APM8X13
Boundary Value Problem assignment

Consider the boundary value problem (BVP)

$$y'' = y\left(y'\right)^2 - \left(1 + \cos^2 x\right) y$$
$$x \in \left[0, \frac{\pi}{2}\right]$$

with solution

$$y\left(x\right) = \sin x.$$

Assume that $\left[0, \frac{\pi}{2}\right]$ is discretized according to the scheme

$$x_i = ih,\ i = 0, 1, 2, \ldots, N+1$$

where $h > 0$ denotes the uniform spacing between the nodes x_i.

a) If Newton's method is to be used to determine an approximate solution to the BVP at the nodes $\{x_1, \ldots, x_N\}$, determine the various entries in the relevant Jacobian matrix. Use the given solution to set the boundary values. Use the symbol w to denote the approximate solution. Use central difference approximations for the derivatives.

b) Use your expression for the intermediate rows of the Jacobian to show that, if $h \ll 1$, then the Jacobian should *not* be expected to be strictly diagonally dominant. You may use the exact solution $y\left(x\right)$ to facilitate your analysis, if necessary.

c) Let D_i^1 and D_i^2 denote central-difference formulae (including error terms) for y' and y'' at x_i, respectively, as in

$$D_i^1 = y_i' + \frac{h^2}{6} y_i''' + \ldots \qquad \text{and} \qquad D_i^2 = y_i'' + \frac{h^2}{12} y_i^{(4)} + \ldots.$$

Define the truncation error τ_i at x_i by

$$\tau_i \equiv f\left(x_i, y_i, y_i', y_i''\right) - f\left(x_i, y_i, D_i^1, D_i^2\right), \qquad i = 1, 2, \ldots, N.$$

Show that

$$\tau_i = Ah^2 + Bh^4 + \dots\,,$$

where A and B are coefficients independent of h.

d) Determine the numerical values of A and B at

$$x_i = \frac{\pi}{4}.$$