class A we have eight vectors

$$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\\0\end{pmatrix}.$$

The normalization factor would be $1/\sqrt{2}$. In class B we have the eight vectors

$$\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\-1\\0\end{pmatrix}.$$

The normalization factor is also $1/\sqrt{2}$. In class C we have the eight vectors

$$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\0\end{pmatrix}.$$

Again the normalization factor is $1/\sqrt{2}$. Show that if $\alpha(\mathbf{n}_A, \mathbf{m})$ is the angle between the nth vector of class A and the mth vector of class B, then

$$\alpha(\mathbf{n}_A, \mathbf{m}_B) = \alpha(\mathbf{n}_B, \mathbf{m}_C) = \alpha(\mathbf{n}_C, \mathbf{m}_A)$$

and

$$\alpha(\mathbf{n}_A, \mathbf{m}_A) = \alpha(\mathbf{n}_B, \mathbf{m}_B) = \alpha(\mathbf{n}_C, \mathbf{m}_C).$$

Each class contains four oppositely oriented pairs of vectors. This means that the ordering of the vectors is such that class B is related to class A in exactly the same way as class C is related to B and A is related to C.
(ii) Show that the normalized vectors in class A are maximally entangled.
(iii) Show that the vectors in class B and class C can be written as the Kronecker product of two vectors from $\mathbb{R}^2$.
(iv) The Hadamard gate given by the unitary matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

plays a central role in quantum computing. Consider now the 4×4 matrix

$$R = I_2 \otimes H$$

where $\otimes$ denotes the Kronecker product and I_2 is 2×2 unit matrix. Thus R itself is a unitary matrix. Applying this matrix to the 24 vectors. Discuss.
(v) Show that the construction given above can be extended to higher dimensional cases. For example in $\mathbb{R}^8$ we would start with

$$\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)^T$$

and all permutations. Show that this provides us with the GHZ-state

$$\frac{1}{\sqrt{2}}(1\,0\,0\,0\,0\,0\,0\,1)^T$$

which is fully entangled when we use the three tangle (based on the hyperdeterminant) as measure of entanglement. Show that we find a set of unentangled states, for example

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

Show that using the three-tangle as measure for entanglement we find 0 for these vectors.

Problem 23. The associative algebra $M_d(\mathbb{C})$ of $d \times d$ matrices can be considered as a C^* algebra with the square of the norm $\|\,.\,\|$ defined by $(A \in M_d(\mathbb{C}))$

$$\|A\|^2 := \text{largest eigenvalue of the (normal) matrix } A^*A.$$

Let $d = 2$ and

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} .$$

Find the norm.

Problem 24. Consider the Hilbert space $\mathbb{C}^d$. Let $|j\rangle$ $(j = 1, \ldots, d)$ be an orthonormal basis in $\mathbb{C}^d$. Then a $d \times d$ matrix A acting in $\mathbb{C}^d$ can be written as

$$A = \sum_{j,k=1}^{d} a_{jk} |j\rangle\langle k|$$

with $a_{jk} \in \mathbb{C}$. Obviously A depends on the underlying orthonormal basis. If we have the standard basis, then A reduces to the matrix $A = (a_{jk})$. We can associate a vector $|\psi_A\rangle$ in the Hilbert space $\mathbb{C}^{d^2}$ with the matrix A via

$$|\psi_A\rangle = \sum_{j,k=1}^{d} a_{jk}|j\rangle \otimes |k\rangle.$$

(i) Let $d = 2$ and consider the standard basis

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find A and $|\psi_A\rangle$.
(ii) Let $d = 2$ and consider the Hadamard basis

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find A and $|\psi_A\rangle$.
(iii) Let $d = 3$ and consider the basis

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Find A and $|\psi_A\rangle$.
(iv) Describe the connection of the map $A \mapsto |\psi_A\rangle$ with the vec-operator.

Problem 25. Let $\phi_1, \phi_2 \in \mathbb{R}$. From the Bell basis

$$\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_1} \\ 0 \\ 0 \\ e^{i\phi_1} \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{i\phi_2} \\ e^{i\phi_2} \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{i\phi_2} \\ -e^{i\phi_2} \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_1} \\ 0 \\ 0 \\ -e^{i\phi_1} \end{pmatrix}$$

we form the matrix

$$M(\phi_1, \phi_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_1} & 0 & 0 & e^{i\phi_1} \\ 0 & e^{i\phi_2} & e^{i\phi_2} & 0 \\ 0 & e^{i\phi_2} & -e^{i\phi_2} & 0 \\ e^{i\phi_1} & 0 & 0 & -e^{i\phi_1} \end{pmatrix}.$$

Is $M(\phi_1, \phi_2)$ an element of the Lie group $SU(4)$?

Problem 26. (i) Let $x_1, x_2, x_3 \in \mathbb{R}$. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Show that

$$e^{i(x_1\sigma_1+x_2\sigma_2+x_3\sigma_3)} = \cos(r)I_2 + \frac{\sin(r)}{r} i(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$$
$$= \begin{pmatrix} \cos(r) + ix_3\sin(r)/r & i(x_1 - ix_2)\sin(r)/r \\ i(x_1 + ix_2)\sin(r)/r & \cos(r) - ix_3\sin(r)/r \end{pmatrix}$$

where $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$.
(ii) Let $y_1, y_2, y_3 \in \mathbb{R}$ and

$$X := x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3, \qquad Y := y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3.$$

Consider the maps

$$X \leftrightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad Y \leftrightarrow \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Let $\mathbf{x} \cdot \mathbf{y} := x_1y_1 + x_2y_2 + x_3y_3$ (scalar product). Show that

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}\mathrm{tr}(XY).$$

(iii) Show that

$$-\frac{i}{2}[X, Y] \leftrightarrow \mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Problem 27. Let $\mathbf{s}$ be a spin with a fixed total angular momentum quantum number

$$s \in \{1/2, 1, 3/2, 2, \ldots\}.$$

The (normalized) eigenstates of x_3-angular momentum $|s, m\rangle$ form a ladder with

$$m = -s, -s + 1, \ldots, s - 1, s.$$

The eigenstates $|s, m\rangle$ form an orthonormal basis in a $2s + 1$ dimensional Hilbert space. For example if $s = 1/2$ we have the two states $|1/2, -1/2\rangle$, $|1/2, 1/2\rangle$ and can identify

$$|1/2, 1/2\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1/2, -1/2\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus we have the Hilbert space $\mathbb{C}^2$. For $s = 1$ we have the three states $|1,-1\rangle$, $|1,0\rangle$, $|1,1\rangle$ and can identify

$$|1,-1\rangle \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |1,0\rangle \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,1\rangle \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

A *spin coherent state* $|s,\theta,\phi\rangle$ for $s = 1/2, 1, 3/2, \ldots$ can be given by

$$|s,\theta,\phi\rangle = \sum_{m=-s}^{m=s} \sqrt{\frac{(2s)!}{(s+m)!(s-m)!}} (\cos(\theta/2))^{s+m} (\sin(\theta/2))^{s-m} e^{-im\phi} |s,m\rangle.$$

(i) Find $|1/2,\theta,\phi\rangle$ and write it as a vector in $\mathbb{C}^2$.
(ii) Find $|1,\theta,\phi\rangle$ and write it as a vector in $\mathbb{C}^3$.
(iii) For a given s find the scalar product $\langle s,m|s,\theta,\phi\rangle$.

Problem 28. (i) Consider the Pauli spin matrix σ_2 and the Lie group $SL(2,\mathbb{C})$. Let $S \in SL(2,\mathbb{C})$. Show that

$$S\sigma_2 S^T = \sigma_2$$

where ^{T} denotes the transpose.
(ii) Show that

$$(S \otimes S)(\sigma_2 \otimes \sigma_2)(S^T \otimes S^T) = \sigma_2 \otimes \sigma_2.$$

Problem 29. Let $|1\rangle$, $|2\rangle$, ..., $|d\rangle$ be an orthonormal basis in the Hilbert space $\mathbb{C}^d$. Consider the matrix

$$S = \sum_{j,k=1}^{d} (|j\rangle\langle k| \otimes |k\rangle\langle j|).$$

Is S independent of the chosen orthonormal basis?

Problem 30. (i) Let $\phi_1, \phi_2 \in \mathbb{R}$. Show that

$$U(\phi_1,\phi_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & -e^{-i\phi_2} \\ e^{i\phi_2} & e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $U(\phi_1,\phi_2)$ an element of $SU(2)$? Find the eigenvalues of $U(\phi_1,\phi_2)$.

(ii) Let $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathbb{R}$. Show that

$$U(\phi_1, \phi_2, \phi_3, \phi_4) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & 0 & 0 & -e^{-i\phi_2} \\ 0 & e^{i\phi_3} & -e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_4} & e^{-i\phi_3} & 0 \\ e^{i\phi_2} & 0 & 0 & e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $U(\phi_1, \phi_2, \phi_3, \phi_4)$ an element of $SU(4)$? Find the eigenvalues of $U(\phi_1, \phi_2, \phi_3, \phi_4)$.
(iii) Let $\phi_1, \phi_2 \in \mathbb{R}$. Show that

$$U(\phi_1, \phi_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & 0 & -e^{-i\phi_2} \\ 0 & \sqrt{2} & 0 \\ e^{i\phi_2} & 0 & e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $U(\phi_1, \phi_2)$ an element of $SU(2)$? Find the eigenvalues of $U(\phi_1, \phi_2)$.

Problem 31. Consider the Hadamard matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues of the Hadamard matrix are given by $+1$ and -1 with the corresponding normalized eigenvectors

$$\frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{4+2\sqrt{2}} \\ \sqrt{4-2\sqrt{2}} \end{pmatrix}, \qquad \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{4-2\sqrt{2}} \\ -\sqrt{4+2\sqrt{2}} \end{pmatrix}.$$

How can this information be used to find the eigenvalues and eigenvectors of the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Problem 32. (i) Consider the Hilbert space $\mathbb{C}^4$. Do the vectors

$$\mathbf{v}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

form an orthonormal basis in $\mathbb{C}^4$. Prove or disprove.

(ii) Can the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be written as Kronecker products of vectors in $\mathbb{C}^2$. Prove or disprove.
(iii) Consider the 4×4 matrices

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of the two matrices. Compare to (i). Disucss.
(iv) Find the commutator of S and T, i.e. $[T, S]$. What can be said about eigenvectors of such a pair of matrices? Discuss. Hint. Look at your result from (iii).

Problem 33. Consider the matrix

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

(i) Is the matrix unitary?
(ii) Find the eigenvalues and nonnormalized eigenvectors of U. Use this information to write down the spectral decomposition of U.
(iii) Find a skew-hermitian matrix K such that $U = \exp(K)$. One can utilize the results from (ii).
(iv) Apply the unitary matrix to the normalized state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the state $U|\psi\rangle$ and calculate the probability $|\langle\psi|K|\psi\rangle|^2$.

Problem 34. Let $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathbb{R}$. Consider the 2×2 matrix

$$U(\phi_1, \phi_2, \phi_3, \phi_4) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & e^{i\phi_2} \\ e^{i\phi_3} & e^{i\phi_4} \end{pmatrix}.$$

The matrix contains the two column vector

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} \\ e^{i\phi_2} \end{pmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_3} \\ e^{i\phi_4} \end{pmatrix}.$$

Find the conditions on $\phi_1, \phi_2, \phi_3, \phi_4$ such that

$$\langle \mathbf{v}_1 | \mathbf{v}_2 \rangle = 0.$$

Is the matrix unitary if this condition is satisfied?

Problem 35. (i) An $n \times n$ matrix $H = (h_{jk})$ over $\mathbb{C}$ is called a complex Hadamard matrix if $|h_{jk}| = 1$ for $j, k = 1, \ldots, n$ and $HH^* = nI_n$. Note that $\frac{1}{\sqrt{n}}H$ is then a unitary matrix. Let $\phi \in [0, \pi)$. Let $n = 4$. Show that

$$H(\phi) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & ie^{i\phi} & -1 & -ie^{i\phi} \\ 1 & -1 & 1 & -1 \\ 1 & -ie^{i\phi} & -1 & ie^{i\phi} \end{pmatrix}$$

is a complex Hadamard matrix.
(ii) Given two complex Hadamard matrices H_1 and H_2. Is $H_1 \otimes H_2$ a complex Hadamard matrix?

Problem 36. Consider the Hamilton operator

$$\hat{\gamma}_3 = i\hbar\omega A, \qquad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where $\hbar$ and ω (frequency) are constants.
(i) Find

$$\exp(-i\hat{\gamma}_3 t/\hbar).$$

(ii) Let

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

be the initial state in the Hilbert space $\mathbb{C}^4$. Calculate

$$|\psi(t)\rangle = \exp(-i\hat{\gamma}_3 t/\hbar)|\psi(t=0)\rangle$$

and thus solve the Schrödinger equation.
(iii) If we know the eigenvalues of γ_3 what can be said about the eigenvalues of $\exp(-i\hat{\gamma}_3 t/\hbar)$?

Problem 37. Let $\alpha, \theta, \phi \in \mathbb{R}$. Consider the vector in $\mathbb{C}^4$

$$\mathbf{v} = \begin{pmatrix} \sinh(\alpha)\sin(\theta)\cos(\phi) \\ \sinh(\alpha)\sin(\theta)\sin(\phi) \\ \cosh(\alpha)\cos(\theta) \\ \cosh(\alpha)\sin(\theta) \end{pmatrix}.$$

(i) Normalize the vector.
(ii) Apply the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

to the normalized vector. Calculate $\mathbf{v}^* B \mathbf{v}$. Discuss.

Problem 38. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. For the Dirac equation the following 4×4 matrices play a central role. We define

$$\beta := \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0_2 & \sigma_k \\ \sigma_k & 0_2 \end{pmatrix} \quad k = 1, 2, 3.$$

Let $\gamma_k = i\beta\alpha_k$ for $k = 1, 2, 3$, $\gamma_0 = -i\beta$ and $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0$. Find the gamma matrices and calculate their anticommutators.

Problem 39. (i) Consider the finite-dimensional Hilbert space $\mathbb{C}^d$. A symmetric informatially complete positive operator valued measure (SIC-POVM) consists of d^2 outcomes that are subnormalized projection matrices Π_j onto pure states

$$\Pi_j = \frac{1}{d} |\psi_j\rangle\langle\psi_j|$$

for $j, k = 1, \ldots, d^2$ such that

$$|\langle\psi_k|\psi_k\rangle|^2 = \frac{1 + d\delta_{jk}}{d+1}.$$

Consider the case $d = 2$. Show that the normalized vectors

$$|\psi_1\rangle = \begin{pmatrix} \sqrt{(3+\sqrt{3})/6} \\ e^{i\pi/4}\sqrt{(3-\sqrt{3})/6} \end{pmatrix}$$

$$|\psi_2\rangle = \begin{pmatrix} \sqrt{(3+\sqrt{3})/6} \\ -e^{i\pi/4}\sqrt{(3-\sqrt{3})/6} \end{pmatrix}$$

$$|\psi_3\rangle = \begin{pmatrix} e^{i\pi/4}\sqrt{(3-\sqrt{3})/6} \\ \sqrt{(3+\sqrt{3})/6} \end{pmatrix}$$

$$|\psi_4\rangle = \begin{pmatrix} -e^{i\pi/4}\sqrt{(3-\sqrt{3})/6} \\ \sqrt{(3+\sqrt{3})/6} \end{pmatrix}$$

satisfy this condition.
(ii) Consider the matrices σ_1, $-i\sigma_2$, σ_3. Find

$$\sigma_1|\psi_1\rangle, \quad -i\sigma_2|\psi_1\rangle, \quad \sigma_3|\psi_1\rangle.$$

(iii) Let $d = 2$. Let

$$S_d := \sum_{j=1}^{d} |j\rangle\otimes|j\rangle\otimes\langle j|\otimes\langle j| + \sum_{k>j=1} \frac{1}{\sqrt{2}}(|j\rangle\otimes|k\rangle+|k\rangle\otimes|j\rangle)\otimes\frac{1}{\sqrt{2}}(\langle j|\otimes\langle k|+\langle k|\otimes\langle j|)$$

where $|1\rangle$, $|2\rangle$ denotes the standard basis in $\mathbb{C}^2$, i.e.

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that

$$\sum_{j=1}^{d^2} |\psi_j\rangle \otimes |\psi_j\rangle\langle\psi_j| \otimes \langle\psi_j| = \frac{2d}{d+1}.$$

(iv) Can one find a SIC-POVM in $\mathbb{C}^4$ using the states from (i) and the Kronecker product?

Problem 40. Let a_1, a_2, b_1, b_2 be real. Find the normalization factors for the vector in $\mathbb{C}^4$

$$|\psi\rangle = \begin{pmatrix} a_1\cos(\phi/2) + b_1\sin(\phi/2) \\ a_2\cos(\phi/2) + b_2\sin(\phi/2) \\ ia_1\sin(\phi/2) - ib_1\cos(\phi/2) \\ ia_2\sin(\phi/2) - ib_2\cos(\phi/2) \end{pmatrix}.$$

Problem 41. Any 2×2 matrix can be written as a linear combination of the Pauli spin matrices and the 2×2 identity matrix

$$A = aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where $a, b, c, d \in \mathbb{C}$.
(i) Find A^2 and A^3.
(ii) Use the result from (i) to find all matrices A such that $A^3 = \sigma_1$.

Problem 42. Let $r, s, \theta \in \mathbb{R}$. Consider the Hamilton operator given by the 2×2 matrix

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}.$$

(i) Is the matrix a normal matrix?
(ii) Is the matrix hermitian?
(iii) Find the eigenvalues and eigenvectors of $\hat{K}$.

Problem 43. Consider the vector space of $n \times n$ matrices over $\mathbb{C}$. Let $B_1, B_2, \ldots, B_{n^2}$ be a basis. Assume that all B's are invertible. Is B_1^{-1}, $B_2^{-1}, \ldots, B_{n^2}^{-1}$ also a basis for the vector space?

Problem 44. What can be said about the eigenvalues of an $n \times n$ matrix which is unitary and skew-hermitian? Give an example of such a matrix.

Problem 45. Let $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$. Consider the matrix

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_{11}} & e^{i\phi_{12}} \\ e^{i\phi_{21}} & e^{i\phi_{22}} \end{pmatrix}.$$

(i) What are the conditions on ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} such that the matrix is unitary?
(ii) What are the conditions on ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} such that the matrix is hermitian?
What are the conditions on ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} such that $V = V^{-1}$?

Problem 46. Is the 3×3 matrix

$$V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/2 & -1/(2\sqrt{3}) & -\sqrt{2/3} \\ 1/2 & \sqrt{3}/2 & 0 \end{pmatrix}$$

unitary?

Problem 47. Let A be an $n \times n$ matrix over $\mathbb{C}$. An $n \times n$ matrix B is called a square root of A if $B^2 = A$. Find the square roots of the 2×2 identity matrix applying the spectral theorem. The eigenvalues of I_2 are $\lambda_1 = 1$ and $\lambda_2 = 1$. As normalized eigenvectors choose

$$\begin{pmatrix} e^{i\phi_1}\cos(\theta) \\ e^{i\phi_2}\sin(\theta) \end{pmatrix}, \quad \begin{pmatrix} e^{i\phi_1}\sin(\theta) \\ -e^{i\phi_2}\cos(\theta) \end{pmatrix}$$

which form an orthonormal basis in $\mathbb{C}^2$. Four cases $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (1,1)$, $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (1,-1)$, $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (-1,1)$, $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (-1,-1)$ have to be studied. The first and last case are trivial. So study the second case $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (1,-1)$. The second case and the third case are "equivalent".

Problem 48. Let $|j\rangle$ $(j = 1, \ldots, d)$ be an orthonormal basis in $\mathbb{C}^d$ and $\langle k|$ $(k = 1, \ldots, d)$ be the dual basis. We define

$$R_{jk} = |j\rangle\langle k|, \quad j, k = 1, \ldots, d.$$

Show that

$$R_{jk}R_{\ell m} = R_{jm}\delta_{\ell k}, \quad [R_{jk}, R_{\ell m}] = R_{jm}\delta_{\ell k} - R_{\ell k}\delta_{jm}, \quad \sum_{j=1}^{d} R_{jj} = I_d.$$

Hint. Utilize

Problem 49. Let $|0\rangle$, $|1\rangle$, $\ldots |d-1\rangle$ be an orthonormal basis in $\mathbb{C}^d$. Let $T_{jk} \in \mathbb{C}$ with $j, k = 0, 1, \ldots, d-1$. Consider the linear operator

$$T = \sum_{j=0}^{d-1}\sum_{k=0}^{d-1} T_{jk}|j\rangle\langle k|.$$

(i) Let $d = 2$ and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find T.
(ii) Let $d = 2$ and

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find T.

Problem 50. Let $\hat{H}$ be a hermitian $n \times n$ matrix describing the Hamilton operator and acting in the Hilbert space $\mathbb{C}^n$. Let A, B be $n \times n$ hermitian matrices and $|\psi\rangle \in \mathbb{C}^n$. One defines (*quantum correlation function*)

$$Q(|\psi\rangle) := \frac{1}{2}\langle\psi|(A(t)B - AB(t) + BA(t) - B(t)A)|\psi\rangle$$

where

$$A(t) = e^{i\hat{H}t/\hbar}Ae^{-i\hat{H}t/\hbar}, \quad B(t) = e^{i\hat{H}t/\hbar}Be^{-i\hat{H}t/\hbar}.$$

(i) Let

$$\hat{H} = \hbar\omega\sigma_2, \quad A = \sigma_1, \quad B = \sigma_3, \quad |\psi\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

(ii) Let

$$\hat{H} = \hbar\omega\sigma_2\otimes\sigma_2, \quad A = \sigma_1\otimes\sigma_1, \quad B = \sigma_3\otimes\sigma_3, \quad |\psi\rangle = \begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1)\cos(\phi_2) \\ \sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\ \sin(\phi_1)\sin(\phi_2)\sin(\phi_3) \end{pmatrix}$$

Problem 51. Let H be a hermitian matrix. Then all unitary matrices $U_0 = I_n$, U_1, $\ldots$, U_k with $U_j H U_j^* = H$ form a group under matrix multiplication, where $j = 0, 1, \ldots, k$. Note that depending on H is group could consist only of $U_0 = I_n$. From $U_j H U_j^* = H$ it follows that $[H, U_j] = 0_n$. If $\mathbf{v}$ is an (normalized) eigenvector of H with eigenvalue λ, then $U_j \mathbf{v}$ is also an eigenvector of H with the same eigenvalue since

$$HU_j\mathbf{v} = U_j H \mathbf{v} = \lambda U_j \mathbf{v}.$$

The eigenvectors of H are bases of the irreducible representations of the group of H and can be classified according to them. Let

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(i) Find all 3×3 permutation matrices P such that $PHP^{-1} = H$.
(ii) Find the eigenvalues λ_1, λ_2, λ_3 and the corresponding normalized eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of H.
(iii) Find $P\mathbf{v}_j$ for $j = 1, 2, 3$ and the permutation matrices found in (i)

Problem 52. Consider the Bell basis

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

which form an orthonormal basis in $\mathbb{C}^4$. We can form $24 = 4!$ unitary matrices

$$(\mathbf{v}_{j_1}\ \mathbf{v}_{j_2}\ \mathbf{v}_{j_3}\ \mathbf{v}_{j_4}) \quad j_k \neq j_\ell \text{ (pairwise)}$$

with the lexigographical order $(\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3\, \mathbf{v}_4)$, $(\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_4\, \mathbf{v}_3)$, $\ldots$, $(\mathbf{v}_4\, \mathbf{v}_3\, \mathbf{v}_2\, \mathbf{v}_1)$. Apply the 24 unitary matrices to the normalized vector

$$\mathbf{w} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Discuss.

Problem 53. Can the *Bell matrix* be written as

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} = c_0 \sigma_0 \otimes \sigma_0 + c_1 \sigma_1 \otimes \sigma_1 + c_2 \sigma_2 \otimes \sigma_2 + c_3 \sigma_3 \otimes \sigma_3$$

where $\sigma_0 = I_2$ and σ_1, σ_2, σ_3 are the Pauli spin matrices.

Problem 54. Let S_1, S_2, S_3 be the spin matrices for spin

$$s = \frac{1}{2},\ 1,\ \frac{3}{2},\ 2, \ldots$$

These matrices are $(2s+1)\times(2s+1)$ hermitian matrices with trace equal to 0 satisfying the commutation relations

$$[S_1, S_2] = iS_3, \quad [S_2, S_3] = iS_1, \quad [S_3, S_1] = iS_2.$$

The eigenvalues of these three matrices are $s, s-1, \ldots, -s$ for a given s. Furthermore one has

$$S_1^2 + S_2^2 + S_3^2 = s(s+1)I_{2s+1}$$

where I_{2s+1} is the $(2s+1)\times(2s+1)$ identity matrix.
(i) Show that $\mathrm{tr}(S_j^2) = \frac{1}{3}s(s+1)(2s+1)$.
(ii) Show that $\mathrm{tr}(S_jS_k) = 0$ for $j \neq k$ and $j, k = 1, 2, 3$.
(iii) Show that the hermitian $(2s+1)^2 \times (2s+1)^2$ matrices

$$\widetilde{H} = \frac{H}{\hbar\omega} = S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3$$

$$\widetilde{K} = \frac{K}{\hbar\omega} = S_1 \otimes S_2 + S_2 \otimes S_3 + S_3 \otimes S_1$$

admit the same spectrum for all s.
(iv) Find the commutator

$$[S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3, S_1 \otimes S_2 + S_2 \otimes S_3 + S_3 \otimes S_1]$$

and anticommutator

$$[S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3, S_1 \otimes S_2 + S_2 \otimes S_3 + S_3 \otimes S_1]_+.$$

(v) For the case $s = \frac{1}{2}$, $s = 1$, $s = \frac{3}{2}$ find the eigenvalues and the normalized eigenvectors.
(vi) Calculate $\exp(z\widetilde{H})$ and $\exp(z\widetilde{K})$.

Problem 55. (i) Consider the hermitian 3×3 matrices to describe a particle with *spin-1*

$$S_1 := \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 := \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 := \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

With $S_+ := S_1 + iS_2$, $S_- := S_1 - iS_2$ we find

$$S_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

An example of a spin-1 particle is the photon. Let $\mathbf{m}$, $\mathbf{n}$ be normalized vectors in $\mathbb{R}^3$ which are orthogonal, i.e. $\mathbf{m}^T\mathbf{n} = 0$. Find the eigenvalues of the 3×3 matrix

$$K = (\mathbf{m} \cdot \mathbf{S})^2 - (\mathbf{n} \cdot \mathbf{S})^2$$

where $\mathbf{m} \cdot \mathbf{S} = m_1 S_1 + m_2 S_2 + m_3 S_3$.
(ii) Show that

$$P_{\mathbf{m}} = I_3 - (\mathbf{m} \cdot \mathbf{S})^2$$

is a projection operator.

Problem 56. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$. Let $\langle A \rangle := \mathbf{v}^* A \mathbf{v}$ and $\langle B \rangle := \mathbf{v}^* B \mathbf{v}$. We have the identity

$$AB \equiv (A - \langle A \rangle I_n)(B - \langle B \rangle I_n) + A\langle B \rangle + B\langle A \rangle - \langle A \rangle \langle B \rangle I_n.$$

We approximate AB as $AB \approx A\langle B \rangle + B\langle A \rangle - \langle A \rangle \langle B \rangle I_n$.
(i) Let

$$A = \sigma_1, \quad B = \sigma_2, \quad \mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find AB and $A\langle B \rangle + B\langle A \rangle - \langle A \rangle \langle B \rangle I_n$ and the distance (Frobenius norm) between the two matrices.
(ii) Apply the result to the case $n = 2$ and

$$A = \sigma_1, \quad B = \sigma_2, \quad \mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(iii) Consider the case

$$A = \sigma_1, \quad B = \sigma_2, \quad \mathbf{v} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Problem 57. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and σ_1, σ_2, σ_3 be the Pauli spin matrices. We define

$$\mathbf{a} \cdot \boldsymbol{\sigma} := a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3.$$

What is the condition on $\mathbf{a}$, $\mathbf{b}$ such that

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) \equiv (\mathbf{a} \cdot \mathbf{b}) I_2 + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}?$$

Here $\times$ denotes the vector product and I_2 is the 2×2 identity matrix.

Problem 58. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 2×2 matrix over the complex numbers

$$\Pi(\mathbf{n}) := \frac{1}{2}\left(I_2 + \sum_{j=1}^{3} n_j \sigma_j\right)$$

where $\mathbf{n} := (n_1, n_2, n_3)$ ($n_j \in \mathbb{R}$) is a unit vector, i.e. $n_1^2 + n_2^2 + n_3^2 = 1$.
(i) Describe the property of $\Pi(\mathbf{n})$, i.e. find $\Pi^*(\mathbf{n})$, $\text{tr}(\Pi(\mathbf{n}))$ and $\Pi^2(\mathbf{n})$, where tr denotes the trace. The trace is the sum of the diagonal elements of a square matrix.
(ii) Find the vector

$$\Pi(\mathbf{n}) \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Discuss.

Problem 59. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let A, B be two arbitrary 2×2 matrices. Is

$$\frac{1}{2}\text{tr}(AB) \equiv \sum_{j=1}^{3} \left(\frac{1}{2}\text{tr}(\sigma_j A)\right)\left(\frac{1}{2}\text{tr}(\sigma_j B)\right) ?$$

Problem 60. Let σ_j ($j = 0, 1, 2, 3$) be the Pauli spin matrices, where σ_0 is the 2×2 identity matrix. Form the four 4×4 matrices

$$\gamma_k = \begin{pmatrix} 0_2 & \sigma_k \\ -\sigma_k & 0_2 \end{pmatrix}, \quad k = 0, 1, 2, 3$$

where 0_2 is the 2×2 identity matrix.
(i) Are the matrices γ_k linearly independent?
(ii) Find the eigenvalues and eigenvectors of the γ_k's.
(iii) Are the matrices γ_k invertible. Use the result from (ii). If so, find the inverse.
(iv) Find the commutators $[\gamma_k, \gamma_\ell]$ for $k, \ell = 0, 1, 2, 3$. Find the anticommutators $[\gamma_k, \gamma_\ell]_+$ for $k, \ell = 0, 1, 2, 3$.
(v) Can the matrices γ_k be written as the Kronecker product of two 2×2 matrices?

Problem 61. Consider the 4×4 matrices

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_2$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3.$$

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$, $\mathbf{d} = (d_1, d_2, d_3)$ be elements in $\mathbb{R}^3$ and

$$\mathbf{a} \cdot \boldsymbol{\alpha} := a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3.$$

Calculate the traces

$$\mathrm{tr}((\mathbf{a} \cdot \boldsymbol{\alpha})(\mathbf{b} \cdot \boldsymbol{\alpha})), \quad \mathrm{tr}((\mathbf{a} \cdot \boldsymbol{\alpha})(\mathbf{b} \cdot \boldsymbol{\alpha})(\mathbf{c} \cdot \boldsymbol{\alpha})(\mathbf{d} \cdot \boldsymbol{\alpha})).$$

Problem 62. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 4×4 gamma matrices

$$\gamma_1 = \begin{pmatrix} 0_2 & \sigma_1 \\ -\sigma_1 & 0_2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0_2 & \sigma_2 \\ -\sigma_2 & 0_2 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ -\sigma_3 & 0_2 \end{pmatrix}$$

and

$$\gamma_0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}.$$

Find $\gamma_1\gamma_2\gamma_3\gamma_0$ and $\mathrm{tr}(\gamma_1\gamma_2\gamma_3\gamma_0)$.

Problem 63. Find the eigenvalues and eigenvectors of $\sigma_1\sigma_2\sigma_3$.

Problem 64. Consider the spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$S_{\alpha,1} = S_\alpha \otimes I_3 \otimes I_3, \quad S_{\alpha,2} = I_3 \otimes S_\alpha \otimes I_3, \quad S_{\alpha,3} = I_3 \otimes I_3 \otimes S_\alpha$$

with $\alpha = 1, 2, 3$. Let

$$\mathbf{S}_1 = \begin{pmatrix} S_1 \otimes I_3 \otimes I_3 \\ S_2 \otimes I_3 \otimes I_3 \\ S_3 \otimes I_3 \otimes I_3 \end{pmatrix}, \quad \mathbf{S}_2 = \begin{pmatrix} I_3 \otimes S_1 \otimes I_3 \\ I_3 \otimes S_2 \otimes I_3 \\ I_3 \otimes S_3 \otimes I_3 \end{pmatrix}, \quad \mathbf{S}_3 = \begin{pmatrix} I_3 \otimes I_3 \otimes S_1 \\ I_3 \otimes I_3 \otimes S_2 \\ I_3 \otimes I_3 \otimes S_3 \end{pmatrix}$$

and

$$\mathbf{S}_2 \times \mathbf{S}_3 = \begin{pmatrix} I_3 \otimes S_2 \otimes S_3 - I_3 \otimes S_3 \otimes S_2 \\ I_3 \otimes S_3 \otimes S_1 - I_3 \otimes S_1 \otimes S_3 \\ I_3 \otimes S_1 \otimes S_2 - I_3 \otimes S_2 \otimes S_1 \end{pmatrix}$$

Thus

$$\mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3) = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3 - \mathbf{S}_1 \otimes \mathbf{S}_3 \otimes \mathbf{S}_2 + \mathbf{S}_2 \otimes \mathbf{S}_3 \otimes \mathbf{S}_1 - \mathbf{S}_2 \otimes \mathbf{S}_1 \otimes \mathbf{S}_3 + \mathbf{S}_3 \otimes \mathbf{S}_1 \otimes \mathbf{S}_2 - \mathbf{S}_3 \otimes \mathbf{S}_2 \otimes \mathbf{S}_1.$$

Find the eigenvalues of $\mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$.

Problem 65. (i) Find the eigenvalues and eigenvectors of

$$\sigma_1 \otimes \sigma_2 \otimes \sigma_3.$$

Can one find entangled eigenvectors?

(ii) Find the eigenvalues and eigenvectors of the Hamilton operator

$$\hat{H} = \epsilon_1(\sigma_1 \otimes I_2 \otimes I_2) + \epsilon_2(I_2 \otimes \sigma_2 \otimes I_2) + \epsilon_3(I_2 \otimes I_2 \otimes \sigma_3) + \gamma(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \gamma \in \mathbb{R}$.

Problem 66. Find the nonzero (column) vectors $\mathbf{u} \in \mathbb{C}^{16}$ such that

$$\begin{aligned} (\sigma_1 \otimes \sigma_3 \otimes I_2 \otimes \sigma_3)\mathbf{u} &= \mathbf{u} \\ (\sigma_3 \otimes \sigma_1 \otimes \sigma_3 \otimes I_2)\mathbf{u} &= \mathbf{u} \\ (I_2 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_3)\mathbf{u} &= \mathbf{u} \\ (\sigma_3 \otimes I_2 \otimes \sigma_3 \otimes \sigma_1)\mathbf{u} &= \mathbf{u}. \end{aligned}$$

Problem 67. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Find the skew-hermitian matrices Σ_1, Σ_2, Σ_3 such that

$$\sigma_1 = \exp(\Sigma_1), \quad \sigma_2 = \exp(\Sigma_2), \quad \sigma_3 = \exp(\Sigma_3).$$

Find the commutators $[\Sigma_1, \Sigma_2]$, $[\Sigma_2, \Sigma_3]$, $[\Sigma_3, \Sigma_1]$ and compare with the commutators $[\sigma_1, \sigma_2]$, $[\sigma_2, \sigma_3]$, $[\sigma_3, \sigma_1]$.

Problem 68. (i) Consider the three (hermitian) spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

all with the eigenvalues $+1$, 0 and -1. Show that $S_j^3 = S_j$.
(ii) Let $\phi \in \mathbb{R}$. Show that

$$\exp(i\phi S_j) = I_3 + i\sin(\phi)S_j - (1-\cos(\phi))S_j^2$$

which is a unitary matrix.

Problem 69. (i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices and $z_1, z_2, z_3 \in \mathbb{C}$. Calculate

$$\exp(z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3).$$

(ii) Calculate the matrix

$$\exp(z_1\sigma_1 \otimes \sigma_1 + z_2\sigma_2 \otimes \sigma_2 + z_3\sigma_3 \otimes \sigma_3).$$

Problem 70. Find the unitary matrix

$$U(t) = e^{i\phi \sin(\omega t)\sigma_1}$$

with the Pauli spin matrix σ_1.

Problem 71. Let $z \in \mathbb{C}$. Calculate

$$\exp(-z(\sigma_1 \otimes \sigma_1))(\sigma_2 \otimes \sigma_2)\exp(z(\sigma_1 \otimes \sigma_1))$$

$$\exp(-z(\sigma_2 \otimes \sigma_2))(\sigma_3 \otimes \sigma_3)\exp(z(\sigma_2 \otimes \sigma_2))$$

$$\exp(-z(\sigma_3 \otimes \sigma_3))(\sigma_1 \otimes \sigma_1)\exp(z(\sigma_3 \otimes \sigma_3)).$$

Problem 72. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Calculate the commutator

$$[\sigma_j \otimes \sigma_k, \sigma_\ell \otimes \sigma_m]$$

where $j \neq \ell$ and $k \neq m$.

Problem 73. Let σ_3, σ_1 be the Pauli spin matrices. Find the commutators

$$[\sigma_1 \otimes \sigma_1 \otimes \sigma_1, \sigma_3 \otimes \sigma_3 \otimes \sigma_3]$$

and

$$[\sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3].$$

Discuss the general case with n Kronecker products.

Problem 74. Consider the spin-1 matrices S_1, S_2, S_3. Find the eigenvalues of the hermitian matrix

$$K = \frac{H}{\hbar\omega} = \cos(\theta)(S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3) + \sin(\theta)(S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3)^2.$$

Problem 75. Consider the Pauli spin matrices σ_1, σ_2, σ_3 and the 4×4 matrices

$$\sigma_1 \otimes \sigma_2, \quad \sigma_2 \otimes \sigma_3, \quad \sigma_3 \otimes \sigma_2.$$

Find the commutators

$$[\sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_3], \quad [\sigma_2 \otimes \sigma_3, \sigma_3 \otimes \sigma_1], \quad [\sigma_3 \otimes \sigma_2, \sigma_1 \otimes \sigma_2].$$

Discuss

Problem 76. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Calculate the commutators

$$[\sigma_1 \otimes I_2 + I_2 \otimes \sigma_2, \sigma_2 \otimes I_2 + I_2 \otimes \sigma_3]$$
$$[\sigma_2 \otimes I_2 + I_2 \otimes \sigma_3, \sigma_3 \otimes I_2 + I_2 \otimes \sigma_1]$$
$$[\sigma_3 \otimes I_2 + I_2 \otimes \sigma_1, \sigma_1 \otimes I_2 + I_2 \otimes \sigma_2].$$

Problem 77. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 4×4 matrices

$$\sigma_1 \otimes I_2 + I_2 \otimes \sigma_1, \quad \sigma_2 \otimes I_2 + I_2 \otimes \sigma_2, \quad \sigma_3 \otimes I_2 + I_2 \otimes \sigma_3.$$

Find the commutators. Discuss.

Problem 78. Consider the $2^6 \times 2^6$ unitary and hermitian matrices

$$X = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_3, \qquad S = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3.$$

Find the commutator $[X, S]$ and anticommutator $[X, S]_+$.

Problem 79. Consider the *Bell matrix*

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

which is a unitary matrix. Each column vector of the matrix is a fully entangled state. Are the normalized eigenvectors of B are also fully entangled states?

Problem 80. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Let $x_1, x_2, x_3 \in \mathbb{R}$. Show that

$$e^{i(x_1\sigma_1+x_2\sigma_2+x_3\sigma_3)} = I_2\cos(r) + \frac{\sin(r)}{r} i(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$.

Problem 81. Consider the Pauli spin matrices to describe a spin-$\frac{1}{2}$ particle. In the square array of 4×4 matrices

$$\begin{array}{ccc} I_2 \otimes \sigma_3 & \sigma_3 \otimes I_2 & \sigma_3 \otimes \sigma_3 \\ \sigma_1 \otimes I_2 & I_2 \otimes \sigma_1 & \sigma_1 \otimes \sigma_1 \\ \sigma_1 \otimes \sigma_3 & \sigma_3 \otimes \sigma_1 & \sigma_2 \otimes \sigma_2 \end{array}$$

each row and each column is a triad of commuting operators. Consider the hermitian 3×3 matrices to describe a particle with *spin-1*

$$S_1 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 := \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Is in the square array of 9×9 matrices

$$\begin{array}{ccc} I_3 \otimes S_3 & S_3 \otimes I_3 & S_3 \otimes S_3 \\ S_1 \otimes I_3 & I_3 \otimes S_1 & S_1 \otimes S_1 \\ S_1 \otimes S_3 & S_3 \otimes S_1 & S_2 \otimes S_2 \end{array}$$

each row and each column a triad of commuting operators?

Problem 82. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3, \quad \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}.$$

Calculate $(\mathbf{a} \times \sigma)^T) \cdot (\mathbf{b} \times \sigma)$, where $\times$ denotes the vector product and $\cdot$ the scalar product.

Problem 83. Consider the Pauli spin matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Let $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ be unit vectors in $\mathbb{R}^3$. We define

$$Q := \mathbf{q} \cdot \boldsymbol{\sigma}, \quad R := \mathbf{r} \cdot \boldsymbol{\sigma}, \quad S := \mathbf{s} \cdot \boldsymbol{\sigma}, \quad T := \mathbf{t} \cdot \boldsymbol{\sigma}$$

where $\mathbf{q} \cdot \boldsymbol{\sigma} := q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3$. Calculate

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2.$$

Express the result using commutators.

Problem 84. Let σ_1, σ_2, σ_3 be the *Pauli spin matrices*.
(i) Find

$$R_{1x}(\alpha) := \exp(-i\alpha(\sigma_1 \otimes I_2)), \qquad R_{1y}(\alpha) := \exp(-i\alpha(\sigma_2 \otimes I_2))$$

where $\alpha \in \mathbb{R}$ and I_2 denotes the 2×2 unit matrix.
(ii) Consider the case $R_{1x}(\alpha = \pi/2)$ and $R_{1y}(\alpha = \pi/4)$. Calculate $R_{1x}(\pi/2)R_{1y}(\pi/4)$. Discuss.

Problem 85. Let σ_1, σ_2 and σ_3 be the Pauli spin matrices. We define $\sigma_+ := \sigma_1 + i\sigma_2$ and $\sigma_- := \sigma_1 - i\sigma_2$. Let

$$c_k^* := \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \left(\frac{1}{2}\sigma_+\right) \otimes I_2 \otimes I_2 \otimes \cdots \otimes I_2$$

where σ_+ is on the kth position and we have $N-1$ Kronecker products. Thus c_k^* is a $2^N \times 2^N$ matrix.
(i) Find c_k.
(ii) Find the anticommutators $[c_k, c_j]_+$ and $[c_k^*, c_j]_+$.
(iii) Find $c_k c_k$ and $c_k^* c_k^*$.

Problem 86. Using the definitions from the previous problem we define

$$s_{-,j} := \frac{1}{2}(\sigma_{x,j} - i\sigma_{y,j}) = \frac{1}{2}\sigma_{-,j}, \qquad s_{+,j} := \frac{1}{2}(\sigma_{x,j} + i\sigma_{y,j}) = \frac{1}{2}\sigma_{+,j}$$

and

$$c_1 = s_{-,1}$$
$$c_j = \exp\left(i\pi \sum_{\ell=1}^{j-1} s_{+,\ell}s_{-,\ell}\right) s_{-,j} \quad \text{for} \quad j = 2, 3, \ldots$$

(i) Find c_j^*.
(ii) Find the inverse transformation.
(iii) Calculate $c_j^* c_j$.

Problem 87. Find the conditions on $c_1, c_2, c_3 \in \mathbb{C}$ such that

$$(c_1\sigma_1 \otimes \sigma_1 + c_2\sigma_2 \otimes \sigma_2 + c_3\sigma_3 \otimes \sigma_3)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Problem 88. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Find the 3×3 matrix R defined by

$$R_{jk} = \mathrm{tr}(\sigma_j \otimes \sigma_k), \qquad j,k = 1,2,3.$$

Problem 89. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Calculate the trace of σ_1, σ_2, σ_3, $\sigma_1\sigma_2$, $\sigma_1\sigma_3$, $\sigma_2\sigma_3$, $\sigma_1\sigma_2\sigma_3$.

Problem 90. Consider

$$\exp\left(\epsilon \sum_{j=1}^{n} s_j^- s_{j+1}^-\right)$$

with $s_{n+1}^- = s_1^-$ (cyclic boundary condition) and the term

$$Y = \frac{1}{m!}\left(\sum_{j=1}^{n} s_j^- s_{j+1}^-\right)^m .$$

Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Calculate

$$Y(\mathbf{u} \otimes \mathbf{u} \otimes \cdots \otimes \mathbf{u}), \qquad Y(\mathbf{v} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{v}).$$

Problem 91. Let $z \in \mathbb{C}$. Calculate the commutator

$$[\sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_1 \otimes \sigma_1]$$

and

$$\exp(-z\sigma_2 \otimes \sigma_2 \otimes \sigma_2)(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)\exp(z\sigma_2 \otimes \sigma_2 \otimes \sigma_2).$$

Problem 92. Find the eigenvalues of the unitary operator

$$U = \exp\left(-i\frac{\pi}{4} b^\dagger b \otimes \sigma_3\right).$$

Note that $e^{i\pi/4} = (1+i)/\sqrt{2}$, $e^{-i\pi/4} = (1-i)/\sqrt{2}$.

Problem 93. Consider the *Bell matrix*

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Write the matrix B in the form

$$B = \sum_{j_1,j_2=0}^{3} c_{j_1,j_2} \sigma_{j_1} \otimes \sigma_{j_2}.$$

Problem 94. Consider the 4×4 matrix

$$M = \begin{pmatrix} 0 & 0 & -a & -b \\ 0 & 0 & b & -a \\ -a & b & 0 & 0 \\ -b & -a & 0 & 0 \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Show that $B^T M B$ can be written as the direct sum of two 2×2 matrices.

Problem 95. Consider the *Bell matrix*

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

(i) Find all matrices A such that $BAB^* = A$.
(ii) Find all matrices A such that BAB^* is a diagonal matrix.

Problem 96. Consider the standard basis in the vector space of 2×2 matrices

$$E_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{01} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

and the mutually unbiased basis

$$\mu_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Express the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

with the basis given by $\mu_j \otimes \mu_k$ $(j,k=0,1,2,3)$.

Problem 97. Let $\sigma_0 = I_2$, σ_1, σ_2, σ_3 be the Pauli spin matrices. Find

$$(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2)^2, \quad (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2)^3.$$

Problem 98. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let

$$R := \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3.$$

(i) Find $\mathrm{tr}(R)$. Using this result what can be said about the eigenvalues of R.
(ii) Find R^2. Using this result and the result from (i) derive the eigenvalues of the matrix R.
(iii) Find $\frac{1}{4}(I_4+R)^2$.

Problem 99. Given the 4×4 matrix

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Are the four column vectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

eigenvectors of A? Are the vectors entangled?

Problem 100. Find 4×4 matrices A, B consisting of Kronecker products of σ_1, σ_2, σ_3, I_2 such that

$$[A,B] = \sigma_2 \otimes \sigma_3 - \sigma_3 \otimes \sigma_2.$$

Problem 101. Let A, B be hermitian $n \times n$ matrices. Consider the Hamilton operator

$$\hat{H} = A \otimes I_n + I_n \otimes B + \epsilon(A \otimes B)$$

where $\epsilon \in \mathbb{R}$. Let $\hat{H}_0 = A \otimes I_n + I_n \otimes B$. Find the Moller operator

$$\hat{\Omega}_{\pm} := \lim_{T\to\mp\infty} \exp(-i\hat{H}T/\hbar)\exp(i\hat{H}_0 T/\hbar).$$

Problem 102. Calculate

$$\exp(-i\pi(\sigma_3 \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes I_2)(I_2 \otimes \sigma_3 \otimes I_2 + I_2 \otimes I_2 \otimes I_2)(I_2 \otimes I_2 \otimes \sigma_1/8).$$

This is the *Toffoli gate*.

Problem 103. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $\alpha \in \mathbb{R}$. Calculate

$$\exp(\alpha(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3))$$

and the trace of this expression

$$\operatorname{tr}\exp(\alpha(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)).$$

Problem 104. Every 4×4 unitary matrix U can be written as

$$U = (U_1 \otimes U_2)\exp(i(\alpha\sigma_1 \otimes \sigma_1 + \beta\sigma_2 \otimes \sigma_2 + \gamma\sigma_3 \otimes \sigma_3))(U_3 \otimes U_4)$$

where $U_j \in U(2)$ $(j = 1,2,3,4)$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Calculate

$$\exp(i(\alpha\sigma_1 \otimes \sigma_1 + \beta\sigma_2 \otimes \sigma_2 + \gamma\sigma_3 \otimes \sigma_3)).$$

Problem 105. Let $\omega = \exp(2\pi i/4) \equiv \exp(\pi i/2)$. Consider the four 64×64 invertible matrices

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^3 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \otimes I_4$$

$$S_2 = I_4 \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \otimes I_4$$

$$S_3 = I_4 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^3 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

$$S_4 = I_4 \otimes I_4 \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find S_j^4 for $j = 1,2,3,4$. Find

$$S_1 S_2 S_1^{-1} S_2^{-1}, \quad S_2 S_3 S_2^{-1} S_3^{-1}, \quad S_3 S_4 S_3^{-1} S_4^{-1}, \quad S_4 S_1 S_4^{-1} S_1^{-1}.$$

Find the commutators $[S_1, S_2]$, $[S_2, S_3]$, $[S_3, S_4]$, $[S_4, S_1]$.

Problem 106. Consider the Pauli spin matrices σ_1, σ_2 and σ_3. Can one find an $\alpha \in \mathbb{R}$ such that

$$\exp(i\alpha\sigma_3)\sigma_1 \exp(-i\alpha\sigma_3) = \sigma_2\,?$$

Problem 107. Consider the 3×3 matrix

$$S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Is the matrix hermitian? Find the eigenvalues and eigenvectors of S_2.
(ii) Show that $S_2^3 = S_2$.
(iii) Let $\phi \in \mathbb{R}$. Find $\exp(i\phi S_2)$.

Problem 108. (i) Let $\tau = (\sqrt{5} - 1)/2$ be the golden mean number. Consider the 2×2 matrices

$$B_1 = \begin{pmatrix} e^{-i7\pi/10} & 0 \\ 0 & -e^{-i3\pi/10} \end{pmatrix}, \quad B_2 = \begin{pmatrix} -\tau e^{-i\pi/10} & -i\sqrt{\tau} \\ -i\sqrt{\tau} & -\tau e^{i\pi/10} \end{pmatrix}.$$

The matrices are invertible. Are the matrices unitary? Is $B_1 B_2 B_1 = B_2 B_1 B_2$?
(ii) Show that using computer algebra

$$B_2^{-2} B_1^4 B_2^{-1} B_1 B_2^{-1} B_1 B_2 B_1^{-2} B_2 B_1^{-1} B_2^{-5} B_1 B_2^{-1} \approx \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Problem 109. Consider the Hamilton operator $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_0 = \hbar\omega\sigma_3, \qquad \hat{H}_1 = \hbar\omega\sigma_1.$$

Let U and U_0 be the unitary matrices

$$U = \exp(-i\hat{H}t/\hbar), \qquad U_0 = \exp(-i\hat{H}_0 t/\hbar).$$

Let n be a positive integer. The *Moller wave operators*

$$\Omega_\pm := \lim_{n \to \pm\infty} U^n U_0^{-n}.$$

Owing to their intertwining property the Moller wave operators transform the eigenvectors of the free dynamics $U_0 = \exp(-i\hat{H}_0 t/\hbar)$ into eigenvectors of the interacting dynamics $U = \exp(-i\hat{H}t/\hbar)$. Find $\Omega_\pm$.

Problem 110. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Find the conditions on $\alpha_1, \alpha_2, \alpha_3$ such that

$$U = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

is a unitary matrix. Note that

$$\begin{aligned}UU^* &= (\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)(\alpha_1^*\sigma_1 + \alpha_2^*\sigma_2 + \alpha_3^*\sigma_3)\\ &= (\alpha_1\alpha_1^* + \alpha_2\alpha_2^*\alpha_3\alpha_3)I_2 + (\alpha_1\alpha_2^* - \alpha_2\alpha_1^*)\sigma_1\sigma_2 + (\alpha_3\alpha_1^* - \alpha_1\alpha_3^*)\sigma_3\sigma_1 + (\alpha_2\alpha_3^* - \alpha_3\alpha_2^*)\sigma_2\sigma_3.\end{aligned}$$

Problem 111. Is the 8×8 matrix

$$U = \frac{1}{\sqrt{3}}(I_2 \otimes I_2 \otimes I_2 + i\sigma_1 \otimes \sigma_1 \otimes \sigma_1 + i\sigma_3 \otimes \sigma_3 \otimes \sigma_3)$$

unitary?

Problem 112. Consider the Pauli spin matrices $\sigma_0 = I_2$, σ_1, σ_2, σ_3. The matrices are unitary and hermitian.
(i) Is the 4×4 matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_0 & \sigma_1 \\ \sigma_2 & \sigma_3 \end{pmatrix}$$

unitary?
(ii) Is the 4×4 matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{pmatrix}$$

unitary?

Problem 113. Consider the two spin-1 matrices

$$L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\theta, \phi \in \mathbb{R}$. Calculate $T(\theta, \phi) = \exp(-i\phi L_3)\exp(-i\theta L_2)$. Is $T(\theta, \phi)$ an elements of $SO(3, \mathbb{R})$?

Problem 114. The following states form an orthonormal basis in the Hilbert space $\mathbb{C}^3$

$$|\pi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |\pi^0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\pi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

These states play a role for the π-mesons. Show that the states

$$|\pi^+\rangle \otimes |\pi^+\rangle, \qquad |\pi^-\rangle \otimes |\pi^-\rangle$$

$$\frac{1}{\sqrt{2}}(|\pi^+\rangle \otimes |\pi^0\rangle + |\pi^0\rangle \otimes |\pi^+\rangle), \quad \frac{1}{\sqrt{2}}(|\pi^0\rangle \otimes |\pi^-\rangle + |\pi^-\rangle \otimes |\pi^0\rangle)$$

$$\frac{1}{\sqrt{2}}(|\pi^+\rangle\otimes|\pi^0\rangle-|\pi^0\rangle\otimes|\pi^+\rangle), \quad \frac{1}{\sqrt{2}}(|\pi^+\rangle\otimes|\pi^-\rangle-|\pi^-\rangle\otimes|\pi^+\rangle), \quad \frac{1}{\sqrt{2}}(|\pi^0\rangle\otimes|\pi^-\rangle-|\pi^-\rangle\otimes|\pi^0\rangle)$$

$$\frac{1}{\sqrt{6}}(2|\pi^0\rangle\otimes|\pi^0\rangle+|\pi^+\rangle\otimes|\pi^-\rangle+|\pi^-\rangle\otimes|\pi^+\rangle), \quad \frac{1}{\sqrt{3}}(|\pi^+\rangle\otimes|\pi^-\rangle+|\pi^-\rangle\otimes|\pi^+\rangle-|\pi^0\rangle\otimes|\pi^0\rangle)$$

form an orthonormal basis in the Hilbert space $\mathbb{C}^9$. Which of these states are entangled?

Problem 115. (i) The electronic scattering matrix has the form

$$S(\phi_1,\phi_2,\phi_3,\gamma) = e^{i\phi_1\sigma_0}e^{i\phi_2\sigma_3}e^{i\gamma\sigma_2}e^{i\phi_3\sigma_3}$$

where $\phi_1,\phi_2,\phi_3 \in [0,2\pi)$, $\gamma \in [0,\pi/2)$. Find $S(\phi_1,\phi_2,\phi_3,\gamma)$.
(ii) Find

$$T(\phi_1,\phi_2,\phi_3,\gamma) = e^{i\phi_1\sigma_0} \otimes e^{i\phi_2\sigma_3} \otimes e^{i\gamma\sigma_2} \otimes e^{i\phi_3\sigma_3}.$$

Problem 116. Consider the $2^7 \times 2^7$ matrices

$$\begin{aligned}
&\quad\;\; -3 \;\; -2 \;\; -1 \quad 0 \quad\; 1 \quad\; 2 \quad\; 3\\
X_{02} &= I_2 \otimes I_2 \otimes I_2 \otimes \sigma_z \otimes I_2 \otimes \sigma_z \otimes I_2\\
X_{-20} &= I_2 \otimes \sigma_z \otimes I_2 \otimes \sigma_z \otimes I_2 \otimes I_2 \otimes I_2\\
X_{12} &= I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_z \otimes \sigma_z \otimes I_2\\
X_{-2-1} &= I_2 \otimes \sigma_z \otimes \sigma_z \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2\\
X_{23} &= I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_z \otimes \sigma_z\\
X_{-3-2} &= \sigma_z \otimes \sigma_z \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2.
\end{aligned}$$

Let

$$K = \lambda(X_{02} + X_{-20} + X_{12} + X_{-2-1} + X_{23} + X_{-3-2})$$

where $\lambda \in \mathbb{R}$. Calculate $\mathrm{tr}(\exp(K))$ and discuss the behaviour on λ.

Problem 117. Consider the two 4×4 matrices $\sigma_1 \otimes \sigma_3$, $\sigma_3 \otimes \sigma_1$.
(i) Find the eigenvalues.
(ii) Show that the eigenvectors can be given as product states (unentangled states), but also as entangled states (i.e. they cannot be written as product states). Explain.

Problem 118. Find a unitary matrix U which can be written as a direct sum of two 2×2 matrices and

$$U\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \equiv U\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}.$$

Problem 119. A wave-scattering problem can be described by its scattering matrix U. In a stationary problem, U relates the outgoing-wave to the ingoing-wave amplitudes. The condition of flux conservation implies unitary of U, i.e.

$$UU^\dagger = I$$

where I is the identity operator. If, additionally, the scattering problem is invariant under the operation of time reversal, we also have $U = U^T$, i.e. U is symmetric. Find all 2×2 unitary matrices that also satisfy $U = U^T$. Do these matrices form a subgroup of the Lie group $U(2)$?

Problem 120. Is

$$(\sigma_3\otimes\sigma_2\otimes\sigma_1)(\sigma_1\otimes\sigma_2\otimes\sigma_3)(\sigma_3\otimes\sigma_2\otimes\sigma_1) = \sigma_1\otimes\sigma_2\otimes\sigma_3\,?$$

Is

$$(\sigma_3\otimes I_2\otimes\sigma_1)(\sigma_1\otimes\sigma_2\otimes\sigma_3)(\sigma_3\otimes I_2\otimes\sigma_1) = \sigma_1\otimes\sigma_2\otimes\sigma_3\,?$$

Problem 121. Let n be odd and $n\geq 3$. Consider the matrices

$$A_3 = \begin{pmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\end{pmatrix},\quad A_5 = \begin{pmatrix}1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2}\end{pmatrix}$$

and generally

$$A_n = \begin{pmatrix}1/\sqrt{2} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & \dots & 0 & 0 & 0 & \dots & 1/\sqrt{2} & 0\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & \dots & 1/\sqrt{2} & 0 & 1/\sqrt{2} & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 1/\sqrt{2} & 0 & -1/\sqrt{2} & \dots & 0 & 0\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 1/\sqrt{2} & \dots & 0 & 0 & 0 & \dots & -1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -1/\sqrt{2}\end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A_3, A_5. Then solve the general case. The rank of the matrix A_n is n. Thus the matrices are invertible. The determinant of A_n is -1. Since $\mathrm{tr}(A_n) = 1$ the sum of the eigenvalues is 1.

Problem 122. Let σ_1, σ_2, σ_3 be the Pauli spin matrices.
(i) Let $z \in \mathbb{C}$. Calculate $\cosh(z\sigma_j)$, $\sinh(z\sigma_j)$, $j = 1, 2, 3$. Note that $\sigma_j^2 = I_2$.
(ii) Show that $\sin(\theta\sigma_j) = \sin(\theta)\sigma_j$.
(iii) Find the matrix

$$U = \exp\left(i\frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)\right).$$

Is the matrix U unitary? Prove or disprove. If so find the group generated by U.

Problem 123. Let $\mathbf{v}$ be a normalized vector in $\mathbb{C}^n$. Do all $n \times n$ matrices M which satisfy

$$M\mathbf{v} = \mathbf{v}, \qquad \det(M) = 1$$

form a group under matrix multiplication?

Problem 124. Let $c_1^\dagger, c_2^\dagger, \ldots, c_n^\dagger$ be Fermi creation operators and $c_1, c_2, \ldots, c_n$ be Fermi annihilation operators with the anticommutation relations

$$[c_j^\dagger, c_k]_+ = \delta_{jk} I.$$

(i) Consider the unitary matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Is the operator

$$\hat{K}_U = \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1^\dagger c_2 + c_2^\dagger c_1$$

unitary?
(ii) Consider the hermitian matrix

$$U = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Is the operator

$$\hat{K}_H = \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -ic_1^\dagger c_2 + ic_2^\dagger c_1$$

hermitian?
(iii) Consider the nonnormal matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Is the operator

$$\hat{K}_N = \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1^\dagger c_2 + c_2^\dagger c_1$$

nonnormal?

Problem 125. Let $\Gamma = (\Gamma_{jk})$ $(j,k = 1,\ldots,n)$, Γ_1, Γ_2 be $n \times n$ skew-hermitian matrices. Then $V = \exp(\Gamma)$, $V_1 = \exp(\Gamma_1)$, $V_2 = \exp(\Gamma_2)$ are unitary matrices. Let $c_1^\dagger$, ..., $c_n^\dagger$ be Fermi creation operators and c_1, ..., c_n be Fermi annihilation operators. Then

$$U(V) = \exp(\sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{jk} c_j^\dagger c_k)$$

is a unitary operator with $V = \exp(\Gamma)$. The commutation relation for the operators are $c_j^\dagger c_k$ are

$$[c_j^\dagger c_k, c_\ell^\dagger c_m] = \delta_{k\ell} c_j^\dagger c_m - \delta_{jm} c_\ell^\dagger c_k.$$

Show that owing to these commutation relations we have

$$U(V_1)U(V_2) = U(V_1 V_2), \quad U(V^{-1}) = U^{-1}(V) = U^\dagger(V), \quad U(I_n) = I$$

where I is the identity operator.

Problem 126. Consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $[A, B] = A$ and A, B form a basis of a two-dimensional non-abelian Lie algebra. Let $c_1^\dagger$, $c_2^\dagger$ be Fermi creation operators and c_1, c_2 be Fermi annihilation operators. We define

$$\hat{A} := \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1^\dagger c_2$$

$$\hat{B} := \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_2^\dagger c_2.$$

Find the commutator $[\hat{A}, \hat{B}]$. Discuss.

Problem 127. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and $[A, B]$ be the commutator of A and B. Let $c_1^\dagger, \dots, c_n^\dagger$ be Fermi creation operators and $c_1, \dots, c_n$ be Fermi annihilation operators. We define the operators

$$\hat{A} = \begin{pmatrix} c_1^\dagger & \dots & c_n^\dagger \end{pmatrix} A \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} c_1^\dagger & \dots & c_n^\dagger \end{pmatrix} B \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Show that

$$[\hat{A}, \hat{B}] = \widehat{[A, B]}.$$

Utilize that

$$[c_i^\dagger c_j, c_k^\dagger c_\ell] = c_i^\dagger c_\ell \delta_{jk} - c_k^\dagger c_j \delta_{i\ell}.$$

Problem 128. Let $c_1^\dagger$, $c_2^\dagger$ be Fermi creation operators and annihilation and c_1, c_2 be Fermi annihilation operators. Consider the operators

$$F_1 = c_1^\dagger c_2^\dagger, \quad F_2 = c_2 c_1, \quad F_3 = c_1^\dagger c_2, \quad F_4 = c_2^\dagger c_1$$

i.e. $F_2 = F_1^\dagger$ and $F_4 = F_1^\dagger$. Find the commutators $[F_j, F_k]$ and anti-commutators $[F_j, F_k]_+$.

Problem 129. Consider a vector $\mathbf{a}$ in $\mathbb{C}^4$ and the corresponding 2×2 matrix A via the vec^{-1} operator

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$$

and analogously

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}.$$

Show that

$$\mathbf{a}^* \mathbf{b} = \text{tr}(A^* B).$$

Problem 130. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be elements of $\mathbb{C}^2$. Find the conditions on $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ such that

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 = \mathbf{v}_3 \otimes \mathbf{v}_2 \otimes \mathbf{v}_1.$$

Problem 131. Let $n \geq 1$ and $\{|0\rangle\rangle, |1\rangle, \ldots, |n\rangle\}$ be an orthonormal basis in $\mathbb{C}^{n+1}$. Consider the linear operators ($(n+1) \times (n+1)$ matrices)

$$a_n = \sum_{j=1}^{n} \sqrt{j}|j-1\rangle\langle j|, \qquad a_n^\dagger = \sum_{k=1}^{n} \sqrt{k}|k\rangle\langle k-1|.$$

Find the commutator $[a_n, a_n^\dagger]$. Note that

$$\sum_{\ell=0}^{n} |\ell\rangle\langle\ell| = I_{n+1}.$$

Problem 132. Consider the vector $\mathbf{v} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}^T \in \mathbb{C}^6$. Find a Schmidt decomposition of $\mathbf{v}$ over $\mathbb{C}^6 = \mathbb{C}^2 \otimes \mathbb{C}^3$ and over $\mathbb{C}^6 = \mathbb{C}^3 \otimes \mathbb{C}^2$.

Problem 133. Consider the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$. Let $u_0, u_1, u_2, u_3 \in \mathbb{R}$ and

$$u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1.$$

Is

$$U = u_0 I_2 + \sum_{j=1}^{3} u_j \sigma_j$$

a unitary matrix?

Problem 134. Consider the spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that the eigenvalues of the matrix

$$M(k) = S_3^2 + k^2 S_2^2$$

are given by 1, k^2, $1 + k^2$.

Problem 135. Find all 2×2 matrices A and B over $\mathbb{C}$ such that the 4×4 matrix

$$U = A \otimes I_2 + I_2 \otimes B$$

is unitary. Start of with

$$A = \begin{pmatrix} r_{11}e^{i\alpha_{11}} & r_{12}e^{i\alpha_{12}} \\ r_{21}e^{i\alpha_{21}} & r_{22}e^{i\alpha_{22}} \end{pmatrix}, \quad A = \begin{pmatrix} s_{11}e^{i\beta_{11}} & s_{12}e^{i\beta_{12}} \\ s_{21}e^{i\beta_{21}} & s_{22}e^{i\beta_{22}} \end{pmatrix},$$

Problem 136. Let A be a 2×2 matrix which admit the eigenvalues $\lambda_1 = +1$ and $\lambda_2 = -1$ with the corresponding normalized eigenvectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi/2}(\cos(\theta/2)-\sin(\theta/2)) \\ e^{i\phi/2}(\cos(\theta/2)+\sin(\theta/2)) \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{-i\phi/2}(\cos(\theta/2)+\sin(\theta/2)) \\ e^{i\phi/2}(\cos(\theta/2)-\sin(\theta/2)) \end{pmatrix}.$$

The eigenvectors form an orthonormal basis in $\mathbb{C}^2$. Reconstruct the matrix A from this imformation applying the spectral theorem.

Problem 137. Let S_1, S_2, S_3 be the spin matrices for spin $s = \frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, ... Thus the size of the matrices is $(2s+1)\times(2s+1)$. Find the eigenvalues and eigenvectors of the $(2s+1)^3 \times (2s+1)^3$ matrix

$$S_1 \otimes S_3 \otimes S_2.$$

Problem 138. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the five gamma matrices

$$\Gamma_1 = \begin{pmatrix} \sigma_1 & 0_2 \\ 0_2 & \sigma_1 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} \sigma_2 & 0_2 \\ 0_2 & \sigma_2 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ \sigma_3 & 0_2 \end{pmatrix},$$

$$\Gamma_4 = \begin{pmatrix} 0_2 & -i\sigma_3 \\ i\sigma_3 & 0_2 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} -\sigma_3 & 0_2 \\ 0_2 & \sigma_3 \end{pmatrix}$$

Note that

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = \delta_{\mu\nu}I_4.$$

Find the ten matrices $\Gamma_{[\mu,\nu]}$ defined by

$$\Gamma_{[\mu,\nu]} := \frac{1}{2}i(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu).$$

Do these fifteen matrices together with the 4×4 identity matrix form an orthogonal basis in the vector space of 4×4 matrices over $\mathbb{C}$?

Problem 139. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Show that

$$\frac{1}{2}\mathrm{tr}(e^{i\pi\sigma_1/2}e^{i\pi\sigma_2}e^{i\pi\sigma_3}) = 1.$$

Problem 140. Let $a, b \in \mathbb{R}$ and $a \neq b$. Find all unitary matrices U such that

$$U\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}U^{-1} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.$$

Problem 141. Let J_2 be the 2×2 matrix with all entries 1 and I_2 the 2×2 identity matrix. Find the eigenvalues and normalized eigenvectors of

$$I_2 \otimes J_2 + J_2 \otimes I_2.$$

Extend to J_n and I_n.

Chapter 4

Density Operators

Problem 1. Consider the 2×2 matrix

$$\rho = \begin{pmatrix} 3/4 & \sqrt{2}e^{-i\phi}/4 \\ \sqrt{2}e^{i\phi}/4 & 1/4 \end{pmatrix}.$$

(i) Is the matrix a density matrix?
(ii) If so do we have a pure state or a mixed state?
(iii) Find the eigenvalues of ρ.
(iv) Find $\text{tr}(\sigma_1 \rho)$, where σ_1 is the first Pauli spin matrix.

Problem 2. Let $\epsilon \in [0, 1]$. Is

$$\rho_\epsilon = \begin{pmatrix} \epsilon & \sqrt{\epsilon(1-\epsilon)}e^{-i\phi} \\ \sqrt{\epsilon(1-\epsilon)}e^{i\phi} & 1-\epsilon \end{pmatrix}$$

with $0 \leq \phi < 2\pi$ a density matrix?

Problem 3. (i) Find a normalized state $|\phi\rangle$ in the Hilbert space $\mathbb{C}^2$ such that we have the density matrix

$$|\phi\rangle\langle\phi| = \frac{1}{2}\left(I_2 + \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)\right).$$

(ii) Find a normalized state $|\psi\rangle$ in the Hilbert space $\mathbb{C}^2$ such that we have the density matrix

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(I_2 + \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)\right).$$

Problem 4. Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli spin matrices. Find the conditions on the coefficients a_j, b_j and c_{jk} such that ρ

$$\rho = \frac{1}{4}(I_4 + (\sum_{j=1}^{3} a_j \sigma_j) \otimes I_2 + I_2 \otimes (\sum_{j=1}^{3} b_j \sigma_j) + \sum_{j,k=1}^{3} c_{jk} \sigma_j \otimes \sigma_k)$$

is a density matrix.

Problem 5. Let $\mathbf{m}, \mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{m}\| = \|\mathbf{n}\| = 1$. Is the 4×4 matrix

$$\rho(\mathbf{m}, \mathbf{n}) = \frac{1}{4}(I_4 + (\mathbf{n} \cdot \boldsymbol{\sigma}) \otimes I_2 + I_2 \otimes (\mathbf{m} \cdot \boldsymbol{\sigma}) + (\mathbf{n} \cdot \boldsymbol{\sigma}) \otimes (\mathbf{m} \cdot \boldsymbol{\sigma}))$$

a density matrix?

Problem 6. Consider the 3×3 matrix

$$\rho = \begin{pmatrix} 1/2 & 0 & 1/4 \\ 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \end{pmatrix}.$$

(i) Find the eigenvalues of ρ.
(ii) Is ρ a density matrix? Prove or disprove. If so, is ρ a mixed or pure state?

Problem 7. Consider the normalized state

$$|\psi\rangle = e^{-i\phi} \begin{pmatrix} e^{i(\alpha+\gamma)} \cos(\beta) \sin(\theta) \\ e^{-i(\alpha-\gamma)} \sin(\beta) \sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$

Find the density matrix $\rho = |\psi\rangle\langle\psi|$ and the eigenvalues of ρ.

Problem 8. Let $\epsilon \in \mathbb{R}$ and $|\epsilon| < 1$. Is the 4×4 matrix

$$\rho(\epsilon) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1-\epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-\epsilon & 0 & 0 & 1 \end{pmatrix}$$

a density matrix?

Problem 9. Show that the 4×4 matrices

$$\rho^- = \frac{1}{4}(I_2 \otimes I_2 - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3)$$

$$\omega^- = \frac{1}{4}(I_2 \otimes I_2 - \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)$$
$$\omega^+ = \frac{1}{4}(I_2 \otimes I_2 + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)$$
$$\rho^+ = \frac{1}{4}(I_2 \otimes I_2 + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3)$$

are density matrices. How they are related to the 4 *Bell states*

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad |\phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad |\phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}?$$

Problem 10. Let ρ_1 and ρ_2 be density matrices is a finite-dimensional Hilbert space. Let $\lambda \in [0,1]$. Is

$$\lambda\rho_1 + (1-\lambda)\rho_2$$

a density matrix?

Problem 11. Show that

$$\rho = \begin{pmatrix} \varepsilon_1 & 0 & 0 & \sqrt{\varepsilon_1\varepsilon_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\varepsilon_1-\varepsilon_2 & 0 \\ \sqrt{\varepsilon_1\varepsilon_2} & 0 & 0\varepsilon_2 & \end{pmatrix}$$

where $0 \le \varepsilon_1, \varepsilon_2 \le 1$ and $\varepsilon_1 + \varepsilon_2 \le 1$ is a density matrix.

Problem 12. Consider the density matrix

$$\rho = \sum_{j=1}^{4} p_j |\psi_j\rangle\langle\psi_j|, \quad 0 \le p_j \le 1, \quad \sum_{j=1}^{4} p_j = 1$$

where the $|\psi_j\rangle$ are the Bell states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

Write ρ using the Pauli spin matrices σ_1, σ_2, σ_3, the 2×2 identity matrix I_2 and the Kronecker product.

Problem 13. Consider the Hilbert space $\mathbb{C}^n$. Let ρ be a density matrix, i.e. $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$. The mean value of an observable A (hermitian $n \times n$ matrix) is given by

$$\langle A \rangle = \mathrm{tr}(\rho A).$$

If the density ρ is unkown, then it may be determined using n^2 mean values $\langle A^{(k)} \rangle$ $(k = 1, 2, \ldots, n^2)$ obtained from measurement if the set $\{A^{(k)}\}$ is a basis in the space of all hermitian $n \times n$ matrices.
(i) Let $n = 2$,

$$A = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\mathrm{tr}(\rho A) = 0, \quad \mathrm{tr}(\rho A^2) = 1, \quad \mathrm{tr}(\rho A^3) = 0, \quad \mathrm{tr}(\rho A^4) = 1.$$

Find the density matrix.
(ii) Let $n = 2$ and

$$\mathrm{tr}(\rho I_2) = 1, \quad \mathrm{tr}(\rho \sigma_1) = -1, \quad \mathrm{tr}(\rho \sigma_2) = 0, \quad \mathrm{tr}(\rho \sigma_3) = 0.$$

Find ρ.

Problem 14. (i) Let $x_1, x_2, x_3 \in \mathbb{R}$. Consider the hermitian matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & 1 - x_3 \end{pmatrix}.$$

Find the condition on x_1, x_2, x_3 such that $\rho^2 = \rho$. Is this matrix then a density matrix?
(ii) Let $\epsilon \in [0, 1]$. Consider the hermitian matrix

$$\rho = \frac{1}{2} \begin{pmatrix} \epsilon + x_3 & 0 & x_1 - ix_2 \\ 0 & 2 - 2\epsilon & 0 \\ x_1 + ix_2 & 0 & \epsilon - x_3 \end{pmatrix}.$$

Find the condition on x_1, x_2, x_3 and ϵ such that $\rho^2 = \rho$.

Problem 15. Consider the density matrix

$$\rho = \sum_{j=1}^{4} p_j |\psi_j\rangle\langle\psi_j|, \quad 0 \leq p_j \leq 1, \quad \sum_{j=1}^{4} p_j = 1$$

where the $|\psi_j\rangle$ are the Bell states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

Write ρ using the Pauli spin matrices and the 2×2 identity matrix I_2.

Problem 16. Let A be a nonzero $n \times n$ matrix over $\mathbb{C}$. Consider the map

$$A \to \rho = \frac{AA^*}{\mathrm{tr}(AA^*)}.$$

(i) Show that ρ is a density matrix.
(ii) Show that ρ is invariant under the map $A \to AU$, where U is an $n \times n$ unitary matrix.
(iii) Is $AA^* = A^*A$ in general?
(iv) Consider the map

$$A \to \sigma = \frac{A^*A}{\mathrm{tr}(A^*A)}.$$

Is $\sigma = \rho$? Prove or disprove.

Problem 17. Consider the state

$$|\psi\rangle = \begin{pmatrix} \cos(\theta) \\ e^{i\phi}\sin(\theta) \end{pmatrix}$$

and the density matrix

$$\rho(0) = |\psi\rangle\langle\psi|.$$

Given the Hamilton operator

$$\hat{H} = \hbar\omega\sigma_1.$$

Solve the von Neumann equation for the given $\rho(0)$ and the given $\hat{H}$. The von Neumann equation is given by

$$i\hbar\frac{d\rho}{dt} = [\hat{H}, \rho](t)$$

with the solution

$$\rho(t) = e^{-i\hat{H}t/\hbar}\rho(0)e^{i\hat{H}t/\hbar}.$$

Problem 18. Consider the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and the density matrix

$$\rho = |\psi\rangle\langle\psi|.$$

Given the Hamilton operator

$$\hat{H} = \hbar\omega\sigma_1 \otimes \sigma_1.$$

Solve the von Neumann equation for given ρ and the given $\hat{H}$. The von Neumann equation is given by

$$i\hbar\frac{d\rho}{dt} = [\hat{H}, \rho](t)$$

with the solution

$$\rho(t) = e^{-i\hat{H}t/\hbar}\rho(0)e^{i\hat{H}t/\hbar}.$$

Problem 19. (i) Is the 2×2 matrix

$$\rho = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}$$

a density matrix?
(ii) Can one find a state $|\psi\rangle$ in $\mathbb{C}^2$ such that

$$\rho = |\psi\rangle\langle\psi|\,?$$

(iii) Are the 4×4 matrices

$$\rho \otimes \rho, \quad \rho \oplus \rho, \quad \rho \star \rho$$

density matrices? Here $\otimes$ denotes the Kronecker product, $\oplus$ the direct sum and $\star$ operation which is defined for two 2×2 matrices A and B as

$$A \star B = \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Problem 20. Let $|0\rangle$, $|1\rangle$ be the standard basis in $\mathbb{C}^2$. Consider the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

with the density matrix $\rho = |\psi\rangle\langle\psi|$. Find the reduced density matrix ρ_1. Discuss.

Problem 21. Is the 2×2 matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + r\cos(\theta) & r\sin(\theta)e^{-i\phi} \\ r\sin(\theta)e^{i\phi} & 1 - r\cos(\theta) \end{pmatrix}$$

a density matrix? What are the conditions on r, θ, ϕ?

Problem 22. Consider a finite dimensional Hilbert space of dimension d on which the density matrix ρ acts. A quantum operation is represented by a completely positive and trace preserving map Λ which takes the form

$$\Lambda(\rho) = \sum_{j=1}^{d^2} V_j \rho V_j^*.$$

Show that the trace preserving condition $\text{tr}(\Lambda(\rho)) = \text{tr}(\rho)$ is equivalent to the equality

$$\sum_{j=1}^{d^2} V_j^* V_j = I.$$

Problem 23. Let S be the set of unit vectors in the Hilbert space $\mathbb{C}^n$. Let $\mathbf{u} \in S$. A function $\mu(\mathbf{u})$ from S to $\mathbb{R}$ is called a generalized probability measure if the following two conditions hold: (i) for $\mathbf{u} \in S$, $0 \le \mu(\mathbf{u}) \le 1$, (ii) if $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form an orthonormal basis in the Hilbert space $\mathbb{C}^n$, then $\sum_{j=1}^n \mu(\mathbf{u}_j) = 1$.

Let $n \ge 3$. Then any generalized probability measure μ on $\mathbb{C}^n$ has the form

$$\mu(\rho) = \text{tr}(\rho \mathbf{u}\mathbf{u}^*)$$

for a uniquely defined density matrix ρ. (Gleason 1957)

(i) Consider the Hilbert space $\mathbb{C}^3$, the orthonormal basis

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

and the density matrix

$$\rho = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find $\mu(\mathbf{u}_1)$, $\mu(\mathbf{u}_2)$, $\mu(\mathbf{u}_3)$.

(ii) Consider the Hilbert space $\mathbb{C}^4$, the orthonormal basis

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} \\ 0 \\ 0 \\ e^{i\phi} \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} \\ 0 \\ 0 \\ -e^{i\phi} \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i\phi} \\ e^{i\phi} \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i\phi} \\ -e^{i\phi} \\ 0 \end{pmatrix}$$

and the density matrix

$$\rho = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Find $\mu(\mathbf{u}_1)$, $\mu(\mathbf{u}_2)$, $\mu(\mathbf{u}_3)$, $\mu(\mathbf{u}_4)$.

Problem 24. Consider the two 2×2 density matrices

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Is the 4×4 matrix

$$\rho \star \sigma = \frac{1}{2} \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{12} \\ 0 & \sigma_{11} & \sigma_{12} & 0 \\ 0 & \sigma_{21} & \sigma_{22} & 0 \\ \rho_{21} & 0 & 0 & \rho_{22} \end{pmatrix}$$

a density matrix?

Problem 25. Consider the density matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and let σ_1, σ_2, σ_3 be the Pauli spin matrices. Calculate the commutators $[\rho, \sigma_1]$, $[\rho, \sigma_2]$, $[\rho, \sigma_3]$ and discuss.

Problem 26. Consider the density matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find the *Cayley transform*

$$U = (\rho - iI_2)(\rho + iI_2)^{-1}$$

and then the commutator $[\rho, U]$. Discuss

Problem 27. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Find the normalized eigenvectors

$$\mathbf{v}_{11}, \ \mathbf{v}_{12}, \quad \mathbf{v}_{21}, \ \mathbf{v}_{22}, \quad \mathbf{v}_{31}, \ \mathbf{v}_{32}$$

and construct the six density matrices (pure states)

$$\rho_{jk} = \mathbf{v}_{jk}\mathbf{v}_{jk}^*$$

where $j = 1, 2, 3$ and $k = 1, 2$. Calculate commutators $[\rho_{jk}, \rho_{j'k'}]$ and anti-commutators $[\rho_{jk}, \rho_{j'k'}]_+$ and compare to the commutators $[\sigma_j, \sigma_k]$ and anti-commutators $[\sigma_j, \sigma_k]_+$.

Problem 28. Does the density matrix

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

represent a pure or mixed state?

Problem 29. Consider the Hamilton operator acting in the Hilbert space $\mathbb{C}^4$

$$\hat{H} = \hbar\omega_1(\sigma_3 \otimes I_2 + I_2 \otimes \sigma_3) + \hbar\omega_2(\sigma_1 \otimes \sigma_1)$$

where $\omega_1, \omega_2 > 0$.
(i) Find the (real) eigenvalues (the matrix $\hat{H}$ is hermitian) E_0, E_1, E_2, E_3 with the ordering $E_0 \leq E_1 \leq E_2 \leq E_3$.
(ii) Find the corresponding normalized eigenvectors $|E_0\rangle$, $|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$. Are the eigenvectors separable?
(iii) Calculate the partition function $Z(\beta)$ ($\beta = 1/(k_B T)$) defined by

$$Z(\beta) := \sum_{j=0}^{3} \exp(-\beta E_j).$$

(iv) We define

$$p_j(\beta) := \frac{e^{-\beta E_j}}{Z(\beta)}, \qquad j = 0, 1, 2, 3.$$

Calculate the density matrix

$$\rho(\beta) = \sum_{j=0}^{3} p_j(\beta)|E_j\rangle\langle E_j|.$$

Do we have a mixed or pure state? Study the cases $\rho(\infty)$ and $\rho(0)$.

Problem 30. Let $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$. Consider the hermitian matrix

$$\rho(\epsilon_1, \epsilon_2, \epsilon_3) = \frac{1}{4} \begin{pmatrix} 1+\epsilon_3 & 0 & 0 & \epsilon_1+\epsilon_2 \\ 0 & 1-\epsilon_3 & \epsilon_1-\epsilon_2 & 0 \\ 0 & \epsilon_1-\epsilon_2 & 1-\epsilon_3 & 0 \\ \epsilon_1+\epsilon_2 & 0 & 0 & 1+\epsilon_3 \end{pmatrix}.$$

What is the condition such that $\rho(\epsilon_1, \epsilon_2, \epsilon_3)$ is a density matrix? For the eigenvalues of the matrix $\rho(\epsilon_1, \epsilon_2, \epsilon_3)$ we find

$$\lambda_1 = \frac{1}{4}(1+\epsilon_1+\epsilon_2+\epsilon_3), \quad \lambda_2 = \frac{1}{4}(1+\epsilon_1-\epsilon_2-\epsilon_3),$$

$$\lambda_3 = \frac{1}{4}(1-\epsilon_1+\epsilon_2-\epsilon_3), \quad \lambda_4 = \frac{1}{4}(1-\epsilon_1-\epsilon_2+\epsilon_3).$$

Problem 31. Consider the Hilbert space $\mathbb{C}^n$. Let ρ be a density matrix in this Hilbert space and H and K be two hermitian $n \times n$ matrices. One defines

$$\langle H \rangle := \mathrm{tr}(\rho H), \quad \langle H^2 \rangle := \mathrm{tr}(\rho H^2)$$

and analogously for K. Let

$$\Delta H := \sqrt{\langle H^2 \rangle - \langle H \rangle^2}, \qquad \Delta K := \sqrt{\langle K^2 \rangle - \langle K \rangle^2}.$$

Then we have the uncertainty relation

$$(\Delta H)(\Delta K) \geq \frac{1}{2} \left| \langle i[H, K] \rangle \right|.$$

Let

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that the uncertainty relation becomes an equality for the given ρ, H and K.

Problem 32. Let I_d be the $d \times d$ identity matrix. Consider the matrix

$$\rho = \frac{1}{d}(I_d + K)$$

where K is a hermitian $d \times d$ matrix with all diagonal entries equal to 0. What is the condition on such a K such that ρ is a density matrix?

Problem 33. Let $\alpha \in [0,1]$. Show that

$$\rho(\alpha) = \frac{1}{4}\begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1+\alpha & -2\alpha & 0 \\ 0 & -2\alpha & 1+\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix}$$

is a density matrix (so-called *Werner state*). Find the eigenvalues and eigenvectors of ρ.

Problem 34. Are the matrices

$$\rho(\theta) = \frac{1}{2}\begin{pmatrix} 2\sin^2\theta & 0 & 0 & 0 \\ 0 & \cos^2\theta & \cos^2\theta & 0 \\ 0 & \cos^2\theta & \cos^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\rho(\theta) = \frac{1}{2}\begin{pmatrix} 2\cos^2\theta & 0 & 0 & 0 \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

density matrices? Prove or disprove. If so, do we have a mixed or pure state?

Problem 35. Is the matrix

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \equiv \frac{1}{4}(I_2 \otimes I_2 + \sigma_1 \otimes \sigma_1)$$

a density matrix?

Problem 36. Can the density matrix

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

be written as a Kronecker product of two 2×2 density matrices?

Problem 37. Let ρ be a density matrix given as an $n \times n$ matrix and U be an $n \times n$ unitary matrix. Then $U\rho U^{-1}$ is again a density matrix. Let

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the density matrix $U\rho U^{-1}$.

Problem 38. Let

$$\rho_1^{\mp} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \mp 1 & 0 \\ 0 & \mp 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_2^{\mp} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \mp 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mp 1 & 0 & 0 & 1 \end{pmatrix}$$

be the four density matrices for the Bell states.
(i) Let $t \in [0,1]$. Is the convex combination

$$\rho = t\rho_1 + (1-t)\rho_2$$

a density matrix?
(ii) The Hilbert-Schmidt distance $d(\rho_1, \rho_2)$ is given by

$$d(\rho_1, \rho_2) := \sqrt{\operatorname{tr}((\rho_1 - \rho_2)^2)}.$$

Find $d(\rho_1, \rho_2)$ for the given density matrices.

Problem 39. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Find the conditions on the real coefficients r_j, u_j, t_{jk} $(j, k = 1, 2, 3)$ such that

$$\rho = \frac{1}{4}(I_2 \otimes I_2 + \sum_{j=1}^{3} r_j \sigma_j \otimes I_2 + \sum_{j=1}^{3} u_j I_2 \otimes \sigma_j + \sum_{j=1}^{3}\sum_{k=1}^{3} t_{jk}\sigma_j \otimes \sigma_k)$$

is a density matrix. Note that since $\operatorname{tr}(\sigma_j) = 0$ for $j = 1, 2, 3$ we have $\operatorname{tr}(\rho) = 1$.

Problem 40. The variance of an observable A and a density operator ρ in a Hilbert space $\mathcal{H}$ is defined as

$$V(\rho, A) := \operatorname{tr}(\rho A^2) - (\operatorname{tr}(\rho A))^2.$$

Let $|\psi\rangle$ be a normalized state in the Hilbert space $\mathcal{H}$. Show that if $\rho = |\psi\rangle\langle\psi|$ (pure state) we obtain

$$V(|\psi\rangle\langle\psi|, A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2.$$

Problem 41. (i) Consider the spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which are hermitian and traceless. Let I_3 be the 3×3 unit matrix. Let

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

be a vector in $\mathbb{R}^3$ with $\|\mathbf{v}\| \leq 1$. Is the matrix

$$\rho = \frac{1}{3}(I_3 + \sum_{j=1}^{3} v_j S_j)$$

a density matrix. Obviously this matrix is hemitian and has trace 1, but are all the eigenvalues are non-zero?
(ii) Is the matrix

$$\rho = \frac{1}{9}(I_3 \otimes I_3 + \sum_{j=1}^{3} v_j S_j \otimes S_j)$$

a density matrix?

Problem 42. (i) Consider the three 3×3 matrices

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_2 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \rho_3 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which of these matrices are density matrices?
(ii) For the matrices which represent density matrices found out whether it represents of pure state or mixed state. If it is pure state find the state $|\psi\rangle$ in the Hilbert space $\mathbb{C}^3$ such that $\rho = |\psi\rangle\langle\psi|$.

Problem 43. Consider a mixture of 25% of the pure state $(1,0)^T$, 25% of the pure state $(0,1)^T$ and 50% of the pure state $\frac{1}{\sqrt{2}}(1,1)^T$ described by the density matrix

$$\rho = \frac{1}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \quad 0) + \frac{1}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \quad 1) + \frac{1}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} (1 \quad 1).$$

Find the spectral representation of ρ. Use the spectral representation of ρ to find another mixture of pure states with the same (measurement) statistical properties as ρ.

Problem 44. Consider the state

$$|\psi\rangle = \begin{pmatrix} \cos(\theta) \\ e^{i\phi}\sin(\theta) \end{pmatrix}$$

in the Hilbert space $\mathbb{C}^2$, where $\phi, \theta \in \mathbb{R}$. Let $\rho(t=0) = \rho(0) = |\psi\rangle\langle\psi|$ be a density matrix at time $t = 0$. Given the Hamilton operator $\hat{H} = \hbar\omega\sigma_1$. Solve the von Neumann equation to find $\rho(t)$.

Problem 45. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $\mathcal{H}_1 \otimes \mathcal{H}_2$ be the product Hilbert space. Let ρ be a density operators of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Show that if one of the reduced density operators $\mathrm{tr}_{\mathcal{H}_2}(\rho) = \rho_1$ or $\mathrm{tr}_{\mathcal{H}_1}(\rho) = \rho_2$ is pure, then $\rho = \rho_1 \otimes \rho_2$. If both ρ_1 and ρ_2 are pure, then ρ is pure too.

Problem 46. Let $\alpha \in [0,1]$ and $\phi \in \mathbb{R}$. Is

$$\rho(\alpha, \phi) = \frac{1}{2}\begin{pmatrix} \alpha & 0 & e^{-i\phi} \\ 0 & 2-2\alpha & 0 \\ e^{i\phi} & 0 & \alpha \end{pmatrix}$$

a density matrix?

Problem 47. Let $|\phi_j\rangle$ $(j = 1, \ldots, d)$ be an orthonormal basis in the Hilbert space $\mathbb{C}^d$. Is

$$\rho = \frac{1}{d}\sum_{j,k=1}^{d} |\phi_j\rangle\langle\phi_k|$$

a density matrix.

Problem 48. (i) Consider the density matrix (pure state)

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\, 1 \quad 0 \,).$$

Apply the Cayley transform to find the corresponding unitary matrix. Discuss.

(ii) Consider the density matrix (pure state)

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}} (\, 1 \quad -i \,).$$

Apply the Cayley transform to find the corresponding unitary matrix. Discuss.

(iii) Consider the $n \times n$ density matrix (pure state)

$$\rho = \frac{1}{n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Apply the Cayley transform to find the corresponding unitary matrix. Discuss.
(iv) Consider the mixed state

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Apply the Cayley transform to find the corresponding unitary matrix. Discuss.

Problem 49. Let $|n\rangle$ $(n = 0, 1, \ldots, N)$ be the standard basis in $\mathbb{C}^{N+1}$. Consider the states

$$|\theta, \phi\rangle = \sum_{n=0}^{N} \binom{N}{n}^{1/2} (\cos(\theta/2))^{N-n} (\sin(\theta/2))^n e^{-in\phi} |n\rangle.$$

Consider the density matrix

$$\rho(t) = \sum_{n=0}^{N} \sum_{m=0}^{N} C_m^*(t) C_n(t) |n\rangle\langle m|.$$

Show that

$$\begin{aligned} Q(\theta, \phi, t) &:= \frac{N+1}{4\pi} \langle \theta, \phi | \rho(t) | \theta, \phi \rangle \\ &= \frac{N+1}{4\pi} \sum_{m=0}^{N} \sum_{n=0}^{N} \binom{N}{m}^{1/2} \binom{N}{n}^{1/2} C_m^*(t) C_n(t) \\ &\quad \times (\cos(\theta/2))^{2N-m-n} (\sin(\theta/2))^{m+n} e^{-i(m-n)\phi}. \end{aligned}$$

Problem 50. A quantum system is described by the density matrix ρ a positive semi-definite operator with $\operatorname{tr}(\rho) = 1$. The observable is described by self-adjoint operators A and their expectation values are given by $\operatorname{tr}(A\rho)$. Consider the Hilbert space $\mathbb{C}^2$, the density matrices

$$\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

and the hermitian 2×2 matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find

$$\mathrm{tr}(\rho_1 \sigma_2), \quad \mathrm{tr}(\rho_2 \sigma_2), \quad \mathrm{tr}((\rho_1 \otimes \rho_2)(\sigma_2 \otimes \sigma_2)).$$

Problem 51. Let $t \in [0,1]$. Let ρ_1, ρ_2 be two density matrices.
(i) Is the convex combination

$$\rho = t\rho_1 + (1-t)\rho_2$$

a density matrix.
(ii) If so apply it to the density matrices which are related to the Bell states

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

(iii) The Hilbert-Schmidt distance $d(\rho_1, \rho_2)$ is given by

$$d(\rho_1, \rho_2) = \sqrt{\mathrm{tr}((\rho_1 - \rho_2)^2)}.$$

Find the distance for the two density matrices given in (ii).

Problem 52. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and I_2 the 2×2 identity matrix.
(i) Show that the four matrices

$$\rho_1 = \frac{1}{2}(I_2 + \sigma_3), \quad \rho_2 = \frac{1}{2}(I_2 - \sigma_3), \quad \rho_3 = \frac{1}{2}(I_2 + \sigma_1), \quad \rho_4 = \frac{1}{2}(I_2 + \sigma_2)$$

are density matrices.
(ii) Show that the four matrices ρ_1, ρ_2, ρ_3, ρ_4 form a basis in the Hilbert space $M_2(\mathbb{C})$ with the scalar product $\langle A, B \rangle := \mathrm{tr}(AB^*)$.

Problem 53. Consider the Hilbert space $\mathbb{C}^2$ and the projection matrices

$$\Pi_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Pi_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Find $\Pi_1 \Pi_2$ and $\Pi_1 + \Pi_2$. Let

$$\rho(\theta) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix}.$$

Find

$$\mathrm{tr}(\rho\Pi_1), \qquad \mathrm{tr}(\rho\Pi_2).$$

Problem 54. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and I_2 be the 2×2 identity matrix.
(i) Show that the four 2×2 matrices

$$\rho_1 = \frac{1}{2}(I_2 + \sigma_3), \quad \rho_2 = \frac{1}{2}(I_2 - \sigma_3), \quad \rho_3 = \frac{1}{2}(I_2 + \sigma_1), \quad \rho_4 = \frac{1}{2}(I_2 + \sigma_2)$$

are density matrices in the Hilbert space $\mathbb{C}^2$.
(ii) Show that the four matrices form a basis in the Hilbert space $M_2(\mathbb{C})$ with the scalar product $\langle A, B\rangle := \mathrm{tr}(AB^*)$.
(iii) Are the matrices $\rho_1 \otimes \rho_1$, $\rho_2 \otimes \rho_2$, $\rho_3 \otimes \rho_3$ density matrices in the Hilbert space $M_2(\mathbb{C})$.

Problem 55. Let A, B be hermitian $n \times n$ matrices and $\langle A\rangle = \mathrm{tr}(A\rho)$ with ρ an $n \times n$ density matrix or $\langle A\rangle = \langle\psi|A|\psi\rangle$ with $|\psi\rangle$ a normalized state in $\mathbb{C}^n$. Let

$$\sigma_A^2 := \langle A^2\rangle - \langle A\rangle^2, \qquad \sigma_B^2 := \langle B^2\rangle - \langle B\rangle^2.$$

Then

$$\sigma_A^2\sigma_B^2 \geq \left|\frac{1}{2}\langle [A,B]_+\rangle - \langle A\rangle\langle B\rangle\right|^2 + \left|\frac{1}{2}\langle [A,B]\rangle\right|^2.$$

(i) Let $n = 2$ and $A = \sigma_1$, $B = \sigma_2$ and

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find the left and right hand side of the inequality.
(ii) Let $n = 2$ and $A = \sigma_1$, σ_2 and

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the left and right hand side of the inequality.

Problem 56. Let

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Calculate

$$\rho = (|\psi_0\rangle\langle\psi_0|) \otimes (|\psi_1\rangle\langle\psi_1|).$$

Is ρ a density matrix?

Problem 57. Let A, B be positive semidefinite $n \times n$ matrices. Then

$$\det(A+B) \geq \det(A) + \det(B), \qquad \mathrm{tr}(AB) \leq \mathrm{tr}(A)\mathrm{tr}(B).$$

Let

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both matrices are density matrices, with A a pure state and B a mixed state. Calculate the left- and right-hand side of the two inequality. Discuss.

Problem 58. Let $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ be an orthonormal basis in $\mathbb{C}^n$ and μ_1, μ_2, …, μ_n be nonnegative numbers such that $\sum_{j=1} \mu_j = 1$. Is

$$\rho = \sum_{j=1}^{n} \mu_j \mathbf{v}_j \mathbf{v}_1^*$$

a density matrix? If so would it cover pure and mixed states?

Problem 59. Let

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

be a density matrix. Is

$$\rho = \begin{pmatrix} \rho_{11} & 0 & \rho_{12} \\ 0 & 0 & 0 \\ \rho_{21} & 0 & \rho_{22} \end{pmatrix}$$

a density matrix?

Problem 60. Let $\mathbf{v}_1$, …, $\mathbf{v}_n$ (column vectors) be an orthonormal basis in $\mathbb{C}^n$ and let λ_1, …, λ_n be nonnegative real numbers with

$$\sum_{j=1}^{n} \lambda_j = 1.$$

Is

$$\rho = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \mathbf{v}_j^*$$

a density matrix?

Problem 61. Consider the Hilbert space $\mathbb{C}^4$ and the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Let $\gamma \in [-1, 1]$. Consider the density matrix

$$\rho = \frac{1}{2}(1+\gamma)|\psi_1\rangle\langle\psi_2|.$$

Let

$$M = \begin{pmatrix} 0 & e^{-i\theta_1} \\ e^{i\theta_1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{-i\theta_2} \\ e^{i\theta_2} & 0 \end{pmatrix}.$$

Problem 62. Consider the Hilbert space $\mathbb{C}^n$. Let ρ be a density matrix. Then the diagonal part σ of ρ is also a density matrix. Let f be a convex function on the interval $[0, 1]$. Then (*Klein inequality*)

$$\mathrm{tr}(f(\rho)) \geq \mathrm{tr}(f(\sigma)).$$

Let

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad f(x) = x^2.$$

Calculate the right-hand side and left-hand side of the inequality.

Problem 63. Let ρ_1, ρ_2 be density matrices. Then one has (*Klein inequality*)

$$\mathrm{tr}(f(\rho_1) - f(\rho_2) - (\rho_1 - \rho_2)f'(\rho_2)) \geq 0$$

where $f : (0, \infty)$ is a convex function. Consider

$$\rho_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \rho_2 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and $f(x) = x^2$. Calculate the left-hand side of the inequality.

Problem 64. Show that

$$\rho = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

is a density matrix (pure state). Find the normalized vector $\mathbf{v}$ in $\mathbb{C}^3$ such that

$$\rho = \mathbf{v}\mathbf{v}^*.$$

Problem 65. Consider the density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and let A be an 2×2 real symmetric matrix. Assume that

$$\mathrm{tr}(\rho A) = -1, \qquad \mathrm{tr}(\rho A^2) = 1.$$

Reconstruct the matrix from this information.

Problem 66. Let $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ be a probability vector, i.e. $p_j \geq 0$ and $\sum_{j=1}^N p_j = 1$. Let U be a unitary $N \times N$ matrix. Then

$$\rho(U, \mathbf{p}) = U \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & p_N \end{pmatrix} U^*$$

is a density matrix. Find the density matrix for the case $N = 2$ and with $\mathbf{p} = (1/4, 3/4)$ and

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 67. Let $x_1, x_2, x_3, x_4 \in [-1, 1]$. Is

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + x_3 & x_1 - ix_2 \\ x_1 - ix_2 & 1 - x_3 \end{pmatrix}$$

a density matrix?

Chapter 5

Partial Trace

Problem 1. Consider the finite-dimensional Hilbert spaces $\mathcal{H}_1 = \mathbb{C}^{n_1}$ and $\mathcal{H}_2 = \mathbb{C}^{n_2}$. Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be the product Hilbert space. Let $|\psi\rangle$ and $|\phi\rangle$ be states in the product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Show that if

$$\mathrm{tr}_{\mathcal{H}_2}(|\psi\rangle\langle\psi|) = \mathrm{tr}_{\mathcal{H}_2}(|\phi\rangle\langle\phi|)$$

then there exists a unitary matrix U acting in the Hilbert space $\mathcal{H}_2$ such that

$$|\psi\rangle = (I_{n_1} \otimes U)|\phi\rangle$$

where I_{n_1} is the identity matrix in the Hilbert space $\mathcal{H}_1$.

Problem 2. Consider the GHZ-state in the Hilbert space $\mathbb{C}^8$ ($\mathbb{C}^8 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$)

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right).$$

Then the density matrix is given by the 8×8 matrix

$$\rho = |GHZ\rangle\langle GHZ| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Calculate the partial trace $\rho_{AB} = \mathrm{tr}_C(\rho)$ with the basis

$$I_4 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad I_4 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(ii) Calculate the partial trace $\rho_A = \mathrm{tr}_B(\rho_{AB})$ with the basis

$$I_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad I_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem 3. Consider the product Hilbert space $\ell_2(\mathbb{N}_0) \otimes \mathbb{C}^{2s+1}$, where $s = 1/2, 1, 3/2, 2, \ldots$ is the spin. Find the partial trace over $\mathbb{C}^{2s+1}$.

Chapter 6

Reversible Logic Gates

Problem 1. Find the truth table for the boolean function

$$f(a, a', b, b') = (a \cdot b') \oplus (a' \cdot b).$$

Problem 2. The *Feynman gate* is a 2 input/2 output gate given by

$$x_1' = x_1$$
$$x_2' = x_1 \oplus x_2$$

(i) Give the truth table for the Feynman gate.
(ii) Show that copying can be implemented using the Feynman gate.
(iii) Show that the complement can be implemented using the Feynman gate.
(iv) Is the Feynman gate invertible?

Problem 3. Consider the 3-input/3-output gate given by

$$x_1' = x_1$$
$$x_2' = x_1 \oplus x_2$$
$$x_3' = x_1 \oplus x_2 \oplus x_3.$$

(i) Give the truth table.
(ii) Is the transformation invertible.

Problem 4. Consider the 3-input/3-output gate given by

$$x_1' = x_1$$
$$x_2' = x_1 \oplus x_2$$
$$x_3' = x_3 \oplus (x_1 \cdot x_2).$$

(i) Give the truth table.
(ii) Is the gate invertible?

Problem 5. Consider the 3-input/3-output gate given by

$$x_1' = x_1 \oplus x_3$$
$$x_2' = x_1 \oplus x_2$$
$$x_3' = (x_1 \cdot x_2) \oplus (x_1 \cdot x_3) \oplus (x_2 \cdot x_3).$$

(i) Give the truth table.
(ii) Is the gate invertible?

Problem 6. Consider the *Toffoli gate*

$$T : \{0,1\}^3 \to \{0,1\}^3, \qquad T(a,b,c) := (a, b, (a \cdot b) \oplus c)$$

where $\bar{a}$ is the NOT operation, $+$ is the OR operation, $\cdot$ is the AND operation and $\oplus$ is the XOR operation.

1. Express $NOT(a)$ exclusively in terms of the TOFFOLI gate.
2. Express $AND(a,b)$ exclusively in terms of the TOFFOLI gate.
3. Express $OR(a,b)$ exclusively in terms of the TOFFOLI gate.
4. Show that the TOFFOLI gate is invertible.

Thus the TOFFOLI gate is universal and reversible (invertible).

Problem 7. Consider the *Fredkin gate*

$$F : \{0,1\}^3 \to \{0,1\}^3, \qquad F(a,b,c) := (a, a \cdot b + \bar{a} \cdot c, a \cdot c + \bar{a} \cdot b)$$

where $\bar{a}$ is the NOT operation, $+$ is the OR operation, $\cdot$ is the AND operation and $\oplus$ is the XOR operation.

1. Express $NOT(a)$ exclusively in terms of the FREDKIN gate.
2. Express $AND(a,b)$ exclusively in terms of the FREDKIN gate.

3. Express $OR(a,b)$ exclusively in terms of the FREDKIN gate.
4. Show that the FREDKIN gate is invertible.

Thus the FREDKIN gate is universal and reversible (invertible).

Problem 8. The *Toffoli gate* $\mathrm{T}(x_1, x_2; x_3)$ has 3 inputs (x_1, x_2, x_3) and three outputs (y_1, y_2, y_3) and is given by

$$(x_1, x_2, x_3) \to (x_1, x_2, x_3 \oplus (x_1 \cdot x_2))$$

where $x_1, x_2, x_3 \in \{0, 1\}$, $\oplus$ is the XOR-operation and $\cdot$ the AND-operation. Give the truth table.

Problem 9. A *generalized Toffoli gate* $\mathrm{T}(x_1, x_2, \ldots, x_n; x_{n+1})$ is a gate that maps a boolean pattern $(x_1, x_2, \ldots, x_n, x_{n+1})$ to

$$(x_1, x_2, \ldots, x_n, x_{n+1} \oplus (x_1 \cdot x_2 \cdot \ldots \cdot x_n))$$

where $\oplus$ is the XOR-operation and $\cdot$ the AND-operation. Show that the generalized Toffoli gate includes the NOT-gate, CNOT-gate and the original Toffoli gate.

Problem 10. The *Fredkin gate* $\mathrm{F}(x_1; x_2, x_3)$ has 3 inputs (x_1, x_2, x_3) and three outputs (y_1, y_2, y_3). It maps boolean patterns

$$(x_1, x_2, x_3) \to (x_1, x_3, x_2)$$

if and only if $x_1 = 1$, otherwise it passes the boolean pattern unchanged. Give the truth table.

Problem 11. The *generalized Fredkin gate* $\mathrm{F}(x_1, x_2, \ldots, x_n; x_{n+1}, x_{n+2})$ is a gate is the mapping of the boolean pattern

$$(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}) \to (x_1, x_2, \ldots, x_n, x_{n+2}, x_{n+1})$$

if and only if the boolean product $x_1 \cdot x_2 \cdot \ldots \cdot x_n = 1$ ($\cdot$ is the bitwise AND operation), otherwise the boolean pattern passes unchanged. Let $n = 2$ and $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$. Find the output.

Problem 12. Is the gate $(a, b, c \in \{0, 1\})$

$$(a, b, c) \to (a, a \cdot b \oplus c, \overline{a} \cdot \overline{c} \oplus \overline{b})$$

reversible?

Problem 13. Prove that the *Fredkin gate* is universal. A set of gates is called universal if we can build any logic circuits using these gates assuming bit setting gates are given.

Problem 14. The *half-adder* is given by

$$S = A \oplus B$$
$$C = A \cdot B.$$

Construct a half-adder using two Toffoli gates.

Problem 15. Consider the 3-input/3-output gate given by

$$x_1' = x_1 \oplus x_3$$
$$x_2' = x_1 \oplus x_2$$
$$x_3' = (x_1 + x_2) \oplus (x_1 + x_3) \oplus (x_2 + x_3).$$

(i) Give the truth table.
(ii) Is the gate invertible?

Problem 16. Consider the 4-input/4-output gate given by

$$x_1' = x_1$$
$$x_2' = x_2$$
$$x_3' = x_3$$
$$x_4' = x_4 \oplus x_1 \oplus x_2 \oplus x_3.$$

(i) Give the truth table.
(ii) Is the gate invertible?

Problem 17. Consider the 4-input/4-output gate given by

$$x_1' = x_1 \oplus x_3$$
$$x_2' = x_2 \oplus x_3 \oplus (x_1 \cdot x_2) \oplus (x_2 \cdot x_3)$$
$$x_3' = x_1 \oplus x_2 \oplus x_3$$
$$x_4' = x_4 \oplus x_3 \oplus (x_1 \cdot x_2) \oplus (x_2 \cdot x_3)\,.$$

(i) Give the truth table.
(ii) Is the gate invertible?

Problem 18. Show that one Fredkin gate

$$(a, b, c) \to (a,\ \overline{a} \cdot b + a \cdot c,\ \overline{a} \cdot c + a \cdot b)$$

is sufficient to implement the XOR gate. Assume that either $\bar{b}$ or $\bar{c}$ are available.

Problem 19. Show that the map $\mathbf{f} : \{0,1\}^3 \to \{0,1\}^3$

```
abc    xyz
000 -> 000
100 -> 100
010 -> 101
110 -> 011
001 -> 001
101 -> 010
011 -> 110
111 -> 111
```

is invertible. The map describes a reversible half-adder. If $c = 0$, then x is the first digit of the sum $a + b$ and y is the carry bit. If $c = 1$, then z is the first digit of the sum $a + b + c$ and y is the carry bit.

Problem 20. Show that the Toffoli gate which maps

$$|a\rangle \otimes |b\rangle \otimes |c\rangle \mapsto |a\rangle \otimes |b\rangle \otimes |c \oplus (a \cdot b)\rangle$$

can simulate the FANOUT and the NAND gate.

Problem 21. (i) Let $x_1, x_2 \in \{0,1\}$. Let $\oplus$ be the XOR operation. Show that

$$(x_1, x_2) \mapsto (x_1 \oplus 1, x_1 \oplus x_2)$$

is a 2-bit reversible gate.
(ii) Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the 4×4 permutation matrix P such that

$$P(|x_1\rangle \otimes |x_2\rangle) = |x_1 \oplus 1\rangle \otimes |x_1 \oplus x_2\rangle .$$

(iii) Show that

$$(x_1, x_2) \mapsto (x_1 \oplus x_2, x_2 \oplus 1)$$

is a 2-bit reversible gate.
(iv) Find the 4×4 permutation matrix P such that

$$P(|x_1\rangle \otimes |x_2\rangle) = |x_1 \oplus x_2\rangle \otimes |x_2 \oplus 1\rangle.$$

(v) Given a 4×4 permutation matrix (as a quantum gate). How can one construct a corresponding 2-bit reversible gate? Apply it to the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 22. The NOT, AND and OR gate form a universal set of operations (gates) for boolean algebra. The NAND operation is also universal for boolean algebra. However these sets of operations are not reversible sets of operations. Consider the Toffoli and Fredkin gates

$$TOFFOLI : \{0,1\}^3 \to \{0,1\}^3, \quad TOFFOLI(a,b,c) = (a,b,(a \cdot b) \oplus c)$$

$$FREDKIN : \{0,1\}^3 \to \{0,1\}^3, \quad FREDKIN(a,b,c) = (a, a{\cdot}c{+}\bar{a}{\cdot}b, a{\cdot}b{+}\bar{a}{\cdot}c)$$

where $\bar{a}$ is the NOT operation, $+$ is the OR operation, $\cdot$ is the AND operation and $\oplus$ is the XOR operations.

1. Express NOT(a) exclusively in terms of the TOFFOLI gate.
2. Express NOT(a) exclusively in terms of the FREDKIN gate.
3. Express AND(a,b) exclusively in terms of the TOFFOLI gate.
4. Express AND(a,b) exclusively in terms of the FREDKIN gate.
5. Express OR(a,b) exclusively in terms of the TOFFOLI gate.
6. Express OR(a,b) exclusively in terms of the FREDKIN gate.
7. Show that the TOFFOLI gate is reversible.
8. Show that the FREDKIN gate is reversible.

Thus the TOFFPLI and FREDKIN gates are eachuniversal and reversible (invertible).

Chapter 7

Unitary Transformations and Quantum Gates

Problem 1. Consider the compact Lie group $SU(4)$. Let $U \in SU(4)$. Then the 4×4 matrix U can be factorized as follows

$$U = (V_1 \otimes V_2) \exp\left(\frac{i}{2}\sum_{j=1}^{3} \theta_j \sigma_j \otimes \sigma_j\right)(V_3 \otimes V_4)$$

where $V_1, V_2, V_3, V_4 \in SU(2)$ and $\theta_j \in \mathbb{R}$. Let

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Show that $S \in SU(4)$. Find the factorization given above for S.

Hint. Since $[\sigma_j \otimes \sigma_j, \sigma_k \otimes \sigma_k] = 0_4$ we can write

$$\exp\left(\frac{i}{2}\sum_{j=1}^{3} \theta_j \sigma_j \otimes \sigma_j\right) \equiv \exp\left(\frac{i\theta_1}{2}\sigma_1 \otimes \sigma_1\right)\exp\left(\frac{i\theta_2}{2}\sigma_2 \otimes \sigma_2\right)\exp\left(\frac{i\theta_3}{2}\sigma_3 \otimes \sigma_3\right).$$

Problem 2. Consider the *Bell matrix*

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Show that B is invertible and find B^{-1}. Is B unitary?
(ii) Express B^2 using the Pauli spin matrices, an overall phase and the Kronecker product.
(iii) Find a 4×4 matrix A such that $B = \exp(iA)$.
(iv) Can one find a positive integer n such that $B^n = I_4$?
(v) Show that

$$B = \frac{1}{\sqrt{2}}(I_4 + B^2).$$

Problem 3. Consider the state $|\psi\rangle$ in the Hilbert space $\mathbb{C}^9$

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

Is the state invariant under $U \otimes U$, where U is the 3×3 unitary matrix

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 4. (i) The *Schrödinger equation* is given by

$$i\hbar \frac{d\psi}{dt} = H\psi(t) \tag{1}$$

with $\psi(0)$ the initial value. The evolution of $\psi(t)$ is determined by

$$\psi(t) = U(t)\psi(0) \tag{2}$$

where $U(t)$ is a unitary evolution operator and $U(0) = I$. Show that

$$i\hbar \frac{dU(t)}{dt}\psi(0) = HU(t)\psi(0). \tag{3}$$

(ii) Assume that $H = \hbar\omega\sigma_3$. Find $U(t)$.

Problem 5. Let I_2 be the 2×2 identitity matrix and σ_1 be the Pauli spin matrix and $|0\rangle$, $|1\rangle$ be the standard basis. The CNOT-gate can be represented as

$$U_{CNOT} = |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes \sigma_1.$$

Is U_{CNOT} hermitian? Is $U_{CNOT}^2 = I_4$?

Problem 6. Let U be an $n \times n$ unitary matrix. Show that if the bipartite states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ satisfy

$$|\phi\rangle = (U \otimes I_m)|\psi\rangle$$

then the ranks of the corresponding reduced density matrices satisfy

$$r(\rho_1^\psi) \geq r(\rho_1^\phi), \qquad r(\rho_2^\psi) \geq r(\rho_2^\phi).$$

Problem 7. Consider the unitary matrices

$$V_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes I_2, \qquad V_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_2$$

$$V_C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_2 + \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2$$

where U_2 is an arbitrary 2×2 unitary matrix and $\chi \in \mathbb{R}$. Consider the 4×4 density matrix

$$\rho_{in} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \rho_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \rho_2$$

where ρ_2 is an arbitrary 2×2 density matrix. Find the density matrix

$$\rho_{out} = V_H V_M V_C V_H \rho_{in} V_H^* V_C^* V_M^* V_H^*.$$

Problem 8. The n-qubit *Pauli group* is defined by

$$\mathcal{P}_n := \{\, I_2,\, \sigma_1,\, \sigma_2,\, \sigma_3 \,\}^{\otimes n} \otimes \{\, \pm 1,\, \pm i \,\}$$

where σ_1, σ_2, σ_3 are the 2×2 Pauli matrices and I_2 is the 2×2 identity matrix. The dimension of the Hilbert space under consideration is $\dim \mathcal{H} = 2^n$. Thus each element of the Pauli group $\mathcal{P}_n$ is (up to an overall phase ± 1, $\pm i$) a Kronecker product of Pauli matrices and 2×2 identity matrices acting on n qubits.

The n-qubit *Clifford group* $\mathcal{C}_n$ is the normalizer of the Pauli group. A $2^n \times 2^n$ unitary matrix U acting on n qubits is an element of the Clifford group iff

$$UMU^* \in \mathcal{P}_n \quad \text{for each} \quad M \in \mathcal{P}_n.$$

This means the unitary matrix U acting by conjugation takes a Kronecker product of Pauli matrices to Kronecker product of Pauli matrices. An element of the Clifford group is defined as this action by conjugation, so that the overall phase of the unitary matrix U is not relevant. In other words the Clifford group is the group of all matrices that leave the Pauli group invariant.
(i) What is order of the n-qubit Pauli group?
(ii) Show that the single-qubit Hadamard gate

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the single-qubit phase gate

$$U_P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

are elements of the Clifford group $\mathcal{C}_1$.
(iii) Show that the CNOT-gate

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is an element of $\mathcal{C}_2$.
(iv) Is the Fredkin gate an element of $\mathcal{C}_3$?

Problem 9. Find the 4×4 matrix

$$U = e^{-i\pi(\sigma_1 \otimes I_2)/4} e^{-i\pi(\sigma_3 \otimes \sigma_3)/4} e^{-i\pi(\sigma_2 \otimes I_2)/4}.$$

Is the matrix U unitary?

Problem 10. Consider the four unary gates (2×2 unitary matrices)

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

and the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Calculate the state $NHVW|\psi\rangle$ and the expection value $\langle\psi|NHVW|\psi\rangle$.

Problem 11. (i) Let U be an $n \times n$ unitary matrix. Then the eigenvalues take the form $e^{i\phi}$, where $\phi \in \mathbb{R}$. Let $e^{i\phi_1}, \ldots, e^{i\phi_n}$ be the eigenvalues of U with the corresponding normalized eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ which form an orthonormal basis in $\mathbb{C}^n$. Then one has (spectral decomposition)

$$U = \sum_{j=1}^{n} e^{i\phi_j}\mathbf{u}_j\mathbf{u}_j^*.$$

Then the unitary matrix

$$V = \sum_{j=1}^{n} e^{i\phi_j/2}\mathbf{u}_j\mathbf{u}_j^*$$

satisfies $V^2 = U$ and can be viewed as the square root of U. Show that $[U, V] = 0_n$.

(ii) Let U_1, U_2 be two unitary matrices with the spectral representation

$$U_1 = \sum_{j=1}^{n} e^{i\phi_{1j}}\mathbf{u}_{1j}\mathbf{u}_{1j}^*$$

$$U_2 = \sum_{j=1}^{n} e^{i\phi_{2j}}\mathbf{u}_{2j}\mathbf{u}_{2j}^*$$

where $e^{i\phi_{1j}}$, $e^{i\phi_{2j}}$ $(j = 1, \ldots, n)$ are the eigenvalues of U_1 and U_2, respectively and $\mathbf{u}_{1j}$, $\mathbf{u}_{2j}$ $(j = 1, \ldots, n)$ are the corresponding normalized eigenvectors of U_1 and U_2, respectively. Let the unitary matrices

$$V_1 = \sum_{j=1}^{n} e^{i\phi_{1j}/2}\mathbf{u}_{1j}\mathbf{u}_{1j}^*$$

$$V_2 = \sum_{j=1}^{n} e^{i\phi_{2j}/2}\mathbf{u}_{2j}\mathbf{u}_{2j}^*$$

be the square roots of U_1 and U_2, respectively. Find the commutators $[U_1, U_2]$ and $[V_1, V_2]$.

(iii) Study the question from (ii) under the condition that the bases $\mathbf{u}_{1j}$ and $\mathbf{u}_{2j}$ $(j = 1, \ldots, n)$ are mutually unbiased bases, i.e.

$$|\langle\mathbf{u}_{1j}|\mathbf{u}_{2k}\rangle|^2 = \frac{1}{n}, \quad j, k = 1, \ldots, n.$$

Problem 12. (i) Consider the Hadamard gate

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)$$

with the eigenvalues $+1$ and -1. Find a square root of the Hadamard gate.
(ii) The star product of the Hadamard gate with itself provides the Bell matrix

$$B = U_H \star U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Use the result from (i) to find a square root of the Bell matrix. Note that the eigenvalues of the Bell matrix are $+1$ (twice) and -1 (twice).

Problem 13. In the Hilbert space $\mathbb{C}^4$ the Bell states

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

(i) Let $\omega = e^{2\pi i/4}$. Apply the Fourier transformation

$$U_F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix}$$

to the Bell states and study the entanglement of these states.
(ii) Apply the Haar wavelet transformation

$$U_H = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

to the Bell states and study the entanglement of these states.
(iii) Apply the Walsh-Hadamard transformation

$$U_W = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

to the Bell states and study the entanglement of these states.

Extend to the Hilbert space $\mathbb{C}^{2^n}$ with the first Bell state given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & \cdots & 0 & 1\end{pmatrix}^T$$

Problem 14. Consider the *Bell matrix* B and the normalized vector $\mathbf{v}$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 1 & 0 & 0 & -1\end{pmatrix}, \qquad \mathbf{v} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

Is the normalized vector $B\mathbf{v}$ entangled?

Problem 15. Find a 4×4 unitary matrix U such that

$$U\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad U\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},$$

$$U\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \quad U\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}.$$

Problem 16. Find a 4×4 matrix U such that

$$U\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}.$$

Problem 17. Apply the quantum Fourier transform to the state

$$\frac{1}{2}\sum_{j=0}^{7}\cos(2\pi j/8)|j\rangle$$

where the quantum Fourier transform is given by

$$U_{QFT} = \frac{1}{2\sqrt{2}}\sum_{j,k=0}^{7} e^{-i2\pi kj/8}|k\rangle\langle j|.$$

Is the operator U_{QFT} unitary? Prove or disprove. Remember that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\sum_{k=0}^{N-1} e^{i2\pi k(n-m)/N} = N\delta_{nm}.$$

We use

$$\{|j\rangle,\ j = 0, 1, \ldots, 7\}$$

as an orthonormal basis in $\mathbf{C}^8$.

Problem 18. Write the Bell matrix

$$U_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

as a linear combination of Kronecker products of Pauli spin matrices.

Problem 19. Let $\alpha, \beta, \gamma \in \mathbb{R}$. Show that any $U \in SU(2)$ can be written as

$$U = \exp(i\gamma\sigma_3)\exp(i\beta\sigma_1)\exp(i\alpha\sigma_3).$$

Problem 20. (i) Let A, B be $n \times n$ matrices over $\mathbb{R}$. Show that one can find a $2n \times 2n$ unitary matrix U such that

$$U \begin{pmatrix} A & B \\ -B & A \end{pmatrix} U^* = \begin{pmatrix} A + iB & 0_n \\ 0_n & A - iB \end{pmatrix}.$$

Here 0_n denotes the $n \times n$ zero matrix.
(ii) Use the result from (i) to show that

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(A + iB)\overline{\det(A + iB)} \geq 0.$$

Problem 21. Let $\mathbf{u}$ be a column vector in $\mathbb{C}^n$ with $\mathbf{u}^*\mathbf{u} = 1$, i.e. the vector is normalized. Consider the matrix

$$U = I_n - 2\mathbf{u}\mathbf{u}^*.$$

(i) Show that U is hermitian.
(ii) Show that U is unitary.

Problem 22. Can one find a 2×2 unitary matrix such that

$$U \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Problem 23. (i) Do the 2×2 unitary matrices

$$A = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & ie^{-i\pi/4} \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

satisfy the *braid-like relation*

$$ABA = BAB.$$

(ii) Find the smallest $n \in \mathbb{N}$ such that $A^n = I_2$.
(iii) Find the smallest $m \in \mathbb{N}$ such that $B^m = I_2$.

Problem 24. Find all $(n+1) \times (n+1)$ matrices A such that

$$A^* U A = U$$

where U is the unitary matrix

$$U = \begin{pmatrix} 0 & 0 & i \\ 0 & I_{n-1} & 0 \\ -i & 0 & 0 \end{pmatrix}$$

and $\det(A) = 1$. Consider first the case $n = 2$.

Problem 25. Consider the 2×2 hermitian matrices A and B with $A \neq B$ with the eigenvalues λ_1, λ_2; μ_1, μ_2; and the corresponding normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$; $\mathbf{v}_1$, $\mathbf{v}_2$, respectively. Form from the normalized eigenvectors the 2×2 matrix

$$\begin{pmatrix} \mathbf{u}_1^* \mathbf{v}_1 & \mathbf{u}_1^* \mathbf{v}_2 \\ \mathbf{u}_2^* \mathbf{v}_1 & \mathbf{u}_2^* \mathbf{v}_2 \end{pmatrix}.$$

Is this matrix unitary? Find the eigenvalues of this matrix and the corresponding normalized eigenvectors of the 2×2 matrix. How are the eigenvalues and eigenvectors are linked to the eigenvalues and eigenvectors of A and B?

Problem 26. Let I_n be the $n \times n$ unit matrix. Is the $2n \times 2n$ matrix

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix}$$

unitary?

Problem 27. Consider the two 2×2 unitary matrices

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Can one find a unitary 2×2 matrix V such that

$$U_1 = VU_2V^* \,?$$

Problem 28. Let U be an $n \times n$ unitary matrix.
(i) Is $U + U^*$ invertible?
(ii) Is $U + U^*$ hermitian?
(iii) Calculate $\exp(\epsilon(U + U^*))$, where $\epsilon \in \mathbb{R}$

Problem 29. Let U be an $n \times n$ unitary matrix. Then $U + U^*$ is a hermitian matrix. Can any hermitian matrix represented in this form?

Problem 30. (i) Find the condition on the $n \times n$ matrix A over $\mathbb{C}$ such that $I_n + A$ is a unitary matrix.
(ii) Let B be an 2×2 matrix over $\mathbb{C}$. Find all solutions of the equation

$$B + B^* + BB^* = 0_2.$$

Problem 31. Find all 2×2 invertible matrices A such that

$$A + A^{-1} = I_2.$$

Problem 32. Let $z_1, z_2, w_1, w_2 \in \mathbb{C}$. Consider the 2×2 matrices

$$U = \begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & w_1 \\ w_2 & 0 \end{pmatrix}$$

where $z_1\bar{z}_1 = 1$, $z_2\bar{z}_2 = 1$, $w_1\bar{w}_1 = 1$, $w_2\bar{w}_2 = 1$. This means the matrices U, V are unitary. Find the condition on z_1, z_2, w_1, w_2 such that the commutator $[U, V]$ is again a unitary matrix.

Problem 33. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Find the conditions on $\alpha_1, \alpha_2, \alpha_3$ such that

$$U = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

is a unitary matrix.

Problem 34. Consider the unitary matrix

$$U = \frac{1}{\sqrt{2}}(I_2 \otimes I_2 + i\sigma_1 \otimes \sigma_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate

$$U(\sigma_1 \otimes I_2)U^{-1}, \quad U(\sigma_2 \otimes I_2)U^{-1}, \quad U(\sigma_3 \otimes I_2)U^{-1},$$

$$U(\sigma_1 \otimes \sigma_1)U^{-1}, \quad U(\sigma_2 \otimes \sigma_2)U^{-1}, \quad U(\sigma_3 \otimes \sigma_3)U^{-1}.$$

Problem 35. (i) What are the conditions on $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$ such that

$$U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_{11}} & e^{i\phi_{12}} \\ e^{i\phi_{21}} & e^{i\phi_{22}} \end{pmatrix}$$

is a unitary matrix?
(ii) What are the condition on $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$ such that $U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22})$ is an element of $SU(2)$?

Chapter 8

Entropy

Problem 1. An $n \times n$ density matrix ρ is a positive semidefinite matrix such that $\mathrm{tr}(\rho) = 1$. The nonnegative eigenvalues of ρ are the probabilities of the physical states described by the corresponding eigenvectors. The entropy of the statistical state described by the density matrix ρ is defined by

$$S(\rho) := -\mathrm{tr}(\rho \ln(\rho))$$

with the convention $0 \ln(0) = 0$. For the $n \times n$ hermitian matrix H (energy operator) the statistical average of the energy E is defined by

$$E := \mathrm{tr}(H\rho).$$

Let

$$\psi(\rho) := \mathrm{tr}(H\rho) - \mathrm{tr}(\rho \ln(\rho)).$$

(i) Show that

$$\ln \mathrm{tr}(e^H) = \max \{ \mathrm{tr}(H\rho) + S(\rho) \}.$$

(ii) Show that

$$-S(\rho) = \max \{ \mathrm{tr}(H\rho) - \ln \mathrm{tr}(e^H) \}.$$

Problem 2. The von Neumann entropy, the standard measure of randomness of a statistical ensemble described by a $n \times n$ density matrix ρ, is defined by

$$S(\rho) = -\mathrm{tr}(\rho \log(\rho)) = -\sum_{j=1}^{n} \lambda_j \log(\lambda_j)$$

where λ_j $(j = 1, 2, \ldots, n)$ are the eigenvalues of the density matrix ρ and the log is taken to base n, the dimension of the Hilbert space $\mathbb{C}^n$. Consider the density matrix in $\mathbb{C}^4$

$$\rho = \begin{pmatrix} 1/3 & 0 & 0 & 1/6 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 1/6 & 0 \\ 1/6 & 0 & 0 & 1/3 \end{pmatrix}.$$

Find the eigenvalues of ρ and then the von Neumann entropy $S(\rho)$.

Problem 3. Consider the normalized states $|\psi_k\rangle$, $k = 0, 1, \ldots, N-1$ in the Hilbert space $\mathbb{C}^N$. A positive operator valued measure is specified by a decomposition of the identity matrix I_N into M positive semidefinite matrices P_m, i.e.

$$I_N = \sum_{m=0}^{M-1} P_m.$$

The mutual information is defined by

$$I = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} p_{nm} \log_N \left(\frac{p_{nm}}{p_{n\cdot} p_{\cdot m}} \right)$$

where

$$p_{nm} := \langle \psi_n | P_m | \psi_n \rangle$$

are the joint probabilities and

$$p_{n\cdot} := \sum_{m=0}^{M-1} p_{nm}, \qquad p_{\cdot m} := \sum_{n=0}^{N-1} p_{nm}$$

are their marginals. Let $M = N = 2$ and

$$P_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad P_1 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad |\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find p_{nm}, $p_{n\cdot}$, $p_{\cdot m}$ and then I.

Problem 4. Let A, B be $n \times n$ hermitian matrices acting in the Hilbert space $\mathbb{C}^n$. Assume that the eigenvalues of A are pairwise different and analogously for B. Then the normalized eigenvectors $|\alpha_j\rangle$ $(j = 1, \ldots, n)$ of A form an orthonormal basis in $\mathbb{C}^n$ and analogously for B the normalized eigenvectors $|\beta_j\rangle$ $(j = 1, \ldots, n)$ form an orthonormal basis in $\mathbb{C}^n$. Let $|\psi\rangle$ be a normalized state in $\mathbb{C}^n$. Then there are n possible outcomes for measurements of each observable and the probabilties $p_j(A, |\psi\rangle)$, $p_j(B, |\psi\rangle)$ $(j = 1, \ldots, n)$ are given by

$$p_j(A, |\psi\rangle) := |\langle\psi|\alpha_j\rangle|^2, \qquad p_j(B, |\psi\rangle) := |\langle\psi|\beta_j\rangle|^2.$$

Let $H_{|\psi\rangle}(X)$ be the *Shannon information entropy*

$$H_{|\psi\rangle}(X) := -\sum_{j=1}^{n} p_j(X, |\psi\rangle) \ln(p_j(X, |\psi\rangle))$$

corresponding to the probability distribution $\{p_j(X, |\psi\rangle)\}$ $(j = 1, \ldots, n)$. The (Maassen-Uffink) *entropic uncertainty relation* is given by

$$H_{|\psi\rangle}(A) + H_{|\psi\rangle}(B) \geq -2\ln(\max_{1\leq j,k\leq n} |\langle\alpha_j|\beta_k\rangle|) > 0.$$

Note that the right-hand side does not involve the state $|\psi\rangle$.
(i) Let

$$A = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad |\psi\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Calculate the left and right-hand side of the entropic uncertainty relation. Is the entropic uncertainty relation tight for this case?
(ii) The (Landau-Pollak) uncertainty relation states that

$$\arccos(\sqrt{P_A}) + \arccos(\sqrt{P_B}) \geq \arccos(\max_{1\leq j,k\leq n} |\langle\alpha_j|\beta_k\rangle|)$$

where

$$P_A := \max_{1\leq j\leq n} p_j(A, |\psi\rangle), \qquad P_B := \max_{1\leq j\leq n} p_j(B, |\psi\rangle).$$

Calculate the left-hand and right-hand side of this uncertainty relation for A and B given in (i).

Problem 5. Consider the Hilbert space $\mathbb{C}^n$. Let A, B be two hermitian $n \times n$ matrices (observable). Assume that A and B have non-degenerate eigenvalues with the corresponding normalized eigenvectors $|a_1\rangle$, $|a_2\rangle$, $\ldots$, $|a_n\rangle$ and $|b_1\rangle$, $|b_2\rangle$, $\ldots$, $|b_n\rangle$, respectively. The entropic uncertainty relation is an inequality given by

$$S^{(A)} + S^{(B)} \geq S^{(AB)}$$

where

$$S^{(A)} = -\sum_{j=1}^{n} |\langle\psi|a_j\rangle|^2 \ln(|\langle\psi|a_j\rangle|^2), \quad S^{(B)} = -\sum_{j=1}^{n} |\langle\psi|b_j\rangle|^2 \ln(|\langle\psi|b_j\rangle|^2),$$

and $S^{(AB)}$ is a positive constant which gives the lower bound of the right-hand side of the inequality. Consider the Hilbert space $\mathbb{C}^2$. Let

$$A = \sigma_1, \quad B = \sigma_2, \quad |\psi\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Find $S^{(A)}$, $S^{(B)}$ and $S^{(A)} + S^{(B)}$.

Problem 6. Consider the Hilbert space $\mathbb{C}^n$ and $|\psi\rangle \in \mathbb{C}^n$. Let A and B $n \times n$ hermitian matrices (observable) with non-degenerate eigenvalues and corresponding normalized eigenvectors $|u_j\rangle$, $|v_j\rangle$ ($j = 1, \ldots, n$). The entropic uncertainty relation is an inequality of the form

$$S^{(A)} + S^{(B)} \geq S_{AB}$$

where

$$S^{(A)} = -\sum_{j=1}^{n} |\langle\psi|u_j\rangle|^2 \ln(|\langle\psi|u_j\rangle|^2), \quad S^{(A)} = -\sum_{j=1}^{n} |\langle\psi|v_j\rangle|^2 \ln(|\langle\psi|v_j\rangle|^2)$$

and S_{AB} is a positive constant providing the lower bound of the right-hand side of the inequality. Let

$$A = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$|\psi\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Calculate $S^{(A)}$ and $S^{(B)}$.

Chapter 9

Measurement

Problem 1. Consider the tripartite states

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \qquad |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

Find the probability

$$p = |\langle W|GHZ\rangle|^2.$$

Problem 2. Consider the W-state

$$|W\rangle = \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle).$$

Apply the invertible local operator

$$\hat{L} = \begin{pmatrix} \sqrt{a} & \sqrt{d} \\ 0 & \sqrt{c} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3b}/\sqrt{a} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

to the W-state, where $a, b, c > 0$ and $d = 1 - (a + b + c) \geq 0$. Calculate the probability $|\langle W|LW\rangle|^2$.

Problem 3. Consider the single qubit state

$$|\psi\rangle := a|0\rangle + b|1\rangle, \qquad |a|^2 + |b|^2 = 1.$$

Rewrite the first two qubits of the state

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

in terms of the Bell basis

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Describe how to obtain $|\psi\rangle$ as the state of the last qubit by measuring the first two qubits in the Bell basis. Suppose that the only errors which can occur to three qubits are described by the transforms

$$\{I \otimes I \otimes I, I \otimes U_{NOT} \otimes U_{NOT}, I \otimes U_P \otimes U_P, I \otimes (U_P U_{NOT}) \otimes (U_P U_{NOT})\}.$$

Describe how an arbitrary error

$$\alpha I \otimes I \otimes I + \beta I \otimes U_{NOT} \otimes U_{NOT} + \delta I \otimes U_P \otimes U_P + \gamma I \otimes (U_P U_{NOT}) \otimes (U_P U_{NOT})$$

on the state

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes |\psi\rangle$$

can be corrected to obtain the correct $|\psi\rangle$ as the last qubit.

Problem 4. Consider the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right).$$

and the 2×2 unitary matrices

$$U_A = \begin{pmatrix}\cos(\pi/8) & -\sin(\pi/8)\\ \sin(\pi/8) & \cos(\pi/8)\end{pmatrix}, \qquad U_B = U_A^{-1} = \begin{pmatrix}\cos(\pi/8) & \sin(\pi/8)\\ -\sin(\pi/8) & \cos(\pi/8)\end{pmatrix},$$

Note that $\cos(\pi/8) = \frac{1}{2}(\sqrt{2+\sqrt{2}}$. $\sin(\pi/8) = \frac{1}{2}(\sqrt{2+\sqrt{2}}$. Let I_2 be the 2×2 identity matrix. Find

$$\langle\psi|(U_A \otimes I_2)|\psi\rangle, \quad \langle\psi|(I_2 \otimes U_B)|\psi\rangle, \quad \langle\psi|(U_A \otimes U_B)|\psi\rangle$$

and the probabilities

$$|\langle\psi|(U_A \otimes I_2)|\psi\rangle|^2, \quad |\langle\psi|(I_2 \otimes U_B)|\psi\rangle|^2, \quad |\langle\psi|(U_A \otimes U_B)|\psi\rangle|^2.$$

These probabilities play a role for the CHSH game.

Problem 5. Consider the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.$$

Alice has the first qubit and Bob has the second qubit. Let I_2 be the 2×2 identity matrix and U_H be the Hadamard matrix

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}.$$

If Alice receives the bit $a=0$ and Bob receives the bit $b=0$, then Alice applies I_2 to her qubit and Bob applies I_2 to his quibit, i.e. $I_2\otimes I_2$ to the Bell states. If Alice receives the bit $a=1$ and Bob receives the bit $b=0$, then Alice applies U_H to her qubit and Bob applies I_2 to his quibit, i.e. $U_H\otimes I_2$ is applied to the Bell states. If Alice receives the bit $a=0$ and Bob receives the bit $b=1$, then Alice applies I_2 to her qubit and Bob applies U_H to his quibit, i.e. $I_2\otimes U_H$ is applied to the Bell states. If Alice receives the bit $a=1$ and Bob receives the bit $b=1$, then Alice applies U_H to her qubit and Bob applies U_H to his quibit, i.e. $U_H\otimes U_H$ is applied to the Bell states. Find the states

$$|\psi_1\rangle = (I_2\otimes I_2)|\psi\rangle,\ |\psi_2\rangle = (U_H\otimes I_2)|\psi\rangle,\ |\psi_3\rangle = (I_2\otimes U_H)|\psi\rangle,\ |\psi_4\rangle = (U_H\otimes U_H)|\psi\rangle$$

and the probabilities $|\langle\psi_j|\psi\rangle|^2$ for $j=1,2,3,4$.

Chapter 10

Entanglement

Problem 1. Consider the singlet state (Bell state)

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Show that the matrices $I_2 \otimes I_2$, $-\sigma_1 \otimes \sigma_1$, $-\sigma_2 \otimes \sigma_2$, $-\sigma_3 \otimes \sigma_3$ leave the state $|\psi\rangle$ invariant.

Problem 2. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Calculate the commutators

$$[\sigma_1 \otimes \sigma_1, \sigma_2 \otimes \sigma_2], \quad [\sigma_1 \otimes \sigma_1, \sigma_3 \otimes \sigma_3], \quad [\sigma_2 \otimes \sigma_2, \sigma_3 \otimes \sigma_3].$$

Problem 3. Let $|0\rangle$, $|1\rangle$ be the standard basis in the Hilbert space $\mathbb{C}^2$. Consider the GHZ-state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0|\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle).$$

Find the expectation values

$$\langle\psi|\sigma_1 \otimes \sigma_2 \otimes \sigma_2|\psi\rangle, \qquad \langle\psi|\sigma_2 \otimes \sigma_1 \otimes \sigma_2|\psi\rangle,$$

$$\langle\psi|\sigma_2 \otimes \sigma_2 \otimes \sigma_1|\psi\rangle, \qquad \langle\psi|\sigma_1 \otimes \sigma_1 \otimes \sigma_1|\psi\rangle.$$

Problem 4. Consider the state in the Hilbert space $\mathbb{C}^4$

$$\begin{pmatrix} n_0(\tau,\phi,\theta) \\ n_1(\tau,\phi,\theta) \\ n_2(\tau,\phi,\theta) \\ n_3(\tau,\phi,\theta) \end{pmatrix} = \begin{pmatrix} \sin((\tau-\phi)/2)\sin(\theta/2) \\ \sin((\tau+\phi)/2)\cos(\theta/2) \\ \cos((\tau-\phi)/2)\sin(\theta/2) \\ \cos((\tau+\phi)/2)\cos(\theta/2) \end{pmatrix}.$$

The state is obviously normalized, i.e. $n_0^2+n_1^2+n_2^2+n_3^2=1$. Find the conditions on ϕ, τ, θ such that $n_0n_3=n_1n_2$ (separability condition). Show that in this case the state can be written as product state.

Problem 5. Let $|0\rangle$, $|1\rangle$ be an arbitrary orthonormal basis. Can the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle + \frac{1}{\sqrt{8}}|0\rangle\otimes|1\rangle + \frac{1}{\sqrt{8}}|1\rangle\otimes|0\rangle + \frac{1}{\sqrt{4}}|1\rangle\otimes|1\rangle$$

be written as a product state?

Problem 6. Consider the Hamilton operator

$$H = \hbar\omega(\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_2+\sigma_3\otimes\sigma_3).$$

(i) Is the 4×4 matrix H hermitian? Find the trace of H. What can be said about the eigenvalues of H.
(ii) Find the eigenvalues and normalized eigenvectors of H.
(iii) Calculate $\exp(-iHt/\hbar)$.

Problem 7. Consider the unitary matrices

$$U_1 = e^{i\pi\sigma_1/4}\otimes e^{i\pi\sigma_1/4}, \qquad U_2 = e^{i\pi\sigma_2/4}\otimes e^{i\pi\sigma_2/4}.$$

Calculate

$$U_1^*(\sigma_3\otimes\sigma_3)U_1, \qquad U_2^*(\sigma_3\otimes\sigma_3)U_2.$$

Problem 8. Consider the state

$$|\psi\rangle = \frac{1}{2}(|0\rangle\otimes|0\rangle + e^{i\phi_1}|0\rangle\otimes|1\rangle + e^{i\phi_2}|1\rangle\otimes|0\rangle + e^{i\phi_3}|1\rangle\otimes|1\rangle).$$

(i) Let $\phi_3=\phi_1+\phi_2$. Is the state $|\psi\rangle$ a product state?
(ii) Let $\phi_3=\phi_1+\phi_2+\pi$. Is the state $|\psi\rangle$ a product state?

Problem 9. There are six different types of quark known as flavor: up, down, charm, strange, top, bottom. Consider the two equations for states

$$\cos\theta\left(\frac{1}{\sqrt{3}}(|u\bar{u}\rangle+|d\bar{d}\rangle+|s\bar{s}\rangle)\right)+\sin\theta\left(\frac{1}{\sqrt{6}}(|u\bar{u}\rangle+|d\bar{d}\rangle-2|s\bar{s}\rangle)\right) = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle+|d\bar{d}\rangle)$$

$$\cos\theta\left(\frac{1}{\sqrt{6}}(|u\overline{u}\rangle + |d\overline{d}\rangle - 2|s\overline{s}\rangle)\right) - \sin\theta\left(\frac{1}{\sqrt{3}}(|u\overline{u}\rangle + |d\overline{d}\rangle + |s\overline{s}\rangle)\right) = -|s\overline{s}\rangle$$

where $|u\overline{u}\rangle \equiv |u\rangle \otimes |\overline{u}\rangle$ etc. Find $\cos\theta$ and $\sin\theta$ from this two equations.

Problem 10. Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consider the normalized state (*Aharonov state*)

$$|\psi\rangle = \frac{1}{\sqrt{6}}(|012\rangle - |021\rangle + |120\rangle - |102\rangle + |201\rangle - |210\rangle)$$

where $|012\rangle = |0\rangle \otimes |1\rangle \otimes |2\rangle$ etc and

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Is $|\psi\rangle$ an eigenstate of $S_z \otimes S_z \otimes S_z$?

Problem 11. Consider the 4×4 matrix

$$H = \frac{1}{2}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3 + I_2 \otimes I_2).$$

(i) Is H hermitian? Find the trace of H.
(ii) Calculate H^2 and $\mathrm{tr}(H^2)$.
(iii) Using the result from (ii) calculate $\exp(i\theta H)$, $\exp(-i\pi H/4)$ and $\exp(-i\pi H/2)$.
(iv) Using the results from (i) and (ii) find the eigenvalues of H.
(v) Find the normalized eigenstates of H.

Problem 12. Consider the normalized state

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle)$$

where we used the notation $|0000\rangle \equiv |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$ etc. and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Calculate the states

$$(\sigma_1 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1)|\psi\rangle, \quad (\sigma_1 \otimes \sigma_1 \otimes I_2 \otimes \sigma_3)|\psi\rangle, \quad (I_2 \otimes \sigma_1 \otimes \sigma_1 \otimes I_2)|\psi\rangle$$

and

$$(I_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2)|\psi\rangle, \quad (\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1)|\psi\rangle, \quad (I_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_2)|\psi\rangle.$$

Problem 13. The *hyperdeterminant* of a $2 \times 2 \times 2$ hypermatrix $C = (c_{ijk})$ $(i, j, k \in \{ 0, 1 \})$ is defined by

$$\text{Det}C := -\frac{1}{2} \sum_{i,j,k,m,n,p=0}^{1} \sum_{i',j',k',m',n',p'=0}^{1} \epsilon_{ii'}\epsilon_{jj'}\epsilon_{kk'}\epsilon_{mm'}\epsilon_{nn'}\epsilon_{pp'}c_{ijk}a_{i'j'm}c_{npk'}c_{n'p'm'}$$

where $\epsilon_{00} = \epsilon_{11} = 0$, $\epsilon_{01} = 1$, $\epsilon_{10} = -1$.
(i) Calculate Det$C$.
(ii) Consider the three qubit state

$$|\psi\rangle = \sum_{i,j,k=0}^{1} c_{ijk}|i\rangle \otimes |j\rangle \otimes |k\rangle.$$

The *three tangle* τ_3 is a measure of entanglement and is defined for the three qubit state $|\psi\rangle$ as

$$\tau_{123} := 4|\text{Det}C|$$

where $C = (c_{ijk})$. Find the three tangle for the GHZ-state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)$$

and the W-state

$$|W\rangle = \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle).$$

Problem 14. Calculate the product of the unitary matrices

$$\exp(i\pi(\sigma_2 \otimes I_2)/4) \exp(-i\pi(\sigma_3 \otimes \sigma_3)/4) \exp(-i\pi(\sigma_1 \otimes I_2)/4).$$

Problem 15. Let $|0\rangle$, $|1\rangle$ be an orthonormal basis in $\mathbb{C}^2$. Consider the normalized state

$$|\psi\rangle = \sum_{j,k=0}^{1} c_{jk}|j\rangle \otimes |k\rangle$$

in the Hilbert space $\mathbb{C}^4$ and the 2×2 matrix $C = (c_{jk})$. Using the 4 coefficients c_{jk}, $j, k \in \{0, 1\}$) we form a multilinear polynomial p in two variables x_1, x_2

$$p(x_1, x_2) = c_{00} + c_{01}x_1 + c_{10}x_2 + c_{11}x_1x_2. \tag{1}$$

Show that determinant $\det C = c_{00}c_{11} - c_{01}c_{10}$ is the unique irreducible polynomial (up to sign) of content one in the 4 unkowns c_{jk} that vanishes whenever the system of equations

$$p = \frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = 0 \tag{2}$$

has a solution (x_1^*, x_2^*) in $\mathbb{C}^2$.

Problem 16. Let $|0\rangle$, $|1\rangle$ be an orthonormal basis in $\mathbb{C}^2$. Consider the normalized state

$$|\psi\rangle = \sum_{j,k,\ell=0}^{1} c_{jk\ell}|j\rangle \otimes |k\rangle \otimes |\ell\rangle$$

in the Hilbert space $\mathbb{C}^8$ and the $2 \times 2 \times 2$ array $C = (c_{jk\ell})$ $(j, k, \ell \in \{0, 1\}$. Using the 8 coefficients $c_{jk\ell}$ we form a multilinear polynomial in three variables x_1, x_2, x_3

$$p(x_1, x_2, x_3) = c_{000} + c_{001}x_1 + c_{010}x_2 + c_{100}x_3 + c_{011}x_1x_2 + c_{101}x_1x_3 + c_{110}x_2x_3 + c_{111}x_1x_2x_3. \tag{1}$$

Show that the hyperdeterminant

$$\begin{aligned}
\mathrm{Det}C = & c_{000}^2c_{111}^2 + c_{001}^2c_{110}^2 + c_{010}^2c_{101}^2 + c_{100}^2c_{011}^2 \\
& -2(c_{000}c_{001}c_{110}c_{111} + c_{000}c_{010}c_{101}c_{111} \\
& +c_{000}c_{100}c_{011}c_{111} + c_{001}c_{010}c_{101}c_{110} \\
& +c_{001}c_{100}c_{011}c_{110} + c_{010}c_{100}c_{011}c_{101}) \\
& +4(c_{000}c_{011}c_{101}c_{110} + c_{001}c_{010}c_{100}c_{111})
\end{aligned}$$

is the unique irreducible polynomial (up to sign) of content one in the 8 unkowns $c_{jk\ell}$ that vanishes whenever the system of equations

$$p = \frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} = 0 \tag{2}$$

has a solution (x_1^*, x_2^*, x_3^*) in $\mathbb{C}^3$.

Problem 17. Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|H\rangle \otimes |V\rangle - |V\rangle \otimes |H\rangle).$$

We define a polarization state that is rotated by an angle α from the horizontal axis as

$$|\alpha\rangle = \cos(\alpha)|H\rangle + \sin(\alpha)|V\rangle$$

and analogously

$$|\beta\rangle = \cos(\beta)|H\rangle + \sin(\beta)|V\rangle.$$

Calculate the probability

$$p(\alpha,\beta) = |(\langle\alpha| \otimes \langle\beta|)|\psi\rangle|^2 .$$

Problem 18. Consider the state

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the unitary operator (4×4 matrix)

$$U = e^{-i\pi\sigma_2/4} \otimes I_2.$$

Find the state $U|\psi\rangle$.

Problem 19. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the Hamilton operator

$$\hat{H} = J(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3).$$

(i) Let $\epsilon = J\beta \equiv J/(k_B T)$, where k_B is the Boltzmann constant and T the absolute temperature and $J > 0$. Calculate

$$\rho(\epsilon) = \frac{1}{Z(\epsilon)} \exp(-\beta\hat{H}) \equiv \frac{1}{Z(\epsilon)} \exp(-\epsilon(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3))$$

where $Z(\epsilon)$ is the partition function

$$Z(\epsilon) = \mathrm{tr}\exp(-\epsilon(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)).$$

(ii) The concurrence $C(\rho(\epsilon))$ is defined by

$$C(\rho(\epsilon)) = \max(0, \mu_1(\epsilon) - \mu_2(\epsilon) - \mu_3(\epsilon) - \mu_4(\epsilon))$$

where the μ_j's are the square roots of the eigenvalues of the 4×4 matrix

$$\rho(\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)$$

in decreasing order. Calculate $C(\rho(\epsilon))$ and discuss the result as function of $\epsilon \equiv J\beta$.

Problem 20. Can we find 2×2 matrices S_1 and S_2 such that

$$(S_1 \otimes S_2)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

Problem 21. Let N be an integer larger than 5. Consider the following state

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} |j \bmod N\rangle \otimes |3j \bmod N\rangle \otimes |5j \bmod N\rangle.$$

Let U be the quantum Fourier transform. Calculate $(U \otimes U \otimes U)|\psi\rangle$. Write the answer in the basis $\{\,|0\rangle, |1\rangle, \ldots, |N-1\rangle\}^{\otimes 3}$. Show that it is the superposition of equally probable states. Find the probability.

Problem 22. Consider the Hamilton operator

$$\hat{H} = \hbar\omega\sigma_1 \otimes \sigma_3 \otimes \sigma_1$$

and the corresponding unitary operator

$$U(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega t\sigma_1 \otimes \sigma_3 \otimes \sigma_1}.$$

(i) Calculate $\hat{H}^2$ and $U(t)$.
(ii) Show that $U(t)$ can be written as, i.e. we decompose $U(t)$ into elementary gates of one qubit rotations and two qubits interactions,

$$U(t) = e^{-i\pi I \otimes I \otimes \sigma_2/4} e^{i\pi\sigma_3 \otimes I \otimes \sigma_3/4} e^{i\pi\sigma_1 \otimes I \otimes I/4} e^{-i\omega t\sigma_3 \otimes \sigma_3 \otimes I} e^{-i\pi\sigma_1 \otimes I \otimes I/4} e^{-i\pi\sigma_3 \otimes I \otimes \sigma_3/4} e^{i\pi I \otimes I \otimes \sigma_2/4}$$

where I is the 2×2 unit matrix.

Problem 23. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Find the 4×4 matrix

$$U = e^{-i\pi(\sigma_1 \otimes I_2)/4} e^{-i\pi(\sigma_3 \otimes \sigma_3)/4} e^{-i\pi(\sigma_2 \otimes I_2)/4}.$$

Is the matrix unitary?

Problem 24. Consider the Hamilton operator

$$H = \frac{1}{2}\left(-\hbar\omega_1\sigma_3 \otimes I_2 - \hbar\omega_2 I_2 \otimes \sigma_3 + \hbar\gamma\sigma_3 \otimes \sigma_3\right).$$

Find

$$U = e^{-i\pi(\sigma_1 \otimes I_2)/2} e^{-iHt/\hbar} e^{-i\pi(\sigma_1 \otimes I_2)/2} e^{-iHt/\hbar}.$$

Give an interpretation of the result.

Problem 25. Consider the normalized state

$$|\psi\rangle = \cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle, \qquad 0 < \alpha < \pi/4$$

where α is called the Schmidt angle.
(i) Find the eigenvalues of the density matrix $|\psi\rangle\langle\psi|$.
(ii) Find the partically traced density matrix (we find when we trace over one of the subsystems).
(iii) Show that the partically traced has two unequal and non-zero eigenvalues $\lambda_1 = \cos^2(\alpha)$ and $\lambda_2 = \sin^2(\alpha)$.
(iv) Calculate the von Neumann entropy for the corresponding density matrix. Show that the entropy grows monotonically with the Schmidt angle.

Problem 26. Consider the Hadamard matrix

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Is $U_H \in SU(2)$? Is $iU_H \in SU(2)$?

Problem 27. Consider the finite-dimensional Hilbert space

$$\mathcal{H}_N := \text{span}\{\, |n\rangle \; : \; n = 0, 1, \ldots, N-1\,\}$$

i.e. $\dim(\mathcal{H})_N = N$ and $\langle n|m\rangle = \delta_{nm}$ with $m = 0, 1, \ldots, N-1$. Let

$$|\phi_\ell\rangle := \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\exp(in\phi_\ell)|n\rangle, \qquad \phi_\ell := \phi_0 + 2\pi\frac{\ell}{N}$$

for $\ell \in \mathbb{Z}_N$. We define a self-adjoint *phase operator* as

$$\hat{\phi}_N := \sum_{\ell=0}^{N-1}\phi_\ell|\phi_\ell\rangle\langle\phi_\ell|.$$

Find the matrix elements of the phase operator $\hat{\phi}_N$ in the occupation number basis $|n\rangle$ with $n = 0, 1, \ldots, N-1$.

Problem 28. Calculate the *three-tangle* for the W-state

$$|W\rangle = \frac{1}{\sqrt{3}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right).$$

Problem 29. Summarize the requirements for quantum computation.

Problem 30. Find the eigenvalues and normalized eigenvectors of the Hamilton operator

$$\hat{H} = \hbar\omega(\sigma_3 \otimes \sigma_3) + \Delta(\sigma_1 \otimes \sigma_1).$$

Calculate $\exp(-i\hat{H}t/\hbar)$.

Problem 31. Consider the XX-model described by the Hamilton operator

$$H_{XX} = \sum_{j=1}^{N} (J(\sigma_{x,j}\sigma_{x,j+1}) + B\sigma_{z,j})$$

with the periodic boundary conditions $\sigma_{1,N+1} = \sigma_{1,1}$, $\sigma_{3,N+1} = \sigma_{3,1}$. We have

$$\sigma_{1,j} = I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1 \otimes I_2 \otimes \cdots \otimes I_2$$

where σ_1 is at the j-position with $j = 1, 2, \ldots, N$. Let

$$\Sigma_3 := \sum_{j=1}^{N} \sigma_{3,j}.$$

Calculate the commutator $[H_{XX}, \Sigma_3]$. Discuss.

Problem 32. Consider the Hamilton operator (so-called transverse XY-model in one dimension)

$$\hat{H} = -g\sum_{j=0}^{L-1} \sigma_j^z - \sum_{j=0}^{L-1} \left(\frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y \right)$$

where σ^x, σ^y, σ^z are the Pauli spin matrices, $0 \le \gamma \le 1$, g is a constant and we impose cyclic boundary conditions. This means $\sigma_L^x = \sigma_0^x$, $\sigma_L^y = \sigma_0^y$, $\sigma_L^z = \sigma_0^z$.

(i) Find the commutator $[\hat{H}, \hat{C}]$, where

$$\hat{C} := \prod_{j=0}^{L-1} \sigma_j^z.$$

(ii) Calculate $\hat{C}^2$. Show that $\hat{C}$ and $\hat{C}^2$ form a group under matrix multiplication. Give the character table. What are the eigenvalues of $\hat{C}$? We define

$$\hat{Q} := \frac{1}{2}(I - \hat{C})$$

where I is the unit operator ($2^L \times 2^L$ identity matrix). Calculate the eigenvalues of $\hat{Q}$.
(iii) Let $L = 4$. Calculate the eigenvalues of $\hat{H}$.
(iv) Let $\gamma = 0$. Calculate $[\hat{Z}, \hat{H}]$, where

$$\hat{Z} = \sum_{j=0}^{L-1} \sigma_j^z.$$

Discuss.

Problem 33. Let $|0\rangle$, $|1\rangle$ be an orthonormal basis in $\mathbb{C}^2$. Consider the normalized state

$$|\psi\rangle = \sum_{j,k=0}^{1} c_{jk} |j\rangle \otimes |k\rangle.$$

Using the four coefficients c_{jk} we form the polynomial p in the two variables x_1, x_2

$$p(x_1, x_2) = c_{00} + c_{01}x_1 + c_{10}x_2 + c_{11}x_1x_2.$$

Consider the three equations $p = 0$, $\partial p/\partial x_1 = 0$, $\partial p/\partial x_2 = 0$, i.e.

$$p(x_1, x_2) = c_{00} + c_{01}x_1 + c_{10}x_2 + c_{11}x_1x_2 = 0$$

and

$$\frac{\partial p}{\partial x_1} = c_{01} + c_{11}x_2 = 0$$
$$\frac{\partial p}{\partial x_2} = c_{10} + c_{11}x_1 = 0.$$

Show that this system of three equations with two unkowns x_1, x_2 only admits solutions if

$$\det(C) \equiv c_{00}c_{11} - c_{01}c_{10} = 0$$

where C is the 2×2 matrix

$$C = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}.$$

Problem 34. Consider the finite dimensional Hilbert space

$$\mathcal{H}_N := \text{span}\{\, |n\rangle \,:\, n = 0, 1, \ldots, N-1 \,\}$$

where $\langle n'|n\rangle = \delta_{nn'}$. Thus $\dim\mathcal{H}_N = N$. We define the state

$$|\phi_\ell\rangle := \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\exp(in\phi_\ell)|n\rangle, \qquad \phi_\ell := \phi_0 + 2\pi\frac{\ell}{N}$$

for $\ell \in \mathbb{Z}_N$. We define the linear operator

$$\hat{\phi}_N := \sum_{n=0}^{N-1}\phi_\ell|\phi_\ell\rangle\langle\phi_\ell|.$$

Find the matrix elements of this linear operator in the occupation number basis $|n\rangle$.

Problem 35. We consider the finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{2^n}$ and the normalized state

$$|\psi\rangle = \sum_{j_1,j_2,\dots,j_n=0}^{1} c_{j_1,j_2,\dots,j_n}|j_1\rangle\otimes|j_2\rangle\otimes\cdots\otimes|j_n\rangle$$

in this Hilbert space. Here $|0\rangle$, $|1\rangle$ denotes the standard basis. Let ϵ_{jk} $(j,k=0,1)$ be defined by $\epsilon_{00}=\epsilon_{11}=0$, $\epsilon_{01}=1$, $\epsilon_{10}=-1$. Let n be even or $n=3$. Then an n-tangle can be introduced by

$$\tau_{1\dots n} = 2\left|\sum_{\substack{\alpha_1,\dots,\alpha_n=0\\ \dots\\ \delta_1,\dots,\delta_n=0}}^{1} c_{\alpha_1\dots\alpha_n}c_{\beta_1\dots\beta_n}c_{\gamma_1\dots\gamma_n}c_{\delta_1\dots\delta_n}\right.$$
$$\left.\times\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\cdots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\cdots\epsilon_{\gamma_{n-1}\delta_{n-1}}\epsilon_{\alpha_n\gamma_n}\epsilon_{\beta_n\delta_n}\right|.$$

(i) Consider the case $n=4$ and a state $|\psi\rangle$ with $c_{0000}=1/\sqrt{2}$, $c_{1111}=1/\sqrt{2}$ and all other coefficients are 0. Find τ_{1234}.
(ii) Consider the case $n=4$ and a state $|\psi\rangle$ with $c_{0000}=1/\sqrt{2}$, $c_{1111}=-1/\sqrt{2}$ and all other coefficients are 0. Find τ_{1234}.
(iii) Consider the case $n=4$ and a state $|\psi\rangle$ with $c_{0001}=1/\sqrt{2}$, $c_{1110}=1/\sqrt{2}$ and all other coefficients are 0. Find τ_{1234}.
(iv) Consider the case $n=4$ and a state $|\psi\rangle$ with $c_{0001}=1/\sqrt{2}$, $c_{1110}=-1/\sqrt{2}$ and all other coefficients are 0. Find τ_{1234}.

Problem 36. The n-qubit *Pauli group* is defined by

$$\mathcal{P}_n := \{\, I_2,\, \sigma_1,\, \sigma_2,\, \sigma_3\,\}^{\otimes n}\otimes\{\,\pm 1,\, \pm i\,\}$$

where σ_1, σ_2, σ_3 are the 2×2 Pauli matrices and I_2 is the 2×2 identity matrix. The dimension of the Hilbert space under consideration is $\dim\mathcal{H} =$

2^n. Thus each element of the Pauli group $\mathcal{P}_n$ is (up to an overall phase ± 1, $\pm i$) a Kronecker product of Pauli matrices and 2×2 identity matrices acting on n qubits. What is the order of the n-qubit Pauli group?

Problem 37. Consider the Hamilton operator

$$\hat{H} = \hbar\omega(\sigma_3 \otimes \sigma_3) + \Delta_1 \sigma_1 \otimes \sigma_1 + \Delta_2 \sigma_2 \otimes \sigma_2.$$

(i) Find the eigenvalues. Discuss energy level crossing. Find the normalized eigenvectors.
(ii) Calculate the commutators

$$[\sigma_1 \otimes \sigma_1, \sigma_2 \otimes \sigma_2], \qquad [\sigma_2 \otimes \sigma_2, \sigma_3 \otimes \sigma_3], \qquad [\sigma_3 \otimes \sigma_3, \sigma_2 \otimes \sigma_2]$$

(iii) Use the result from (ii) to calculate $\exp(-i\hat{H}t/\hbar)$.

Problem 38. Let $\sigma_1 = \sigma_1$, $\sigma_2 = \sigma_2$, $\sigma_3 = \sigma_3$ be the Pauli spin matrices. We form the nine 4×4 matrices

$$\Sigma_{jk} := \sigma_j \otimes \sigma_k, \qquad j,k = 1,2,3.$$

Note that $[\Sigma_{jk}, \Sigma_{mn}] = 0$. The *variance* of an hermitian operator $\hat{O}$ and a wave vector $|\phi\rangle$ is defined by

$$V_{\hat{O}}(|\phi\rangle) := \langle\phi|(\hat{O})^2|\phi\rangle - (\langle\phi|\hat{O}|\phi\rangle)^2.$$

The *remoteness* for a given normalized state $|\psi\rangle$ in $\mathbb{C}^4$ is defined by

$$R(|\psi\rangle) = \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\langle\psi|(\Sigma_{jk})^2|\psi\rangle - (\langle\psi|\Sigma_{jk}|\psi\rangle)^2\right).$$

Find the remoteness for the Bell states

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \qquad |\phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \qquad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

Problem 39. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the standard basis in the Hilbert space $\mathbb{C}^3$. Are the states in the Hilbert space $\mathbb{C}^{27}$ are entangled

$$\frac{1}{\sqrt{6}}(\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_2\otimes\mathbf{e}_3\otimes\mathbf{e}_1+\mathbf{e}_3\otimes\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_1\otimes\mathbf{e}_3\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_2\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_3)$$

$$\frac{1}{\sqrt{6}}(\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_2\otimes\mathbf{e}_3\otimes\mathbf{e}_1+\mathbf{e}_3\otimes\mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_1\otimes\mathbf{e}_3\otimes\mathbf{e}_2-\mathbf{e}_3\otimes\mathbf{e}_2\otimes\mathbf{e}_1-\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_3)$$

$$\frac{1}{\sqrt{6}}((\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_3)+\varepsilon(\mathbf{e}_2\otimes\mathbf{e}_3\otimes\mathbf{e}_1+\mathbf{e}_1\otimes\mathbf{e}_3\otimes\mathbf{e}_2)+\varepsilon^*(\mathbf{e}_3\otimes\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_2\otimes\mathbf{e}_1).$$

Problem 40. Find the entanglement (three tangle) as a function of θ of the normalized state in $\mathbb{C}^8$

$$|\psi\rangle = \cos(\theta)\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 - i\sin(\theta)\mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2$$

where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and $0 < \theta < \pi/4$.

Problem 41. Given the eigenvalue equations $A\mathbf{x} = \lambda\mathbf{x}$, $A\mathbf{y} = \lambda\mathbf{y}$ and $\mathbf{x}^*\mathbf{y} = 0$. Then $A(\mathbf{x}+\mathbf{y}) = \lambda(\mathbf{x}+\mathbf{y})$. Thus $\mathbf{x}+\mathbf{y}$ is also an eigenvector with eigenvalue λ. Consider the 4×4 matrix

$$\sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are $+1$ (twice) and -1 (twice). The normalized eigenvectors for $+1$ are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

These two states are orthonormal to each other and obviously not entangled. The normalized eigenvectors for the eigenvalue -1 are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

These two states are orthonormal to each other and obviously not entangled. All four vectors form an orthonomal basis in $\mathbb{C}^4$. Find linear combinations of the two cases so that the eigenvectors are entangled and still form an orthonormal basis in $\mathbb{C}^4$.

Problem 42. Are the states in $\mathbb{C}^4$

$$|\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} \pm \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} \pm \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}$$

entangled?

Problem 43. (i) Consider the two states in $\mathbb{C}^4$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad |\phi\rangle = \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\otimes\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix} \equiv \begin{pmatrix}\cos(\alpha)\cos(\beta)\\ \cos(\alpha)\sin(\beta)\\ \sin(\alpha)cos(\beta)\\ \sin(\alpha)\sin(\beta)\end{pmatrix}.$$

One defines

$$G(|\psi\rangle) = \max_{\alpha,\beta} |\langle\phi|\psi\rangle|$$

as the maximum overlap between $|\psi\rangle$ and the product state $|\phi\rangle$. Find $G(|\psi\rangle)$.
(ii) Given the state

$$|\chi\rangle = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

Find $G(|\chi\rangle)$ with the product state given at (i). Discuss.

Problem 44. Consider a bipartite system and the product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $|\psi\rangle \in \mathcal{H}$ and normalized. Then a density matrix (pure state)

$$\rho_{12} := |\psi\rangle\langle\psi|$$

is entangled when the density matrices

$$\rho_j = \mathrm{tr}_k(\rho_{12}), \qquad j,k = 1,2, \;\; j \neq k$$

provided by partial tracing as non-zero von Neumann entropy, i.e.

$$S(\rho_j) = -\mathrm{tr}(\rho_j \log(\rho_j)) \neq 0, \quad j = 1,2.$$

There is no entanglement if $S(\rho_j) = 0$. Consider the Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^3$ and $\mathcal{H} = \mathbb{C}^9$. Is the normalized state in $\mathbb{C}^9$

$$|\psi\rangle = \frac{1}{\sqrt{3}}(1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1)^T$$

entangled?

Problem 45. An entanglement measure is the relative entropy of entanglement. It is defined for a density matrix σ as

$$E_R(\sigma) := \min_{\rho \in \mathcal{D}} S(\sigma \| \rho)$$

where $\mathcal{D}$ is the set of density matrices with positive partial transpose (PPT states) and

$$S(\sigma\|\rho) := \mathrm{tr}(\sigma \log_2(\sigma) - \sigma \log_2(\rho)).$$

Find $S(\sigma\|\rho)$ for the density matrix (one of the Werner states)

$$\sigma = \frac{1}{6}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Problem 46. Let

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that the normalized state

$$\frac{1}{2}(|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle+|1\rangle\otimes|0\rangle\otimes|0\rangle\otimes|1\rangle+|0\rangle\otimes|0\rangle\otimes|1\rangle\otimes|0\rangle+|1\rangle\otimes|0\rangle\otimes|1\rangle\otimes|1\rangle)$$

in the Hilbert space $\mathbb{C}^{16}$ is three-separable and thus biseparable.

Problem 47. Are the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \cos(\pi/4) \\ \cos(\pi/2) \\ \cos(3\pi/4) \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ \sin(\pi/4) \\ \sin(\pi/2) \\ \sin(3\pi/4) \end{pmatrix}$$

entangled?

Problem 48. Can the normalized vector in $\mathbb{C}^{16}$

$$\frac{1}{2}(|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle+|0\rangle\otimes|1\rangle\otimes|0\rangle\otimes|1\rangle+|1\rangle\otimes|0\rangle\otimes|1\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle\otimes|1\rangle\otimes|1\rangle)$$

be written as Kronecker product of lower dimensional vectors?

Problem 49. Let $\mathcal{H}$ be the finite dimensional Hilbert space $\mathbb{C}^d$. Let I_d be the $d \times d$ identity matrix and A an arbitrary $d \times d$ matrix over $\mathbb{C}$. We call a vector $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ maximally entangled, if it normalized, and its reduced density matrix is maximally mixed, i.e., a multiple of I_d

$$\langle \Psi(A \otimes I_d)|\Psi\rangle = \frac{1}{d}\mathrm{tr}(A).$$

(i) Let $d = 2$. Consider the normalized state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Calculate $\langle \Psi|(A \otimes I_2)|\Psi\rangle$ and $\frac{1}{d}\mathrm{tr}(A)$.
(ii) Let $d = 2$. Consider the normalized state

$$|\Psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Calculate $\langle \Psi|(A \otimes I_2)|\Psi\rangle$ and $\frac{1}{d}\mathrm{tr}(A)$.

Problem 50. (i) Is the state in $\mathbb{C}^4$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

entangled?
(ii) Is the state in $\mathbb{C}^4$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \\ -i & \\ 0 & i \end{pmatrix}$$

entangled?

Problem 51. We set

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Can the state in $\mathbb{C}^{16}$

$$\frac{1}{2}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$$

be written as the Kronecker product of 2×8, 8×2, 4×4 normalized vectors?

Problem 52. Consider the three spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Apply the vec-operator to these matrices and then normalize them. Can the vectors be written as Kronecker products of vectors in $\mathbb{C}^3$?

Chapter 11

Bell Inequality

Problem 1. Consider four observers: Alice (A), Bob (B), Charlie (C) and Dora (D) each having one of the qubits. Every observer is allowed to choose between two dichotomic observables. Denote the outcome of observer X's measurement by X_i ($X = A, B, C, D$) with $i = 1, 2$. Under the assumption of local realism, each outcome can either take the value $+1$ or -1. The correlations between the measurement outcomes of all four observers can be represented by the product $A_iB_jC_kD_l$, where $i, j, k, l = 1, 2$. In a local realistic theory, the correlation function of the measurement performed by all four observers is the average of $A_iB_jC_kD_l$ over many runs of the experiment

$$Q(A_iB_jC_kD_l) := \langle\psi|A_iB_jC_kD_l|\psi\rangle$$

The Mermin-Ardehali-Belinskii-Klyshko inequality is given by

$$\begin{aligned}
&Q(A_1B_1C_1D_1) - Q(A_1B_1C_1D_2) - Q(A_1B_1C_2D_1) - Q(A_1B_2C_1D_1) \\
&-Q(A_2B_1C_1D_1) - Q(A_1B_1C_2D_2) - Q(A_1B_2C_1D_2) - Q(A_2B_1C_1D_2) \\
&-Q(A_1B_2C_2D_1) - Q(A_2B_1C_2D_1) - Q(A_2B_2C_1D_1) + Q(A_2B_2C_2D_2) \\
&+Q(A_2B_2C_2D_1) + Q(A_2B_2C_1D_2) + Q(A_2B_1C_2D_2) + Q(A_1B_2C_2D_2) \le 4\,.
\end{aligned}$$

Each observer X measures the spin of each qubit by projecting it either along $\mathbf{n}_1^X$ or $\mathbf{n}_2^X$. Every observer can independently choose between two arbitrary directions. For a four qubit state $|\psi\rangle$, the correlation functions are thus given by

$$Q(A_iB_jC_kD_l) = \langle\psi|(\mathbf{n}_i^A \cdot \boldsymbol{\sigma}) \otimes (\mathbf{n}_j^B \cdot \boldsymbol{\sigma}) \otimes (\mathbf{n}_k^C \cdot \boldsymbol{\sigma}) \otimes (\mathbf{n}_l^D \cdot \boldsymbol{\sigma})|\psi\rangle\,.$$

where $\cdot$ denotes the scalar product, i.e. $\mathbf{n}_j^X \cdot \boldsymbol{\sigma} := n_{j1}^X \sigma_1 + n_{j2}^X \sigma_2 + n_{j3}^X \sigma_3$. Let

$$\mathbf{n}_1^A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{n}_2^A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{n}_1^B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{n}_2^B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{n}_1^C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{n}_2^C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{n}_1^D = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{n}_2^D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that the Mermin-Ardehali-Belinskii-Klyshko inequality is violated for the state

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle)$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

and $|0000\rangle \equiv |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$ etc..

Chapter 12

Quantum Channels

We consider the Hilbert space $\mathcal{H}$ of $n \times n$ matrices over $\mathbb{C}$ with the scalar product (Frobenius inner product)

$$\langle A, B \rangle := \mathrm{tr}(AB^*)$$

with $A, B \in \mathcal{H}$. A state is described using $n \times n$ density matrices ρ, i.e. $\mathrm{tr}(\rho) = 1$ and $\rho \geq 0$ (positive semidefinite). The space of trace-class operators acting in this Hilbert space is denoted by $S(\mathcal{H})$. A quantum channel from a Hilbert space $\mathcal{H}_A$ to a Hilbert space $\mathcal{H}_B$ is represented by a completely positive trace-preserving map $\Phi : S(\mathcal{H}_A) \to S(\mathcal{H}_B)$. Such a positive trace-preserving map can be represented in Stinespring representation, Kraus operator representation and Choi-Jamiolkowski representation.

Let H_n denote the vector space of $n \times n$ Hermitian matrices over the real numbers. We say that $\rho \in H_n$ is positive semi-definite (or $\rho \geq 0$) if $\mathbf{x}^* \rho \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{C}^n$, or equivalently: all of the eigenvalues of ρ are non-negative. A linear map $\psi : H_n \to H_p$ is TPCP (trace-preserving completely positive) if

1. TP (trace-preserving): $\forall \rho \in H_n$, $\mathrm{tr}\rho = \mathrm{tr}\psi(\rho)$
2. CP (completely positive): $\forall m \in \mathbb{N}$, $\rho \in H_{mn}$,

$$\rho \geq 0 \quad \Rightarrow \quad (\psi \otimes I_{m \times m})(\rho) \geq 0$$

where $I_{m \times m}$ is the identity operator on $m \times m$ matrices.

Problem 1. Let $\mathbb{H}_n$ be the vector space of $n \times n$ hermitian matrices. The adjoint (conjugate transpose) of a matrix $A \in \mathbb{C}^{n \times n}$ is denoted by A^*, Consider a family $V_1, V_2, \ldots, V_m$ of $n \times n$ matrices over $\mathbb{C}$. We associate with this family the completely positive map $\psi : \mathbb{H}_n \to \mathbb{H}_n$ defined by

$$\psi(X) = \sum_{j=1}^{m} V_j X V_j^* .$$

The map ψ is said to be a *Kraus map* if $\psi(I_n) = I_n$, i.e.

$$\sum_{j=1}^{m} V_j V_j^* = I_n$$

and the matrices $V_1, V_2, \ldots, V_m$ are called *Kraus operators*.

Let $m = n = 2$ and

$$V_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Show that V_1 and V_2 are Kraus operators and find the associated Kraus map.

Problem 2. Let $\psi : \mathbb{H}_n \to \mathbb{H}_n$ be a Kraus map. Thus ψ is linear. Show that there exists $\Psi \in \mathbb{C}^{n \times n}$ such that for all $X \in \mathbb{H}_n$

$$\mathrm{vec}(\psi(X)) = \Psi \, \mathrm{vec}(X)$$

where 1 is an eigenvalue of Ψ. What is a corresponding eigenvector?

Problem 3. Find all Kraus maps $\psi : \mathbb{H}_2 \to \mathbb{H}_2$, associated with families of 2 Kraus operators (V_1 and V_2), which provide the transformation

$$\psi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate

$$\psi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Is there a Kraus map associated with a single Kraus operator which also provides this transformation?

Problem 4. Let $p \in [0, 1]$ and $\sigma_1, \sigma_2, \sigma_3, \sigma_0 = I_2$ be the Pauli spin matrices.

(i) Show that the four 2×2 matrices

$$K_0 = \frac{\sqrt{1+3p}}{2}\sigma_0, \quad K_1 = \frac{\sqrt{1-p}}{2}\sigma_1, \quad K_2 = \frac{\sqrt{1-p}}{2}\sigma_2, \quad K_3 = \frac{\sqrt{1-p}}{2}\sigma_3$$

are Kraus operators.
(ii) Show that the sixteen 4×4 matrices

$$K_j \otimes K_\ell, \quad j, \ell = 0, 1, 2, 3$$

are Kraus operators, where $\otimes$ denotes the Kronecker product.
(iii) Show that the sixteen 4×4 matrices

$$K_j \star K_\ell, \quad j, \ell = 0, 1, 2, 3$$

are Kraus operators, where $\star$ denotes the star product.

Problem 5. Let K_j $(j = 1, \ldots, m)$ be $n \times n$ matrices over $\mathbb{C}$ with

$$\sum_{j=1}^{m} K_j K_j^* = I_n.$$

Show that

$$\sum_{j=1}^{m}\sum_{\ell=1}^{m}(K_j \otimes K_\ell)(K_j^* \otimes K_\ell^*) = I_n \otimes I_n \equiv I_{n^2}.$$

Problem 6. Let K_j $(j = 1, \ldots, m)$ be 2×2 matrices over $\mathbb{C}$ with

$$\sum_{j=1}^{m} K_j K_j^* = I_2.$$

Show that

$$\sum_{j=1}^{m}\sum_{\ell=1}^{m}(K_j \star K_\ell)(K_j^* \star K_\ell^*) = I_2 \otimes I_2 = I_4.$$

Problem 7. (i) Let A be an $n \times n$ matrix over $\mathbb{C}$. Let G be a finite group given by $n \times n$ matrices over $\mathbb{C}$ and $g \in G$. Consider the linear map

$$A \mapsto \widetilde{A} = \frac{1}{|G|}\sum_{g \in G} gAg^{-1}$$

where $|G|$ denotes the number of elements in the finite group G. Show that $\mathrm{tr}(A) = \mathrm{tr}(\widetilde{A})$.
(ii) Is the determinant preserved under the linear map?
(iii) Let A be positive semi-definite. Is $\widetilde{A}$ positive semi-definite?
(iv) Apply it to the case of 4×4 matrices with

$$A = \rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and the group is given by the 4×4 permutation matrices with $|G| = 4! = 24$.

Problem 8. Let $\rho_1, \rho_2 \in H_n$ be positive semi-definite matrices.

(i) Is $\rho_3 = \rho_1 + \rho_2$ positive semi-definite?

(ii) Is $\rho_4 = k\rho_1$ ($k \in \mathbb{C}$) positive semi-definite?

(iii) Let $\rho_5 \in H_n$ such that $\rho_1 + \rho_5$ is positive semi-definite. Is ρ_5 positive semi-definite?

Problem 9. Show that a linear map $\psi : H_n \to H_p$ is a TP map if and only if $\psi^*(I_n) = I_n$ where * denotes the adjoint with respect to the Frobenius inner product and I_n is the $n \times n$ identity matrix.

Problem 10. Show that

$$\rho_0 := \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} \in H_{n^2}$$

is positive semi-definite, where E_{ij} is the $n \times n$ matrix with a 1 in row i and column j and 0 elsewhere.

Problem 11. An orthonormal basis, with respect to the Frobenius inner product, for H_n ($n \geq 2$) is given by $B = B_1 \cup B_2$ where

$$B_1 = \left\{ \frac{1}{\sqrt{2}} (E_{jk} + E_{kj}) \; : \; j, k = 1, \ldots, n, \; j \leq k \right\}$$
$$B_2 = \left\{ \frac{i}{\sqrt{2}} (E_{jk} - E_{kj}) \; : \; j, k = 1, \ldots, n, \; j < k \right\}.$$

Express

$$\rho_0 := \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} \in H_{n^2}$$

in terms of this basis.

Problem 12. Show that a linear map $\psi : H_n \to H_p$ is a CP map if and only if $(\psi \otimes I_{n\times n})(\rho_0)$ is positive semi-definite where

$$\rho_0 := \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} \in H_{n^2}.$$

Problem 13. Is the map $\psi : H_n \to H_p$ given by $\psi(\rho) = \rho^T$ completely positive?

Problem 14. Let $\psi : H_n \to H_p$ given by

$$\psi(\rho) := \sum_{k=1}^{n^2} V_k \rho V_k^*$$

be a CP map. Find the condition on $V_1, \dots, V_{n^2}$ such that ψ is TP (and hence TPCP).

Problem 15. Let $\psi : H_n \to H_p$ given by

$$\psi(\rho) := \sum_{k=1}^{n^2} V_k \rho V_k^*$$

be a CP map. Show that there exists a matrix V such that

$$\psi(\rho) = \mathrm{tr}_m V(\rho \otimes I_{n^2})V^*.$$

Problem 16. A minimal *Stinespring representation* of a CP map $\psi : H_n \to H_p$ is a represenatation

$$\psi(\rho) = \mathrm{tr}_m V(\rho \otimes I_m)V^*$$

where m is minimal. This corresponds to minimizing the number of non-zero Kraus operators V_k in a Kraus representation

$$\psi(\rho) := \sum_{k=1}^{m} V_k \rho V_k^*.$$

Given a Kraus representation

$$\psi(\rho) := \sum_{k=1}^{n^2} \widetilde{V}_k \rho \widetilde{V}_k^*.$$

Consider Let

$$A = \sum_{k=1}^{n^2} (\text{vec}\widetilde{V}_k)(\text{vec}\widetilde{V}_k)^*.$$

The matrix A is positive definite and thus has a spectral decomposition

$$A = \sum_{k=1}^{m} \lambda_k \mathbf{v}_k \mathbf{v_k}^*$$

where m is the rank of A and $\lambda_1, \ldots, \lambda_m > 0$. We find the Kraus operators V_k for the minimal representation from $\text{vec}V_k = \sqrt{\lambda_k}\mathbf{v}_k$. Find a minimal representation for the completely map on H_2 given by

$$\widetilde{V}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{V}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \widetilde{V}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{V}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 17. Consider the Kraus operators K_1 and K_2

$$K_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow K_1^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow K_2^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and an arbitrary 2×2 matrix $A = (a_{jk})$. Then

$$K_1 A K_1^* + K_2 A K_2^* = \begin{pmatrix} a_{22} & 0 \\ 0 & a_{11} \end{pmatrix}.$$

So the trace is preserved under this transformation. Let $c_1^\dagger$, $c_2^\dagger$, c_1, c_2 be Fermi creation and annihilatin operators, respectively. Consider the operators

$$\hat{K}_1 = \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = c_1^\dagger c_2 \;\; \hat{K}_1^\dagger = c_2^\dagger c_1$$

$$\hat{K}_2 = \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = c_2^\dagger c_1 \;\; \hat{K}_2^\dagger = c_1^\dagger c_2$$

and

$$\hat{A} = \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = a_{11} c_1^\dagger + a_{12} c_1^\dagger c_2 + a_{21} c_2^\dagger c_2^\dagger c_1 + a_{22} c_2^\dagger c_2.$$

Find the operator

$$\hat{K}_1 \hat{A} \hat{K}_1^\dagger + \hat{K}_2 \hat{A} \hat{K}_2^\dagger.$$

Chapter 13

Miscellaneous

Problem 1. Let H_0 and V be $n \times n$ hermitian matrices and $\epsilon \in \mathbb{R}$. Consider the hermitian matrix $H = H_0 + \epsilon V$. Let

$$U(\beta) = e^{-\beta(H_0+\epsilon V)}$$

with $\beta \geq 0$. Then

$$\frac{dU(\beta)}{d\beta} = -(H_0 + \epsilon V)e^{-\beta(H_0+\epsilon V)} = -(H_0 + \epsilon V)U(\beta)$$

where $U(\beta = 0) = I_n$. Let

$$U(\beta) = e^{-\beta H_0}W(\beta).$$

(i) Show that $W(\beta)$ is given by

$$W(\beta) = \sum_{k=0}^{\infty}(-1)^k\epsilon^k \int_0^{\beta}\int_0^{\beta_1}\cdots\int_0^{\beta_{k-1}} d\beta_1 d\beta_2 \cdots d\beta_k \widetilde{V}(\beta_1)\widetilde{V}(\beta_2)\cdots\widetilde{V}(\beta_k).$$

where $\widetilde{V}(\beta) := e^{\beta H_0}Ve^{-\beta H_0}$.
(ii) Apply (i) to

$$H_0 = \hbar\omega\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V = \Delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 2. Let H, A, B be hermitian matrices. Let

$$A(t) := e^{iHt}Ae^{-iHt}, \qquad B(s) := e^{iHs}Be^{-iHs}$$

where $s, t \in \mathbb{R}$. Show that

$$\mathrm{tr}(A(t)B(s)e^{-\beta H}) = \mathrm{tr}(e^{iH(t-s)}Ae^{-iH(t-s)}Be^{-\beta H}).$$

Problem 3. Let $\hat{H}(t)$ be a given time-dependent hermitian Hamilton operator given as an $n \times n$ matrix. We assume that $\hat{H}(t)$ depends smoothly on t. Find the solution of the initial value problem of the matrix differential equation

$$\frac{dU(t)}{dt} = -\frac{i}{\hbar}\hat{H}(t)U(t), \qquad U(0) = I_n$$

where I_n is the $n \times n$ identity matrix. Apply the ansatz (*Magnus expansion*)

$$U(t) = \exp(\Omega(t))$$

and $\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t)$. Find the first two terms in the expansion, i.e. find $\Omega_1(t)$ and $\Omega_2(t)$.

Problem 4. Consider the Hamilton operator

$$\hat{H} = \hbar\omega \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \equiv \hbar\omega(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3).$$

Find the eigenvalues and normalized eigenvectors of $\hat{H}$.

Problem 5. Let A, H be $n \times n$ hermitian matrices, where H plays the role of the Hamilton operator. The Heisenberg equations of motion is given by

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[H, A(t)].$$

with $A = A(t = 0) = A(0)$. Let E_j ($j = 1, 2, \ldots, n^2$) be an orthonormal basis in the Hilbert space $\mathcal{H}$ of the $n \times n$ matrices with scalar product

$$\langle X, Y \rangle := \mathrm{tr}(XY^*), \qquad X, Y \in \mathcal{H}.$$

Now $A(t)$ can be expanded using this orthonormal basis as

$$A(t) = \sum_{j=1}^{n^2} c_j(t) E_j$$

and H can be expanded as

$$H = \sum_{j=1}^{n^2} h_j E_j.$$

Find the time evolution for the coefficients $c_j(t)$, i.e. dc_j/dt, where $j = 1, 2, \ldots, n^2$.

Problem 6. Consider the standard basis in the Hilbert space $\mathbb{C}^9$

$$|00\rangle, \quad |01\rangle, \quad |02\rangle, \quad |10\rangle, \quad |11\rangle, \quad |11\rangle, \quad |12\rangle, \quad |20\rangle, \quad |21\rangle, \quad |22\rangle$$

where $|00\rangle \equiv |0\rangle \otimes |0\rangle$, and $|0\rangle$, $|1\rangle$, $|2\rangle$ is the standard basis in $\mathbb{C}^3$. Show that the normalized states

$$|\psi\rangle_{nm} = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} e^{2\pi ijn/3} |j\rangle \otimes |(j+m) \bmod 3\rangle$$

i.e.

$$\begin{aligned}
|\psi\rangle_{00} &= \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \\
|\psi\rangle_{10} &= \frac{1}{\sqrt{3}}(|00\rangle + e^{2\pi i/3}|11\rangle + e^{4\pi i/3}|22\rangle) \\
|\psi\rangle_{20} &= \frac{1}{\sqrt{3}}(|00\rangle + e^{4\pi i/3}|11\rangle + e^{2\pi i/3}|22\rangle) \\
|\psi\rangle_{01} &= \frac{1}{\sqrt{3}}(|01\rangle + |12\rangle + |20\rangle) \\
|\psi\rangle_{11} &= \frac{1}{\sqrt{3}}(|01\rangle + e^{2\pi i/3}|12\rangle + e^{4\pi i/3}|20\rangle) \\
|\psi\rangle_{21} &= \frac{1}{\sqrt{3}}(|01\rangle + e^{4\pi i/3}|12\rangle + e^{2\pi i/3}|20\rangle) \\
|\psi\rangle_{02} &= \frac{1}{\sqrt{3}}(|02\rangle + |10\rangle + |21\rangle) \\
|\psi\rangle_{12} &= \frac{1}{\sqrt{3}}(|02\rangle + e^{2\pi i/3}|10\rangle + e^{4\pi i/3}|21\rangle) \\
|\psi\rangle_{22} &= \frac{1}{\sqrt{3}}(|02\rangle + e^{4\pi i/3}|10\rangle + e^{2\pi i/3}|21\rangle)
\end{aligned}$$

form an orthonormal basis in the Hilbert space $\mathbb{C}^9$.

Problem 7. Consider the state

$$|\psi\rangle = -\mathcal{E}_1\mathcal{E}_2 \left(e^{i(\mathbf{k}_1\cdot\mathbf{r}+\mathbf{k}_2\cdot\mathbf{r}') - i(\omega_1 t + \omega_2 t')} + e^{i(\mathbf{k}_2\cdot\mathbf{r}+\mathbf{k}_1\cdot\mathbf{r}') - i(\omega_2 t + \omega_1 t')} \right) |0\rangle \otimes |0\rangle.$$

Find

$$w \propto \langle\psi|\psi\rangle.$$

Problem 8. Let $\hat{A}$ be a nonzero bounded linear operator in a Hilbert space $\mathcal{H}$. Let $|n\rangle$, $|m\rangle$ be normalized states in the Hilbert space $\mathcal{H}$. We define

$$S(|m\rangle, |n\rangle, \hat{A}) := \frac{\langle m|\hat{A}\hat{A}^\dagger|n\rangle}{\sqrt{\langle m|\hat{A}\hat{A}^\dagger|m\rangle}\sqrt{\langle n|\hat{A}\hat{A}^\dagger|n\rangle}}.$$

Consider the Hilbert space $\mathbb{C}^2$. Calculate S for $\hat{A} = \sigma_1$ and the normalized states

$$|\mathbf{u}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\mathbf{v}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 9. Consider the state $|\psi\rangle$

$$|\psi\rangle = \sum_{j_0, j_1, \ldots, j_n = 0}^{1} c_{j_0, j_1, \ldots, j_{N-1}} |j_0\rangle \otimes |j_1\rangle \otimes \cdots \otimes |j_{N-1}\rangle$$

in the Hilbert space $\mathbb{C}^{2^n}$. The bitstring $j_0 j_1 \ldots j_{N-1}$ can be mapped one-to-one into a non-negative integer j

$$j = \sum_{k=0}^{N-1} j_k 2^k$$

where $j_k \in \{0, 1\}$. Thus we can write the state as

$$|\psi\rangle = \sum_{j=0}^{2^{N-1}} c_j |j\rangle.$$

We can associate a polynomial with the state $|\psi\rangle$ via

$$p(|\psi\rangle, x) = \sum_{j=0}^{2^{N-1}} c_j x^j.$$

(i) Consider the Bell state ($N = 2$)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

Find the polynomial of $|\psi\rangle$ and calculate the roots.
(ii) Consider the state ($N = 2$)

$$|\phi\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle - |1\rangle \otimes |1\rangle).$$

Find the polynomial of $|\phi\rangle$ and calculate the roots.

Problem 10. Let A, B be observable, i.e. hermitian matrices. Then the *uncertainty relation* is given by

$$\Delta^2 A \cdot \Delta^2 B \geq \frac{1}{4} |\langle [A, B] \rangle|^2 + \mathrm{cov}(A, B)$$

where $[\, ,\,]$ denotes the commutator,

$$\mathrm{cov}(A, B) := \frac{1}{2}(\langle AB \rangle + \langle BA \rangle) - \langle A \rangle \langle B \rangle$$

and

$$\Delta^2 A := \mathrm{cov}(A, A).$$

This inequality can generalized to $2n$ observable A_1, A_2, $\dots$, A_{2n}. We have

$$\det(\Sigma) \geq \det(C)$$

where

$$\Sigma_{k\ell} = \mathrm{cov}(A_k, A_\ell), \qquad C_{k\ell} = -\frac{i}{2} \langle [A_k, A_\ell] \rangle.$$

Let

$$A = \sigma_1, \qquad B = \sigma_2, \quad |\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the left-hand side and right-hand side of the inequality.

Problem 11. The most general real three-qubit state can be written as

$$\begin{aligned} |\psi\rangle = & -c_3 \cos^2 \theta |0\rangle \otimes |0\rangle \otimes |1\rangle - c_2 |0\rangle \otimes |1\rangle \otimes |0\rangle + c_3 \sin(\theta) \cos(\theta) |0\rangle \otimes |1\rangle \otimes |1\rangle \\ & - c_1 |1\rangle \otimes |0\rangle \otimes |0\rangle - c_3 \sin(\theta) \cos(\theta) |1\rangle \otimes |0\rangle \otimes |1\rangle + (c_0 + c_3 \sin^2(\theta)) |1\rangle \otimes |1\rangle \otimes |1\rangle \end{aligned}$$

where $c_0, c_1, c_2, c_3, \theta$ are real parameters. Classify the state with respect to entanglement.

Problem 12. Let A, B be $n \times n$ matrices acting in the Hilbert space $\mathbb{C}^n$. Then A, B can be considered as observable. The two overvable A and B are called *complementary* if their eigenvalues are non-degenerate and any two normalized eigenvectors $\mathbf{a}_j$ of A and $\mathbf{b}_j$ of B satisfy

$$|\mathbf{a}_j^* \mathbf{b}_k| = \frac{1}{\sqrt{n}}$$

where * means transpose and conjugate complex. Give an example for such hermitian matrices in $\mathbb{C}^2$.

Problem 13. Two orthonormal bases

$$\{\, \mathbf{u}_j \,:\, j = 1, 2, \ldots, n \,\}, \qquad \{\, \mathbf{v}_k \,:\, k = 1, 2, \ldots, n \,\}$$

in the Hilbert space $\mathbb{C}^n$ are called mutually unbiased if

$$\mathbf{u}_j^* \mathbf{v}_k = \frac{1}{\sqrt{n}} \quad \text{for all} \quad j, k \in \{\, 1, 2, \ldots, n \,\}.$$

(i) Give an example in $\mathbb{C}^2$.
(ii) Give an example in $\mathbb{C}^4$.

Problem 14. Solve the initial value problem of the *optical Bloch equations.*

$$\frac{d\rho_{11}}{dt} = -\frac{d\rho_{22}}{dt} = i\frac{b}{2}(e^{-i(\omega-\alpha)t}\rho_{12} - e^{i(\omega-\alpha)t}\rho_{21})$$

$$\frac{d\rho_{12}}{dt} = \frac{d\rho_{21}^*}{dt} = i\frac{b}{2}e^{i(\omega-\alpha)t}(\rho_{11} - \rho_{22})$$

with the initial conditions

$$\rho_{11}(0) = \rho_{12}(0) = \rho_{21}(0) = 0, \qquad \rho_{22}(0) = 1.$$

Problem 15. Consider a finite dimensional Hilbert space. Let $\hat{H}$ be a hermitian Hamilton operator. Let $|\psi\rangle$ be the normalized ground state of the system, i.e. $\hat{H}|\psi\rangle = E_0|\psi\rangle$. Let $|\phi\rangle$ be another normalized state. Let $\hat{F}$ be a positive semidefinite operator and $\langle\phi|\hat{F}|\phi\rangle > 0$. Then we have the inequality

$$\langle\psi|\hat{F}|\psi\rangle \geq \frac{(\langle\phi|\psi\rangle\langle\phi|\hat{F}|\phi\rangle - (\Delta\hat{F})(1 - \langle\phi|\psi\rangle^2)^{1/2})^2}{\langle\phi|\hat{F}|\phi\rangle}$$

where

$$(\Delta F)^2 := \langle\phi|\hat{F}^2|\phi\rangle - \langle\phi|\hat{F}|\phi\rangle^2.$$

The inequality follows from the non-negativity of the *Gramian determinant* of the vectors $|\psi\rangle$, $|\phi\rangle$, and $\hat{F}|\phi\rangle$. Consider the Hilbert space $\mathbb{C}^2$ and the Hamilton operator

$$\hat{H} = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the positive semidefinite operator

$$\hat{F} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and the normalized states

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad |\phi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with $\hat{H}|\psi\rangle = E_0|\psi\rangle$ and $E_0 = -\hbar\omega$. Apply the inequality to these operators and states, i.e. calculate the left and right-hand side of the inequality.

Problem 16. Consider the normalized states in the Hilbert space $\mathbb{C}^3$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Find the unitary matrices U_{12}, U_{23}, U_{31} such that

$$|\psi_2\rangle = U_{12}|\psi_1\rangle, \quad |\psi_3\rangle = U_{23}|\psi_2\rangle, \quad |\psi_1\rangle = U_{31}|\psi_3\rangle.$$

Problem 17. Consider the Hadamard matrix

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Is $U_H \in SU(2)$? Is $iU_H \in SU(2)$?

Problem 18. The available *uncertainty relations* in finite-dimensional Hilbert spaces are those of Robertson and Schrödinger. Let ρ be the state of the quantum system (density matrix), i.e. a positive semi-definite, self-adjoint linear operator with $\mathrm{tr}(\rho) = 1$. The mean value functional is

$$\langle \cdot \rangle := \mathrm{tr}(\rho \cdot).$$

Then for two self-adjoint operators, A and B, the variance is defined by

$$(\Delta A)^2 := \langle A^2 \rangle - \langle A \rangle^2, \qquad (\Delta B)^2 := \langle B^2 \rangle - \langle B \rangle^2.$$

We have the inequalities

$$(\Delta A)(\Delta B) \geq |\langle AB \rangle - \langle A \rangle \langle B \rangle| \geq \frac{1}{2} |\langle [A, B] \rangle|.$$

Note that we have the identity

$$|\langle AB \rangle|^2 = \frac{1}{4} |\langle [A, B]_+ \rangle + \langle [A, B] \rangle|^2 = \frac{1}{4} |\langle [A, B]_+ \rangle|^2 + \frac{1}{4} |\langle [A, B] \rangle|^2.$$

(i) Let

$$A = \sigma_1, \quad B = \sigma_2, \quad \rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Calculate the left-hand side of the inequalty and the right-hand sides of the inequality. Discuss.
(ii) Let

$$A = \sigma_1 \otimes \sigma_1, \quad B = \sigma_2 \otimes \sigma_2, \quad \rho = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}.$$

Calculate the left-hand side of the inequalty and the right-hand sides of the inequality. Discuss.

Problem 19. A spin-$\frac{1}{2}$ system in a time-dependent magnetic fields $S(t)$ is described by the Hamilton operator

$$\hat{H}(t) = \frac{1}{2}\hbar\omega\sigma_3 + \frac{1}{2}\hbar\gamma S(t)\sigma_1$$

where σ_1 and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}(t)\psi$$

for the spinor $\psi = (\psi_1, \psi_2)^T$ takes the form

$$i\frac{d\psi_1}{dt} = -\frac{1}{2}\omega\psi_1 + \frac{1}{2}\gamma S(t)\psi_2, \qquad i\frac{d\psi_2}{dt} = \frac{1}{2}\omega\psi_2 + \frac{1}{2}\gamma S(t)\psi_1.$$

Rewrite this system in terms of the observable *Bloch variables*

$$A(t) := |\psi_2|^2 - |\psi_1|^2, \quad B(t) := i(\psi_2\psi_1^* - \psi_1\psi_2^*), \quad C(t) := \psi_2\psi_1^* + \psi_1\psi_2^*.$$

Problem 20. Consider the operators

$$H_1 = 1, \quad H_2 = x, \quad H_3 = \frac{\partial^2}{\partial x^2}, \quad H_4 = i\frac{\partial}{\partial x}.$$

(i) Show that we have a nilpotent Lie algebra under the commutator.

(ii) Let

$$\alpha_1(t) = cf(t), \quad \alpha_2(t) = c, \quad \alpha_3(t) = -\frac{1}{2}, \quad \alpha_4(t) = \frac{df}{dt}.$$

Consider the Hamilton operator

$$K = \sum_{j=1}^{4} \alpha_j(t) H_j$$

and the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = K\psi.$$

We write the solution of the Schrödinger equation in the form

$$\psi(x,t) = U(t,0)\psi(x,0)$$

where the unitary time evolution operator is give by

$$U(t,0) = \exp(\beta_1(t)H_1)\exp(\beta_2(t)H_2)\exp(\beta_3(t)H_3)\exp(\beta_4(t)H_4).$$

Find the system of ordinary differential equations for $\beta_j(t)$ $(j = 1,2,3,4)$ and solve them.

Problem 21. Consider the spin Hamilton operator

$$\hat{H} = \Delta_1 \sigma_1 \otimes \sigma_1 + \Delta_2 \sigma_2 \otimes \sigma_2 + \Delta_3 \sigma_3 \otimes \sigma_3.$$

Let $\hat{K} = \beta \hat{H}$. The *partition function* is

$$Z(\beta) = \mathrm{tr}\exp(-K).$$

The logarithm of the partition function is given by the cumulant expansion

$$\begin{aligned}\ln(Z(\beta)) = &\ln \mathrm{tr}(I) - \langle K\rangle + \frac{1}{2!}(\langle K^2\rangle - \langle K\rangle^2)\\ &-\frac{1}{3!}(\langle K^3\rangle - 3\langle K^2\rangle\langle K\rangle + 2\langle K\rangle^3)\\ &+\frac{1}{4!}(\langle K^4\rangle - 4\langle K^3\rangle\langle K\rangle - 3\langle K^2\rangle^2 + 12\langle K^2\rangle\langle K\rangle^2 - 6\langle K\rangle^4)\\ &-\cdots\end{aligned}$$

Here I is the identity operator given by $I_n \otimes I_n$ with $n = 2$ and

$$\langle \cdots \rangle := \frac{\mathrm{tr}(\cdots)}{\mathrm{tr} I}.$$

Calculate the function $\ln(Z(\beta))$ up to this order.

Problem 22. Let $|\psi\rangle$, $|\phi\rangle$ be normalized states in the Hilbert space $\mathbb{C}^n$. Let K be a positive semi-definite matrix in $\mathbb{C}^n$. Show that

$$G := \det \begin{pmatrix} 1 & \langle\phi|\psi\rangle & \langle\phi|K|\psi\rangle \\ \langle\phi|\psi\rangle & 1 & \langle\phi|K|\phi\rangle \\ \langle\phi|K|\psi\rangle & \langle\phi|K|\phi\rangle & \langle\phi|K^2|\phi\rangle \end{pmatrix} \geq 0.$$

G is called the *Gramian*. Apply it to the Hilbert space $\mathbb{C}^2$ and

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 23. Let $C = (c_{jk})$ $(j, k = 1, \ldots, n)$ be an $n \times n$ matrix with real entries. Then C is called a *quantum correlation matrix* if there are self-adjoint operators A_j, B_k $(j, k = 1, \ldots, n)$ on a Hilbert space $\mathcal{H}$ with $\|A_j\| \leq 1$, $\|B_k\| \leq 1$ and $\mathbf{u}$ in the unit sphere of $\mathcal{H} \otimes \mathcal{H}$ such that

$$c_{jk} = \langle (A_j \otimes B_k)\mathbf{u}, \mathbf{u} \rangle$$

where $\langle \, , \, \rangle$ denotes the scalar product. If the self-adjoint operators A_j, B_k $(j, k = 1, \ldots, n)$ commute the matrix C is called a classical correlation matrix. Consider the case with $n = 3$, the Hilbert space $\mathbb{C}^2$, $A_1 = B_1 = \sigma_1$, $A_2 = B_2 = \sigma_2$, $A_3 = B_3 = \sigma_3$ and the Bell state

$$\mathbf{u}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the correlation matrix C. What is the significance of this matrix?

Problem 24. Let

$$\hat{T}(X, P) = \exp(i(P\hat{x} - X\hat{p})/\hbar)$$

be the phase space *translation operator*. Show that

$$\hat{T}(X_1, P_1)\hat{T}(X_2, P_2) = \exp(i(X_2 P_1 - X_1 P_2)/(2\hbar))\hat{T}(X_1 + X_2, P_1 + P_2).$$

Problem 25. The group generator of the compact Lie group $SU(2)$ can be written as

$$J_1 = \frac{1}{2}\left(z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_1}\right), \quad J_2 = \frac{i}{2}\left(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2}\right), \quad J_3 = \frac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}\right).$$

(i) Find

$$J_+ = J_1 + iJ_2, \qquad J_- = J_1 - iJ_2.$$

(ii) Let $j = 0, 1, 2, \ldots$ and $m = -j, -j+1, \ldots, 0, \ldots, j$. We define

$$e_m^j(z_1, z_2) = \frac{1}{\sqrt{(j+m)!(j-m)!}} z_1^{j+m} z_2^{j-m}.$$

Find

$$J_+ e_m^j(z_1, z_2), \quad J_- e_m^j(z_1, z_2), \quad J_3 e_m^j(z_1, z_2)$$

(iii) Let

$$J^2 = J_1^2 + J_2^2 + J_3^2 \equiv \frac{1}{2}(J_+ J_- + J_- J_+) + J_3^2.$$

Find

$$J^2 e_m^j(z_1, z_2).$$

Problem 26. Show that the operators

$$L_+ = \overline{z} z, \qquad L_- = -\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}$$

$$L_3 = -\frac{1}{2}\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}} + 1\right), \qquad L_0 = -\frac{1}{2}\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}} + 1\right).$$

form a basis for the Lie algebra $su(1,1)$ under the commutator.

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