CLASSICAL AND QUANTUM COMPUTING
EXERCISE III

1) Consider the 3×3 matrix (Beskou die 3×3 matriks)

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) The matrix A can be considered as an element of the Hilbert space of the 3×3 matrices with the scalar product $\langle A, B \rangle := \mathrm{tr}(AB^T)$. Die matriks A kan as 'n element van die Hilbertruimte van 3×3 matrikse beskou word met die skalaarprodukt $\langle A, B \rangle := \mathrm{tr}(AB^T)$. Find the norm of A with respect to this Hilbert space. Vind die norm van A met betrekking tot hierdie Hilbertruimte.

b) On the other hand A can be considered as a linear operator in the Hilbert space $\mathbf{R}^3$. Andersins kan A as 'n lineêre operator in die Hilbertruimte $\mathbf{R}^3$ beskou word. Find die norm (Vind die norm)

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|, \qquad \mathbf{x} \in \mathbf{R}^3$$

c) Find the eigenvalues of A and AA^T. Vind die eiewaardes van A en AA^T. Compare the result with a) and b). Vergelyk die resultaat met a) en b).

2) Consider the Hilbert space $L_2[0,1]$. Beskou die Hilbertruimte $L_2[0,1]$. Find a non-trivial function f such that (Vind 'n nie-triviale funksie f sodanig dat

$$\langle f(x), x \rangle = 0, \qquad \langle f(x), x^2 \rangle = 0, \qquad \langle f(x), x^3 \rangle = 0$$

where $\langle \, , \, \rangle$ denotes the scalar product (waar $\langle \, , \, \rangle$ die skalaarprodukt aandui).

3) Consider the function $f \in L_2[0,1]$ (Beskou die funksie $f \in L_2[0,1]$)

$$f(x) = \begin{cases} x & \text{for} \quad 0 \leq x \leq 1/2 \\ 1 - x & \text{for} \quad 1/2 \leq x \leq 1 \end{cases}$$

A basis in the Hilbert space is given by ('n basis in die Hilbertruimte word gegee deur)

$$\mathcal{B} := \left\{ 1, \sqrt{2}\cos(\pi n x) \quad : \quad n = 1, 2, \ldots \right\} .$$

Find the Fourier expansion of f with respect to this basis. Vind die Fourieruitbreiding van f met betrekking tot hierdie basis. From this expansion show that (Uit hierdie uitbreiding, toon aan dat)

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$