Problems and Solutions in Optimization

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Preface

The purpose of this book is to supply a collection of problems in optimization theory.

Prescribed book for problems.

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Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\|z\|$	modulus of complex number z
	$\|x+iy\| = (x^2+y^2)^{1/2}, \ \ x, y \in \mathbb{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbb{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\\| \, . \, \\|$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbb{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbb{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of matrix A
A^T	transpose of matrix A
$\overline{A}$	conjugate of matrix A

A^*	conjugate transpose of matrix A
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
H	Hamilton function
L	Lagrange function

Chapter 1

General

1.1 One-Dimensional Functions

1.1.1 Solved Problem

Problem 1. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 4x(1-x).$$

(i) The fixed points of the function f are the solutions of the equation $f(x^*) = x^*$. Find the fixed points.
(ii) The critical points of f are the solutions of the equation $df(x)/dx = 0$. Find the critical points of f. If there are critical points determine whether they relate to minima or maxima.
(iii) The roots of the function f are the solutions of $f(x) = 0$. Find the roots of f.
(iv) Find the fixed points of the analytic function $g(x) = f(f(x))$.
(v) Find the critical points of the analytic function $g(x) = f(f(x))$. If there are critical points of g determine whether they relate to minima and maxima.
(vi) Find the roots of the analytic function $g(x) = f(f(x))$.

Problem 2. Let $x \in (-\infty, \infty)$. Find

$$f(x) = \min_{x \in \mathbb{R}}(1, e^{-x}).$$

Is the function f differentiable? Discuss.

Problem 3. Consider the analytic functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2 + 1, \qquad g(x) = 2x + 2.$$

Find the function

$$h(x) = f(g(x)) - g(f(x)).$$

Find the extrema of h.

Problem 4. Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = x^x.$$

(i) Find the extrema of f.
(ii) Can f be defined for $x = 0$?

Problem 5. Let $\omega > 0$, $c > 0$ and $g > 0$. Find the minima and maxima of the function $V : \mathbb{R} \to \mathbb{R}$

$$V(x) = \omega^2 x^2 + \frac{cx^2}{1 + gx^2}.$$

The function (potential) plays a role in quantum mechanics.

Problem 6. Find the minima and maxima of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sin(2^x).$$

Problem 7. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \cos(\sin(x)) - \sin(\cos(x)).$$

Show that f admits at least one critical point. By calculating the second order derivative find out whether this critical point refers to a maxima or minima.

Problem 8. Let $x \in \mathbb{R}$ and $x > 0$. Show that

$$\sqrt[e]{e} \geq \sqrt[x]{x}.$$

Is $f(x) = \sqrt[x]{x}$ convex? Calculate $\lim_{x \to \infty} f(x)$.

Problem 9. (i) Find the minima of the function $f : \mathbb{R} \to [0, \infty)$

$$f(x) = \ln(\cosh(2x)).$$

(ii) Let $\epsilon > 0$. Find the maxima and minima of the function $f_\epsilon : \mathbb{R} \to \mathbb{R}$

$$f_\epsilon(x) = \ln(\cosh(\epsilon x)).$$

Problem 10. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sin(x) \sinh(x).$$

Show that $x = 0$ provides a local minimum.

Problem 11. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1 - \cos(2\pi x)}{x}.$$

(i) Define $f(0)$ using L'Hosptial rule.
(ii) Find all the critical points.
(iii) Find the maxima and minima of the function.

Problem 12. (i) Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) := \exp(-1/x^2).$$

Find the minima and maxima of f.
(ii) Calculate the Gateaux derivative of f.

Problem 13. Find the minima and maxima of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x^2 e^{-x} & \text{for } x \geq 0 \end{cases}$$

Problem 14. Let $x \geq 0$ and $c > 0$. Find the minima and maxima of the analytic function

$$f(x) = xe^{-cx}.$$

Problem 15. Let $x > 0$. Is

$$\ln(x) \leq x - 1 \quad \text{for all} \quad x > 0 \ ?$$

Prove or disprove by constructing a function f from the inequality and finding the extrema of this function by calculating the first and second order derivatives.

Problem 16. Let $x < 0$. Find the maxima of the function

$$f(x) = x + \frac{1}{x}.$$

Problem 17. Let $a > 0$ and $x > 0$. Find the minima of

$$f(x) = x^2 + \frac{a^2}{x^2}.$$

Problem 18. Let $\ell = 10\,meter$, $b = 4\,meter$ and $\alpha \in [0, \pi/2)$. Find the maximum of the function

$$h(\alpha) = \ell \sin(\alpha) - b \tan(\alpha).$$

Problem 19. Find the minima and maxima of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f_{\sigma,\omega}(t) = \text{sech}(\sigma t)\cos(\omega t).$$

In practial application one normally has $\omega \geq \sigma$.

Problem 20. The intensity of a Laguerre-Gaussian laser mode of indices ℓ and p is given by

$$I(r,z) = \frac{2p!}{\pi(p+|\ell|)!}\frac{P_0}{w^2(z)} \times \exp\left(-2\frac{r^2}{w^2(z)}\right)\left(\frac{2r^2}{w^2(z)}\right)^{|\ell|}\left(L_p^{|\ell|}\left(\frac{2r^2}{w^2(z)}\right)\right)^2$$

and the waist size at any position is given by

$$w(z) = w_0\sqrt{1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2}.$$

Here z is the distance from the beam waist, P_0 is the power of the beam, λ is the wavelength, $w(z)$ is the radius at which the Gaussian term falls to $1/e^2$ of its on-axis value, w_0 is the beam waist size and L_p^ℓ is the generalized Laguerre polynomial. The generalized Laguerre polynomials are defined by

$$L_p^{(\alpha)}(x) := \frac{x^{-\alpha}e^x}{p!}\frac{d^p}{dx^p}(e^{-x}x^{p+\alpha}), \qquad p = 0, 1, 2, \ldots$$

Find the maximum intensity of beam as function of z for $p = 0$ and $\ell > 0$.

Problem 21. Consider the function $f : (0, \infty) \to \mathbb{R}$

$$f(x) = x + \frac{1}{x}.$$

Find the minimum of this function using the following considerations. If x is large then the term x dominates and we are away from the minimum. If x is small (close to 0) then the term $1/x$ dominates and we are away from the minimum. Thus for the minimum we are looking for a number which is not to large and not to small (to close to 0). Compare the result from these considerations by differentiating the function f finding the critical points and check whether there is a minimum.

Problem 22. Let $r \geq 0$ and $k > 0$. Consider the potential

$$V(r) = \frac{1}{r} - \frac{e^{-kr}}{r}.$$

Find $V(0)$. Find the maximum of the potential $V(r)$.

Problem 23. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable and let $x \in \mathbb{R}$ with $f'(x) \neq 0$. Let $\epsilon \in \mathbb{R}$ and $\epsilon > 0$ such that $f'(x)f'(y) > 0$ for $y \in (x - \epsilon, x + \epsilon)$. Show that for

$$0 < \eta < \frac{\epsilon}{|f'(x)|}$$

it follows that

$$f\left(x - \eta\frac{df}{dx}\right) < f(x),$$

and

$$f\left(x + \eta\frac{df}{dx}\right) > f(x).$$

Hint: Apply the mean-value theorem.

Problem 24. Let $x \in \mathbb{R}$. Show that $e^x \geq 1 + x$ by considering the function

$$f(x) = e^x - (1 + x).$$

Problem 25. Can one find polynomials $p : \mathbb{R} \to \mathbb{R}$ such that one critical point of p and one fixed point of p coincide?

Problem 26. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and let $x \in \mathbb{R}$ with $f'(x) \neq 0$. Let $\epsilon \in \mathbb{R}$ and $\epsilon > 0$ such that $f'(x)f'(y) > 0$ for $y \in (x - \epsilon, x + \epsilon)$. Show that for

$$0 < \eta < \frac{\epsilon}{|f'(x)|}$$

it follows that

$$f\left(x - \eta\frac{df}{dx}\right) < f(x)$$

and

$$f\left(x + \eta\frac{df}{dx}\right) > f(x).$$

Hint: Apply the mean-value theorem.
(ii) Consider the sequence

$$x_0 \in \mathbb{R}, \quad x_{j+1} = x_j - \eta f'(x_j)$$

where $\eta > 0$. Assume that

$$\lim_{j\to\infty} x_j = x_*$$

exists and

$$\lim_{j\to\infty} f(x_j) = f(x_*), \qquad \lim_{j\to\infty} f'(x_j) = f'(x_*).$$

Show that $f'(x_*) = 0$.

Problem 27. (i) Give a smooth function $f : \mathbb{R} \to \mathbb{R}$ that has no fixed point and no critical point. Draw the function f and the function $g(x) = x$. Find the inverse of f.
(ii) Give a smooth function $f : \mathbb{R} \to \mathbb{R}$ that has exactly one fixed point and no critical point. Draw the function f and the function $g(x) = x$. Is the fixed point stable? Find the inverse of f.

Problem 28. Consider the continuous function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{|x|}{1 + |x|}.$$

Find the minima and maxima of f.

Problem 29. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2 + x + 1.$$

Find the minima and maxima of

$$g(x) = f(f(x)) - f(x)f(x).$$

Problem 30. Let $x \geq 0$. Study minima and maxima for the functions

$$f_1(x) = 2(\sqrt{\ln(2)} - \sqrt{x - \ln(\cosh(x))}), \quad f_2(x) = 2\sqrt{\ln(2)\cosh(x) - x\tanh(x)}.$$

Problem 31. Study minima and maxima of the analytic functions

$$f_1(x) = e^x - 1 - x$$

$$f_2(x) = e^x - 1 - x - \frac{x^2}{2}$$

$$f_3(x) = e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!}.$$

Problem 32. Let $c_1 > 0$, $c_2 > 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2 + \frac{c_1 x^2}{1 + c_1 x^2}.$$

Find the minima and maxima. Find the fixed points.

Problem 33. Let $c > 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f_c(x) = \frac{4}{x^2} \sin^2\left(\frac{cx}{2}\right).$$

Find $f_c(0)$. Show that f_c has a maximum at $x = 0$.

Problem 34. (i) Find the minima and maxima of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \cosh(x)\cos(x).$$

Find the fixed points.
(ii) Find the minima and maxima of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sinh(x)\sin(x).$$

Find the fixed points.

Problem 35. Find the minima of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 100(y - x^2)^2 + (1 - x^2).$$

Problem 36. Find the minima and maxima of the analytic functions

$$f_1(x) = \sinh(x)\sin(x), \quad f_2(x) = \cosh(x)\cos(x), \quad f_3(x) = \sinh(x)\cos(x), \quad f_4(x) = \cosh(x)\sin(x).$$

Problem 37. Let $\epsilon > 0$ and fixed. Find the minima and maxima of

$$f_\epsilon(x) = \frac{1}{\epsilon}\max\left(1 - \frac{|x|}{|\epsilon|}, 0\right).$$

Draw the function for $\epsilon = 2, 1, 1/2, 1/4$.

Problem 38. Let $n \geq 1$ and $c_1, \ldots, c_n$ be constants. Find the minima of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{j=1}^{n} (x - c_j)^2.$$

Problem 39. Find the extrema of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \ln(4\cosh(2x)).$$

Problem 40. Find the minima and maxima of the functions

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy, \qquad \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy = 1 - \text{erf}(x).$$

Problem 41. Consider the function $f : \mathbb{R} \setminus \{0\} \to [-1, 1]$

$$f(x) = \cos(\pi/x).$$

Find the minima and maxima of f.

1.1.2 Supplementary Problem

Problem 42. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(\alpha) = \cos(\alpha) + 2\sin(\alpha).$$

Find the maxima and minima of the function. Note that

$$\frac{df}{d\alpha} = -sin(\alpha) + 2\cos(\alpha), \quad \frac{d^2 f}{d\alpha^2} = -\cos(\alpha) + 2\sin(\alpha).$$

Problem 43. Find the minima of the functions

$$f_1(x) = \cosh(x), \quad f_2(x) = \cosh(x) - 1 - \frac{x^2}{2!}, \quad f_3(x) = 4\arctan(\exp(x)).$$

1.2 Curves

Problem 44. Find the shortest distance in the plane $\mathbb{R}^2$ between the curves

$$x^{2/3} + y^{2/3} = 1, \qquad x^2 + y^2 = 4.$$

A graphical solution would be fine. Find a parameter representation for the two curves.

Problem 45. Consider the subset of $\mathbb{R}^2$

$$S = \{ (x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 x_2 = 1 \}.$$

It actually consists of two non-intersecting curves. Find the shortest distance between these two curves. A graphical solution would be fine. Find the normal vectors at $(1, 1)$ and $(-1, -1)$. Discuss.

Problem 46. Consider the *elliptic curve*

$$y^2 = x^3 - x$$

over the real numbers. Actually the elliptic curve consists of two non-intersecting components. Find the shortest distance between these two components. Draw a picture.

Problem 47. (i) Apply *polar coordinates* to describe the set in the plane $\mathbb{R}^2$ given by the equation

$$(x^2 - y^2)^{2/3} + (2xy)^{2/3} = (x^2 + y^2)^{1/3}.$$

(ii) What is the longest possible distance between two points in the set?
(iii) Is the set convex? A subset of $\mathbb{R}^n$ is said to be convex if for any $\mathbf{a}$ and $\mathbf{b}$ in S and any θ in $\mathbb{R}$, $0 \le \theta \le 1$, the n-tuple $\theta \mathbf{a} + (1 - \theta)\mathbf{b}$ also belongs to S.

Problem 48. An *elliptic curve* is the set of solutions (x, y) to an equation of the form

$$y^2 = x^3 + ax + b$$

with $4a^3 + 27b^2 \neq 0$. Consider $y^2 = x^3 - 4x$. We find two curves which do not intersect. Find the shortest distance between these two curves.

Problem 49. Find the shortest distance between the curves

$$x^3 + y^3 - 3xy = 0, \qquad (x - 4)^2 + (y - 4)^2 = 1.$$

Perhaps the symmetry of the problem could be utilizied.

Problem 50. Find the shortest distance between the curves

$$(x^2 + y^2 - 1)^3 + 27x^2y^2 = 0, \qquad x^2 + y^2 = 4$$

in the plane $\mathbb{R}^2$. A graphical solution would be fine.

1.3 Two-Dimensional Functions

Problem 51. Consider the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^2 + xy + y^2.$$

(i) Find the critical points.
(ii) Find minima and maxima.
(iii) Solve $f(x, y) = 0$.

Problem 52. Find the minima of the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^2 x_2^2 e^{-(x_1^2+x_2^2)}$$

by inspection of the function. Study the cases $|x_1| \to \infty$ and $|x_2| \to \infty$. Does the function have maxima?

Problem 53. Consider the Euclidean plane $\mathbb{R}^2$. Let

$$A = \begin{pmatrix} 0 \\ -10 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Find the point(s) C on the parabola $y = x^2$ which minimizes the area of the triangle ABC. Draw a picture.

Problem 54. Let $z = x + iy$ with $x, y \in \mathbb{R}$. Find the maxima and minima of the functions

$$f_1(x, y) = \frac{z + \bar{z}}{1 + z\bar{z}}$$

$$f_2(x, y) = \frac{i(\bar{z} - z)}{1 + z\bar{z}}$$

$$f_3(x, y) = \frac{1 - z\bar{z}}{1 + z\bar{z}}.$$

Are the functions bounded?

Problem 55. Let $c > 0$. Find the minima and maxima of the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = |x_1 - x_2| e^{-c|x_1 - x_2|}.$$

Problem 56. Show that the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(\alpha, \beta) = \sin(\alpha)\sin(\beta)\cos(\alpha - \beta)$$

is bounded between $-1/8$ and 1.

Problem 57. Find the minima of the analytic function

$$V(x,y) = 2 + \cos(x) + \cos(y).$$

Problem 58. Find the largest triangle that can be inscribed in the *ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Problem 59. Find all extrema (minima, maxima etc) for the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1 x_2.$$

Problem 60. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x_1, x_2, x_3) = \sin(x_1^2 + x_2^2 + x_3^2) + x_1 x_2 + x_1 x_3 + x_2 x_3.$$

(i) Show that the function f has a local minimum at $(0,0,0)$.
(ii) Does the function f have other extrema?

Problem 61. Let $x = x(s)$ and $y = y(s)$ for $s \in S \subseteq \mathbb{R}$. S is an open interval and $s^* \in S$. Let $f(x,y)$ be a function where

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

exist for all $x(s)$ and $y(s)$ where $s \in S$. Suppose $\frac{\partial f}{\partial x}(x^*, y^*) = \frac{\partial f}{\partial y}(x^*, y^*) = 0$. Show that

$$\left. \frac{d^2}{ds^2} f(x(s), y(s)) \right|_{x=x^*, y=y^*} = \left. \mathbf{z}^T H_f \, \mathbf{z} \right|_{x=x^*, y=y^*}$$

where (H_f is the *Hessian* matrix) $x^* = x(s^*)$, $y^* = y(s^*)$

$$\mathbf{z} = \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix}, \qquad H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$

What are the constraints on $\mathbf{z}$? Can we determine whether f has a minimum or maximum at (x^*, y^*)?

Problem 62. Find minima and maxima of the function

$$f(x,y) = \ln \left(\frac{\cosh(\pi x) + \cos(\pi y)}{\cosh(\pi x) - \cos(\pi y)} \right).$$

Problem 63. Let $x = x(s)$ and $y = y(s)$ for $s \in S \subseteq \mathbb{R}$. S is an open interval and $s^* \in S$. Let $f(x, y)$ and $g(x, y)$ be functions where

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial g}{\partial x} \neq 0, \quad \frac{\partial g}{\partial y} \neq 0$$

exist for all $x(s)$ and $y(s)$ where $s \in S$. Suppose

$$g(x(s), y(s)) \equiv 0.$$

Let (treat λ as independent of s)

$$F(x, y) := f(x, y) + \lambda g(x, y).$$

Suppose

$$\frac{\partial f}{\partial x}(x^*, y^*) + \lambda \frac{\partial g}{\partial x}(x^*, y^*) = \frac{\partial f}{\partial y}(x^*, y^*) + \lambda \frac{\partial g}{\partial y}(x^*, y^*) = 0.$$

Show that

$$\frac{d^2}{ds^2} F(x(s), y(s)) \Big|_{x=x^*, y=y^*} = \mathbf{z}^T H_F \, \mathbf{z} \Big|_{x=x^*, y=y^*}$$

where (H_F is the *Hessian matrix*) $x^* = x(s^*)$, $y^* = y(s^*)$,

$$\mathbf{z} = \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix}, \qquad H_F = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix}.$$

What are the constraints on $\mathbf{z}$? Can we determine whether f has a minimum or maximum at (x^*, y^*) subject to $g(x, y) = 0$?

Problem 64. Find the maxima and minima of the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f((x_1, x_2) = x_1^2 - x_2^2.$$

Problem 65. Determine the locations of the extrema of

$$f(x, y) = x^2 y^2 - 9x^2 - 4y^2, \qquad x, y \in \mathbb{R}.$$

Classify the extrema as maxima or minima using the Hessian criteria.

Problem 66. Determine the locations of the extrema of

$$f(x, y) = \sin(x^2 + y^2), \qquad x, y \in \mathbb{R}.$$

Classify the extrema as maxima or minima using the Hessian criteria. Draw the graph of $f(x, y)$.

Problem 67. (i) Is the set

$$S := \{ (x_1, x_2) \, : \, -x_1 + 2x_2 \leq 6, \quad x_1 + x_2 \leq 5, \quad x_1 \geq 0, \;\; x_2 \geq 0 \}$$

convex?
(ii) Let

$$M := \left\{ (x_1, x_2) \, : \, x_1 - 3x_2 = -\frac{29}{3} \right\}.$$

Find $M \cap S$. Is $M \cap S$ convex?

Problem 68. Consider the differentiable function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1 - \cos(2\pi x)}{x}$$

where using L'Hosptial $f(0) = 0$.
(i) Find the zeros of f.
(ii) Find the maxima and minima of f.

Problem 69. Let the function f be of class $C^{(2)}$ on an open set O ($O \subset \mathbb{R}^n$) and $\mathbf{x}_0 \in O$ a critical point. Then

$$\left(\frac{\partial^2 f}{\partial x_i x_j} \right) \Bigg|_{\mathbf{x}_0} \geq 0$$

is necessary for a relative minimum at $\mathbf{x}_0$,

$$\left(\frac{\partial^2 f}{\partial x_i x_j} \right) \Bigg|_{\mathbf{x}_0} > 0$$

is sufficient for a strict relative minimum at $\mathbf{x}_0$,

$$\left(\frac{\partial^2 f}{\partial x_i x_j} \right) \Bigg|_{\mathbf{x}_0} \leq 0$$

is necessary for a relative maximum at $\mathbf{x}_0$,

$$\left(\frac{\partial^2 f}{\partial x_i x_j} \right) \Bigg|_{\mathbf{x}_0} < 0$$

is sufficient for a strict relative maximum at $\mathbf{x}_0$. Apply it to the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^4 - 2x_1^2 x_2^2 + x_2^4.$$

Problem 70. Determine the locations of the extrema of

$$f(x, y) = x^2 - y^2, \qquad x^2 + y^2 \leq 1$$

using *polar coordinates* $x = r\cos\theta$ and $y = r\sin\theta$. Classify the extrema as maxima or minima.

Problem 71. Consider the curve given by

$$x_1(t) = \cos(t)(2\cos(t) - 1)$$
$$x_2(t) = \sin(t)(2\cos(t) - 1)$$

where $t \in [0, 2\pi]$. Draw the curve with GNUPLOT. Find the longest distance between two points on the curve.

Problem 72. Consider the two hyperplane ($n \geq 1$)

$$x_1 + x_2 + \cdots + x_n = 2, \qquad x_1 + x_2 + \cdots + x_n = -2.$$

The hyperplanes do no intersect. Find the shortest distance between the hyperplanes. What happens if $n \to \infty$? Discuss first the cases $n = 1$, $n = 2$, $n = 3$.

Problem 73. (i) Consider the two-dimensional Euclidean space and let $\mathbf{e}_1$, $\mathbf{e}_2$ be the standard basis

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider the vectors

$$\mathbf{v}_0 = \mathbf{0}, \quad \mathbf{v}_1 = \frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_2 = -\frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2.$$

$$\mathbf{v}_3 = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_4 = \frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_5 = -\mathbf{e}_1, \quad \mathbf{v}_6 = \mathbf{e}_1.$$

Find the distance between the vectors and select the vectors pairs with the shortest distance.

Problem 74. Let $x = x(s)$ and $y = y(s)$ for $s \in S \subseteq \mathbb{R}$. S is an open interval and $s^* \in S$. $f(x, y)$ be a function where

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

exist for all $x(s)$ and $y(s)$ where $s \in S$. Suppose $\frac{\partial f}{\partial x}(x^*, y^*) = \frac{\partial f}{\partial y}(x^*, y^*) = 0$. Show that

$$\frac{d^2}{ds^2} f(x(s), y(s))\Big|_{x=x^*, y=y^*} = \mathbf{z}^T H_f\, \mathbf{z}\Big|_{x=x^*, y=y^*}$$

where (H_f is the *Hessian* matrix) $x^* = x(s^*)$, $y^* = y(s^*)$

$$\mathbf{z} = \begin{pmatrix} \dfrac{dx}{ds} \\ \dfrac{dy}{ds} \end{pmatrix}, \qquad H_f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$

What are the constraints on $\mathbf{z}$? Can we determine whether f has a minimum or maximum at (x^*, y^*)?

Problem 75. Find the maxima and minima of the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(\alpha, \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

Problem 76. Let $x, y \in \mathbb{R}$ and $z = x + iy$, $\bar{z} = x - iy$. Find the extrema of the analytic function

$$f(x, y) = z^2 \bar{z}^2 - (z\bar{z}).$$

Problem 77. Find the extrema of the function

$$H(\theta_1, \theta_2) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \equiv \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2).$$

Problem 78. (i) Consider the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = 1 + x_1 + x_1 x_2.$$

Find the extrema of the function.
(i) Consider the analytic function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = 1 + x_1 + x_1 x_2 + x_1 x_2 x_3.$$

Find the extrema of the function. Extend to n dimensions.

Problem 79. Let $x_1, x_2 > 0$. Find the minima of the function

$$f(x_1, x_2) = \frac{1}{x_1} + \frac{1}{x_2} + x_1 + x_2.$$

Problem 80. Study the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{|\sin(2x)|}{|\sin(x)|}.$$

Problem 81. Study the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = \begin{cases} \cos(x_1)\sin(x_2) \text{ for } x_1 > x_2 \\ \sin(x_1)\cos(x_2) \text{ for } x_1 < x_2 \end{cases}$$

Problem 82. (i) Find the minima and maxima of the function

$$E(\theta_1, \theta_2) = \cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_1).$$

(ii) Find the minima and maxima of the function

$$E(\theta_1, \theta_2, \theta_3) = \cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1).$$

Problem 83. Let $z \in \mathbb{C}$ and $z = x + iy$ with $x, y \in \mathbb{R}$. Find the minima and maxima of the functions

$$f(z) = \frac{1}{(1 + z\bar{z})^2}, \qquad g(z) = \frac{1}{(z + \bar{z})^2}.$$

Problem 84. Consider the two lines in $\mathbb{R}^2$

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = 1, \qquad -\frac{1}{2}x_1 - \frac{1}{2}x_2 = 1.$$

Obviously the two lines do not intersect. Find the largest circle that can be inscribed between the two lines with the center of the circle at $(0, 0)$. Give the radius of the circle.

Problem 85. Let $c > 0$. Find maxima and minima of the analytic function

$$V(x) = cx^2(x_2 - 1).$$

Problem 86. Consider the analytic function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(\gamma, x, y) = \cos(\gamma)xy + \sin(\gamma)(xy)^2.$$

Find minima and maxima of f.

Problem 87. Find the extrema of the function analytic $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^2 + x_2^2$$

subject to the constraint $x_1^2 - x_2^2 = 1$. Apply

$$x_1(r, \phi) = r\cosh(\phi), \qquad x_2(r, \phi) = r\sinh(\phi).$$

Problem 88. Find the minima and maxima of the functions

$$f_2(x_1, x_2) = 1 + x_1 + x_2 + x_1x_2$$
$$f_3(x_1, x_2, x_3) = 1 + x_1 + x_2 + x_3 + x_1x_2x_3$$
$$f_4(x_1, x_2, x_3, x_4) = 1 + x_1 + x_2 + x_3 + x_4 + x_1x_2x_3x_4.$$

Problem 89. Find the minima and maxima of the functions

$$\begin{aligned} f_2(x_1,x_2) &= 1 + x_1 + x_1x_2 \\ f_3(x_1,x_2,x_3) &= 1 + x_1 + x_1x_2 + x_1x_2x_3 \\ f_4(x_1,x_2,x_3,x_4) &= 1 + x_1 + x_1x_2 + x_1x_2x_3 + x_1x_2x_3x_4. \end{aligned}$$

Problem 90. (i) Find the minima and maxima of the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \frac{1}{2}(x + y + |x - y|).$$

(ii) Find the minima and maxima of the function $g : \mathbb{R}^2 \to \mathbb{R}$

$$g(x,y) = \frac{1}{2}(x + y - |x - y|).$$

Problem 91. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x_1,x_2) = 2x_1^2 - x_1^4 - 2x_1^2x_2^2 - x_2^2 - x_2^4.$$

Find minima and maxima for f.

Problem 92. Let $x_1, x_2, x_3 \in \mathbb{R}$. Show that

$$x_1^2 + x_2^2 + x_3^2 \geq x_1x_2 + x_2x_3 + x_3x_1.$$

Why could it be useful to consider the analytic function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1\,?$$

1.4 Problems with Matrices

Problem 93. Find the extrema of the analytic functions

$$f_2(x) = \det \begin{pmatrix} 1 & x \\ -x & x^2 \end{pmatrix}, \qquad f_3(x) = \det \begin{pmatrix} 1 & x & x^2 \\ -x & x^2 & x^3 \\ -x^2 & x^3 & x^4 \end{pmatrix}.$$

Problem 94. Let $x, y \in \mathbb{R}$. Consider the 2×2 matrix

$$A(x,y) = \begin{pmatrix} x-1 & y \\ x & -xy \end{pmatrix}.$$

Find the minima and maxima of the analytic function

$$f(x,y) = \operatorname{tr}(A(x,y)A^T(x,y))$$

where tr denotes the trace and ^{T} denotes transpose.

Problem 95. For the vector space of all $n \times n$ matrices over $\mathbb{R}$ we can introduce the scalar product

$$\langle A, B \rangle := \operatorname{tr}(AB^T).$$

This implies a norm $\|A\|^2 = \operatorname{tr}(AA^T)$. Let $\epsilon \in \mathbb{R}$. Consider the 3×3 matrix

$$M(\epsilon) = \begin{pmatrix} \epsilon & 0 & \epsilon \\ 0 & 1 & 0 \\ \epsilon & 0 & -\epsilon \end{pmatrix}.$$

Find the minima of the function

$$f(\epsilon) = \operatorname{tr}(M(\epsilon)M^T(\epsilon)).$$

Problem 96. Let A be an $n \times n$ positive semidefinite matrix over $\mathbb{R}$. Show that the positive semidefinite quadratic form

$$f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

is a convex function throughout $\mathbb{R}^n$.

Problem 97. The vector space of all $n \times n$ matrices over $\mathbb{C}$ form a Hilbert space with the scalar product defined by

$$\langle A, B \rangle := \operatorname{tr}(AB^*).$$

This implies a norm $\|A\|^2 = \operatorname{tr}(AA^*)$.
(i) Find two unitary 2×2 matrices U_1, U_2 such that $\|U_1 - U_2\|$ takes a maximum.

(ii) Are the matrices

$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

such a pair?

Problem 98. Consider the symmetric 3×3 matrix

$$A(\epsilon) = \begin{pmatrix} \epsilon & 1 & 1 \\ 1 & \epsilon & 1 \\ 1 & 1 & \epsilon \end{pmatrix}, \qquad \epsilon \in \mathbb{R}.$$

(i) Find the maxima and minima of the function

$$f(\epsilon) = \det(A(\epsilon)).$$

(ii) Find the maxima and minima of the function

$$g(\epsilon) = \mathrm{tr}(A(\epsilon)A^T(\epsilon)).$$

(iii) For which values of ϵ is the matrix noninvertible?

Problem 99. Let $\alpha \in \mathbb{R}$. Consider the matrix

$$A(\alpha) = \begin{pmatrix} \alpha & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 1 & 0 & 0 & \alpha \end{pmatrix}.$$

Find the maxima and minima of the function $f(\alpha) = \det(A(\alpha))$. Does the inverse of $A(\alpha)$ exists at the minima and maxima?

Problem 100. Let A be an $m \times m$ symmetric positive-semidefinite matrix over $\mathbb{R}$. Let $\mathbf{x}, \mathbf{b} \in \mathbb{R}^m$, where $\mathbf{x}$, $\mathbf{b}$ are considered as column vectors. A and $\mathbf{b}$ are given. Consider the quadratic functional

$$E(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x} - \mathbf{x}^T \mathbf{b}$$

where ^{T} denotes transpose.
(i) Find the minimum of $E(\mathbf{x})$. Give a geometric interpretation.
(ii) Solve the linear equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Discuss the problem in connection with (i).

Problem 101. (i) Consider the matrix

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find the analytic function (characteristic polynomial)

$$p(\lambda) = \det(A - \lambda I_2).$$

Find the eigenvalues of A by solving $p(\lambda) = 0$. Find the minima of the function

$$f(\lambda) = |p(\lambda)|.$$

Discuss.
(ii) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the analytic function (characteristic polynomial)

$$p(\lambda) = \det(A - \lambda I_3).$$

Find the eigenvalues of A by solving $p(\lambda) = 0$. Find the minima of the function

$$f(\lambda) = |p(\lambda)|.$$

Discuss.

Problem 102. Let $\alpha \in \mathbb{R}$. Consider the matrix

$$A(\alpha) = \begin{pmatrix} \alpha & \alpha \\ \alpha & -\alpha \end{pmatrix}.$$

(i) Find the extrema of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(\alpha) = \operatorname{tr}(A(\alpha)A^T(\alpha)).$$

Is $A(\alpha)A^T(\alpha) = A^T(\alpha)A(\alpha)$?
(ii) Find the extrema of the analytic function $g : \mathbb{R} \to \mathbb{R}$

$$g(\alpha) = \det(A(\alpha)).$$

Find the values of α where the matrix $A(\alpha)$ is invertible. Give the inverse of the matrix for these values.

Problem 103. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the functions

$$E_j = \frac{1}{2}(A\mathbf{c}_j - \mathbf{e}_j)^T(A\mathbf{c}_j - \mathbf{e}_j)$$

where $j = 1, \dots, n$, $\mathbf{c}_j$ is the j-th column of the inverse matrix of A, $\mathbf{e}_j$ is the j-th column of the $n \times n$ identity matrix. This means $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis (as column vectors) in $\mathbb{R}^n$. The $\mathbf{c}_j$ are determined by minimizing the E_j with respect to the $\mathbf{c}_j$. Apply this method to find the inverse of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Problem 104. (i) Find the minima and maxima of

$$f(x_1, x_2) = (\, x_1 \quad x_2 \,) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(i) Find the minima and maxima of

$$f(x_1, x_2, x_3, x_4) = (\, x_1 \quad x_2 \quad x_3 \quad x_4 \,) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where $\otimes$ denotes the Kronecker product.

1.5 Problems with Integrals

Problem 105. Find the solution of the initial value problem of the first order differential equation

$$\frac{du}{dx} = \min(x, u(x))$$

with $u(0) = 2/3$.

Problem 106. Let $a, b \in \mathbb{R}$. Let

$$f(a,b) := \int_0^{2\pi} [\sin(x) - (ax^2 + bx)]^2 dx.$$

Find the minima of f with respect to a, b. Hint.

$$\int x\sin(x)dx = -x\cos(x) + \sin(x)$$
$$\int x^2\sin(x)dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x).$$

Problem 107. Let $\alpha \in \mathbb{R}$ and

$$f(\alpha) := \int_0^{2\pi} \sin(\alpha x)dx.$$

(i) Find $f(\alpha = 0)$. Apply Hospital's rule. Is $\alpha = 0$ a critical point? We have

$$\int \sin(\alpha x)dx = -\frac{\cos(\alpha x)}{\alpha}.$$

(ii) Find the minima and maxima of f.
(iii) Find

$$\lim_{\alpha\to+\infty} f(\alpha), \qquad \lim_{\alpha\to-\infty} f(\alpha).$$

Problem 108. Let $\alpha, \beta \in \mathbb{R}$ and

$$f(\alpha,\beta) = \int_0^{2\pi} \sin(\alpha\beta x)dx.$$

Find the minima and maxima of f.

Problem 109. Let

$$f(\alpha) := \int_0^{2\pi} \sin(\alpha x)dx.$$

(i) Is $\alpha = 0$ a critical point of f?
(ii) Find

$$\lim_{\alpha \to +\infty} f(\alpha), \qquad \lim_{\alpha \to -\infty} f(\alpha).$$

Problem 110. Find the minima and maxima of the function

$$f(t) = \exp\left(-\int_0^t \frac{1-e^{-s}}{s} ds\right).$$

Problem 111. Let $a \geq 0$. Find the minima and maxima for the function

$$f(a) = \int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}.$$

Note that

$$f(a) = \frac{\pi(a+1)e^{-a}}{4}$$

with $f(0) = \pi/4$.

Problem 112. Find the minima and maxima of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = e^{-2x} + \int_0^x e^{-s} ds.$$

1.6 Problems with Constraints

Problem 113. A square with length a has an area of $A_S = a^2$. Let P_S be the perimeter of the square. Suppose that a circle of radius r has the area $A_C = a^2$, i.e. the square and the circle have the same area. Is $P_S > P_C$? Prove or disprove.

Problem 114. A cube with length a has a volume of $A_C = a^3$. Let A_C be the surface of the cube. Suppose that a sphere of radius r has the volume $V_C = V_S$. Let A_S be the surface of the sphere. Is $A_S > A_C$? Prove or disprove.

Problem 115. Let r be the radius and h the height of a cylindrical metal can. Design such a can having a cost of α Euro per unit area of the metal, with a prescribed volume. Minimize the material cost.

Problem 116. Find the extrema of

$$f(x_1, x_2) = x_1 + x_2$$

subject to $x_1^2 + x_2^2 = 1$. Apply polar coordinates

$$x_1(r, \phi) = r\cos(\phi), \quad x_2(r, \phi) = r\sin(\phi)$$

where $r \geq 0$ and $\phi \in [0, 2\pi)$.

Problem 117. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2|x|.$$

Find the minima of f.

Problem 118. (i) Find the extrema of

$$f(x_1, x_2) = x_1 + x_2$$

subject to $x_1^2 + x_2^2 = 1$. Apply polar coordinates.
(ii) Find the extrema of

$$f(x_1, x_2) = x_1^2 + x_2^2$$

subject to $x_1 + x_2 = 1$. Apply polar coordinates.

Problem 119. Let $A = (a_{jk})_{j,k=1}^{n}$ be an $n \times n$ symmetric matrix over $\mathbb{R}$. The eigenvalue problem is given by $A\mathbf{x} = \lambda\mathbf{x}$ ($\mathbf{x} \neq \mathbf{0}$) where we assume that the eigenvectors are normalized, i.e. $\mathbf{x}^T\mathbf{x} = 1$. Thus

$$\sum_{k=1}^{n} a_{jk}x_k = \lambda x_j, \quad j = 1, \ldots, n.$$

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \sum_{j,k=1}^{n} a_{jk} x_j x_k \equiv \mathbf{x}^T A \mathbf{x}.$$

To obtain its maximum and minimum of f subject to the constraint (compact manifold) $\sum_{j=1}^{n} x_j^2 = 1$ we consider the Lagrange function

$$L(\mathbf{x}) = f(\mathbf{x}) - \lambda \sum_{j=1}^{n} x_j^2.$$

Show that the largest and smallest eigenvalue of A are the maximum and minimum of the function f subject to the constraint $\sum_{j=1}^{n} x_j^2 = 1$.

Problem 120. Consider the two circles in $\mathbb{R}^3$

$$S_1 = \{ (x_1, x_2, x_3) \, : \, x_1^2 + x_2^2 = 1, \; x_3 = 0 \}$$

$$S_2 = \{ (x_1, x_2, x_3) \, : \, (x_1 + 1)^2 + x_3^2 = 1, \; x_2 = 0 \}.$$

Is the point $p = (1, 1, 1) \in \mathbb{R}^3$ closer to circle 1 or circle 2?

Problem 121. Let $c > 0$. Find the minima and maxima of the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = |x_1 - x_2| \exp(-c|x_1 - x_2|).$$

Problem 122. Find $n \times n$ matrices A, B such that

$$\| [A, B] - I_n \| \to \min$$

where $\|.\|$ denotes the norm and $[,]$ denotes the commutator.

Problem 123. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = 2x_1 x_2 + 2x_1 x_3 + 2x_2 x_3$$

subject to the constraint (manifold) $x_1^2 + x_2^2 + x_3^2 = 1$. Find the extrema.

Problem 124. Let $a_1, a_2, a_3 > 0$. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1 x_2 x_3.$$

Find the extrema subject to the constraint (manifold, plane in $\mathbb{R}^3$)) $x_1/a_1 + x_2/a_2 + x_3/a_3 = 1$.

Problem 125. Let $a_1, a_2, a_3 > 0$. Consider the analytic function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_3 x_1$$

Show that the extrema value of f subject to the constraint (compact manifold)

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

are given by the largest and smallest eigenvalues of the 3×3 matrix

$$\begin{pmatrix} 0 & a_1^2 & a_1^2 \\ a_2^2 & 0 & a_2^2 \\ a_3^2 & 0 & a_3^2 \end{pmatrix}.$$

Problem 126. Let $a_1, a_2, a_3 > 0$. Consider the analytic function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1 x_2 + 2x_2 x_3 + 2x_3 x_1$$

Show that the extrema value of f subject to the constraint (compact manifold)

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

are given by the largest and smallest eigenvalues of the 3×3 matrix

$$\begin{pmatrix} 0 & a_1^2/2 & a_1^2 \\ a_2^2/2 & 0 & a_2^2 \\ a_3^2 & 0 & a_3^2 \end{pmatrix}.$$

Chapter 2

Lagrange Multiplier Method

2.1 Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions, where $j = 1, 2, \ldots, k$. Suppose that $\mathbf{x}^*$ is a local maximum or minimum of f on the set

$$\mathcal{D} = U \cap \{\, \mathbf{x} \,:\, g_j(\mathbf{x}) = 0 \;\; j = 1, 2, \ldots, k \,\}$$

where $U \subset \mathbb{R}^n$ is open. Let

$$\nabla f := \begin{pmatrix} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_n \end{pmatrix}, \qquad \nabla g_j := \begin{pmatrix} \partial g_j/\partial x_1 \\ \vdots \\ \partial g_j/\partial x_n \end{pmatrix}.$$

Suppose that

$$\operatorname{rank}(\nabla g(\mathbf{x}^*)) = k.$$

Then there exists a vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*) \in \mathbb{R}^k$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{j=1}^{k} \lambda_j^* \nabla g_j(\mathbf{x}) = \mathbf{0}.$$

Consider the problem of maximizing or minimizing $f : \mathbb{R}^n \to \mathbb{R}$ over the set

$$\mathcal{D} = U \cap \{\, \mathbf{x} \,:\, g_j(\mathbf{x}) = 0, \;\; j = 1, 2, \ldots, k \,\}$$

where $\mathbf{g}:\mathbb{R}^n\to\mathbb{R}^k$ ($\mathbf{g}=(g_1,\ldots,g_k)$) and $U\subset\mathbb{R}^n$ is open. Assume that f and $\mathbf{g}$ are both C^2 functions. Given any $\boldsymbol{\lambda}\in\mathbb{R}^k$ define the function L (Lagrangian) on $\mathbb{R}^n$ by

$$L(\mathbf{x},\boldsymbol{\lambda}):=f(\mathbf{x})+\sum_{j=1}^{k}\lambda_j g_j(\mathbf{x}).$$

The second derivative $\nabla^2 L(\mathbf{x},\boldsymbol{\lambda})$ of L with respect to the $\mathbf{x}$ variable is the $n\times n$ matrix defined by

$$\nabla^2 L(\mathbf{x},\boldsymbol{\lambda})=\nabla^2 f+\sum_{j=1}^{k}\lambda_j\nabla^2 g_j(\mathbf{x}).$$

Since f and g are both C^2 functions of $\mathbf{x}$, so is the Lagrangian L for any given value of $\boldsymbol{\lambda}\in\mathbb{R}^k$. Thus $\nabla^2 L$ is a symmetric matrix and defines a quadratic form on $\mathbb{R}^n$. Suppose there exist points $\mathbf{x}^*\in\mathcal{D}$ and $\boldsymbol{\lambda}^*\in\mathbb{R}^k$ such that

$$\mathrm{rank}(\nabla\mathbf{g}(\mathbf{x}^*))=k$$

and

$$\nabla f(\mathbf{x}^*)+\sum_{j=1}^{k}\lambda_j^*\nabla g_j(\mathbf{x}^*)=\mathbf{0}$$

where $\mathbf{0}$ is the (column) zero vector. Let

$$Z(\mathbf{x}^*):=\{\,\mathbf{z}\in\mathbb{R}^n\ :\ \nabla\mathbf{g}(\mathbf{x}^*)\mathbf{z}=\mathbf{0}\,\}$$

where $\nabla\mathbf{g}$ is the $k\times n$ Jacobian matrix and $\mathbf{0}$ is the zero vector with k rows. Let $\nabla^2 L^*$ denote the $n\times n$ matrix

$$\nabla^2 L(\mathbf{x},\boldsymbol{\lambda}^*)=\nabla^2 f(\mathbf{x}^*)+\sum_{j=1}^{k}\lambda_j^*\nabla^2 g_j(\mathbf{x}^*).$$

1) If f has a local maximum on $\mathcal{D}$ at $\mathbf{x}^*$, then $\mathbf{z}^T\nabla^2 L^*\mathbf{z}\le 0$ for all $\mathbf{z}\in Z(\mathbf{x}^*)$.
2) If f has a local minimum on $\mathcal{D}$ at $\mathbf{x}^*$, then $\mathbf{z}^T\nabla^2 L^*\mathbf{z}\ge 0$ for all $\mathbf{z}\in Z(\mathbf{x}^*)$.
3) If $\mathbf{z}^T\nabla^2 L^*\mathbf{z}<0$ for all $\mathbf{z}\in Z(\mathbf{x}^*)$ with $\mathbf{z}\ne\mathbf{0}$, then $\mathbf{x}^*$ is a strict local maximum of f on $\mathcal{D}$.
4) If $\mathbf{z}^T\nabla^2 L^*\mathbf{z}>0$ for all $\mathbf{z}\in Z(\mathbf{x}^*)$ with $\mathbf{z}\ne\mathbf{0}$, then $\mathbf{x}^*$ is a strict local minimum of f on the domain $\mathcal{D}$.

2.2 Solved Problems

Problem 1. Find the extrema (minima and maxima) of the function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x_1, x_2) = x_1^2 + x_2^2$$

subject to the constraint (ellipse)

$$\{ (x_1, x_2) \in \mathbb{R}^2 \, : \, g(x_1, x_2) = x_1^2 + 2x_2^2 - 1 = 0 \}.$$

Use the Lagrange multiplier method.

Problem 2. Find the shortest distance from the line $x_1 + x_2 = 1$ to the point $(1, 1)$ in the Euclidean space $\mathbb{R}^2$. Apply the Lagrange multiplier method.

Problem 3. Consider the Euclidean space $\mathbb{R}^2$. Find the shortest distance from the origin $(0, 0)$ to the curve

$$x_1 + x_2 = 1.$$

Apply the Lagrange multiplier.

Problem 4. Consider the two-dimensional Euclidean space. Find the shortest distance from the origin, i.e. $(0, 0)$, to the hyperbola

$$x^2 + 8xy + 7y^2 = 225.$$

Use the method of the Lagrange multiplier.

Problem 5. Consider the Euclidean space $\mathbb{R}^2$. Find the shortest distance between the curves

$$x + y = 1, \qquad x + y = 0.$$

Apply the Lagrange multiplier method. Note that there are two constraints.

Problem 6. Find the area of the largest rectangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Problem 7. Find the area of the largest isosceles triangle lying along the x-axis that can be inscribed in the ellipse

$$\frac{x^2}{36} + \frac{y^2}{12} = 1.$$

Problem 8. Use the Lagrange multiplier method to mimimize

$$f(\mathbf{x}) = x_1 + x_2$$

subject to the constraint $x_1^2 + x_2^2 = 2$.

Problem 9. Find the shortest distance between the curves (circles)

$$x_1^2 + y_1^2 = 1, \qquad x_2^2 + y_2^2 = 2.$$

Obviously the solution can be found inspection. Switch to polar coordinates

$$x_1 = r_1 \cos(\phi_1), \quad y_1 = r_1 \sin(\phi_1)$$

$$x_2 = r_2 \cos(\phi_2), \quad y_2 = r_2 \sin(\phi_2).$$

to solve the problem. Thus the constraints become $r_1^2 = 1$, $r_2^2 = 2$.

Problem 10. Find the shortest distance between the curves

$$x_1^2 + y_1^2 = 1, \qquad x_2^2 + 2y_2^2 = 4$$

applying the Lagrange multiplier method. Obviously the first curve is a circle and the second an ellipse.

Problem 11. Find the largest area of a rectangle with vertices at the orign of a Cartesian coordinate system on the x-axis, on the y-axis and on the parabola $y = 4 - x^2$.

Problem 12. Find the shortest and longest distance from the origin $(0, 0)$ to the circle given by

$$(x_1 - 4)^2 + (x_2 - 3)^2 = 1.$$

Apply the Lagrange multiplier method.

Problem 13. Consider two smooth non-intersecting curves

$$f(x, y) = 0, \qquad g(x, y) = 0$$

in the plane $\mathbb{R}^2$. Find the conditions (equations) for the shortest distance between the two curves. The Lagrange multiplier method must be used. Apply the equations to the curves

$$f(x, y) = x^2 + y^2 - 1 = 0, \qquad g(x, y) = (x - 3)^2 + (y - 3)^2 - 1 = 0.$$

Problem 14. An ellipse at the origin and parallel to the axes is characterized by the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

A rotated ellipse is given by

$$\left(\frac{x\cos(\theta) - y\sin(\theta)}{a}\right)^2 + \left(\frac{x\sin(\theta) + y\cos(\theta)}{b}\right)^2 = 1.$$

An rotated ellipse at any position (x_c, y_c) is given by

$$\left(\frac{(x-x_c)\cos(\theta)-(y-y_c)\sin(\theta)}{a}\right)^2+\left(\frac{(x-x_c)\sin(\theta)+(y-y_c)\cos(\theta)}{b}\right)^2=1. \quad (1)$$

Given a point $(x, y) \in \mathbb{R}^2$ find the shortest distance to the ellipse given by (1).

Problem 15. Find the shortest distance from the origin $(0, 0)$ of the 2-dimensional Euclidean space to the curve $y - \ln(x) = 0$ with $x > 0$.

Problem 16. Find the locations and values of the extrema of $(x, y \in \mathbb{R}^+)$

$$f(x,y) = \ln(xy) \quad \text{subject to} \quad xy > 0 \text{ and } x^2 + y^2 = 1.$$

Problem 17. Find the shortest distance from the line $x_1 + x_2 = 1$ in $\mathbb{R}^2$ to the point $(y_1, y_2) = (1, 1)$.

Problem 18. Let E^2 be the two-dimensional Euclidean space. Let $x > 0$ and $y > 0$. Find the shortest distance from the origin $(0, 0)$ to the curve $xy = 4$. Use the Lagrange muliplier method.

Problem 19. Let r be the radius and h the height of a cylindrical metal can. Design such a can having a cost of α Euro per unit area of the metal, with a prescribed volume. Minimize the material cost. Apply the Lagrange multiplier method.

Problem 20. Find the shortest distance between the two curves

$$y - x = 0, \qquad y - e^x = 0.$$

Use the method of the Lagrange multiplier.

Problem 21. (i) Let $x \in \mathbb{R}$. Show that

$$\sin(\cos(x)) = \cos(\sin(x))$$

has no solution.
(ii) Find the shortest distance between the curves

$$y - \sin(\cos(x)) = 0, \qquad y - \cos(\sin(x)) = 0.$$

Problem 22. (i) Consider two nonintersecting curves

$$f(x,y) = 0, \qquad g(x,y) = 0$$

where f and g are differentiable functions. Find the shortest distance between the curves.
(ii) Then apply it to

$$f(x,y) \equiv xy - 1 = 0, \quad (x > 0, y > 0), \qquad g(x,y) \equiv x + y = 0.$$

Problem 23. Show that

$$f(x,y) = xy^2 + x^2y, \qquad x^3 + y^3 = 3, \quad x, y \in \mathbb{R}$$

has a maximum of $f(x,y) = 3$.

Problem 24. Determine the locations of the extrema of

$$f(x,y) = x^2 - y^2, \qquad x^2 + y^2 = 1, \qquad x, y \in \mathbb{R}$$

using the Lagrange multiplier method. Classify the extrema as maxima or minima using the Hessian criteria.

Problem 25. (i) Determine the domain of $y : \mathbb{R} \to \mathbb{R}$ from

$$(y(x))^2 = x^3 - x^4$$

i.e. determine

$$\{\, x \,|\, x \in \mathbb{R},\ \exists y \in \mathbb{R} : y^2 = x^3 - x^4 \,\}.$$

(ii) Find the locations and values of the extrema of

$$f(x,y) = x \quad \text{subject to} \quad y^2 = x^3 - x^4$$

Problem 26. Consider the Euclidean space $\mathbb{R}^2$. Given the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}.$$

The three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are at 120 degrees of each other and are normalized, i.e. $\|\mathbf{u}_j\| = 1$ for $j = 1, 2, 3$. Every given two-dimensional vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

in many different ways. Given the vector $\mathbf{v}$ minimize

$$\frac{1}{2}(c_1^2 + c_2^2 + c_3^2)$$

subject to the two constraints

$$\mathbf{v} - c_1\mathbf{u}_1 - c_2\mathbf{u}_2 - c_3\mathbf{u}_3 = \mathbf{0}.$$

Problem 27. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be unit vectors in the Euclidean space $\mathbb{R}^2$. Find the maximum of the function

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \|\mathbf{x} - \mathbf{y}\|^2 + \|\mathbf{y} - \mathbf{z}\|^2 + \|\mathbf{z} - \mathbf{x}\|^2$$

with respect to the constraints

$$\|\mathbf{x}\| = \|\mathbf{y}\| = \|\mathbf{z}\| = 1$$

i.e., $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are unit vectors. Give an interpretation of the result. Study the symmetry of the problem.

Problem 28. Consider the Euclidean space $\mathbb{R}^2$. Find the shortest and longest distance from $(0,0)$ to the curve

$$(x-1)^2 + (y-1)^2 = \frac{1}{4}.$$

Give a geometric interpretation.

Problem 29. Consider the Euclidean space $\mathbb{R}^2$. Find the shortest distance between the curves

$$x^2 + (y-5)^2 - 1 = 0, \qquad y - x^2 = 0.$$

Give a geometric interpretation (picture). Utilize the fact that the equations are invariant under $x \to -x$.

Problem 30. Show that

$$f(x,y) = xy^2 + x^2y, \qquad x^3 + y^3 = 3, \quad x, y \in \mathbb{R}$$

has a maximum of $f(x,y) = 3$.

Problem 31. Let E_3 be the three-dimensional Euclidian space. Find the shortest distance from the origin, i.e. $(0,0,0)^T$ to the plane

$$x_1 - 2x_2 - 2x_3 = 3.$$

The method of the Lagrange multilipier method must be used.

Problem 32. (i) In the two-dimensional Euclidean space $\mathbb{R}^2$ find the shortest distance from the origin $(0,0)$ to the surface $x_1x_2 = 1$.
(ii) In the three-dimensional Euclidean space $\mathbb{R}^3$ find the shorest distance from the origin $(0,0,0)$ to the surface $x_1x_2x_3 = 1$.
(iii) In the four-dimensional Euclidean space $\mathbb{R}^4$ find the shorest distance from the origin $(0,0,0,0)$ to the surface $x_1x_2x_3x_4 = 1$. Extend to n-dimension.
Hint. The distance d from the origin to any point in the Euclidean space $\mathbb{R}^n$ is given by $d = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Of course one consider $d^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ instead of d (why?). Thus one also avoids square roots in the calculations.

Problem 33. Find the locations and values of the extrema of $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = x + y + z \quad \text{subject to} \quad xyz = 1$$

using the Lagrange multiplier method. Use a second order condition to classify the extrema as minima or maxima.

Problem 34. The equation of an ellipsoid with centre $(0, 0, 0)$ and semi-axis a, b, c is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Find the largest volume of a rectangular parallelepiped that can be inscribed in the ellipsoid.

Problem 35. (i) The equation of an ellipsoid with center $(0, 0, 0)^T$ and semi-axis a, b, c is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Find the largest volume of a rectangular parallelepiped that can be inscribed in the ellipsoid.
(ii) Evaluate the volume of the ellipsoid. Use

$$x(\phi, \theta) = a\cos(\phi)\cos(\theta), \quad y(\phi, \theta) = b\sin(\phi)\cos(\theta), \quad z(\phi, \theta) = c\sin(\theta).$$

Problem 36. (i) Let $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. Find the maximum of the function

$$f(\mathbf{x}) = x_1 x_2 x_3$$

subject to the constraint

$$g(\mathbf{x}) = x_1 + x_2 + x_3 = c$$

where $c > 0$.
(ii) Let $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. Find the minimum of the function

$$f(\mathbf{x}) = x_1 + x_2 + x_3$$

subject to the constraints

$$g(\mathbf{x}) = x_1 x_2 x_3 = c$$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and $c > 0$.

Problem 37. The two planes in the Euclidean space $\mathbb{R}^3$

$$x_1 + x_2 + x_3 = 4, \qquad x_1 + x_2 + 2x_3 = 6$$

intersect and create a line. Find the shortest distance from the origin $(0, 0, 0)$ to this line. The Lagrange multiplier method must be used.

Problem 38. Show that the angles α, β, γ of a triangle in the plane maximize the function

$$f(\alpha, \beta, \gamma) = \sin(\alpha)\sin(\beta)\sin(\gamma)$$

if and only if the triangle is equilateral. An *equilateral triangle* is a triangle in which all three sides are of equal length.

Problem 39. Find the minima of the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(\mathbf{x}) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

subject to the constraint

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, g(\mathbf{x}) = x_1 + x_2 + x_3 = 3\,\}.$$

Use the Lagrange multiplier method.

Problem 40. Let A be an $n \times n$ positive semidefinite matrix over $\mathbb{R}$. Show that the positive semidefinite quadratic form

$$f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

is a convex function throughout $\mathbb{R}^n$.

Problem 41. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ be given m distinct vectors in $\mathbb{R}^n$. Consider the problem of minimizing

$$f(\mathbf{x}) = \sum_{j=1}^{m} \|\mathbf{x} - \mathbf{a}_j\|^2$$

subject to the constraint

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \tag{1}$$

Consider the center of gravity

$$\mathbf{a}_g := \frac{1}{m}\sum_{j=1}^{m} \mathbf{a}_j$$

of the given vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. Show that if $\mathbf{a}_g \neq \mathbf{0}$, the problem has a unique maximum and a unique minimum. What happens if $\mathbf{a}_g = \mathbf{0}$?

Problem 42. Let A be a given $n \times n$ matrix over $\mathbb{R}$ and $\mathbf{u}$ a column vector in $\mathbb{R}^n$. Minimize

$$f(\mathbf{u}) = \mathbf{u}^T A \mathbf{u}$$

subject to the conditions

$$\sum_{j=1}^{n} u_j = 0, \qquad \sum_{j=1}^{n} u_j^2 = 1.$$

Problem 43. Consider fitting a segment of the cepstral trajectory $c(t)$, $t = -M, -M + 1, \ldots, M$ by a second-order polynomial $h_1 + h_2 t + h_3 t^2$. That is, we choose parameters h_1, h_2 and h_3 such that the fitting error

$$E(h_1, h_2, h_3) = \sum_{t=-M}^{M} (c(t) - (h_1 + h_2 t + h_3 t^2))^2$$

is minimized. Find h_1, h_2, h_3 as function of the given $c(t)$.

Problem 44. Find the maximum and minimum of the function

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

to the contrain conditions

$$\frac{x_1^2}{4} + \frac{x_2^2}{5} + \frac{x_3^2}{25} = 1, \qquad x_3 = x_1 + x_2.$$

Use the method of the Lagrange multiplier.

Problem 45. Let A be a positive definite $n \times n$ matrix over $\mathbb{R}$. Then A^{-1} exists and is also positive definite. Let $\mathbf{b}$ be a column vector in $\mathbb{R}^n$. Minimize the function

$$f(\mathbf{x}) = (A\mathbf{x} - \mathbf{b})^T A^{-1} (A\mathbf{x} - \mathbf{b}) \equiv \mathbf{x}^T A \mathbf{x} - 2\mathbf{b}^T \mathbf{x} + \mathbf{b}^T A^{-1} \mathbf{b}.$$

Problem 46. Find the shortest distance between the unit ball localized at $(2, 2, 2)$

$$(x-2)^2 + (y-2)^2 + (z-2)^2 = 1$$

and the plane $z = 0$.

Problem 47. Given three lines in the plane

$$a_1 x_1^{(1)} + b_1 x_2^{(1)} = c_1, \quad a_2 x_1^{(2)} + b_2 x_2^{(2)} = c_2, \quad a_3 x_1^{(3)} + b_3 x_2^{(3)} = c_3$$

which form a triangle. Find the set of points for which the sum of the distances to the three lines is as small as possible. Apply the Lagrange multiplier method with the three lines as the constraints and

$$d^2 = (x_1 - x_1^{(1)})^2 + (x_2 - x_2^{(1)})^2 + (x_1 - x_1^{(2)})^2 + (x_2 - x_2^{(2)})^2 + (x_1 - x_1^{(3)})^2 + (x_2 - x_2^{(3)})^2.$$

Problem 48. Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$. Find the maximum of

$$f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

subject to the constraint

$$\mathbf{x}^T\mathbf{x} = 1$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$.

Problem 49. Let D be a given symmetric $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{c} \in \mathbb{R}^n$. Find the maximum of the function

$$f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T D\mathbf{x}$$

subject to

$$A\mathbf{x} = \mathbf{b}$$

where A is a given $m \times n$ matrix over $\mathbb{R}$ with $m < n$.

Problem 50. The two planes

$$x_1 + x_2 + x_3 = 4, \qquad x_1 + x_2 + 2x_3 = 6$$

intersect and create a line. Find the shortest distance from the origin $(0,0,0)$ to this line.

Problem 51. Find the critical points of the function

$$f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$$

on the surface

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 1.$$

Determine whether the critical points are local minima or local maxima by calculating the *borderd Hessian matrix*

$$\begin{pmatrix} L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} & L_{x_1\lambda} \\ L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} & L_{x_2\lambda} \\ L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} & L_{x_3\lambda} \\ L_{\lambda x_1} & L_{\lambda x_2} & L_{\lambda x_3} & L_{\lambda\lambda} \end{pmatrix}$$

where $L_{x_1x_1} = \partial^2 L/\partial x_1 \partial x_1$ etc.

Problem 52. Maximize the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(\mathbf{x}) = x_1 + x_2 + x_3$$

subject to

$$g(\mathbf{x}) \equiv x_1^2 + x_2^2 + x_3^3 = c$$

where $c > 0$.

Problem 53. Let A be the following 3×3 matrix acting on the vector space $\mathbb{R}^3$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Calculate $\|A\|$ where

$$\|A\| := \sup_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Problem 54. Let A be an $n \times n$ matrix over $\mathbb{R}$. Then a given *vector norm* induces a *matrix norm* through the definition

$$\|A\| := \max_{\mathbf{x}=1} \|A\mathbf{x}\|$$

where $\mathbf{x} \in \mathbb{R}^n$. We consider the Euclidean norm for the vector norm in the following. Thus $\|A\|$ can be calculated using the Lagrange multiplier method with the constraint $\|\mathbf{x}\| = 1$. The matrix norm given above of the matrix A is the square root of the principle component for the positive semidefinite matrix $A^T A$, where ^{T} denotes tranpose. Thus it is equivalent to the *spectral norm*

$$\|A\| := \lambda_{max}^{1/2}$$

where λ_{max} is the largest eigenvalue of $A^T A$.
(i) Apply the two methods to the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

(ii) Which of the two methods is faster in calculating the norm of A? Discuss.
(iii) Is the matrix A positive definite?

Problem 55. A sphere in $\mathbb{R}^3$ has a diameter of 2. What is the edge length of the largest possible cube that would be able to fit within the sphere?

Matrix Problems

Problem 56. Consider the 2×2 symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues and normalized eigenvectors of A.
(ii) Consider the function (polynomial) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Find the minima and maxima of f subject to the constraint

$$x_1^2 + x_2^2 = 1.$$

Apply the Lagrange muliplier method.
(iii) Look at the results from (i) and (ii) and discuss.

Problem 57. Let A be a nonzero $n \times n$ symmetric matrix over A. Find the maximum of the analytic function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

subject to $\mathbf{x}^T \mathbf{x} = 1$.

Problem 58. Consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the norm of A defined by

$$\|A\| := \max_{\mathbf{x} \in \mathbb{R}^3, \|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

where $\|A\mathbf{x}\|$ is the Euclidean norm in $\mathbb{R}^3$. Apply two different methods, i.e. apply the Lagrange multiplier method and as second method calculate $A^T A$ and find the largest eigenvalue of $A^T A$.

Problem 59. Let S be a given nonzero symmetric $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{c} \in \mathbb{R}^n$. Find the maximum of the analytic function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \mathbf{c}^T \mathbf{x} + \frac{1}{2} \mathbf{x}^T S \mathbf{x}$$

subject to $A\mathbf{x} = \mathbf{b}$, where A is a given $m \times n$ matrix with $m < n$.

Problem 60. Let $f : \mathbb{R}^n \to \mathbb{R}$ and similarly $g_j : \mathbb{R}^n \to \mathbb{R}$ $(j = 1, 2, \ldots, m)$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ $(j = 1, 2, \ldots, m)$ be continuously differentiable at $\mathbf{x} \in \mathbb{R}^n$. If f has a minimum at $\mathbf{x}$ subject to the constraints $g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}) \le 0$ and $h_1(\mathbf{x}), \ldots, h_l(\mathbf{x}) = 0$ then the *Fritz-John conditions*

$$u_0 \nabla f(\mathbf{x}) + \sum_{\substack{j=1 \\ g_j(\mathbf{x})=0}}^{m} u_j \nabla g_j(\mathbf{x}) + \sum_{j=1}^{l} \lambda_j \nabla h_j(\mathbf{x}) = \mathbf{0},$$

$$h_1(\mathbf{x}) = \cdots = h_l(\mathbf{x}) = 0$$

$$-u_0, -u_1, \ldots, -u_m, g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}) \le 0,$$

$$(u_0, u_1, \ldots, u_m, v_1, \ldots, v_l) \ne (0, \ldots, 0)$$

are satisfied. Due to the continuous differentiability of the functions at $\mathbf{x}$, we obtain an equivalent set of conditions

$$u_0 \nabla f(\mathbf{x}) + \sum_{j=1}^{m} u_j \nabla g_j(\mathbf{x}) + \sum_{j=1}^{l} \lambda_j \nabla h_j(\mathbf{x}) = \mathbf{0},$$

$$u_1 g_1(\mathbf{x}) = \cdots = u_m g_m(\mathbf{x}) = h_1(\mathbf{x}) = h_2(\mathbf{x}) = \cdots = h_l(\mathbf{x}) = 0$$

$$u_0, u_1, \ldots, u_m \ge 0,$$

$$(u_0, u_1, \ldots, u_m, v_1, \ldots, v_l) \ne (0, \ldots, 0).$$

Consider the optimization problem (where $f : \mathbb{R}^2 \to \mathbb{R}$):

$$\begin{aligned} \text{minimize} \quad & f(x, y) = (x-1)^2 y^2 \\ \text{subject to} \quad & x^2 \le y \le 1. \end{aligned}$$

Give the Fritz-John conditions at $(x, y) = (1/2, 1)$. Is $(1/2, 1)$ a local minimum to $f(x, y)$?

2.3 Supplementary Problems

Problem 1. Let $c > 0$. Find the minima of the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^4 + y^4$$

subject to the constraint $x^2 + y^2 = c^2$. Apply the Lagrange multiplier method. Apply differential forms.

Problem 2. Let $c \in \mathbb{R}$. Find the minima of the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^3 + y^3$$

subject to the constraint $x+y = c$. Apply the Lagrange multiplier method. Apply differential forms.

Problem 3. The area A of a triangle in the Euclidean space $\mathbb{R}^2$ with vertices at

$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3)$$

is given by

$$A = \pm\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

where the sign is chosen so that the area is nonnegative. If $A = 0$ the three points all lie on a line. Find the area of the largest isosceles triangle lying along the x-axis that can be inscribed in the ellipse

$$\frac{x^2}{36} + \frac{y^2}{12} = 1.$$

Thus the vertices satisfy the conditions

$$\frac{x_1^2}{36} + \frac{y_1^2}{12} = 1, \quad \frac{x_2^2}{36} + \frac{y_2^2}{12} = 1, \quad \frac{x_3^2}{36} + \frac{y_3^2}{12} = 1.$$

Problem 4. Given the two curves

$$x^2 + y^2 = \frac{1}{2}, \qquad \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

Find the shortest and longest between the two curves. The Lagrange multiplier method must be used. Discuss. Draw a picture of the curves.

Problem 5. Calculate the shortest distance (Euclidean) between the curves

$$x^2 + (y-4)^2 = 1, \qquad y = x^2.$$

The Lagrange multiplier method must be used.

Problem 6. Find the plane in $\mathbb{R}^3$ passing through the three points $(1,0,1)$, $(1,1,0)$, $(0,1,1)$. Then find the shortest distance from this plane to the point $(2,2,2)$. Apply the Lagrange multiplier method.

Problem 7. Let $n \geq 2$. Minimize

$$\sum_{j=1}^{n} x_j \quad \text{subject to} \quad \prod_{j=1}^{n} x_j = 1$$

with $x_j \geq 0$ for $j = 1, \ldots, n$. Start of with the Lagrangian

$$L = \sum_{j=1}^{n} - \sum_{j=1} \alpha_j x_j + \beta \prod_{j=1}^{n} x_j .$$

Problem 8. Find the maxima of the function $f : (\mathbb{R}^+)^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1 x_2 x_3$$

subject to $g(x_1, x_2, x_3) = 2(x_1 x_2 + x_2 x_3 + x_3 x_1) - C = 0$, where C is a positive constant.

Chapter 3

Differential Forms and Lagrange Multiplier

3.1 Introduction

The Lagrange multiplier method can fail in problems where there is a solution. In this case applying differential forms provide the solution.

The *exterior product* (also called wegde product or Grassmann product) is denoted by $\wedge$. It is associative and we have

$$dx_j \wedge dx_k = -dx_k \wedge dx_j.$$

Thus $dx_j \wedge dx_j = 0$. The exterior derivative is denoted by d and is linear.

3.2 Solved Problem

Problem 1. Find the maximum and minimum of the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^2 - x_2^2$$

subject to the constraint $g(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$.
(i) Apply the Lagrange muliplier method.
(ii) Apply differential forms.
(iii) Apply the transformation $x_1(r, \phi) = r\cos(\phi)$, $x_2(r, \phi) = r\sin(\phi)$.

Problem 2. Let E_2 be the two-dimensional Euclidean space. Let $x_1 > 0$ and $x_2 > 0$. Find the shortest distance from the origin to the curve

$$x_1 x_2 = 4.$$

Apply three different methods, (i) direct substitution of the constraint, (ii) Lagrange multiplier method, (iii) differential forms.

Problem 3. Find the maximum and minimum of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x, y) = x^2 - y^2$$

subject to the constraint $x^2 + y^2 = 1$ (unit circle).
(i) In the first method use the substitution $y^2 = 1 - x^2$.
(ii) In the second method use the Lagrange multiplier method.
(iii) In the third method use differential forms.

Problem 4. Find the minimum of the function

$$f(x, y) = x$$

subject to the constraint

$$g(x, y) = y^2 + x^4 - x^3 = 0.$$

(i) Show that the Lagrange multiplier method fails. Explain why.
(ii) Apply differential forms and show that there is a solution.

Problem 5. Show that the Lagrange multiplier method fails for the following problem. Maximize

$$f(x, y) = -y$$

subject to the constraint

$$g(x, y) = y^3 - x^2 = 0.$$

Problem 6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = 2x^3 - 3x^2$$

and the constraint

$$g(x,y) = (3-x)^3 - y^2 = 0.$$

We want to find the maximum of f under the constraint. Show that the Lagrange multilier method fails. Find the solution using differential forms by calculating df, dg, $df \wedge dg$ and setting $df \wedge dg = 0$.

Problem 7. Consider the function

$$f(x,y) = \frac{1}{3}x^3 - \frac{3}{2}y^2 + 2x.$$

We want to minimize and maximize f subject to the constraint

$$g(x,y) = x - y = 0.$$

Apply differential forms.

Problem 8. (i) Determine the domain of $y : \mathbb{R} \to \mathbb{R}$ from

$$(y(x))^2 = x^3 - x^4$$

i.e. determine

$$\{\, x \mid x \in \mathbb{R},\ \exists y \in \mathbb{R} : y^2 = x^3 - x^4 \,\}.$$

(b) Find the locations and values of the extrema of

$$f(x,y) = x \quad \text{subject to} \quad y^2 = x^3 - x^4.$$

Problem 9. The two planes

$$x_1 + x_2 + x_3 = 4, \qquad x_1 + x_2 + 2x_3 = 6$$

intersect and create a line. Find the shortest distance from the origin $(0,0,0)$ to this line. Apply differential forms.

Problem 10. Consider the function

$$f(x,y) = x^2 - y^2.$$

We want to minimize and maximize f subject to the constraint

$$g(x,y) = 1 - x - y = 0.$$

Show that applying the Lagrange multiplier method provides no solution. Show that also the method using differential forms provides no solution. Explain.

Problem 11. In $\mathbb{R}^2$ the two lines

$$x_1 + x_2 = 0, \qquad x_1 + x_2 = 1$$

do not intersect. Find the shortest distance between the lines. Apply differential forms. We start of from

$$f(x_1, x_2, y_1, y_2) = (x_1 - y_1)^2 + (x_2 - y_2)^2$$

and the constraints

$$g_1(x_1, x_2, y_1, y_2) = x_1 + x_2 = 0, \qquad g_2(x_1, x_2, y_1, y_2) = y_1 + y_2 - 1 = 0.$$

Problem 12. Let $x_1 > 0$, $x_2 > 0$, $x_3 > 0$. Consider the surface

$$x_1 x_2 x_3 = 2$$

in $\mathbb{R}^3$. Find the shortest distance from the origin $(0, 0, 0)$ to the surface.
(i) Apply the Lagrange multiplier method.
(ii) Apply differential forms.
(iii) Apply symmetry consideration (permutation group).

Problem 13. Find the extrema of the function

$$f(x_1, x_2) = x_1 + x_2$$

subject to the constraint $x_1^2 + x_2^2 = 1$.
(i) Apply the Lagrange multiplier method.
(ii) Apply differential forms
(iii) Switch to the coordinate system $x_1(r, \phi) = r\cos(\phi)$, $x_2(r, \phi) = r\sin(\phi)$ with $r \geq 0$.

Problem 14. Find the maximum and minimum of the function

$$f(x, y) = x^2 - y^2$$

subject to the constraint $x^2 + y^2 = 1$ (unit circle).
(i) In the first method use the substitution $y^2 = 2 - x^2$.
(ii) In the second method use the Lagrange multiplier method.
(iii) In the third method use differential forms.

Problem 15. (i) Find the maximum of the function $f : \mathbb{R}^{+3} \to \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1 x_2 x_3$$

subject to the constraint

$$g(x_1, x_2, x_3) = 2(x_1 x_2 + x_2 x_3 + x_3 x_1) - A = 0, \quad A > 0.$$

Give a geometric interpretation of the problem if x_1, x_2, x_3 have the dimension of a length.
(ii) Solve the problem using differential forms.

Problem 16. Consider the Euclidean space $\mathbb{R}^3$. Find the shortest distance between the plane

$$3x_1 + 4x_2 + x_3 = 1$$

and the point $(-1, 1, 1)$.
(i) Apply the Lagrange multiplier method.
(ii) Apply differential forms.
(iii) Find the normal vector for the plane. Discuss.

Problem 17. Find the extrema of the function

$$f(x_1, x_2) = x_1 + x_2$$

subject to the constraint $x_1^2 - x_2^2 = 1$.
(i) Apply the Lagrange multiplier method.
(ii) Apply differential forms
(iii) Switch to the coordinate system $x_1(r, \alpha) = r\cosh(\alpha)$, $x_2(r, \phi) = r\sinh(\alpha)$.

3.3 Supplementary Problems

Problem 1. Find the extrema of the function

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

subject to constraints

$$x_1 + x_2 + x_3 = 1, \qquad x_1^3 + x_2^3 + x_3^3 = 2.$$

Apply differential forms.

Chapter 4

Penalty Method

Problem 1. Find the minimum of the analytic function $f:\mathbb{R}^2\to\mathbb{R}$

$$f(x,y)=x^2+y^2$$

subject to the constraint $y=x-1$.
(i) Apply the penalty method. In the penalty method we seek the minimum of the function

$$h(x,y)=x^2+y^2+\frac{\gamma}{2}(y-x+1)^2.$$

Note that the constraint must be squared, otherwise the sign would preclude the existence of a minimum. To find the minimum we keep $\gamma>0$ fixed and afterwards we consider the limit $\gamma\to\infty$.
(ii) Apply the Lagrange multiplier method.

Problem 2. Find the local minimum of $f:\mathbb{R}^2\to\mathbb{R}$

$$f(x,y)=x^2+xy-1$$

subject to the constraint $x+y^2-2=0$. Apply the penalty method.

Problem 3. Minimize the function

$$f(x_1,x_2)=-2x_1^2-x_2$$

subject to the constraint

$$x_1^2+x_2^2-1\leq 0.$$

(i) Apply the penalty method.
(ii) Apply the Karush-Kuhn-Tucker condition.

Chapter 5

Simplex Method

Problem 1. Maximize

$$f(x_1, x_2) = 3x_1 + 6x_2$$

subject to

$$x_1 \leq 4, \quad 3x_1 + 2x_2 \leq 18, \quad x_1, x_2 \geq 0.$$

Problem 2. Solve the following optimalization problem (i) graphically and (ii) with the simplex method

$$250x_1 + 45x_2 \quad \to \quad \text{maximize}$$

subject to

$$x_1 \geq 0, \quad x_1 \leq 50, \qquad x_2 \geq 0, \quad x_2 \leq 200$$

$$x_1 + \frac{1}{5}x_2 \leq 72, \qquad 150x_1 + 25x_2 \leq 10000.$$

Problem 3. Maximize the function

$$f(x_1, x_2) = 4x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 6 \\ -3x_1 + 2x_2 &\leq 3 \\ 2x_2 &\leq 5 \\ 2x_1 + x_2 &\leq 4 \end{aligned}$$

and $x_1 \geq 0, x_2 \geq 0$. The Simplex method must be used.

Problem 4. Maximize
$$f(x_1, x_2) = 150x_1 + 450x_2$$
subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, $x_1 \leq 120$, $x_2 \leq 70$, $x_1 + x_2 \leq 140$, $x_1 + 2x_2 \leq 180$.

Problem 5. Maximize $(x_1, x_2, x_3 \in \mathbb{R})$

$$f(x_1, x_2, x_3) = 3x_1 + 2x_2 - 2x_3$$

subject to
$$4x_1 + 2x_2 + 2x_3 \leq 20, \qquad 2x_1 + 2x_2 + 4x_3 \geq 6$$
and $x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 2$.
Hint. Use $x_3 = \widetilde{x}_3 + 2$.

Problem 6. Maximize
$$10x_1 + 6x_2 - 8x_3$$
subject to the constraints

$$5x_1 - 2x_2 + 6x_3 \leq 20, \quad 10x_1 + 4x_2 - 6x_3 \leq 30, \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

The simplex method must be applied. The starting extreme point is $(0, 0, 0)^T$.

Problem 7. Find the minimum of

$$E = -5x_1 - 4x_2 - 6x_3$$

subject to

$$x_1 + x_2 + x_3 \leq 100, \quad 3x_1 + 2x_2 + 4x3 \leq 210, \quad 3x_1 + 2x_2 \leq 150$$

where $x_1, x_2, x_3 \geq 0$.

Problem 8. Maximize $(x_1, x_2, x_3 \in \mathbb{R})$

$$f(x_1, x_2, x_3) = x_1 + 2x_2 - x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 14 \\ 4x_1 + 2x_2 + 3x_3 &\leq 28 \\ 2x_1 + 5x_2 + 5x_3 &\leq 30 \end{aligned}$$

and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

Problem 9. Determine the locations of the extrema of

$$f(x,y) = x^2 - y^2, \qquad x^2 + y^2 \le 1$$

using the the substitution $u := x^2$ and $v := y^2$ and the simplex method. Classify the extrema as maxima or minima.

Problem 10. Maximize

$$E = 10x_1 + 6x_2 - 8x_3$$

subject to

$$5x_1 - 2x_2 + 6x_3 \le 20, \qquad 10x_1 + 4x_2 - 6x_3 \le 30$$

where

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Problem 11. Maximize

$$f(x_1, x_2, x_3) = 2x_1 + 3x_2 + 3x_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\le 60 \\ -x_1 + x_2 + 4x_3 &\le 10 \\ 2x_1 - 2x_2 + 5x_3 &\le 50 \end{aligned}$$

where

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Problem 12. Maximize

$$2x_1 + 3x_2 + 3x_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\le 60 \\ -x_1 + x_2 + 4x_3 &\le 10 \\ 2x_1 - 2x_2 + 5x_3 &\le 50 \end{aligned}$$

where

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Text Problems

Problem 13. A manufacturer builds three types of boats: rowboats, canoes and kayaks. They sell at profits of Rand 200, Rand 150, and Rand 120, respectively, per boat. The boats require aluminium in their construction, and also labour in two sections of the workshop. The following table specifies all the material and labour requirements:

	1 row boat	1 canoe	1 kayak
Aluminum	30 kg	15 kg	10 kg
Section 1	2 hours	3 hours	2 hours
Section 2	1 hour	1 hour	1 hour

The manufacturer can obtain 630 kg of aluminium in the coming month. Section 1 and 2 of the workshop have, respectively, 110 and 50 hours available for the month. What monthly production schedule will maximize total profits, and what is the largest possible profit ? Let x_1 = number of row boats, x_2 = number of canoes and x_3 = number of kayaks.

Problem 14. A company produces two types of cricket hats. Each hat of the first type requires twice as much labor time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second types to 150 and 250 hats, respectively. Assume that profits per hat are \$8 for type 1 and \$5 for type 2. Determine the number of hats to be produced of each type in order to maximize profits. Use the simplex method. Let x_1 be the number of cricket hats of type 1, and x_2 be the number of cricket hats of type 2.

Problem 15. Orania Toys specializes in two types of wooden soldiers: Boer soldiers and British soldiers. The profit for each is R 28 and R 30, respectively. A Boer soldier requires 2 units of lumber, 4 hours of carpentry, and 2 hours of finishing to complete. A British soldier requires 3 units of lumber, 3.5 hours of carpentry, and 3 hours of finishing to complete. Each week the company has 100 units of lumber delivered and there are 120 hours of carpentry time and 90 hours of finishing time available. Determine the weekly production of each type of wooden soldier which maximizes the weekly profit.

Problem 16. A carpenter makes tables and bookcases for a nett unit profit he estimates as R100 and R120, respectively. He has up to 240 board meters of lumber to devote weekly to the project and up to 120 hours of labour. He estimates that it requires 6 board meters of lumber and 5 hours of labour to complete a table and 10 board meters of lumber and 3 hours of labour for a bookcase. He also estimates that he can sell all the tables produced but that his local market can only absorb 20 bookcases. Determine a weekly production schedule for tables and bookcases that maximizes the carpenter's profits. Solve the problem graphically and with the Karush-Kuhn-Tucker condition. Solve it with the simplex method.

Let x_1 be the number of tables and x_2 be the number of bookcases. So he wants to maximize

$$f(x_1, x_2) = 100x_1 + 120x_2.$$

The constraints are

$$6x_1 + 10x_2 \leq 240 \quad (Lumber\ supply)$$

$$5x_1 + 3x_2 \leq 120 \quad (Labour\ supply)$$

$$x_2 \leq 20 \quad (Market)$$

$$x_1 \geq 0, \quad x_2 \geq 0 \quad (Physical)$$

Supplementary Problems

Problem 17. Maximize

$$f(x) = 150x_1 + 100x_2$$

subject to the boundary conditions

$$25x_1 + 5x_2 \leq 60, \quad 15x_1 + 20x_2 \leq 60, \quad 20x_1 + 15x_2 \leq 60$$

and $x_1 \geq 0$, $x_2 \geq 0$.

Problem 18. (i) Find the extreme points of the following set

$$S := \{ (x_1, x_2) : x_1 + 2x_2 \geq 0, \ -x_1 + x_2 \leq 4 \ x_1 \geq 0, \ x_2 \geq 0 \}.$$

(ii) Find the extreme point of the following set

$$S := \{ (x_1, x_2) : -x_1 + 2x_2 \leq 3, \ x_1 + x_2 \leq 2, \ , x_2 \leq 1, \ x_1 \geq 0, \ x_2 \geq 0 \}.$$

Chapter 6

Karush-Kuhn-Tucker Conditions

6.1 Linear Problems

Problem 1. Solve the optimization problem applying the Karush-Kuhn-Tucker condition

$$f(x_1, x_2) = -3x_1 - 6x_2 \to \text{minimum}$$

subject to the constraints

$$x_1 \leq 4, \qquad 3x_1 + 2x_2 \leq 18$$

and $x_1 \geq 0$, $x_2 \geq 0$. First draw the domain and show it is convex.

Problem 2. Consider the following optimization problem. Minimize

$$f(\mathbf{x}) = -5x_1 - 4x_2 - 6x_3$$

subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 100 \\ 3x_1 + 2x_2 + 4x_3 &\leq 210 \\ 3x_1 + 2x_2 &\leq 150. \end{aligned}$$

Solve this problem using the Karush-Kuhn-Tucker conditions. Note that the second block of constraints has to be rewritten as

$$\begin{aligned} 100 - x_1 - x_2 - x_3 &\geq 0 \\ 210 - 3x_1 - 2x_2 - 4x_3 &\geq 0 \\ 150 - 3x_1 - 2x_2 &\geq 0. \end{aligned}$$

Do a proper case study.

Problem 3. Minimize

$$f(x, y) = -3x - 4y$$

subject to $x + y \leq 4$, $2x + y \leq 5$, $x \geq 0$, $y \geq 0$.
(i) Apply the Karush-Kuhn-Tucker method.
(ii) Apply the Simplex method.

Problem 4. Maximize the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = 4x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 6 \\ -3x_1 + 2x_2 &\leq 3 \\ 2x_2 &\leq 5 \\ 2x_1 + x_2 &\leq 4 \end{aligned}$$

and $x_1 \geq 0$, $x_2 \geq 0$. Apply the Karush-Kuhn-Tucker method. Compare to the simplex method.

Problem 5. Apply the Karush-Kuhn-Tucker method to solve the following optimization problem. Minimize

$$f(x_1, x_2) = -x_1 - x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ 2x_1 + 3x_2 &\leq 6 \end{aligned}$$

and $x_1 \geq 0$, $x_2 \geq 0$.

Problem 6. The Karush-Kuhn-Tucker condition can also replace the Simplex method for linear programming problems. Solve the following problem. Minimize

$$f(x_1, x_2) = x_1 + x_2$$

subject to

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 \leq 1, \quad x_2 \leq 1.$$

The last two constraints we have to rewrite as

$$1 - x_1 \geq 0, \qquad 1 - x_2 \geq 0.$$

Obviously the domain is convex.

Problem 7. Consider the linear function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1 - 2x_2.$$

Minimize f subject to the constraints

$$1 \le x_1 \le 2, \qquad -x_1 + x_2 \le 0, \qquad x_2 \ge 0.$$

Apply the Karush-Kuhn-Tucker conditions. All the conditions must be written done. Draw the domain and thus show that it is convex.

Problem 8. Minimize the function

$$f(x_1, x_2) = -x_1 - 4x_2$$

subject to the constraints

$$2x_1 + 3x_2 \le 4, \quad 3x_1 + x_2 \le 3, \quad x_1 \ge 0, \quad x_2 \ge 0.$$

Draw the domain and thus show it is convex.
(ii) Solve the linear equations

$$2x_1 + 3x_2 = 4, \qquad 3x_1 + x_2 = 3.$$

Find f for this solution.

Problem 9. Minimize

$$f(x_1, x_2, x_3) = -5x_1 - 4x_2 - 6x_3$$

subject to the conditions $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$ and

$$100 - x_1 - x_2 - x_3 \ge 0$$
$$210 - 3x_1 - 2x_2 - 4x_3 \ge 0$$
$$150 - 3x_1 - 2x_3 \ge 0.$$

Problem 10. (i) Consider the domain in $\mathbb{R}^2$ given by

$$0 \le x \le 2, \quad 0 \le y \le 2, \quad \frac{1}{3}x + \frac{1}{3}y \le 1.$$

Draw the domain. Is the domain convex? Discuss.
(ii) Minimize the functions

$$f(x, y) = -x - 2y$$

(which means maximize the function $-f(x, y)$) subject to the constraints given in (i). Apply the Karush-Kuhn-Tucker condition.
(iii) Check your result from (ii) by study the boundary of the domain given in (i).

Problem 11. Maximize $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1 + 2x_2$$

subject to the constraints

$$0 \le x_1 \le 2, \quad 0 \le x_2 \le 2, \quad \frac{x_1}{3} + \frac{x_2}{3} \le 1$$

in the Euclidean space $\mathbb{R}^2$. Apply the Karush-Kuhn-Tucker method. First check that the domain is convex.

6.2 Nonlinear Problem

Problem 12. Find the locations and values of the extrema of $(x, y \in \mathbb{R})$

$$f(x, y) = (x + y)^2 \quad \text{subject to} \quad x^2 + y^2 \le 1.$$

Problem 13. Consider the optimization problem (where $f : \mathbb{R}^2 \to \mathbb{R}$):

$$\text{minimize} \quad f(x, y) = (x - 1)^2 y^2 \quad \text{subject to} \quad x^2 \le y \le 1.$$

Give the Fritz-John conditions at $(x, y) = (1/2, 1/4)$.
Is $(1/2, 1/4)$ a local minimum to $f(x, y)$?

Problem 14. Use the Kuhn-Tucker conditions to find the maximum of

$$3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

subject to

$$2x_1 + x_2 \le 10, \quad x_1 \ge 0, \quad x_2 \ge 0.$$

Problem 15. Let

$$C := \{\, \mathbf{x} \in \mathbb{R}^2 \,:\, x_1^2 + x_2^2 \le 1 \,\}$$

be the standard unit disc. Consider the function $f_0 : C \to \mathbb{R}$

$$f_0(x_1, x_2) = x_1 + 2x_2$$

where we want to find the global maximum for the given domain C. To find the global maximum, let $f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$ and analyze the Kuhn-Tucker conditions. This means, find $\lambda^* \in \mathbb{R}$ and $\mathbf{x}^* = (x_1^*, x_2^*) \in \mathbb{R}^2$ such that

$$\lambda^* \ge 0 \tag{1}$$

$$\lambda^*(x_1^{*2} + x_2^{*2} - 1) = 0 \tag{2}$$

and from

$$\nabla f_0(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^*} = \lambda^* \nabla f_1(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^*}$$

we obtain

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda^* \begin{pmatrix} 2x_1^* \\ 2x_2^* \end{pmatrix}. \tag{3}$$

Problem 16. Minimize the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^2 + 2y^2$$

subject to the constraints

$$x + y \geq 2, \qquad x \leq y$$

using the Karush-Kuhn-Tucker conditions. Draw the domain. Is it convex?

Problem 17. Find the maximum of the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^2 - y$$

subject to the constraint

$$g(x, y) = 1 - x^2 - y^2 \geq 0.$$

Thus the set given by the constraint is compact.

Problem 18. Minimize the function

$$f(\mathbf{x}) = (x_1 - 10)^3 + (x_2 - 20)^3$$

subject to

$$g_1(\mathbf{x}) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \leq 0$$
$$g_2(\mathbf{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \leq 0$$

where $13 \leq x_1 \leq 100$ and $0 \leq x_2 \leq 100$.

Problem 19. Find the locations and values of the maxima of $(x, y \in \mathbb{R})$

$$f(x, y) = e^{-(x+y)^2} \quad \text{subject to} \quad x^2 + y^2 \geq 1.$$

Problem 20. (i) Find the minima of the function $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ (i.e. $x, y \geq 0$)

$$f(x, y) = \sqrt{x^2 + y^2} + \sqrt{(130 - x)^2 + y^2} + \sqrt{(60 - x)^2 + (70 - y)^2}$$

subject to the constraints

$$\sqrt{x^2 + y^2} \geq 50, \quad \sqrt{(130 - x)^2 + y^2} \geq 50, \quad \sqrt{(60 - x)^2 + (70 - x)^2} \geq 50$$

using the Karush-Kuhn-Tucker condition.
(ii) Apply the multiplier penalty function method.

Problem 21. Determine the locations of the extrema of

$$f(x,y) = x^2 - y^2, \qquad x^2 + y^2 \leq 1$$

using the Karush-Kuhn-Tucker conditions. Classify the extrema as maxima or minima.

Problem 22. Find the minimum of the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(\mathbf{x}) = (x_1 - 2)^2 + (x_2 - 1)^2$$

under the constraints

$$g_1(\mathbf{x}) = x_2 - x_1^2 \geq 0$$
$$g_2(\mathbf{x}) = 2 - x_1 - x_2 \geq 0$$
$$g_3(\mathbf{x}) = x_1 \geq 0.$$

Apply the Karush-Kuhn-Tucker condition.

Problem 23. Find the locations and values of the maxima of ($x, y \in \mathbb{R}$)

$$f(x,y) = e^{-(x+y)^2} \quad \text{subject to} \quad x^2 + y^2 \geq 1.$$

Problem 24. Determine the locations of the extrema of

$$f(x,y) = x^2 - y^2, \qquad x^2 + y^2 \leq 1$$

using polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Classify the extrema as maxima or minima.

Problem 25. Determine the locations of the extrema of

$$f(x,y) = x^2 - y^2, \qquad x^2 + y^2 \leq 1$$

using the Karush-Kuhn-Tucker conditions. Classify the extrema as maxima or minima.

Problem 26. Determine the locations of the extrema of

$$f(x,y) = x^2 - y^2, \qquad x^2 + y^2 \leq 1$$

using the the substitution $u := x^2$ and $v := y^2$ and the simplex method. Classify the extrema as maxima or minima.

6.3 Support Vector Machine

Problem 27. Let the training set of two separate classes be represented by the set of vectors

$$(\mathbf{v}_0, y_0), \quad (\mathbf{v}_1, y_1), \ \dots \ , (\mathbf{v}_{n-1}, y_{n-1})$$

where $\mathbf{v}_j$ $(j = 0, 1, \dots, n-1)$ is a vector in the m-dimensional real Hilbert space $\mathbb{R}^m$ and $y_j \in \{-1, +1\}$ indicates the class label. Given a weight vector $\mathbf{w}$ and a bias b, it is assumed that these two classes can be separated by two margins parallel to the hyperplane

$$\mathbf{w}^T \mathbf{v}_j + b \geq 1, \quad \text{for} \quad y_j = +1 \tag{1}$$

$$\mathbf{w}^T \mathbf{v}_j + b \leq -1, \quad \text{for} \quad y_j = -1 \tag{2}$$

for $j = 0, 1, \dots, n-1$ and $\mathbf{w} = (w_0, w_1, \dots, w_{m-1})^T$ is a column vector of m-elements. Inequalities (1) and (2) can be combinded into a single inequality

$$y_j(\mathbf{w}^T \mathbf{v}_j + b) \geq 1 \quad \text{for } j = 0, 1, \dots, n-1\,. \tag{3}$$

There exist a number of separate hyperplanes for an identical group of training data. The objective of the *support vector machine* is to determine the optimal weight $\mathbf{w}^*$ and the optimal bias b^* such that the corresponding hyperplane separates the positive and negative training data with maximum margin and it produces the best generation performance. This hyperplane is called an optimal separating hyperplane. The equation for an arbitrary hyperplane is given by

$$\mathbf{w}^T \mathbf{x} + b = 0 \tag{4}$$

and the distance between the two corresponding margins is

$$\gamma(\mathbf{w}, b) = \min_{\{\mathbf{v}\,|\,y=+1\}} \frac{\mathbf{w}^T \mathbf{v}}{\|\mathbf{w}\|} - \max_{\{\mathbf{v}\,|\,y=-1\}} \frac{\mathbf{w}^T \mathbf{v}}{\|\mathbf{w}\|}. \tag{5}$$

The optimal separating hyperplane can be obtained by maximizing the above distance or minimizing the norm of $\|\mathbf{w}\|$ under the inequality constraint (3), and

$$\gamma_{max} = \gamma(\mathbf{w}^*, b^*) = \frac{2}{\|\mathbf{w}\|}. \tag{6}$$

The saddle point of the Lagrange function

$$L_P(\mathbf{w}, b, \boldsymbol{\alpha}) = \frac{1}{2}\mathbf{w}^T \mathbf{w} - \sum_{j=0}^{n-1} \alpha_j (y_j(\mathbf{w}^T \mathbf{v}_j + b) - 1) \tag{7}$$

gives solutions to the minimization problem, where $\alpha_j \geq 0$ are Lagrange multipliers. The solution of this quadratic programming optimization problem requires that the gradient of $L_P(\mathbf{w}, b, \boldsymbol{\alpha})$ with respect to $\mathbf{w}$ and b vanishes, i.e.,

$$\left.\frac{\partial L_P}{\partial \mathbf{w}}\right|_{\mathbf{w}=\mathbf{w}^*} = \mathbf{0}, \qquad \left.\frac{\partial L_P}{\partial b}\right|_{b=b^*} = 0.$$

We obtain

$$\mathbf{w}^* = \sum_{j=0}^{n-1} \alpha_j y_j \mathbf{v}_j \tag{8}$$

and

$$\sum_{j=0}^{n-1} \alpha_j y_j = 0. \tag{9}$$

Inserting (8) and (9) into (7) yields

$$L_D(\boldsymbol{\alpha}) = \sum_{i=0}^{n-1} \alpha_i - \frac{1}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \alpha_i \alpha_j y_i y_j \mathbf{v}_i^T \mathbf{v}_j \tag{10}$$

under the constraints

$$\sum_{j=0}^{n-1} \alpha_j y_j = 0 \tag{11}$$

and

$$\alpha_j \geq 0, \qquad j = 0, 1, \ldots, n-1.$$

The function $L_D(\boldsymbol{\alpha})$ has be maximized. Note that L_P and L_D arise from the same objective function but with different constraints; and the solution is found by minimizing L_P or by maximizing L_D. The points located on the two optimal margins will have nonzero coefficients α_j among the solutions of $\max L_D(\boldsymbol{\alpha})$ and the constraints. These vectors with nonzero coefficients α_j are called support vectors. The bias can be calculated as follows

$$b^* = -\frac{1}{2} \left(\min_{\{\mathbf{v}_j \,|\, y_j = +1\}} \mathbf{w}^{*T} \mathbf{v}_j + \max_{\{\mathbf{v}_j \,|\, y_j = -1\}} \mathbf{w}^{*T} \mathbf{v}_j \right).$$

After determination of the support vectors and bias, the decision function that separates the two classes can be written as

$$f(\mathbf{x}) = \operatorname{sgn} \left(\sum_{j=0}^{n-1} \alpha_j y_j \mathbf{v}_j^T \mathbf{x} + b^* \right).$$

Apply this classification technique to the data set (AND gate)

j	Training set $\mathbf{v}_j$	Target y_j
0	(0,0)	1
1	(0,1)	1
2	(1,0)	1
3	(1,1)	-1

Problem 28. Consider the truth table for a 3-input gate

```
i_1 i_2 i_3  o
 0   0   0   1
 0   0   1   0
 0   1   0   0
 0   1   1   0
 1   0   0   0
 1   0   1   0
 1   1   0   0
 1   1   1   1
```

where `i_1`, `i_2`, `i_3` are the three inputs and `o` is the output. Thus we consider the unit cube in $\mathbb{R}^3$ with the vertices ("corner points")

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),\ (1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1)$$

and the map

$$(0,0,0) \to 1,\ (0,0,1) \to 0,\ (0,1,0) \to 0,\ (0,1,1) \to 0,$$

$$(1,0,0) \to 0,\ (1,0,1) \to 0,\ (1,1,0) \to 0,\ (1,1,1) \to 1.$$

Apply the support vector machine (the nonlinear one) to classify the problem.

Problem 29. Consider the training set

$$\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ y_0 = 1, \qquad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \ y_1 = -1.$$

The support vector machine must be used to find the optimal separating hyperplane (in the present case a line). Write down at the end of the calculation the results for the α's, λ's, μ's, b and the equation for the separating line. Draw a graph of the solution.

Problem 30. The truth table for the 3-input AND-gate is given by

```
i_1 i_2 i_3  o
 0   0   0   0
 0   0   1   0
 0   1   0   0
 0   1   1   0
 1   0   0   0
 1   0   1   0
 1   1   0   0
 1   1   1   1
```

where `i_1`, `i_2`, `i_3` are the three inputs and `o` is the output. Thus we consider the unit cube in $\mathbb{R}^3$ with the vertices ("corner points")

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),\ (1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1)$$

and the map

$$(0,0,0) \to 0,\ (0,0,1) \to 0,\ (0,1,0) \to 0,\ (0,1,1) \to 0,$$

$$(1,0,0) \to 0,\ (1,0,1) \to 0,\ (1,1,0) \to 0,\ (1,1,1) \to 1.$$

Construct a plane in $\mathbb{R}^3$ that separates the seven 0's from the one 1. The method of the construction is your choice.

Problem 31. Let $n \geq 2$. Minimize

$$\sum_{j=1}^{n} x_j$$

subject to

$$\prod_{j=1}^{n} x_j = 1$$

with $x_j \geq 0$ for $j = 1, \ldots, n$.

Chapter 7

Fixed Points

Problem 1. Consider the map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{5}{2}x(1-x).$$

(i) Find all the fixed points and study their stability.
(ii) Let $x = 4/5$. Find $f(x)$, $f(f(x))$. Discuss.
(iii) Let $x = 1/3$. Find $f(x)$, $f(f(x))$, $f(f(f(x)))$.

Problem 2. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x|x|.$$

Find the fixed points of the function and study their stability. Is the function differentiable? If so find the derivative.

Problem 3. Find the fixed point of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sin(x) + x.$$

Study the stability of the fixed points.

Problem 4. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \cos(x) + x.$$

Find all the fixed points and study their stability.

Problem 5. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = e^{-x} + x - 1.$$

Find the fixed points and study their stability. Let $x = 1$. Find $f(x)$, $f(f(x))$. Is $|f(x)| < |x|$? Is $|f(f(x))| < |f(x)|$?

Problem 6. Can one find polynomials $p : \mathbb{R} \to \mathbb{R}$ such that a critical point of p and a fixed point of p coincide?

Problem 7. Consider the map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x(1-x).$$

Find the fixed points of f. Are the fixed points stable? Prove or disprove. Let $x_0 = 1/2$ and iterate

$$f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \quad \ldots$$

Does this sequence tend to a fixed point? Prove or disprove.

Problem 8. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 2x(1-x).$$

(i) Find the fixed points and study their stability.
(ii) Calculate

$$\lim_{n \to \infty} f^{(n)}(1/3).$$

Discuss.

Problem 9. (i) Let $r > 0$ (fixed) and $x > 0$. Consider the map

$$f_r(x) = \frac{1}{2}\left(x + \frac{r}{x}\right)$$

or written as difference equation

$$x_{t+1} = \frac{1}{2}\left(x_t + \frac{r}{x_t}\right), \qquad t = 0, 1, 2, \ldots \quad x_0 > 0.$$

Find the fixed points of f_r. Are the fixed points stable?
(ii) Let $r = 3$ and $x_0 = 1$. Find $\lim_{t \to \infty} x_t$. Discuss.

Problem 10. Find all 2×2 matrices A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $\det(A) = 1$ such that

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Problem 11. Consider the map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |1 - |2 - |3 - x|||.$$

(i) Find the fixed points.
(ii) Iterate

$$f(4), \;\; f(f(4)), \;\; f(f(f(4))), \ldots$$

Discuss.
(iii) Iterate

$$f(5), \;\; f(f(5)), \;\; f(f(f(5))), \ldots$$

Discuss.
(iv) Iterate

$$f(e), \;\; f(f(e)), \;\; f(f(f(e))), \ldots$$

Discuss.

Problem 12. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$. Show that the only solution of the fixed point equation $f(x^*) = x^*$ is $x^* = 0$.

Problem 13. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 1 + x + \cos(x)\cosh(x).$$

Show that the fixed point equation $f(x^*) = x^*$ has infinitely many solutions.

Problem 14. Let

$$D = \{\, (x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 + x_3^2 = 1,\, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \,\}.$$

Let $\mathbf{v}$ be a normalized vector in $\mathbb{R}^3$ with nonnegative entries and A be a 3×3 matrix over $\mathbb{R}$ with strictly positive entries. Show that the map $f : D \to D$

$$f(\mathbf{v}) = \frac{A\mathbf{v}}{\|A\mathbf{v}\|}$$

has a fixed point, i.e. there is a normalized vector $\mathbf{v}_0$ such that

$$\frac{A\mathbf{v}_0}{\|A\mathbf{v}_0\|} = \mathbf{v}_0.$$

Chapter 8

Neural Networks

8.1 Hopfield Neural Networks

Problem 1. (i) Find the weight matrix W for the Hopfield neural network which stores the vectors

$$\mathbf{x}_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 := \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

(ii) Determine the eigenvalues and eigenvectors of W.
(iii) Describe all the vectors $\mathbf{x} \in \{1, -1\}^2$ that are recognized by this Hopfield network, i.e.

$$W\mathbf{x} \xrightarrow{\text{sign}} \mathbf{x}.$$

(iv) Find the asynchronous evolution of the vector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(v) Can the weight matrix W be reconstructed from the eigenvalues and normalized eigenvectors of the weight matrix W?

Problem 2. (i) Determine the weight matrix W for a Hopfield neural network which stores the vectors

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(ii) Are the vectors $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ linearly independent?

(iii) Determine which of these vectors are fixed points of the network, i.e. for which of the above vectors $\mathbf{s}(t)$ does

$$\mathbf{s}(t+1) = \mathbf{s}(t)$$

under iteration of the network.
(iv) Determine the energy

$$E_W(s) := -\frac{1}{2}\mathbf{s}^T W \mathbf{s}$$

for each of the vectors.
(v) Describe the synchronous and asynchronous evolution of the vector

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

under iteration of the network. Find the energy of this vector.

Problem 3. (i) Find the weight matrix for the *Hopfield network* which stores the five patterns

$$\mathbf{x}_0 = (1,1,-1,-1)^T, \quad \mathbf{x}_1 = (1,-1,-1,1)^T, \quad \mathbf{x}_2 = (1,-1,1,-1)^T,$$
$$\mathbf{x}_3 = (1,1,1,-1)^T, \quad \mathbf{x}_4 = (1,1,1,1)^T.$$

(ii) Which of these vectors are orthogonal in the vector space $\mathbb{R}^4$?
(iii) Which of these vectors are fixed points under iteration of the network?
(iv) Determine the energy

$$-\frac{1}{2}\mathbf{s}^T W \mathbf{s}$$

under synchronous evolution and asynchronous evolution of the vector $(1,-1,1,1)^T$.

Problem 4. Given the sets in the Euclidean space $\mathbb{R}^2$

$$N := \left\{ \mathbf{x} = \begin{pmatrix} 2.0 \\ 2.0 \end{pmatrix} \right\}, \qquad P := \left\{ \mathbf{y} = \begin{pmatrix} 3.0 \\ 3.0 \end{pmatrix} \right\}$$

or including the bias

$$\widetilde{N} := \left\{ \mathbf{x} = \begin{pmatrix} 1.0 \\ 2.0 \\ 2.0 \end{pmatrix} \right\}, \qquad \widetilde{P} := \left\{ \mathbf{y} = \begin{pmatrix} 1.0 \\ 3.0 \\ 3.0 \end{pmatrix} \right\}$$

Apply the perceptron learning algorithm to classify the two points in $\mathbb{R}^2$. Start with $\mathbf{w}^T = (-2,1,1)$ and the sequence of vectors

$$\widetilde{\mathbf{x}}, \quad \widetilde{\mathbf{y}}, \quad \widetilde{\mathbf{x}}, \quad \widetilde{\mathbf{y}}, \ldots$$

Draw a figure of the solution.

8.2 Kohonen Network

Problem 5. Let the initial weight matrix for a *Kohonen network* be

$$W = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and the input vectors be

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Present the input vectors in the pattern $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_0$, $\mathbf{x}_1$, etc. Use $d_{00} = 0$, $h(x) = 1 - x$, $\eta_0 = 1$ and $\eta_{j+1} = \eta_j/2$ to train the network for 2 iterations (presentations of all input vectors). Since there is only one neuron, it is always the winning neuron.

Problem 6. Let the initial weight matrix for a *Kohonen network* be

$$W = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the input vectors be

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Present the input vectors in the pattern $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_0$, $\mathbf{x}_1$, etc. Use $d_{00} = 0$, $h(x) = 1 - x$, $\eta_0 = 1$ and $\eta_{j+1} = \eta_j/2$ to train the network for 3 iterations (presentations of all input vectors). What can we conclude? What happens for $j \to \infty$? Since there is only one neuron, it is always the winning neuron.

Problem 7. (i) Train a Kohonen network with two neurons using the inputs

$$(1, 0, 0)^T \qquad (0, 1, 0)^T \qquad (0, 0, 1)^T$$

presented in that order 3 times. The initial weights for the two neurons are

$$(0.1, -0.1, 0.1)^T \qquad (-0.1, 0.1, -0.1)^T.$$

Use the Euclidean distance to find the winning neuron. Only update the winning neuron using $\eta = 0.5$. We use $d_{ij} = 1 - \delta_{ij}$ and $h(x) = 1 - x$. In other words we use 1 for h for the winning neuron and 0 otherwise. What does each neuron represent?

(ii) Repeat the previous question using the initial weights

$$(0.1, 0.1, 0.1)^T \qquad (-0.1, -0.1, 0.1)^T$$

$$(-0.1, 0.1, -0.1)^T \qquad (0.1, -0.1, -0.1)^T.$$

Problem 8. Consider a Kohonen network. The two input vectors in $\mathbb{R}^2$ are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The initial weight vector $\mathbf{w}$ is

$$\mathbf{w} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}.$$

Aply the Kohonen algorithm with "winner takes all". Select a learning rate. Give an interpretation of the result.

Problem 9. Consider the Kohonen network and the four input vectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

The initial weight vector is

$$\mathbf{w} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Apply the Kohonen algorithm with "winner takes all". The learning rate is your choice. Give an interpretation of the result.

Problem 10. Consider the Hilbert space of complex $n \times n$ matrices, where the scalar product of two $n \times n$ matrices A and B is given by

$$(A, B) := \mathrm{tr}(AB^*)$$

where tr denotes the trace and * denotes transpose and conjugate complex. The scalar product implies a norm

$$\|A\|^2 = \mathrm{tr}(AA^*)$$

and thus a distance measure. Let $n = 2$. Consider the Kohonen network and the four input matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The initial weight matrix is

$$W = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Apply the Kohonen algorithm with "winner takes all". The learning rate is your choice. Give an interpretation of the result.

Problem 11. Given a set of M vectors in $\mathbb{R}^N$

$$\{ \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{M-1} \}$$

and a vector $\mathbf{y} \in \mathbb{R}^N$. We consider the Euclidean distance, i.e.,

$$\|\mathbf{u} - \mathbf{v}\| := \sqrt{\sum_{j=0}^{N-1} (u_j - v_j)^2}, \qquad \mathbf{u}, \mathbf{v} \in \mathbb{R}^N.$$

We have to find the vector $\mathbf{x}_j$ ($j = 0, 1, \ldots, M-1$) with the shortest distance to the vector $\mathbf{y}$, i.e., we want to find the index j. Provide an efficient computation. This problem plays a role for the Kohonen network.

8.3 Hyperplanes

Problem 12. Find $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ such that

$$\mathbf{x} \in \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} > \theta$$

and

$$\mathbf{x} \in \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} < \theta$$

or show that no such $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ exists.

Problem 13. Find $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ such that

$$\mathbf{x} \in \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} > \theta$$

and

$$\mathbf{x} \in \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} < \theta$$

or show that no such $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ exists.

Problem 14. Find $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ such that

$$\mathbf{x} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} > \theta$$

and

$$\mathbf{x} \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} < \theta$$

or show that no such $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ exists.

Problem 15. Find $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ such that

$$\mathbf{x} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} > \theta$$

and

$$\mathbf{x} \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} < \theta$$

or show that no such $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ exists.

Problem 16. Find $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ such that

$$\mathbf{x} \in \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} > \theta$$

and

$$\mathbf{x} \in \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} < \theta$$

or show that no such $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ exists.

Problem 17. Find $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ such that

$$\mathbf{x} \in \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} > \theta$$

and

$$\mathbf{x} \in \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \quad \Rightarrow \quad \mathbf{w}^T \mathbf{x} < \theta$$

or show that no such $\mathbf{w} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ exists.

Problem 18. (i) Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, $n > 1$, be distinct with

$$\mathbf{w} \in \mathbb{R}^n, \theta \in \mathbb{R} : \quad \mathbf{w}^T \mathbf{x}_1 \geq \theta \quad \mathbf{w}^T \mathbf{x}_2 \geq \theta.$$

Show that $\{\mathbf{x}_1, \mathbf{x}_2\}$ and $\{\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2\}$ for $\alpha \in [0,1]$ are not linearly separable i.e.

$$\mathbf{w}^T \mathbf{x}_3 \geq \theta, \qquad \mathbf{x}_3 := \alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2.$$

(ii) Are there vectors, other than those described by $\mathbf{x}_3$, with this property?

Problem 19. (i) Given the points

$$(0,0) \qquad (1,1)$$

in the Euclidean space $\mathbb{R}^2$ Construct a line that separates the two points. Of course the line should be somewhere in "the middle" of the two points. The method of construction is your choice. Start your solution with the sentence "My method of construction is as follows:". Draw a graph of your solution. Find the normal vector for your line.
(ii) Given the points

$$(0,0,0) \qquad (1,1,1)$$

in the Euclidean space $\mathbb{R}^3$. Construct a plane that separates the two points. Of course the plane should be somewhere in "the middle" of the two points. The method of construction is your choice. Start your solution with the sentence "My method of construction is as follows:". Find the normal vector for your plane.
(iii) Given the points

$$(0,0,0,0) \qquad (1,1,1,1)$$

in the Euclidean space $\mathbb{R}^4$. Construct a hyperplane that separates the two points. Of course the hyperplane should be somewhere in "the middle" of the two points. The method of construction is your choice. Start your solution with the sentence "My method of construction is as follows:". Find the normal vector for your hyperplane.

8.4 Perceptron Learning Algorithm

Problem 20. Let P and N be two finite sets of points in the Euclidean space $\mathbb{R}^n$ which we want to separate linearly. A weight vector is sought so that the points in P belong to its associated positive half-space and the points in N to the negative half-space. The error of a perceptron with weight vector $\mathbf{w}$ is the number of incorrectly classified points. The learning algorithm must minimize this error function $E(\mathbf{w})$. Now we introduce the *perceptron learning algorithm*. The training set consists of two sets, P and N, in n-dimensional extended input space. We look for a vector $\mathbf{w}$ capable of absolutely separating both sets, so that all vectors in P belong to the open positive half-space and all vectors in N to the open negative half-space of the linear separation.

Algorithm. Perceptron learning
start: The weight vector $\mathbf{w}(t=0)$ is generated randomly
test: A vector $\mathbf{x} \in P \cup N$ is selected randomly,
if $\mathbf{x} \in P$ and $\mathbf{w}(t)^T\mathbf{x} > 0$ goto *test*,
if $\mathbf{x} \in P$ and $\mathbf{w}(t)^T\mathbf{x} \leq 0$ goto *add*,
if $\mathbf{x} \in N$ and $\mathbf{w}(t)^T\mathbf{x} < 0$ goto *test*,
if $\mathbf{x} \in N$ and $\mathbf{w}(t)^T\mathbf{x} \geq 0$ goto *subtract*,
add: set $\mathbf{w}(t+1) = \mathbf{w}(t) + \mathbf{x}$ and $t := t+1$, goto *test*
subtract: set $\mathbf{w}(t+1) = \mathbf{w}(t) - \mathbf{x}$ and $t := t+1$ goto *test*

This algorithm makes a correction to the weight vector whenever one of the selected vectors in P or N has not been classified correctly. The perceptron convergence theorem guarantees that if the two sets P and N are linearly separable the vector $\mathbf{w}$ is updated only a finite number of times. The routine can be stopped when all vectors are classified correctly.

Consider the sets in the extended space

$$P = \{\,(1.0, 2.0, 2.0),\ (1.0, 1.5, 1.5)\,\}$$

and

$$N = \{\,(1.0, 0.0, 1.0),\ (1.0, 1.0, 0.0),\ (1.0, 0.0, 0.0)\,\}.$$

Thus in $\mathbb{R}^2$ we consider the two sets of points

$$\{\,(2.0, 2.0),\ (1.5, 1.5)\,\}, \qquad \{\,(0.0, 1.0),\ (1.0, 0.0),\ (0.0, 0.0)\,\}.$$

Start with the weight vector

$$\mathbf{w}^T(0) = (0, 0, 0)$$

and the first vector $\mathbf{x}^T = (1.0, 0.0, 1.0)$ from set N. Then take the second and third vectors from set N. Next take the vectors from set P in the given order. Repeat this order until all vectors are classified.

Problem 21. Use the perceptron learning algorithm to find the plane that separates the two sets (extended space)

$$P = \{\,(1.0, 2.0, 2.0)^T\,\}$$

and

$$N = \{\,(1.0, 0.0, 1.0),\ \ (1.0, 1.0, 0.0)^T,\ \ (1.0, 0.0, 0.0)^T.$$

Thus in $\mathbb{R}^2$ we consider the two sets of points

$$\{\,(2.0, 2.0)^T\,\}, \qquad \{\,(0.0, 1.0),\ \ (1.0, 0.0)^T,\ \ (0.0, 0.0)^T\,\}.$$

Start with $\mathbf{w}^T = (0.0, 0.0, 0.0)$ and the first vector $\mathbf{x}^T = (1.0, 0.0, 1.0)$ from the set N. Then take the second and third vector from the set N. Next take the vector from the set P. Repeat this order until all vectors are classified. Draw a picture in $\mathbb{R}^2$ of the final result.

Problem 22. Consider the NOR-operation

x_1	x_2	NOR(x_1,x_2)
0	0	1
0	1	0
1	0	0
1	1	0

Apply the preceptron learning algorithm to classify the input points. Let the initial weight vector be

$$\mathbf{w} = (0.882687, 0.114597, 0.0748009)^T$$

where the first value $0.882687 = -\theta$, the threshold. Select training inputs from the table, top to bottom cyclically. In other words train the perceptron with inputs

$$00,\ 01,\ 10,\ 11\ 00,\ 01,\ 10\ 11,\ \ldots$$

Problem 23. Use the perceptron learning algorithm that separates the two sets (extended space)

$$P = \{\,(1.0, 0.0, 0.0)^T, \quad (1.0, -1.0, 1.0)^T\,\}$$

and

$$N = \{\,(1.0, 2.0, 0.0)^T, \quad (1.0, 0.0, 2.0)^T\,\}.$$

Thus in $\mathbb{R}^2$ we consider the two sets of points

$$\{\,(0.0, 0.0)^T,\ \ (-1.0, 1.0)^T\,\}, \qquad \{\,(2.0, 0.0)^T, \quad (0.0, 2.0)^T\,\}.$$

Start with $\mathbf{w}^T = (0.0, 0.0, 0.0)$ and the first vector $\mathbf{x}^T = (1.0, 2.0, 0.0)$ from the set N. Then take the second vector from the set N. Next take the first vector from the set P and then the second vector from the set P. Repeat this order until all vectors are classified. Draw a picture in $\mathbb{R}^2$ of the final result.

Problem 24. Given the boolean function $f : \{0,1\}^3 \to \{0,1\}$ as truth table

```
x_1 x_2 x_3 f(x_1,x_2,x_3)
 0   0   0        0
 0   0   1        0
 0   1   0        0
 0   1   1        0
 1   0   0        0
 1   0   1        1
 1   1   0        1
 1   1   1        1
```

Consider the unit cube in $\mathbb{R}^3$ with vertices (corner points) $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$. Find the plane in $\mathbb{R}^3$ that separates the 0 from the 1's.

8.5 Back-Propagation Algorithm

Problem 25. For the back-propagation algorithm in neural networks we need the derivative of the functions

$$f_\lambda(x) = \frac{1}{1+e^{-\lambda x}}, \quad \lambda > 0$$

and

$$g_\lambda(x) = \tanh(\lambda x), \quad \lambda > 0\,.$$

Find the ordinary differential equations for f_λ and g_λ.

Problem 26. (i) Let $x, \lambda \in \mathbb{R}$ with $x < 0$. Determine

$$\lim_{\lambda\to\infty} \frac{1}{1+e^{-\lambda x}}, \qquad \lim_{\lambda\to\infty} \frac{1-e^{-2\lambda x}}{1+e^{-2\lambda x}}.$$

(ii) Let $x, \lambda \in \mathbb{R}$ with $x > 0$. Determine

$$\lim_{\lambda\to\infty} \frac{1}{1+e^{-\lambda x}}, \qquad \lim_{\lambda\to\infty} \frac{1-e^{-2\lambda x}}{1+e^{-2\lambda x}}.$$

(iii) Let $x, \lambda \in \mathbb{R}$ with $x = 0$. Determine

$$\lim_{\lambda\to\infty} \frac{1}{1+e^{-\lambda x}}, \qquad \lim_{\lambda\to\infty} \frac{1-e^{-2\lambda x}}{1+e^{-2\lambda x}}.$$

Problem 27. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \frac{1}{1+e^{-\lambda x}}.$$

Find the autonomous differential equation for $f(x)$, i.e. find $f'(x)$ in terms of $f(x)$ and λ only.

Problem 28. Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) := \frac{1-e^{-2\lambda x}}{1+e^{-2\lambda x}}.$$

Find the autonomous differential equation for $g(x)$, i.e. find $g'(x)$ in terms of $g(x)$ and λ only.

Problem 29. A Boolean function $f : \{0,1\}^2 \to \{0,1\}$ is constant if for every $x \in \{0,1\}^2$ we have $f(x) = c$, where $c \in \{0,1\}$. A Boolean function is balanced if

$$|\{x \in \{0,1\}^2 \,:\, f(x) = 0\,\}| = |\{x \in \{0,1\}^2 \,:\, f(x) = 1\,\}|$$

i.e. f maps to an equal number of zeros and ones. Find all Boolean functions from $\{0,1\}^2$ to $\{0,1\}$ which are either constant or balanced. Determine which of these functions are separable, i.e. for which function can we find a hyperplane which separates the inputs x which give $f(x)=1$ and the inputs y which give $f(y)=0$.

Problem 30. Consider the addition of two binary digits

x	y	S	C
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

where S denotes the sum and C denotes the carry bit. We wish to simulate the sum and carry using a two layer perceptron network using back-propagation learning. The initial weight matrix for the hidden layer is

$$\begin{pmatrix} 0.1 & -0.1 & 0.1 \\ -0.1 & 0.1 & -0.1 \end{pmatrix}.$$

The initial weight matrix for the output layer is

$$\begin{pmatrix} 0.2 & -0.1 & 0.2 \\ 0.1 & -0.2 & 0.1 \end{pmatrix}.$$

Apply three iterations of the back-propagation learning algorithm. Use 3 neurons in the hidden layer. The third neuron in the hidden layer is always active, i.e. it provides the input 1 for the extended space. Use a learning rate of $\eta=1$ for all layers and the parameter $\lambda=1$ for the activation function. Use the input sequence

$$00, 01, 10, 11, 00, 01, 10, 11, 00\ldots$$

8.6 Hausdorff Distance

Problem 31. Determine the Hausdorff distance between

$$A := \{ (1.0, 1.0)^T, \, (1.0, 0.0)^T \}$$

and

$$B := \{ (0.0, 0.0)^T, \, (0.0, 1.0)^T \}.$$

Problem 32. Consider vectors in $\mathbb{R}^3$. Determine the Hausdorff distance between

$$A := \{ \mathbf{a}_1 = (1, 0, 0)^T, \mathbf{a}_2 = (0, 0.5, 0.5)^T, \mathbf{a}_3 = (-1, 0.5, 1)^T \}$$

and

$$B := \{ \mathbf{b}_1 = (3, 1, 2)^T, \mathbf{b}_2 = (2, 3, 1)^T \}.$$

First tabular all possible (Euclidean)distances.

Problem 33. Calculate the Hausdorff distance $H(A, B)$ between the set of vectors in $\mathbb{R}^3$

$$A := \{ (3, 1, 4)^T \, , \, (1, 5, 9)^T \}$$

and

$$B := \{ (2, 7, 1)^T \, , \, (8, 2, 8)^T \, , \, (1, 8, 2)^T \}.$$

Use the Euclidean distance for $\| \cdot \|$.

Problem 34. Determine the Hausdorff distance between

$$A := \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

and

$$B := \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Problem 35. Find the Hausdorff distance $H(A, B)$ between the set of vectors in $\mathbb{R}^2$

$$A := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, , \, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$B := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \, , \, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Give a geometric interpretation.

Problem 36. Determine the Hausdorff distance between

$$A := \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$$

and

$$B := \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}.$$

Problem 37. Determine the Hausdorff distance between the set of the *standard basis*

$$A := \{ (1,0,0,0)^T, \ (0,1,0,0)^T, \ (0,0,1,0)^T, \ (0,0,0,1)^T \}$$

and the set of the *Bell basis*

$$B := \{ \frac{1}{\sqrt{2}}(1,0,0,1)^T, \ \frac{1}{\sqrt{2}}(1,0,0,-1)^T, \ \frac{1}{\sqrt{2}}(0,1,1,0)^T, \ \frac{1}{\sqrt{2}}(0,1,-1,0)^T \}.$$

Problem 38. Given the two sets of vectors

$$A = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

(i) Are the vectors linearly independent in $\mathbb{R}^3$ for each of the sets?
(ii) Calculate the Hausdorff distance between the two sets of vectors. Give a geometric interpretation of the solution.

8.7 Miscellany

Problem 39. Consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Find the three eigenvalues λ_1, λ_2, λ_3 and the corresponding normalized eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$. Does

$$\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T$$

provide the original matrix A? Discuss.

Problem 40. (i) Consider the traveling saleswoman problem. Given 8 cities with the space coordinates

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (0,1,1),\ (1,0,1),\ (1,1,0),\ (1,1,1).$$

The traveling saleswoman does not like her husband. Thus she wants to find the longest route starting at $(0,0,0)$ and returning to $(0,0,0)$. Is there more than solution of the problem? If so provide all the solutions. The coordinates may be considered as a bitstrings (3 bits). For one of the longest routes give the Hamming distance between each two cities on the route. Is there a connection with the Gray code?
(ii) Consider the traveling salesman problem. Given 8 cities with the space coordinates

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (0,1,1),\ (1,0,1),\ (1,1,0),\ (1,1,1).$$

The traveling salesman does like his girlfriend. Thus he wants to find the shortest route starting at $(0,0,0)$ and returning to $(0,0,0)$. Is there more than solution of the problem? If so provide all the solutions. The coordinates may be considered as a bitstrings (3 bits). For one of the shortest routes give the Hamming distance between each two cities on the route. Is there a connection with the Gray code?

Problem 41. Consider the Hilbert space $\mathbb{R}^4$. Find all pairwise orthogonal vectors (column vectors) $\mathbf{x}_1, \ldots, \mathbf{x}_p$, where the entries of the column vectors can only be $+1$ or -1. Calculate the matrix

$$\sum_{j=1}^{p} \mathbf{x}_j \mathbf{x}_j^T$$

and find the eigenvalues and eigenvectors of this matrix.

Problem 42. In the case of least squares applied to supervised learning with a linear model the function to be minimised is the sum-squared-error

$$S(\mathbf{w}) := \sum_{i=0}^{p-1} (\hat{y}_i - f(\mathbf{x}_i))^2$$

($\mathbf{w} = (w_0, w_1, \ldots, w_{m-1})$) where

$$f(\mathbf{x}) = \sum_{j=0}^{m-1} w_j h_j(\mathbf{x})$$

and the free variables are the weights $\{ w_j \}$ for $j = 0, 1, \ldots, m-1$. The given training set, in which there are p pairs (indexed by j running from 0 up to $p-1$), is represented by

$$T := \{ (\mathbf{x}_j, \hat{y}_j) \}_{j=0}^{j=p-1}.$$

(i) Find the weights $\hat{w}_j$ from the given training set and the given h_j.
(ii) Apply the formula to the training set

$$\{ (1.0, 1.1), \quad (2.0, 1.8), \quad (3.0, 3.1) \}$$

i.e., $p = 3$, $x_0 = 1.0$, $y_0 = 1.1$, $x_1 = 2.0$, $y_1 = 1.8$, $x_2 = 3.0$, $y_2 = 3.1$. Assume that

$$h_0(x) = 1, \quad h_1(x) = x, \quad h_2(x) = x^2$$

i.e., $m = 3$.

Problem 43. Find the shortest path through A, B, C and D returning to A. The distances are tabulated below.

	A	B	C	D
A	0	$\sqrt{2}$	1	$\sqrt{5}$
B	$\sqrt{2}$	0	1	1
C	1	1	0	$\sqrt{2}$
D	$\sqrt{5}$	1	$\sqrt{2}$	0

Problem 44. Let

$$A := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Determine the eigenvalues λ_1 and λ_2. Determine the corresponding orthonormal eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Calculate

$$\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T.$$

Problem 45. The XOR-problem in $\mathbb{R}^2$

$$(0,0) \mapsto 0, \quad (1,1) \mapsto 0, \quad (0,1) \mapsto 1, \quad (1,0) \mapsto 1$$

cannot be classified using a straight line. Consider now an extension to $\mathbb{R}^3$

$$(0,0,0) \mapsto 0, \quad (1,1,0) \mapsto 0, \quad (0,1,1) \mapsto 1, \quad (1,0,1) \mapsto 1.$$

Can this problem be classified with a plane in $\mathbb{R}^3$?

Chapter 9

Genetic Algorithms

Problem 1. We define the number π as the number where the function

$$f(x) = \sin(x)$$

crosses the x-axis in the interval $[2, 4]$. How would we calculate π (of course an approximation of it) using this definition and genetic algorithms?

Problem 2. We define the number $\ln 2$ as the number where the function

$$f(x) = e^{-x} - 1/2$$

crosses the x-axis in the interval $[0, 1]$. How would we calculate $\ln 2$ (of course an approximation of it) using this definition and genetic algorithms?

Problem 3. Consider the polynomial

$$p(x) = x^2 - x - 1.$$

We define the golden mean number g as the number where the polynomial p crosses the x-axis in the interval $[1, 2]$.
(i) How would we calculate g (of course an approximation of it) using this definition and genetic algorithms?
(ii) Solve the quadratic equation $p(x) = 0$.
(iii) Apply Newton's method to solve $p(x) = 0$ with $x_0 = 1$. Calculate the first two steps in the sequence.

Problem 4. (i) Describe a map which maps binary strings of length n into the interval $[2, 3]$ uniformly, including 2 and 3. Determine the smallest n if the mapping must map a binary string within 0.001 of a given value in the interval.

(ii) Which binary string represents e? How accurate is this representation?
(iii) Give a fitness function for a genetic algorithm that approximates e, without explicitly using the constant e.

Problem 5. (i) Consider the traveling saleswoman problem. Given eight cities with the space coordinates

$$(0,0,0), \quad (0,0,1), \quad (0,1,0), \quad (0,1,1),$$
$$(1,0,0), \quad (1,0,1), \quad (1,1,0), \quad (1,1,1).$$

Consider the coordinates as a bitstring. We identify the bitstrings with base 10 numbers, i.e. $000 \to 0$, $001 \to 1$, $010 \to 2$, $011 \to 3$, $100 \to 4$, $101 \to 5$, $110 \to 6$, $111 \to 7$. Thus we set

$$x_0 = (0,0,0), \quad x_1 = (0,0,1), \quad x_2 = (0,1,0), \quad x_3 = (0,1,1)$$
$$x_4 = (1,0,0), \quad x_5 = (1,0,1), \quad x_6 = (1,1,0), \quad x_7 = (1,1,1).$$

The traveling saleswoman does not like her husband. Thus she wants to find the longest route starting at $(0,0,0)$ and returning to $(0,0,0)$ after visiting each city once. Is there more than one solution?
(ii) Consider the traveling salesman problem and the eight cities given in (i). The traveling salesman likes his girlfriend. Thus he wants to find the shortes route starting at $(0,0,0)$ and returning to $(0,0,0)$ after visiting each city once. What is the connection with the *Gray code*? Is there more than one solution?

Problem 6. Describe a map which maps binary strings of length n into the interval $[1,5]$ uniformly, including 1 and 5.

Determine n if the mapping must map a binary string within 0.01 of a given value in the interval.

Which binary string represents π? How accurate is this representation?

Problem 7. Solve the first order linear difference equation with constant coefficients

$$m_{t+1} = \frac{f(H)}{\overline{f}} m_t$$

where $f(H)$ and $\overline{f}$ are constants. Assume $m_t = ca^t$ where c and a are constants.

Problem 8. Consider

$$(3,1,4,7,9,2,6,8,5)$$
$$(4,8,2,1,3,7,5,9,6)$$

Positions are numbered from 0 from the left.

Perform the PMX crossing from position 2 to position 4 inclusive.

Perform the Bac and Perov crossing.

Problem 9. Consider binary strings of length 10 representing positive integers (using the base 2 representation). Determine the Gray code of 777. Determine the inverse Gray code of

```
01001101
```

Which numbers do these binary sequences represent?

Problem 10. Consider

$$(3, 1, 4, 7, 9, 2, 6, 8, 5)$$

$$(4, 8, 2, 1, 3, 7, 5, 9, 6)$$

Positions are numbered from 0 from the left. Perform the PMX crossing from position 2 to position 4 inclusive. Perform the Bac and Perov crossing.

Suppose the distances for a TSP is given by the following table.

	1	2	3	4	5	6	7	8	9
1	0	3	1	2	2	1	3	1	4
2	3	0	1	1	1	5	1	1	1
3	1	1	0	1	1	1	1	1	1
4	2	1	1	0	3	4	4	5	5
5	2	1	1	3	0	9	2	1	3
6	1	5	1	4	9	0	3	2	2
7	3	1	1	4	2	3	0	2	1
8	1	1	1	5	1	2	2	0	3
9	4	1	1	5	3	2	1	3	0

Determine the fitness of the sequences given and of the sequences obtained from crossing. Should the table always be symmetric?

Problem 11. Consider the interval $[0, \pi]$. Use the discrete global optimization method (DGO method) (to the maximum resolution) to determine the global maximum of $f(x) = \cos^2(x)\sin(x)$ using the initial value $\frac{\pi}{3}$. Find the point which achieves the global maximum using an accuracy of 0.1. Number the bit positions from 0 from the right.

Problem 12. Show that the XOR-gate can be built from 4 NAND-gates. Let A_1, A_2 be the inputs and O the output.

Problem 13. Consider the function $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$

$$f(x_1, x_2) = x_1 x_2.$$

We want to find the maxima of the (fitness) function f in the given domain $[-1,1] \times [-1,1]$. Consider the two bitstrings

```
b1 = "0000000011111111"    b2 = "1010101010101010"
```

where the 8 bits on the right-hand side belong to x_1 and the 8 bits on the left-hand side belong to x_2. Find the value of the fitness function for the two bitstring. Apply the NOT-operation to the two bitstring to obtain the new bitstrings $b3 = NOT(b1)$ and $b4 = NOT(b2)$. Select the two fittest of the four for survival.

Problem 14. Consider the domain $[-1,1] \times [-1,1]$ in $\mathbb{R}^2$ and the bitstring (16 bits)

```
0101010100001111
```

The 8 bits on the right-hand side belong to x and the 8 bits on the left-hand side belong to y (counting from right to left starting at 0). Find the real values x^* and y^* for this bitstring. Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$ be given by

$$f(x,y) = x^4 + y^4.$$

Find the value $f(x^*, y^*)$. How far is this value from the minimum of f in the given domain?

Problem 15. Let $\phi = \frac{1}{2}(1+\sqrt{5})$ be the golden mean number. The *dodecahedron* has 20 vertices ("corner points") given by

$$(0,1/\phi,\phi), \quad (0,1/\phi,-\phi), \quad (0,-1/\phi,\phi), \quad (0,-1/\phi,-\phi)$$
$$(\phi,0,1/\phi), \quad (-\phi,0,1/\phi), \quad (\phi,0,-1/\phi), \quad (-\phi,0,-1/\phi)$$
$$(1/\phi,\phi,0), \quad (-1/\phi,\phi,0), \quad (1/\phi,-\phi,0), \quad (-1/\phi,-\phi,0)$$
$$(1,1,1), \quad (1,1,-1), \quad (1,-1,1), \quad (-1,1,1)$$
$$(1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1), \quad (-1,-1,-1).$$

Solve the traveling salesman problem and the traveling saleswoman problem. The start city and end city is $(-1,-1,-1)$. The method is your choice.

Problem 16. Consider the tetrahedron. The four vertices ("corner points") have the coordinates in $\mathbb{R}^3$

$$(1,1,1), \quad (1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1).$$

Consider the map

$$(1,1,1) \mapsto 0, \quad (1,-1,-1) \mapsto 1, \quad (-1,1,-1) \mapsto 1, \quad (-1,-1,1) \mapsto 1.$$

Find a plane in $\mathbb{R}^3$ that separates the 0 from the 1's.

Miscellaneus

Problem 17. Consider a set of overdetermined linear equations $A\mathbf{x} = \mathbf{b}$, where A is a given $m \times n$ matrix with $m > n$, $\mathbf{x}$ is a column vector with n rows and $\mathbf{b}$ is a given column vector with m rows. Write a C++ program that finds the Chebyshev or minmax solution to the set of overdetermined linear equations $A\mathbf{x} = \mathbf{b}$, i.e. the vector $\mathbf{x}$ which minimizes

$$f(\mathbf{x}) = \max_{1 \le i \le m} c_i = \max_{1 \le i \le m} \left| b_i - \sum_{j=1}^{n} a_{ij} x_j \right| .$$

Apply your program to the linear system

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -0.5 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \\ 0.5 \\ 2.0 \end{pmatrix} .$$

Problem 18. The set covering problem is the zero-one integer program which consists of finding the N-tuples $\mathbf{s} = (s_0, s_1, \ldots, s_{n-1})$ that minimize the cost function

$$E = \sum_{j=0}^{n-1} c_j s_j$$

subject to p linear constraints

$$\sum_{j=0}^{N-1} a_{kj} s_j \ge e_k$$

and to the integrality constraints $s_j \in \{0, 1\}$. Here $a_{kj} \in \{0, 1\}$ are the elements of the $p \times n$ incidence matri A, c_j is the cost assigned to element j, and $e_k = 1$ for all $k = 0, 1, \ldots, p-1$. A particular instance of the set covering problem is then solely determined by the matrix A. Provide a genetic algorithms implementation for this problem.

Problem 19. By applying the Simulated Annealing method solve the Travelling Salesman Problem (TSP) for 10 cities with the following matrix of distances:

$$\begin{pmatrix} 0.0 & 7.3 & 8.4 & 18.5 & 9.8 & 9.6 & 8.6 & 9.6 & 8.8 & 15.2 \\ 7.3 & 0.0 & 8.0 & 15.6 & 14.2 & 2.3 & 4.3 & 6.6 & 2.9 & 9.9 \\ 8.4 & 8.0 & 0.0 & 10.0 & 7.8 & 8.6 & 4.4 & 2.8 & 6.2 & 8.4 \\ 18.5 & 15.6 & 10.0 & 0.0 & 15.0 & 14.7 & 11.3 & 9.3 & 12.7 & 7.2 \\ 9.8 & 14.2 & 7.8 & 15.0 & 0.0 & 15.6 & 11.8 & 10.7 & 13.4 & 16.0 \\ 9.6 & 2.3 & 8.6 & 14.7 & 15.6 & 0.0 & 4.2 & 6.5 & 2.4 & 8.3 \\ 8.6 & 4.3 & 4.4 & 11.3 & 11.8 & 4.2 & 0.0 & 2.3 & 1.8 & 6.6 \\ 9.6 & 6.6 & 2.8 & 9.3 & 10.7 & 6.5 & 2.3 & 0.0 & 4.1 & 5.9 \\ 8.8 & 2.9 & 6.2 & 12.7 & 13.4 & 2.4 & 1.8 & 4.1 & 0.0 & 7.0 \\ 15.2 & 9.9 & 8.4 & 7.2 & 16.0 & 8.3 & 6.6 & 5.9 & 7.0 & 0.0 \end{pmatrix}$$

Problem 20. The 0-1 Knapsack problem for a capacity of 100 and 10 items with weights

```
{ 14, 26, 19, 45, 5, 25, 34, 18, 30, 12 }
```

and corresponding values

```
{20, 24, 18, 70, 14, 23, 50, 17, 41, 21 };
```

Problem 21. Consider the domain $[-1,1] \times [-1,1]$ in $\mathbb{R}^2$ and the bitstring (16 bits)

```
0011110011000011
```

The 8 bits on the right-hand side belong to x and the 8 bits on the left-hand side belong to y (counting each subbitstring from right to left starting at 0). Find the values x^* and y^* for the two bistrings. Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$ be given by

$$f(x,y) = x^2 + 2y.$$

Find $f(x^*, y^*)$. Now swap around the two eight bits, i.e.

```
1100001100111100
```

Calculate again x^*, y^* and $f(x^*, y^*)$ for this bitstring. Suppose we want to minimize f in the domain $[-1,1] \times [-1,1]$. Which of the bitstrings would survive in a genetic algorithm?

Problem 22. Determine the *Hamming distance* between the bitstrings of length 9

`101011110` and `010101010`.

Calculate the XOR between the two bitstrings. Then calculate the Hamming distances between this bitstring and the two original ones.

Problem 23. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 2x^2 + x + 4.$$

(i) Display this expression as a binary tree.
(ii) Write down this expression using multiexpression programming.
(iii) Use these two expressions to calculate $f(5)$.

Chapter 10

Euler-Lagrange Equations

Problem 1. Consider the problem of joining two points in a plane with the shortest arc. Show that the length of arc is given by

$$I = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Problem 2. Solve the brachistochrone problem

$$\underbrace{\beta\rho\alpha\chi\iota\sigma\tau o\zeta}_{\text{shortest}} \quad \underbrace{\chi\rho o\nu o\zeta}_{\text{time}}$$

We assume that there is no friction. We have to solve

$$t = \int_0^a \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2gy}} dx \longrightarrow \text{ min}$$

y is the vertical distance from the initial level. g is the acceleration of gravity with $g = 9.81 \text{ m/sec}^2$. The boundary conditions are

$$y(x = 0) = 0, \qquad y(x = a) = b.$$

Problem 3. Consider the problem of minimum surface of revolution. We seek a curve $y(x)$ with prescribed end points such that by revolving this curve around the x-axis a surface of minimal area results. Therefore

$$I = 2\pi \int_{x_0}^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Problem 4. Consider the problem of joining two points in a plane with the shortest arc. Use the parameter representation of the curve.

Problem 5. Let

$$I\left(x, y, \frac{dy}{dx}\right) = \int_0^1 \left[y(x) + y(x)\frac{dy}{dx}\right] dx.$$

Find the Euler-Lagrange equation.

Problem 6. Find the Euler-Lagrange equation for

$$I(x, u, u', u'') = \int_{x_0}^{x_1} L(x, u, u', u'')dx.$$

Problem 7. A uniform elastic cantilever beam with bending rigidity EI and length L is uniformly loaded. Derive the differential equation and boundary conditions. Find the end deflection $y(L)$.

Problem 8. We consider partial differential equations. Let

$$\Omega[u(x, y)] = \int\int_B \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy \to \text{Min}$$

where B is a connected domain in E_2 (two-dimensional Euclidean space). The boundary conditions are: u takes prescribed values at the boundary ∂B. First we find the Gateaux derivative of Ω.

Problem 9. Let

$$\Omega[x(t)] = \int_{t_0}^{t_1} L\left(x(t), \frac{dx(t)}{dt}\right) dt$$

with

$$L(x, \dot{x}) = \frac{m\dot{x}^2}{2} - mgx$$

where m is the mass and g the gravitational acceleration. (i) Calculate the Gateaux derivative of Ω.
(ii) Find the Euler Lagrange equation.

(iii) Solve the Euler Lagrange equation with the inital conditions

$$x(t=0) = 5\,meter, \qquad \frac{dx}{dt}(t=0) = 0\,\frac{meter}{sec}$$

Problem 10. Let

$$I = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt$$

with

$$\dot{x} := \frac{dx}{dt}, \qquad \dot{y} := \frac{dy}{dt}, \qquad \dot{z} := \frac{dz}{dt}$$

Express x, y, z in spherical coordinates r, ϕ, θ, where r is a constant. Find the Euler-Lagrange equation.

Problem 11. Let

$$I = \int_{t_0}^{t_1} L(t, \dot{x}, x)dt$$

with

$$L(t, \dot{x}, x) = \frac{1}{2}e^{\alpha t}(\dot{x}^2 - \beta x^2), \qquad \alpha, \beta \in \mathbb{R}.$$

Find the Euler-Lagrange equation. Solve the Euler-Lagrange equation. Discuss the solution.

Problem 12. Solve the variationial problem

$$\Omega[y(x)] = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \to \text{ Extremum}$$

with the constraint

$$\int_0^a y(x)dx = C = \text{ const.}$$

and the boundary condition $y(0) = y(a) = 0$.

Problem 13. Consider

$$\frac{1}{2}\int_a^b ((y')^2 - y^2)dx.$$

Find the Euler-Lagrange equation using the Gateaux derivative. Solve the differential equation.

Miscellaneous

Problem 14. Find the body in $\mathbb{R}^3$ with a given volume V which has the smallest surface S.

Problem 15. Minimize

$$\int_0^1 (u^2 + (du/dx)^2)dx \quad \text{subject to} \quad u(x=0)=1, \;\; du(x=1)/dx = 0$$

Chapter 11

Numerical Implementations

Problem 1. The bounding phase method is determining a bracketing interval for the minimum/maximum of a continouosly differentiable function $f : (a, b) \to \mathbb{R}$. It guarantees to bracket the minimum/maximum for a unimodal function. The algorithm is as follows:

1) Initialize $x^{(0)}$, Δ (positive), and the iteration step $k = 0$.
2) If $f(x^{(0)} - \Delta) \geq f(x^{(0)}) \geq f(x^{(0)} + \Delta)$ then keep Δ positive.

If $f(x^{(0)} - \Delta) \leq f(x^{(0)}) \leq f(x^{(0)} + \Delta)$ then make Δ negative, $\Delta = -\Delta$.
3) Calculate $x^{(k+1)} = x^{(k)} + 2^k \Delta$.

4) If $f(x^{(k+1)}) < f(x^{(k+1)})$ increment $k = k+1$ and go to 3), else terminate and the minimum

lies in the interval $(x^{(k-1)}, x^{(k+1)})$.

Determine a bracketing interval for the minimum of the function $f : (0, \infty) \to \mathbb{R}$

$$f(x) = x^2 + \frac{54}{x}$$

by applying the bounding phase method. Start with the parameters $x^{(0)} = 0.6$ and $\Delta = 0.5$. Provide a C++ program.

Problem 2. The interval halving method is determining a minimum/maximum of a continuously differentiable function f based only on computing the function's values (without using the derivatives). It guarantees the minimum/maximum for a unimodal function. The algorithm is as follows:

1) Initialize the precision ϵ, the search interval (a, b), the initial minimum/maximum $x_m = \frac{a+b}{2}$, and $L = b - a$.

2) Calculate $x_1 = a + \frac{L}{4}$, $x_2 = b - \frac{L}{4}$, $f(x_1)$ and $f(x_2)$.

3) If $f(x_1) < f(x_m)$ set $b = x_m$, $x_m = x_1$, go to 5;
else continue to 4.

4) If $f(x_2) < f(x_m)$ set $a = x_m$, $x_m = x_2$, go to 5;
Else set $a = x_1$, $b = x_2$, continue to 5.

5) Calculate $L = b - a$.
If $|L| < \epsilon$ terminate;
else go to 2.

Determine the minimum of the function

$$f(x) = x^2 + \frac{54}{x}$$

by applying the interval halving method. The initial sercah interval is $(0, 5)$ and the precision is 10^{-3}. Give a C++ implementation.

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