Runge-Kutta Methods

Exercises

1. Consider the one-step method

$$w_{i+1} = w_i + af\left(t_i, w_i\right) + bf\left(t_i + \alpha, w_i + \beta f\left(t_i, w_i\right)\right).$$

Define the local error for this method and, by means of suitable Taylor expansions of the local error, obtain conditions for the parameters a, b, α and β (in terms of h) such that the local error has a leading term proportional to

$$h^3\left(f_t f_y + f_i f_y^2\right)$$

where

$$f_t \equiv \left.\frac{\partial f}{\partial t}\right|_{(t_i,y_i)} \qquad f_y \equiv \left.\frac{\partial f}{\partial y}\right|_{(t_i,y_i)} \qquad f_i \equiv f\left(t_i, y_i\right).$$

2. Consider the Runge-Kutta method

$$w_{i+1} = w_i + af\left(x_i, w_i\right) + bf\left(x_i + c, w_i + df\left(x_i, w_i\right)\right).$$

Using an appropriate Taylor expansion of the local error, determine the parameters a, b, c and d (in terms of h), such that the method is second-order, and has a local error of the form

$$\left(f_{xx} + 2f_i f_{xy} + f_i^2 f_{yy} + \frac{f_x f_y + f_i f_y^2}{6}\right) h^3,$$

where

$$f_x \equiv \left.\frac{\partial f}{\partial x}\right|_{(x_i,y_i)} \qquad f_y \equiv \left.\frac{\partial f}{\partial y}\right|_{(x_i,y_i)} \qquad f_i \equiv f\left(x_i, y_i\right)$$

$$f_{xx} \equiv \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_i,y_i)} \qquad f_{xy} \equiv \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(x_i,y_i)} \qquad f_{yy} \equiv \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_i,y_i)}$$

and the local error ε is defined as

$$\varepsilon \equiv y_{i+1} - \left[y_i + af\left(x_i, y_i\right) + bf\left(x + c, y_i + df\left(x, y_i\right)\right)\right].$$

3. Determine the order of the method

$$\begin{array}{c|cc} -\frac{10}{3} & -\frac{10}{3} & \\ \hline & \frac{23}{20} & -\frac{3}{20} \end{array}$$

by Taylor expansion of a suitable expression for the local error.

4. Determine the order of the method

$$\begin{array}{c|cc} \frac{2}{3} & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

by appealing to the relevant order conditions.

5. Determine the various coefficients in the tableau

$$\begin{array}{c|cc} c_1 & a_{11} & -\frac{1}{12} \\ 1 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$$

if this Runge-Kutta method is intended to be of order three. Show explicitly that the resulting method cannot be of order four.

6. Assume that RK2 and RK3 are used to control the local error in RK2, via the mechanism of local extrapolation, for the problem

$$y' = \lambda y, \ \lambda > 0$$
$$y(0) = 1.$$

Show that

$$h^* \leqslant \left(\frac{6\delta}{|w_i^3| \lambda^3} \right)^{\frac{1}{3}},$$

where h^* is the adjusted stepsize on the $(i+1)$th step, w_i^3 is the RK3 solution, and δ is the tolerance imposed on the local error.

7. Assume that the method

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

is used to control the local error in Euler's method, via local extrapolation, for the initial-value problem

$$y' = xy, \quad y(2) = y_0.$$

If a tolerance δ is placed on the local error, show that the appropriate stepsize on the first step is given by

$$h \leqslant \sqrt{\frac{2\delta}{5\,|y_0|}}.$$

You may assume $h \ll 1$.

8. Assume that RK1 and RK2 are used to control the local error in RK1, via the mechanism of local extrapolation, for the problem

$$y' = y^2, \qquad y(x_0) = 1.$$

Assume that $h \ll 1$, where $h \equiv x_1 - x_0$. Also, assume that the magnitude of the local error at x_1 is greater than a tolerance δ, so that a stepsize adjustment is required. Show that

$$h^* \leqslant \sqrt{\delta},$$

where h^* is the adjusted stepsize for the first step. Use the RK2 method given by

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

9. Show that

$$\lim_{h\to 0} F(y) = f(y) \quad \text{and} \quad \lim_{h\to 0} F_y = f_y,$$

when the RK2 method

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

is applied to the problem

$$y' = xy^2 + e^y \equiv f(x,y), \qquad y(x_0) = 1.$$

Here, $F(x,y)$ is defined by writing RK2 in the form $w_{i+1} = w_i + hF(x_i, w_i)$.

10. Show that, if $y_1 < -\frac{1}{8}$, then the system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -y_1 + y_2 \\ y_1^2 + 1 \end{bmatrix}$$

has stiff eigenvalues.

11. Show that, if $y_1 > y_2, y_2 \neq 0$, then the system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -y_1 - \ln y_2 \\ \frac{y_1^2+1}{8} \end{bmatrix}$$

has at least one stiff eigenvalue.

12. Show that the system

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -3y_1y_2^2 + \cos(y_3) \\ y_2^3 + 4 \\ e^x + \ln y_2 \end{bmatrix}$$

always has one stiff eigenvalue. Assume that the solutions are always nonzero.

13. Show that the implicit method

$$w_{i+1} = w_i + hf(x_{i+1}, w_{i+1})$$

is A-stable.

(HINT: Apply the method to the Dahlquist equation $y' = \lambda y$ so as to obtain the characteristic polynomial $R(h\lambda)$. Then assume that λ is *complex* and hence show that $|R(h\lambda)| < 1$ whenever $\mathrm{Re}(\lambda) < 0$.)

14. Show explicitly that the Runge-Kutta method

$$\begin{array}{c|cc} \frac{2}{3} & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

has characteristic polynomial

$$R(h\lambda) \equiv \frac{h^2\lambda^2}{2} + h\lambda + 1$$

when applied to the Dahlquist equation $y' = \lambda y$, and then show that $|R(h\lambda)| < 1$ when $-2 < h\lambda < 0$.

15. Show that the Runge-Kutta method

$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

has stability function

$$R(z) = \frac{2z+6}{z^2 - 4z + 6}.$$

Solutions

1. Local error:

$$\varepsilon \equiv y_{i+1} - y_i - af_i - bf\left(t_i + \alpha, y_i + \beta f_i\right).$$

Expand y_{i+1} up to third-order (third-order terms are necessary because the local error is third-order):

$$y\left(t_{i+1}\right) = y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \dots$$

Expand $f\left(t_i + \alpha, y_i + \beta f_i\right)$ up to second-order in α :

$$f\left(t_i + \alpha, y_i + \beta f_i\right) = f\left(t_i, y_i + \beta f_i\right) + \alpha f_t\left(t_i, y_i + \beta f_i\right) + \frac{\alpha^2}{2}f_{tt}\left(t_i, y_i + \beta f_i\right) + \dots$$

Expand each term in the above up to second order in α and β :

$$\begin{aligned} f\left(t_i, y_i + \beta f_i\right) &= f_i + \beta f_i f_y + \frac{\beta^2 f_i^2}{2} f_{yy} + \dots \\ \alpha f_t\left(t_i, y_i + \beta f_i\right) &= \alpha f_t + \alpha\beta f_i f_{ty} + \dots \\ \frac{\alpha^2}{2} f_{tt}\left(t_i, y_i + \beta f_i\right) &= \frac{\alpha^2}{2} f_{tt} + \dots \end{aligned}$$

It is understood that f and its various derivatives here are evaluated at $\left(t_i, y_i\right)$. Note that in the second equation above, the term in $\alpha\beta$ is regarded as second-order in α and β. Second-order terms in α and β will give third-order terms in h, due to multiplication by b (note that a, b, α and β are all linearly proportional to h). Third-order terms in α and β are unnecessary for this derivation. Note that

$$\begin{aligned} y'' &= f_t + f f_y \\ y''' &= f_{tt} + 2 f_i f_{ty} + f_i^2 f_{yy} + f_t f_y + f_i f_y^2. \end{aligned}$$

Substituting yields

$$\begin{aligned}
\varepsilon = {} & hf_i + \frac{h^2}{2}\left(f_t + f_i f_y\right) + \frac{h^3}{6}\left(f_{tt} + 2f_i f_{ty} + f_i^2 f_{yy} + f_t f_y + f_i f_y^2\right) \\
& - af_i - b\left(f_i + \beta f_i f_y + \frac{\beta^2 f_i^2}{2} f_{yy} + \ldots\right) \\
& - b\left(\alpha f_t + \alpha\beta f_i f_{ty} + \ldots\right) - \frac{b\alpha^2}{2} f_{tt} + \ldots \\
= {} & (h - a - b) f_i + \left(\frac{h^2}{2} - b\alpha\right) f_t + \left(\frac{h^2}{2} - b\beta\right) f_i f_y \\
& + \left(\frac{h^3}{6} - \frac{b\alpha^2}{2}\right) f_{tt} + \left(\frac{h^3}{3} - b\alpha\beta\right) f_i f_{ty} \\
& + \left(\frac{h^3}{6} - \frac{b\beta^2}{2}\right) f_i^2 f_{yy} + \frac{h^3}{6}\left(f_t f_y + f_i f_y^2\right) + \ldots
\end{aligned}$$

If

$$h - a - b = 0 \qquad \tfrac{h^2}{2} - b\alpha = 0$$
$$\tfrac{h^2}{2} - b\beta = 0$$

and

$$\tfrac{h^3}{6} - \tfrac{b\alpha^2}{2} = 0 \qquad \tfrac{h^3}{3} - b\alpha\beta = 0$$
$$\tfrac{h^3}{6} - \tfrac{b\beta^2}{2} = 0$$

are simultaneously satisfied, then the local error has leading term

$$\frac{h^3}{6}\left(f_t f_y + f_i f_y^2\right).$$

The first three of these equations are, effectively, the usual order conditions for an explicit second-order RK method. The next three equations arise from the condition we have placed on the form of the local error. It is easily shown that the above equations are satisfied by

$$\begin{aligned}
a &= \frac{h}{4} \\
b &= \frac{3h}{4} \\
\alpha = \beta &= \frac{2h}{3}.
\end{aligned}$$

Note that the method is now given by

$$\begin{aligned} k_1 &= hf\left(t_i, w_i\right) \\ k_2 &= hf\left(t_i + \frac{2h}{3}, w_i + \frac{2k_1}{3}\right) \\ w_{i+1} &= w_i + \frac{k_1}{4} + \frac{3k_2}{4} \end{aligned}$$

which is a two-stage explicit Runge-Kutta method with tableau

$$\begin{array}{c|cc} \frac{2}{3} & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}.$$

Of course, if the local error in an RK method is third-order, then the global error is second order, consistent with the two-stage nature of this method. Effectively, we have derived a two-stage method by imposing a condition on the nature of the local error.

2. Local error:

$$\varepsilon \equiv y_{i+1} - y_i - af_i - bf\left(x_i + c, y_i + df_i\right).$$

Expand y_{i+1} up to third-order (third-order terms are necessary because the local error is third-order):

$$y\left(x_{i+1}\right) = y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \ldots$$

Expand $f\left(x_i + c, y_i + df_i\right)$ up to second-order in c :

$$f\left(x_i + c, y_i + df_i\right) = f\left(x_i, y_i + df_i\right) + cf_x\left(x_i, y_i + df_i\right) + \frac{c^2}{2}f_{xx}\left(x_i, y_i + df_i\right) + \ldots$$

Expand each term in the above up to second order in c and d :

$$\begin{aligned} f\left(x_i, y_i + df_i\right) &= f_i + df_i f_y + \frac{d^2 f_i^2}{2} f_{yy} + \ldots \\ cf_x\left(x_i, y_i + df_i\right) &= cf_x + cdf_i f_{xy} + \ldots \\ \frac{c^2}{2}f_{xx}\left(x_i, y_i + df_i\right) &= \frac{c^2}{2}f_{xx} + \ldots \end{aligned}$$

It is understood that f and its various derivatives here are evaluated at $\left(x_i, y_i\right)$. Note that in the second equation above, the term in cd is regarded as second-order in c and d. Second-order terms in c and d will give third-order terms in h, due to multiplication by b (note that a, b, c and d are all linearly proportional to h). Third-order terms in c and d are unnecessary for this derivation. Note that

$$
\begin{aligned}
y'' &= f_x + f f_y \\
y''' &= f_{xx} + 2 f_i f_{xy} + f_i^2 f_{yy} + f_x f_y + f_i f_y^2 .
\end{aligned}
$$

Substituting yields

$$
\begin{aligned}
\varepsilon =& h f_i + \frac{h^2}{2}\left(f_x + f_i f_y\right) + \frac{h^3}{6}\left(f_{xx} + 2 f_i f_{xy} + f_i^2 f_{yy} + f_x f_y + f_i f_y^2\right) \\
& - a f_i - b\left(f_i + d f_i f_y + \frac{d^2 f_i^2}{2} f_{yy} + \ldots\right) \\
& - b\left(c f_x + c d f_i f_{xy} + \ldots\right) - \frac{b c^2}{2} f_{xx} + \ldots \\
=& \left(h - a - b\right) f_i + \left(\frac{h^2}{2} - bc\right) f_x + \left(\frac{h^2}{2} - bd\right) f_i f_y \\
& + \left(\frac{h^3}{6} - \frac{bc^2}{2}\right) f_{xx} + \left(\frac{h^3}{3} - bcd\right) f_i f_{xy} \\
& + \left(\frac{h^3}{6} - \frac{bd^2}{2}\right) f_i^2 f_{yy} + \frac{h^3}{6}\left(f_x f_y + f_i f_y^2\right) + \ldots
\end{aligned}
$$

If

$$
\begin{aligned}
& h - a - b = 0 && \tfrac{h^2}{2} - bc = 0 \\
& \tfrac{h^2}{2} - bd = 0
\end{aligned}
$$

and

$$
\begin{aligned}
& \tfrac{h^3}{6} - \tfrac{bc^2}{2} = h^3 && \tfrac{h^3}{3} - bcd = 2h^3 \\
& \tfrac{h^3}{6} - \tfrac{bd^2}{2} = h^3
\end{aligned}
$$

are simultaneously satisfied, then the local error has leading term

$$
\left(f_{xx} + 2 f_i f_{xy} + f_i^2 f_{yy} + \frac{f_x f_y + f_i f_y^2}{6}\right) h^3 .
$$

The first three of these equations are, effectively, the usual order conditions for an explicit second-order RK method. The next three equations arise from the condition we have placed on the form of the local error. It is easily shown that the above equations are satisfied by

$$
\begin{aligned}
a &= \frac{23h}{20} \\
b &= -\frac{3h}{20} \\
c = d &= -\frac{10h}{3} .
\end{aligned}
$$

Note that the method is now given by

$$
\begin{aligned}
k_1 &= hf\left(x_i, w_i\right) \\
k_2 &= hf\left(x_i - \frac{10h}{3}, w_i - \frac{10k_1}{3}\right) \\
w_{i+1} &= w_i + \frac{23k_1}{20} - \frac{3k_2}{20}
\end{aligned}
$$

which is a two-stage explicit Runge-Kutta method with tableau

$$
\begin{array}{c|cc}
-\frac{10}{3} & -\frac{10}{3} & \\
\hline
 & \frac{23}{20} & -\frac{3}{20}
\end{array}
$$

Of course, if the local error in an RK method is third-order, then the global error is second order, consistent with the two-stage nature of this method. We have derived a two-stage method by imposing a condition on the nature of the local error.

3. We perform the same Taylor expansions as in the exercise above, to yield

$$
\begin{aligned}
\varepsilon = &\left(h - \frac{23h}{20} + \frac{3h}{20}\right) f_i + \left(\frac{h^2}{2} - \left(-\frac{3}{20}\right)\left(-\frac{10h}{3}\right)\right) f_x \\
&+ \left(\frac{h^2}{2} - \left(-\frac{3h}{20}\right)\left(-\frac{10h}{3}\right)\right) f_i f_y + \left(\frac{h^3}{6} - \frac{\left(-\frac{3h}{20}\right)\left(-\frac{10h}{3}\right)^2}{2}\right) f_{xx} \\
&+ \left(\frac{h^3}{3} - \left(-\frac{3h}{20}\right)\left(-\frac{10h}{3}\right)\left(-\frac{10h}{3}\right)\right) f_i f_{xy} \\
&+ \left(\frac{h^3}{6} - \frac{\left(-\frac{3h}{20}\right)\left(-\frac{10h}{3}\right)^2}{2}\right) f_i^2 f_{yy} + \frac{h^3}{6}\left(f_x f_y + f_i f_y^2\right) + \ldots \\
= &\left(f_{xx} + 2f_i f_{xy} + f_i^2 f_{yy} + \frac{f_x f_y + f_i f_y^2}{6}\right) h^3 + \ldots \\
= &O\left(h^3\right).
\end{aligned}
$$

4. The method has the full tableau

$$
\begin{array}{c|cc}
c_1 & a_{11} & a_{12} \\
c_2 & a_{21} & a_{22} \\
\hline
 & b_1 & b_2
\end{array}
=
\begin{array}{c|cc}
0 & 0 & 0 \\
\frac{2}{3} & \frac{2}{3} & 0 \\
\hline
 & \frac{1}{4} & \frac{3}{4}
\end{array}
$$

so that

$$c_1 = a_{11} = a_{12} = a_{22} = 0$$
$$c_2 = \frac{2}{3} \quad a_{21} = \frac{2}{3} \quad b_1 = \frac{1}{4} \quad b_2 = \frac{3}{4}$$

The second-order conditions are

$$\sum b_i = b_1 + b_2 = 1 \qquad \text{(Euler condition)}$$
$$\sum b_i c_i = b_1 c_1 + b_2 c_2 = \frac{1}{2}$$
$$\sum b_i a_{ij} = b_1 a_{11} + b_1 a_{12} + b_2 a_{21} + b_2 a_{22} = \frac{1}{2}$$

and it is easily verified that these conditions are satisfied by the given values. Hence, the method has $O(h^2)$ global error (and $O(h^3)$ local error).

5. The relevant order conditions are

$$b_1 + b_2 = 1 \tag{1}$$
$$b_1 c_1 + b_2 c_2 = \frac{1}{2} \tag{2}$$
$$c_1 = a_{11} + a_{12} \tag{3}$$
$$c_2 = a_{21} + a_{22} \tag{4}$$

$$b_1 c_1^2 + b_2 c_2^2 = \frac{1}{3} \tag{5}$$
$$b_1 a_{11} c_1 + b_1 a_{12} c_2 + b_2 a_{21} c_1 + b_2 a_{22} c_2 = \frac{1}{6} \tag{6}$$

We are given $c_2 = 1$, so that

$$b_1 c_1 + b_2 = \frac{1}{2}$$
$$b_1 c_1^2 + b_2 = \frac{1}{3}$$

which gives, using (1),

$$b_1 c_1 + (1 - b_1) = \frac{1}{2} \Rightarrow b_1 (c_1 - 1) = -\frac{1}{2}$$
$$b_1 c_1^2 + (1 - b_1) = \frac{1}{3} \Rightarrow b_1 \left(c_1^2 - 1\right) = -\frac{2}{3}$$

and so

$$
\begin{aligned}
\frac{b_1\left(c_1-1\right)}{b_1\left(c_1^2-1\right)} &= \frac{c_1-1}{c_1^2-1} = \frac{3}{4} \\
\Rightarrow 3c_1^2 - 4c_1 + 1 &= 0 \\
\Rightarrow c_1 &= 1 \text{ or } \frac{1}{3}.
\end{aligned}
$$

If we choose $c_1 = 1$, the equation $b_1\left(c_1-1\right) = -\frac{1}{2}$ is contradicted, so we choose $c_1 = \frac{1}{3}$. This gives

$$
\begin{aligned}
b_1 &= \frac{3}{4} \\
b_2 &= \frac{1}{4}
\end{aligned}
$$

and, using (3) with $a_{12} = -\frac{1}{12}$,

$$
a_{11} = \frac{5}{12}.
$$

Equation (6) now gives

$$
a_{21} + 3a_{22} = \frac{3}{2}.
$$

Together with (4),

$$
1 = a_{21} + a_{22},
$$

we find

$$
\begin{aligned}
a_{21} &= \frac{3}{4} \\
a_{22} &= \frac{1}{4}.
\end{aligned}
$$

One of the fourth-order conditions is

$$
b_1c_1^3 + b_2c_2^3 = \frac{1}{4}
$$

but using the above values we find

$$
b_1c_1^3 + b_2c_2^3 = \frac{5}{18} \neq \frac{1}{4}.
$$

Hence, this method cannot be of order four.

6. We know that any explicit RK method applied to the Dahlquist equation gives

$$w_{i+1} = w_i R(h\lambda)$$

where $R(h\lambda)$ is the stability function of the relevant method. For RK2 and RK3 we have

$$\begin{aligned} w_{i+1}^2 &= w_i^3 \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) \\ w_{i+1}^3 &= w_i^3 \left(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6}\right). \end{aligned}$$

Note the factor w_i^3 in the expression for w_{i+1}^2 - this is due to the propagation of the higher order solution in local extrapolation. We estimate the local error in RK2 by

$$\begin{aligned} \beta_{i+1} h^3 &\approx w_{i+1}^2 - w_{i+1}^3 \\ &= -\frac{w_i^3 h^3 \lambda^3}{6}. \end{aligned}$$

We compute the new stepsize as

$$h^* \leqslant \left(\frac{\delta}{|\beta_{i+1}|}\right)^{\frac{1}{3}} = \left(\frac{6\delta}{|w_i^3|\,\lambda^3}\right)^{\frac{1}{3}}.$$

7. The RK1 solution at $x_1 = x_0 + h = 2 + h$ is

$$w_1^1 = y_0 + 2hy_0.$$

The RK2 solution at x_1 is

$$\begin{aligned} w_1^2 &= y_0 + \frac{2hy_0}{2} + \frac{h}{2}(2+h)(y_0 + 2hy_0) \\ &= y_0 + 2hy_0 + \frac{5h^2 y_0}{2} + h^3 y_0. \end{aligned}$$

The local error at x_1, $\beta_1 h^2$, is estimated from

$$\begin{aligned} \beta_1 h^2 &\approx w_1^1 - w_1^2 \\ &= -\left(\frac{5h^2 y_0}{2} + h^3 y_0\right) \\ &\approx -\frac{5h^2 y_0}{2} \end{aligned}$$

since $h \ll 1$. If the magnitude of the local error is to be less than δ, we must have

$$\frac{5h^2 |y_0|}{2} \leqslant \delta \;\Rightarrow\; h \leqslant \sqrt{\frac{2\delta}{5|y_0|}}.$$

8. At x_1 we have, for RK1,

$$w_1^1 = w_0 + hw_0^2 = 1 + h$$

and, for RK2,

$$\begin{aligned} w_1^2 &= w_0 + hw_0^2 + h^2 w_0^3 + \frac{h^3 w_0^4}{2} \\ &= 1 + h + h^2 + \frac{h^3}{2}. \end{aligned}$$

The local error is estimated from

$$\begin{aligned} \beta_1 h^2 &\approx w_1^1 - w_1^2 \\ &= -\left(h^2 + \frac{h^3}{2}\right) \\ &\approx -h^2 \end{aligned}$$

since $h \ll 1$. The new stepsize is given by

$$h^* \leqslant \left(\frac{\delta}{|\beta_1|}\right)^{\frac{1}{2}} = \sqrt{\delta}.$$

9. We write

$$\begin{aligned} w_{i+1} &= w_i + \frac{hk_1}{2} + \frac{hk_2}{2} \\ &= w_i + \frac{hf(x_i, w_i)}{2} + \frac{hf(x_i + h, w_i + hf(x_i, w_i))}{2} \end{aligned}$$

so that

$$\begin{aligned} F(x,y) &\equiv \frac{f(x,y)}{2} + \frac{f(x+h, y+hf(x,y))}{2} \\ &= \frac{1}{2}\left(xy^2 + e^y + (x+h)\left(y + hxy^2 + he^y\right)^2 + e^{y+hxy^2+he^y}\right). \end{aligned}$$

Clearly,

$$\begin{aligned}\lim_{h\to 0} F &= \frac{1}{2}\left(xy^2 + e^y + (x+0)\left(y + 0xy^2 + 0e^y\right)^2 + e^{y+0xy^2+0e^y}\right)\\ &= \frac{1}{2}\left(xy^2 + e^y + (x)(y)^2 + e^y\right)\\ &= xy^2 + e^y = f(x,y).\end{aligned}$$

Furthermore,

$$\begin{aligned}F_y = xy &+ \frac{e^y}{2} + (x+h)\left(y + hxy^2 + he^y\right)(1 + 2hxy + he^y)\\ &+ \frac{1}{2}e^{y+hxy^2+he^y}(1 + 2hxy + he^y)\end{aligned}$$

so that

$$\begin{aligned}\lim_{h\to 0} F_y &= xy + \frac{e^y}{2} + (x)(y)(1) + \frac{1}{2}e^y(1)\\ &= 2xy + e^y = f_y.\end{aligned}$$

10. The Jacobian of the system is given by

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2y_1 & 0 \end{bmatrix}$$

The eigenvalues of this matrix are

$$\phi = \frac{-1 \pm \sqrt{1 + 8y_1}}{2}.$$

When $y_1 < -\frac{1}{8}$ we have $1 + 8y_1 < 0$, and so

$$\begin{aligned}\phi &= \frac{-1 \pm \sqrt{-\left|1 + 8y_1\right|}}{2} = \frac{-1 \pm \mathbf{i}\sqrt{\left|1 + 8y_1\right|}}{2}\\ &= -\frac{1}{2} \pm \left(\frac{\sqrt{\left|1 + 8y_1\right|}}{2}\right)\mathbf{i}.\end{aligned}$$

Clearly, the real part of these eigenvalues is negative.

11. The Jacobian is

$$J = \begin{bmatrix} -1 & -\frac{1}{y_2} \\ \frac{y_1}{4} & 0 \end{bmatrix}.$$

The characteristic polynomial is found from

$$\det\left(\begin{bmatrix} -1-\lambda & -\frac{1}{y_2} \\ \frac{y_1}{4} & -\lambda \end{bmatrix}\right) = \lambda^2 + \lambda + \frac{y_1}{4y_2}.$$

Hence, the eigenvalues of J are

$$\lambda_1 = \frac{-1+\sqrt{1-y_1/y_2}}{2}$$
$$\lambda_2 = \frac{-1-\sqrt{1-y_1/y_2}}{2}.$$

Assume $y_1 > y_2$. If y_1 and y_2 are both greater than zero, then $\sqrt{1-y_1/y_2}$ is imaginary, so that $\mathrm{Re}\,(\lambda_2) = -1/2$ is negative. If y_1 and y_2 are both less than zero, then $\sqrt{1-y_1/y_2} < 1$, so that $\mathrm{Re}\,(\lambda_2)$ is negative. If $y_1 > 0$ and $y_2 < 0$, then $\sqrt{1-y_1/y_2} > 0$, so that $\mathrm{Re}\,(\lambda_2)$ is negative. Lastly, if $y_1 = 0$, $\mathrm{Re}\,(\lambda_2) = -1$ is negative. Hence, λ_2 has negative real part for all cases, and so is a stiff eiegenvalue.

12. The Jacobian is

$$J = \begin{bmatrix} -3y_2^2 & -6y_1y_2 & -\sin(y_3) \\ 0 & 3y_2^2 & 0 \\ 0 & \frac{1}{y_2} & 0 \end{bmatrix}.$$

The eigenvalues ϕ are determined from

$$\det\begin{bmatrix} -3y_2^2-\phi & -6y_1y_2 & -\sin(y_3) \\ 0 & 3y_2^2-\phi & 0 \\ 0 & \frac{1}{y_2} & -\phi \end{bmatrix} = 0$$

which gives

$$\phi^3 - 9y_2^4\phi = 0.$$

Hence, there are three eigenvalues:

$$\phi_1 = 0$$
$$\phi_2 = 3y_2^2$$
$$\phi_3 = -3y_2^2.$$

Since we assume that the solution is always nonzero, we have that $\phi_3 < 0$ for both $y_2 > 0$ and $y_2 < 0$. Hence, a stiff eigenvalue is always present.

13. This is the implicit Euler method. Applying it to the Dahlquist equation gives

$$
\begin{aligned}
w_{i+1} &= w_i + h\lambda w_{i+1} \\
\Rightarrow\ w_{i+1} &= \frac{w_i}{(1-h\lambda)}
\end{aligned}
$$

so that

$$R(h\lambda) = \frac{1}{(1-h\lambda)}.$$

Assume that $\lambda = a + b\mathbf{i}$, with $a < 0$. Hence, we can write

$$R(h\lambda) = \frac{1}{1-ha-hb\mathbf{i}} = \frac{1}{1+h|a|-hb\mathbf{i}}.$$

Now,

$$
\begin{aligned}
|R| = R^*R &= \left(\frac{1}{1+h|a|+hb\mathbf{i}}\right)\left(\frac{1}{1+h|a|-hb\mathbf{i}}\right) \\
&= \frac{1}{(1+h|a|)^2 + h^2b^2} \\
&< 1.
\end{aligned}
$$

14. We have

$$
\begin{aligned}
w_{i+1} = w_i + \frac{hk_1}{4} + \frac{3hk_2}{4} &= w_i + \frac{hk_1}{4} + \frac{3h}{4}\left(\lambda\left(w_i + \frac{2h}{3}k_1\right)\right) \\
&= w_i + \frac{h\lambda w_i}{4} + \frac{3h\lambda}{4}\left(w_i + \frac{2h\lambda w_i}{3}\right) \\
&= w_i\left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right).
\end{aligned}
$$

Using interval arithmetic, with $z \equiv h\lambda \in (-2,0)$,

$$
\begin{aligned}
R(z) &= z\left(\frac{z}{2}+1\right)+1 \\
\Rightarrow R \in (-2,0)&\left(\frac{(-2,0)}{2}+1\right)+1 \\
&= (-2,0)\left((-1,0)+1\right)+1 \\
&= (-2,0)(0,1)+1 \\
&= (-2,0)+1 \\
&= (-1,1)
\end{aligned}
$$

So

$$|R| < 1.$$

15. We use

$$R(z) = 1 + z\mathbf{B}\left(I_2 - zA\right)^{-1}\mathbf{1}_2$$

with

$$\begin{aligned} B &= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix} \\ I_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A &= \begin{bmatrix} \frac{5}{12} & -\frac{1}{12} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \\ \mathbf{1}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} R(z) &= 1 + z\mathbf{B}\left(I_2 - zA\right)^{-1}\mathbf{1}_2 \\ &= 1 + z\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}\left(\begin{bmatrix} 1 - \frac{5z}{12} & \frac{z}{12} \\ -\frac{3z}{4} & 1 - \frac{z}{4} \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 1 + z\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}\underbrace{\left(\frac{1}{1 - \frac{2z}{3} + \frac{z^2}{6}}\right)\left(\begin{bmatrix} 1 - \frac{z}{4} & -\frac{z}{12} \\ \frac{3z}{4} & 1 - \frac{5z}{12} \end{bmatrix}\right)}_{(I_2 - zA)^{-1}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 1 + \frac{z - \frac{z^2}{6}}{1 - \frac{2z}{3} + \frac{z^2}{6}} = \frac{1 - \frac{2z}{3} + \frac{z^2}{6} + z - \frac{z^2}{6}}{1 - \frac{2z}{3} + \frac{z^2}{6}} \\ &= \frac{2z + 6}{z^2 - 4z + 6}. \end{aligned}$$