Problems of the Month: October 2015

International School for Scientific Computing
Mathematical Physics and Scientific Computing

Problem 1. Let $b^\dagger$, b be Bose creation and annihilation operators, respectively, with $[b, b^\dagger] = I$ (I identity operator). Find the Lie algebra generated by

$$b^\dagger b, \quad \sqrt{b^\dagger b}, \quad b^\dagger + b.$$

Problem 2. The $n \times n$ primary permutation matrix P is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The eigenvalues of P are given by $\lambda^0 = 1$, λ^1, λ^2, $\dots$, λ^{n-1} with $\lambda := \exp(2\pi i/n)$. The spectral decomposition of U is

$$P = \sum_{j=0}^{n-1} \lambda^j \Pi_j.$$

The projection matrix Π_j can be expressed using P^k. Find the skew-hermitian matrix K such that $P = \exp(K)$. Then study P^2, P^3 etc.

Problem 3. The logistic map $x_{t+1} = 4x_t(1 - x_t)$ with $x_0 \in [0, 1]$ and $t = 0, 1, \ldots$ is the most studied map with chaotic behaviour. Consider the three coupled logistic maps

$$x_{1,t+1} = 4x_{2,t}(1 - x_{2,t}), \quad x_{2,t+1} = 4x_{3,t}(1 - x_{3,t}), \quad x_{3,t+1} = 4x_{1,t}(1 - x_{1,t})$$

where $t = 0, 1, \ldots$ and $x_{1,0}, x_{2,0}, x_{3,0} \in [0, 1]$. First find the fixed points and study their stability. Can one find hyperchaotic behaviour? Is the system ergodic? Is the system mixing?

Problem 4. Consider the circle around $(0, 0, 0)$ in the $x_1 - x_2$ plane

$$\mathbf{r}_1(t) = \begin{pmatrix} x_{1,1}(t) \\ x_{1,2}(t) \\ x_{1,3}(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}, \quad t \in [0, 2\pi]$$

and the circle around $(0, 1, 0)$ in the $x_2 - x_3$ plane

$$\mathbf{r}_2(s) = \begin{pmatrix} x_{2,1}(s) \\ x_{2,2}(s) \\ x_{2,3}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + \cos(s) \\ \sin(s) \end{pmatrix}, \quad s \in [0, 2\pi].$$

Then the derivatives are

$$\frac{d\mathbf{r}_1(t)}{dt} = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix}, \qquad \frac{d\mathbf{r}_2(s)}{ds} = \begin{pmatrix} 0 \\ -\sin(s) \\ \cos(s) \end{pmatrix}.$$

Calculate (Gauss formula)

$$\frac{1}{4\pi} \oint \oint dt ds \left(\frac{d\mathbf{r}_1(t)}{dt} \times \frac{d\mathbf{r}_2(s)}{ds} \right) \cdot \frac{\mathbf{r}_1(t) - \mathbf{r}_2(s)}{|\mathbf{r}_1(t) - \mathbf{r}_2(s)|^3}$$

where $\times$ denotes the vector product, $\cdot$ denotes the scalar product and contour integrations run from 0 to 2π.