CLASSICAL AND QUANTUM COMPUTING
EXERCISE XI

a) We consider the Hilbert space $L_2[-\frac{1}{2}, \frac{1}{2}]$. Expand the step function

$$f(x) := \begin{cases} -1 & x \in [-\frac{1}{2}, 0] \\ 1 & x \in (0, \frac{1}{2}] \end{cases}$$

with respect to the Fourier basis

$$\{\phi_k(x) := \exp(2\pi i k x)\}$$

where $k \in \mathbf{Z}$. In other words calculate the expansion coefficients $\langle f(x), \phi_k(x)\rangle$ where

$$\langle f(x), g(x)\rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{g(x)}dx.$$

Thus we determine

$$f(x) = \sum_{k \in \mathbf{Z}} \langle f(x), \phi_k(x)\rangle \phi_k(x).$$

Write a computer program which implements $f(x)$ using the expansion for $k \in \{-1, 0, 1\}$. Test the approximation with the inputs -0.5, -0.2, 0.3 and 0.5. For given $\epsilon > 0$ determine for which k, $|\langle f(x), \phi_k(x)\rangle| > \epsilon$.

b) We consider the Hilbert space $L_2(\mathbf{R})$. Expand the step function

$$f(x) := \begin{cases} -1 & x \in [-\frac{1}{2}, 0] \\ 1 & x \in (0, \frac{1}{2}] \end{cases}$$

with respect to the Haar basis

$$\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)\}$$

where $j, k \in \mathbf{Z}$ and

$$\psi(x) = \begin{cases} -1 & x \in [0, \frac{1}{2}] \\ 1 & x \in (\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases} .$$

In other words calculate the expansion coefficients $\langle f(x), \psi_{jk}(x)\rangle$ where

$$\langle f(x), g(x)\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

Thus we determine

$$f(x) = \sum_{j,k\in\mathbf{Z}} \langle f(x), \psi_{jk}(x)\rangle \psi_{jk}(x).$$

Write a computer program which implements $f(x)$ using the expansion for $j, k \in \{-1, 0, 1\}$. Test the approximation with the inputs -0.5, -0.2, 0.3 and 0.5. For given $\epsilon > 0$ determine for which j, k, $|\langle f(x), \psi_{jk}(x)\rangle| > \epsilon$.