Conformally related Petrov type III spacetimes

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We generate new exact solutions to the Einstein field equations for a perfect fluid source by performing a conformal transformation on a known perfect fluid spacetime of type III in the Petrov classification scheme. The Allnut (1981) solution is the only known solution of type III and was obtained using the Newman-Penrose formalism. The simplicity of the solution ostensibly lends itself to the conformal transformation approach to generating new solutions. This approach is algebraically special as it constitutes restrictions on the Weyl conformal tensor which essentially invokes symmetries on the spacetime manifold. Solutions of the Einstein field equations are important in relativistic astrophysics. They provide a means to study holistically the evolution and stability of celestial phenomena.

A theorem due to Defrise-Carter (1975) asserts the following. Suppose that a manifold $(M, \mathbf{g})$ is neither conformally flat nor conformally related to a generalised plane wave. Then a Lie algebra of conformal Killing vectors on M with respect to $\mathbf{g}$ can be regarded as a Lie algebra of Killing vectors with regard to some metric on M conformally related to $\mathbf{g}$. Therefore if a spacetime admits the conformal group C_s, then either it is conformally flat ($s = 15$), conformally related to a generalised plane wave ($s \leq 7$), or the metric $\bar{g}_{ab} = e^{2U} g_{ab}$ where g_{ab} admits an s-dimensional ($s \leq 6$) isometry group. We have previously considered Petrov Type D spacetimes within the above framework and have been successful in generating new exact solutions with the aid of Lie group analysis methods (Hansraj, *et al* 2006). Additionally we have succeeded in obtaining new exact solutions for perfect fluids that are conformal to a vacuum (Ricci flat) spacetime such as the non-conformally flat Schwarzschild exterior solution.

The next stage is to determine whether new perfect fluid solutions can be obtained from known perfect fluid solutions. That is solutions that are non-Ricci flat. This has been achieved with the type III solution of Allnut (1981). This perfect fluid metric has the form

$$ds^2 = z^2 \left[-\frac{dt^2}{1+t^2} + f dx^2 + \frac{t^2\left(1+t^2\right)}{f} dy^2 \right] + dz^2$$

where $f = t^{2\beta}\left(1+t^2\right)^{1-\beta}$ and β is a constant. In the conformally related regime, the conformal factor is computed after several arduous calculations and is given by

$$e^U = -\left(z e^{-\frac{1}{\sqrt{1-\beta}} \arctan \frac{t}{\sqrt{1-\beta}}} \left[\frac{-k \arctan \frac{t}{\sqrt{1-\beta}}}{\sqrt{1-\beta}} + C \right] + kz + C_1 + C_2 \ln z \right)^{-1}$$

The metric $\tilde{ds}^2 = e^{2U} ds^2$ is a new solution to the Einstein field equations for a perfect fluid source. The solution is under investigation for physical plausibility. In particular, we wish to study the pressure and energy density profiles as well as the principle of causality. Additionally if the solution admits an equation of state it will possess the desirable qualitative features of a realistic stellar model.