Assignment - von Neumann stability and truncation error

Consider the scheme

$$w_i^{j+1} = \frac{1}{2} w_n^j + \frac{1}{2} w_{n-1}^j$$

for solving the PDE

$$u_t + a u_x = 0.$$

Here a is a positive constant, $x_n - x_i = (n - i)\, h$ and

$$k = \frac{h\,(1 - 2n + 2i)}{2a}.$$

(a) Use the *von Neumann ansatz*

$$w_i^j = \lambda^j e^{\mathbf{i}\theta i}$$

to show that the above scheme is unconditionally stable. Here i denotes the space index, j denotes the time index (so w_i^j is the approximate solution at the point (x_i, t_j)), λ is an amplitude and θ is a phase angle related to the grid spacing,

(b) Define the *local truncation error* at (x_i, t_{j+1}) as

$$\tau_i^{j+1} \equiv \frac{u\,(x_i, t_{j+1}) - w_i^{j+1}}{k}.$$

Show that

$$\tau_i^{j+1} = \frac{a h u_{xx}\,(x_i, t_j)}{4\,(2n - 1 - 2i)} + \dots$$

for this scheme.

HINTS:

(a) Substitute the von Neumann expression into the difference equation and show that $\lambda^2 = \lambda\lambda^* \leqslant 1$.

(b) It is necessary to assume that the solution at time step j is known exactly at x_n and x_{n-1}, *i.e.* $w_n^j = u\,(x_n, t_j)$ and similarly for w_{n-1}^j. Expand in Taylor series about the point (x_i, t_j). Note that the PDE provides a relationship between the temporal and spatial derivatives of u.