Poisson's equation assignment

a) Set up a linear system whose solution approximates the solution of

$$\nabla^2 u\,(x, y) = 20x^3 + 12xy^2 \tag{1}$$

on

$$A = \{(x, y)\,|\,0 < x < 1,\; 0 < y < 1\}$$

subject to the boundary conditions

$$\begin{aligned} u\,(x, 0) &= x^5 \\ \left.\frac{\partial u}{\partial y}\right|_{(x,1)} &= 4x \\ u\,(1, y) &= 1 + y^4 \\ \left.\frac{\partial u}{\partial x}\right|_{(0,y)} &= y^4. \end{aligned}$$

Use the discretization

$$x_i = \frac{i}{3} \qquad\qquad y_j = \frac{j}{3}$$

for $i, j = 0, 1, \ldots, 3$. Make use of second-order approximations to the derivatives in the PDE and in the boundary conditions. Use a single-index notation for the approximate solution, as in

$$w_k = w_i^j$$

where $k = 13 + i - 4j$.

b) If this linear system is solved, we find that the approximate solution at each $(x, y) \in A$ is, in general, *not* equal to the exact solution evaluated at each $(x, y) \in A$. Explain this given that the analytical solution of (1) is

$$u\,(x, y) = x^5 + xy^4.$$