CLASSICAL AND QUANTUM COMPUTING
EXERCISE II

1) Consider the Hilbert space $L_2[-\pi, \pi]$. Beskou die Hilbertruimte $L_2[-\pi, \pi]$. Given the function (Gegee die funksie)

$$f(x) = \begin{cases} 1 & 0 < x \leq \pi \\ 0 & x = 0 \\ -1 & -\pi \leq x < 0 \end{cases}$$

Find the Fourier expansion of f. Vind die Fourieruitbreiding van f. The basis is given by (Die basis word gegee deur)

$$\mathcal{B} := \left\{ \phi_k(x) = \frac{1}{\sqrt{2\pi}} \exp(ikx) \quad k \in \mathbf{Z} \right\}.$$

Draw the function (Teken die funksie)

$$a_0\phi_0(x) + a_1\phi_1(x) + a_{-1}\phi_{-1}(x)$$

where a_0, a_1, a_{-1} are the Fourier coefficients (waar a_0, a_1, a_{-1} die Fourierkoëffisiente is). Compare to f. Vergelyk met f.

2) Consider the Hilbert space $L_2(0, \pi)$. Beskou die Hilbertruimte $L_2(0, \pi)$.

i) Find (Vind)

$$f(a, b) = \| \sin(x) - (ax^2 + bx)\|^2$$

where (waar) $a, b \in \mathbf{R}$.

ii) Find a, b such that $f(a, b)$ is a minimum. Vind a, b sodanig dat $f(a, b)$ ń minimum is.

3) Consider the function $H \in L_2(\mathbf{R})$ (Beskou die funksie $H \in L_2(\mathbf{R})$)

$$H(x) = \begin{cases} 1 & 0 \le x \le 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let (Laat)

$$H_{mn}(x) := 2^{-m/2} H(2^{-m}x - n)$$

where (waar) $m, n \in \mathbf{Z}$. Draw a picture of H_{11}, H_{21}, H_{12}, H_{22}. Teken 'n skets van H_{11}, H_{21}, H_{12}, H_{22}. Show that (Toon aan dat)

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \delta_{mk}\delta_{nl}, \qquad k, l \in \mathbf{Z}$$

where $\langle\,.\,\rangle$ denotes the scalar product in $L_2(\mathbf{R})$ (waar $\langle\,.\,\rangle$ die skalaarproduk in $L_2(\mathbf{R})$ aandui). Expand the function (Brei die funksie)

$$f(x) = \exp(-|x|)$$

with respect to H_{mn} (met betrekking tot H_{mn} uit). Remark. The functions H_{mn} form an orthonormal basis in $L_2(\mathbf{R})$. Opmerking. Die funksies H_{mn} vorm 'n ortonormal basis in $L_2(\mathbf{R})$.