# Problems and Solutions
# in
# Hilbert space theory,
# wavelets
# and
# generalized functions

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# Preface

The purpose of this book is to supply a collection of problems in Hilbert space theory, wavelets and generalized functions.

**Prescribed books for problems.**

1) Hilbert Spaces, Wavelets, Generalized Functions and Modern Quantum Mechanics

by Willi-Hans Steeb
Kluwer Academic Publishers, 1998
ISBN 0-7923-5231-9

2) Classical and Quantum Computing with C++ and Java Simulations

by Yorick Hardy and Willi-Hans Steeb
Birkhauser Verlag, Boston, 2002
ISBN 376-436-610-0

3) Problems and Solutions in Quantum Computing and Quantum Information, second edition

by Willi-Hans Steeb and Yorick Hardy
World Scientific, Singapore, 2006
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6077.html`

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# Notation

| | |
|---|---|
| $:=$ | is defined as |
| $\in$ | belongs to (a set) |
| $\notin$ | does not belong to (a set) |
| $\cap$ | intersection of sets |
| $\cup$ | union of sets |
| $\emptyset$ | empty set |
| $\mathbb{N}$ | set of natural numbers |
| $\mathbb{Z}$ | set of integers |
| $\mathbb{Q}$ | set of rational numbers |
| $\mathbb{R}$ | set of real numbers |
| $\mathbb{R}^+$ | set of nonnegative real numbers |
| $\mathbb{C}$ | set of complex numbers |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space |
| | space of column vectors with $n$ real components |
| $\mathbb{C}^n$ | $n$-dimensional complex linear space |
| | space of column vectors with $n$ complex components |
| $\mathcal{H}$ | Hilbert space |
| $i$ | $\sqrt{-1}$ |
| $\Re z$ | real part of the complex number $z$ |
| $\Im z$ | imaginary part of the complex number $z$ |
| $\|z\|$ | modulus of complex number $z$ |
| | $\|x+iy\| = (x^2+y^2)^{1/2}, \ x, y \in \mathbf{R}$ |
| $T \subset S$ | subset $T$ of set $S$ |
| $S \cap T$ | the intersection of the sets $S$ and $T$ |
| $S \cup T$ | the union of the sets $S$ and $T$ |
| $f(S)$ | image of set $S$ under mapping $f$ |
| $f \circ g$ | composition of two mappings $(f \circ g)(x) = f(g(x))$ |
| $\mathbf{x}$ | column vector in $\mathbb{C}^n$ |
| $\mathbf{x}^T$ | transpose of $\mathbf{x}$ (row vector) |
| $\mathbf{0}$ | zero (column) vector |
| $\\| . \\|$ | norm |
| $\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$ | scalar product (inner product) in $\mathbb{C}^n$ |
| $\mathbf{x} \times \mathbf{y}$ | vector product in $\mathbb{R}^3$ |
| $A, B, C$ | $m \times n$ matrices |
| $\det(A)$ | determinant of a square matrix $A$ |
| tr(A) | trace of a square matrix $A$ |
| rank$(A)$ | rank of matrix $A$ |
| $A^T$ | transpose of matrix $A$ |

| | |
|---|---|
| $\overline{A}$ | conjugate of matrix $A$ |
| $A^*$ | conjugate transpose of matrix $A$ |
| $A^\dagger$ | conjugate transpose of matrix $A$ (notation used in physics) |
| $A^{-1}$ | inverse of square matrix $A$ (if it exists) |
| $I_n$ | $n \times n$ unit matrix |
| $I$ | unit operator |
| $0_n$ | $n \times n$ zero matrix |
| $AB$ | matrix product of $m \times n$ matrix $A$ and $n \times p$ matrix $B$ |
| $A \bullet B$ | Hadamard product (entry-wise product) of $m \times n$ matrices $A$ and $B$ |
| $[A, B] := AB - BA$ | commutator for square matrices $A$ and $B$ |
| $[A, B]_+ := AB + BA$ | anticommutator for square matrices $A$ and $B$ |
| $A \otimes B$ | Kronecker product of matrices $A$ and $B$ |
| $A \oplus B$ | Direct sum of matrices $A$ and $B$ |
| $\delta_{jk}$ | Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$ |
| $\delta$ | delta function |
| $\Theta$ | Heaviside's function |
| $\lambda$ | eigenvalue |
| $\epsilon$ | real parameter |
| $t$ | time variable |
| $\hat{H}$ | Hamilton operator |

# Chapter 1

# General

**Problem 1.** Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \, , \, \rangle$. Let $u, v \in \mathcal{H}$.
(i) Show that

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|.$$

(ii) Show that

$$\|u + v\| \leq \|u\| + \|v\|.$$

**Problem 2.** Consider a Hilbert space $\mathcal{H}$ with scalar product $\langle \, , \, \rangle$. The scalar product implies a norm via $\|f\|^2 := \langle f, f \rangle$, where $f \in \mathcal{H}$.
(i) Show that

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

(ii) Assume that $\langle f, g \rangle = 0$, where $f, g \in \mathcal{H}$. Show that

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2.$$

**Problem 3.** Let $f, g \in \mathcal{H}$. Use the Schwarz inequality

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \langle g, g \rangle = \|f\|^2 \, \|g\|^2$$

to prove the triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|.$$

**Problem 4.** Let $\mathcal{H}$ be a Hilbert space and $f, g \in \mathcal{H}$. Show that

$$\langle f|f\rangle\langle g|g\rangle \geq \frac{1}{4}(\langle f|g\rangle + \langle g|f\rangle)^2.$$

**Problem 5.** Consider a complex Hilbert space $\mathcal{H}$ and $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$. Let $c_1, c_2 \in \mathbb{C}$. An *antilinear operator* $K$ in this Hilbert space $\mathcal{H}$ is characterized by

$$K(c_1|\phi_1\rangle + c_2|\phi_2\rangle) = c_1^* K|\phi_1\rangle + c_2^* K|\phi_2\rangle.$$

A *comb* is an antilinear operator $K$ with zero expectation value for all states $|\psi\rangle$ of a certain complex Hilbert space $\mathcal{H}$. This means

$$\langle\psi|K|\psi\rangle = \langle\psi|LC|\psi\rangle = \langle\psi|L|\psi^*\rangle = 0$$

for all states $|\psi\rangle \in \mathcal{H}$, where $L$ is a linear operator and $C$ is the complex conjugation.
(i) Consider the two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$. Find a unitary $2 \times 2$ matrix such that

$$\langle\psi|UC|\psi\rangle = 0.$$

(ii) Consider the Pauli spin matrices with $\sigma_0 = I_2$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$. Find

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\langle\psi|\sigma_\mu C|\psi\rangle g^{\mu,\nu}\langle\psi|\sigma_\nu C|\psi\rangle$$

where $g^{\mu,\nu} = \text{diag}(-1, 1, 0, 1)$.

**Problem 6.** Let $P$ be a nonzero projection operator in a Hilbert space $\mathcal{H}$. Show that $\|P\| = 1$.

**Problem 7.** A family, $\{\,\psi_j\,\}_{j\in J}$ of vectors in the Hilbert space, $\mathcal{H}$, is called a *frame* if for any $f \in \mathcal{H}$ there exist two constants $A > 0$ and $0 < B < \infty$, such that

$$A\|f\|^2 \leq \sum_{j\in J}|\langle\psi_j|f\rangle|^2 \leq B\|f\|^2.$$

Consider the Hilbert space $\mathcal{H} = \mathbb{R}^2$ and the family of vectors

$$\left\{ \psi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Show that we have a tight frame.

**Problem 8.** Let $T : X \to Y$ be a linear map between linear spaces (vector spaces) $X$, $Y$. The *null space* or *kernel* of the linear map $T$, denoted by $\ker T$, is the subset of $X$ defined by

$$\ker T := \{\, x \in X \,:\, Tx = 0 \,\}.$$

The *range* of $T$, denoted by $\mathrm{ran} T$, is the subset of $Y$ defined by

$$\mathrm{ran} T := \{\, y \in Y \,:\, \text{there exists } x \in X \text{ such that } Tx = y \,\}.$$

Let $P$ be a projection operator in a Hilbert space $\mathcal{H}$. Show that $\mathrm{ran} P$ is closed and

$$\mathcal{H} = \mathrm{ran} P \oplus \ker P$$

is the orthogonal direct sum of $\mathrm{ran} P$ and $\ker P$.

**Problem 9.** Let $\mathcal{H}$ be an arbitrary Hilbert space with scalar product $\langle\, , \,\rangle$. Show that if $\varphi$ is a bounded linear functional on the Hilbert space $\mathcal{H}$, then there is a unique vector $u \in \mathcal{H}$ such that

$$\varphi(x) = \langle u, x \rangle \qquad \text{for all } \; x \in \mathcal{H}.$$

**Problem 10.** Let $\mathcal{H}$ be an arbitrary Hilbert space. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ satisfies the *Fredholm alternative* if one of the following two alternatives holds:

(i) either $Ax = 0$, $A^*x = 0$ have only the zero solution, and the linear equations $Ax = y$, $A^*x = y$ have a unique solution $x \in \mathcal{H}$ for every $y \in \mathcal{H}$;
(ii) or $Ax = 0$, $A^*x = 0$ have nontrivial, finite-dimensional solution spaces of the same dimension, $Ax = y$ has a (nonunique) solution if and only if $y \perp u$ for every solution $u$ of $A^*u = 0$, and $A^*x = y$ has a (nonunique) solution if and only if $y \perp u$ for every solution $u$ of $Au = 0$.

Give an example of a bounded linear operator that satisfies the Fredholm alternative.

**Problem 11.** Let $(M, d)$ be a complete metric space (for example a Hilbert space) and let $f : M \to M$ be a mapping such that

$$d(f^{(m)}(x), f^{(m)}(y)) \leq k d(x, y), \qquad \forall x, y \in M$$

for some $m \geq 1$, where $0 \leq k < 1$ is a constant. Show that the map $f$ has a unique fixed point in $M$.

**Problem 12.** Let $\mathcal{H}$ be a Hilbert space and let $f : \mathcal{H} \to \mathcal{H}$ be a monotone mapping such that for some constant $\beta > 0$

$$\|f(u) - f(v)\| \le \beta \|u - v\| \qquad \forall\, u, v \in \mathcal{H}.$$

Show that for any $w \in \mathcal{H}$, the equation

$$u + f(u) = w$$

has a unique solution $u$.

**Problem 13.** Let $f, g \in \mathcal{H}$. Find all solutions to the equations

$$\langle f, g \rangle \langle g, f \rangle = i.$$

**Problem 14.** Let $f, g \in \mathcal{H}$. Show that

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

where the norm is implied by the scalar product of the Hilbert space.

**Problem 15.** Show that

$$\langle f, g \rangle = \frac{1}{4}\|f + g\|^2 - \frac{1}{4}\|f - g\|^2$$

or

$$\langle f, g \rangle = \frac{1}{4}\|f + g\|^2 - \frac{1}{4}\|f - g\|^2 + \frac{i}{4}\|f + ig\|^2 - \frac{i}{4}\|f - ig\|^2$$

depending on whether we are dealing with a real and complex Hilbert space.

**Problem 16.** Consider a Hilbert space $\mathcal{H}$ with scalar product $\langle , \rangle$. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be elements of the Hilbert space with $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$. Show that

$$\sqrt{1 - |\langle \mathbf{u}, \mathbf{v} \rangle|^2} \le \sqrt{1 - |\langle \mathbf{u}, \mathbf{w} \rangle|^2} + \sqrt{1 - |\langle \mathbf{w}, \mathbf{v} \rangle|^2}.$$

**Problem 17.** Consider a Hilbert space $\mathcal{H}$. Let $\|.\|$ be the norm implied by the scalar product. Let $\mathbf{u}, \mathbf{v} \in \mathcal{H}$.

(i) Show that

$$\|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v}\| \ge \|\mathbf{u}\|.$$

(ii) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle \le 2\|\mathbf{u}\| \cdot \|\mathbf{v}\|.$$

**Problem 18.** Given a Hilbert space $\mathcal{H}$ and a Hilbert subspace $\mathcal{G}$ of $\mathcal{H}$. The Hilbert space *projection theorem* states that for every $f \in \mathcal{H}$, there exists a unique $g \in G$ such that

$$(i) \quad f - g \in G^{\perp}$$

$$(ii) \quad \|f - g\| = \inf_{h \in \mathcal{G}} \|f - h\|$$

where the space $\mathcal{G}^{\perp}$ is defined by

$$\mathcal{G}^{\perp} := \{ \, k \in \mathcal{H} \, : \, \langle k | u \rangle = 0 \text{ for all } u \in \mathcal{G} \, \}.$$

Show that if $g$ is the minimizer of $\|f - h\|$ over all $h \in \mathcal{G}$, then it is true that $f - g \in \mathcal{G}^{\perp}$.

**Problem 19.** Let $\{ \, \phi_n \, \}_{n \in \mathbb{Z}}$ be an orthonormal basis in a Hilbert space $\mathcal{H}$. Then any vector $f \in \mathcal{H}$ can be written as

$$f = \sum_{n \in \mathbb{Z}} \langle f, \phi_n \rangle \phi_n.$$

Now suppose that $\{ \, \psi_n \, \}_{n \in \mathbb{Z}}$ is also a basis for $\mathcal{H}$, but it is not orthonormal. Show that if we can find a so-called dual basis $\{ \, \chi_n \, \}_{n \in \mathbb{Z}}$ satisfying

$$\langle \psi_n | \chi_m \rangle = \delta(n - m)$$

then for any vector $f \in \mathcal{H}$, we have

$$f = \sum_{n \in \mathbb{Z}} \langle f | \chi_n \rangle \psi_n.$$

Here $\delta(n - m)$ denotes the Kronecker delta with $\delta(n - m) = 0$ if $n = m$ and 1 otherwise.

**Problem 20.** Let $(X_1, \| \cdot \|_1)$ and $(X_2, \| \cdot \|_2)$ be two normed spaces. Show that the product vector spaces $X = X_1 \times X_2$ is also a normed vector space if we define

$$\|x\| := \max(\|x_1\|_1, \|x_2\|_2)$$

with $x = (x_1, x_2)$.

**Problem 21.** Let $A$ be a linear bounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Let $u, v \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Consider the equation

$$Au - \lambda u = v.$$

(i) Show that for $\lambda$ nonreal (i.e. it has an imaginary part) $v$ cannot vanish unless $u$ vanishes.

(ii) Show that for $\lambda$ nonreal we have

$$\|(A-\lambda I)^{-1}v\| \leq \frac{1}{|\Im\lambda|}\|v\|.$$

**Problem 22.** Let $\mathbb{E}$ be the exterior of the unit disc

$$\{\, z \in \mathbb{C} \,:\, |z| > 1 \,\}$$

and $\mathbb{T}$ the unit circle

$$\{\, z \in \mathbb{C} \,:\, |z| = 1 \,\}.$$

Let $\mathcal{H}_2(\mathbb{E})$ be the *Hardy spacee* of square integrable functions on $\mathbb{T}$, analytic in the region $\mathbb{E}$. The inner product for $f(z), g(z) \in \mathcal{H}_2(\mathbb{E})$ is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\omega})^* g(e^{i\omega})d\omega = \frac{1}{2\pi i}\oint_{\mathbb{T}} f^*(1/z^*)g(z)\frac{dz}{z}.$$

Let $f(z) = z^2$ and $g(z) = z + 1$. Find the scalar product $\langle f, g\rangle$.

**Problem 23.** Let $\mathcal{O} = \{\, u_1.u_2, \dots \,\}$ be an orthonormal set in a infinite dimensional Hilbert space. Show that if

$$x = \sum_{j=1}^{\infty} c_j u_j$$

then

$$\|x\|^2 = \sum_{j=1}^{\infty} |c_j|^2.$$

**Problem 24.** Two Cauchy sequences $\{\, x_k \,\}$ and $\{\, y_k \,\}$ are said to be equivalent if for all $\epsilon > 0$, there is a $k(epsilon)$ such that for all $j \geq k(\epsilon)$ we have $d(x_j, y_j) < \epsilon$. One writes $\{\, x_k \,\} \sim \{\, y_k \,\}$. Obviously, $\sim$ is an equivalence relationship. Show that equivalent Cauchy sequences have the same limit.

**Problem 25.** Consider the sequence $\{\, x_k \,\}$, $k = 1, 2, \dots$ in $\mathbb{R}$ defined by $x_k = 1/k^2$ for all $k = 1, 2, \dots$. Show that this sequence is a Cauchy sequence.

**Problem 26.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S}$ be a sub Hilbert space. Show that any element $u$ of $\mathcal{H}$ can be decomposed uniquely

$$u = v + w$$

where $v$ is in $\mathcal{S}$ and $w$ is in $\mathcal{S}^\perp$.

**Problem 27.** Let $u$, $v_1$, $v_2$ be elements of a Hilbert space. Show that

$$2\|u-v_1\|^2 + 2\|u-v_2\|^2 = \|2\left(u - \frac{v_1+v_2}{2}\right)\|^2 + \|v_1 - v_2\|^2.$$

**Problem 28.** Let $P$ be the set of prime numbers. We define the set

$$S := \{(p,q) \,:\, p,q \in P \; p \le q\}.$$

Show that

$$d((p_1,q_1),(p_2,q_2)) := |p_1q_1 - p_2q_2|$$

defines a metric.

**Problem 29.** Consider the vector space of all continuous functions defined on $[a,b]$. We define a metric

$$d(f,g) := \max_{a \le x \le b} |f(x) - g(x)|.$$

Let $a = \pi$, $b = \pi$, $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Find $d(f,g)$.

**Problem 30.** The $n \times n$ matrices over $\mathbb{R}$ form a vector space. Show that

$$d(A,B) := \sum_{j=1}^{n}\sum_{k=1}^{n} |a_{jk} - b_{jk}|$$

defines a metric.

**Problem 31.** Let $n \ge 1$. Consider the continuous function

$$f_n(t) = \begin{cases} 0 & 0 \le t < 1/2 - 1/n \\ 1/2 + \frac{n}{2}(t - 1/2) & 1/2 - 1/n \le t \le 1/2 + 1/n \\ 1 & 1/2 + 1/n \le t \le 1 \end{cases}$$

Show that the sequence $\{ f_n(t) \}$ is not a Cauchy sequence for the uniform norm, but with any of the $L^p$ norms $(1 \le p < \infty)$ it is a Cauchy sequence.

**Problem 32.** The sequence space consists of the set of all (bounded or unbounded) sequences of complex

$$x = (\chi_1, \chi_2, \ldots)$$

Thus we have a vector space. Can we define a metric in this vector space which is implied by a norm?

**Problem 33.** Let $f, g \in \mathcal{H}$. Show that

$$\langle f|f\rangle\langle g|g\rangle \geq \frac{1}{4}(\langle f|g\rangle + \langle g|f\rangle)^2.$$

# Chapter 2

# Finite Dimensional Hilbert Spaces

**Problem 1.** Consider the Hilbert space $\mathbb{R}^4$ and the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(i) Show that the vectors are linearly independent.
(ii) Use the *Gram-Schmidt orthogonalization process* to find mutually orthogonal vectors.

**Problem 2.** Consider the Hilbert space $\mathbb{R}^4$. Show that the vectors (*Bell basis*)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

are linearly independent. Show that they form a orthonormal basis in the Hilbert space $\mathbb{R}^4$.

**Problem 3.** Consider the Hilbert space $\mathbb{R}^4$. Find all pairwise orthogonal vectors (column vectors) $\mathbf{x}_1, \ldots, \mathbf{x}_p$, where the entries of the column vectors can only be $+1$ or $-1$. Calculate the matrix

$$\sum_{i=1}^{p} \mathbf{x}_i \mathbf{x}_i^T$$

and find the eigenvalues and eigenvectors of this matrix

**Problem 4.** A sequence $\{ f_n \}$ ($n \in \mathbb{N}$) of elements in a normed space $E$ is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists a number $M_\epsilon$, such that $\|f_p - f_q\| < \epsilon$ for $p, q > M_\epsilon$. Consider the Hilbert space $\mathbb{R}$. Show that

$$s_n = \sum_{j=1}^{n} \frac{1}{(j-1)!}, \qquad n \geq 1$$

is a Cauchy sequence.

**Problem 5.** Two Cauchy sequences $\{ x_k \}$ and $\{ y_k \}$ are said to be equivalent if for all $\epsilon > 0$, there is a $k(\epsilon)$ such that for all $j \geq k(\epsilon)$ we have $d(x_j, y_j) < \epsilon$. One writes $\{ x_k \} \sim \{ y_k \}$. Obviously, $\sim$ is an equivalence relationship. Show that equivalent Cauchy sequences have the same limit.

**Problem 6.** Consider the sequence $\{ x_k \}$, $k = 1, 2, \ldots$ in $\mathbb{R}$ defined by $x_k = 1/k^2$ for all $k = 1, 2, \ldots$. Show that this sequence is a Cauchy sequence.

**Problem 7.** Consider the Hilbert space $\mathbb{C}^2$ and the vectors

$$|0\rangle = \begin{pmatrix} i \\ i \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Normalize these vectors and then calculate the probability $|\langle 0|1\rangle|^2$.

**Problem 8.** Consider the Hilbert space $\mathbb{R}^n$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 \equiv 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

Note that

$$\|\mathbf{x}\|^2 := \langle \mathbf{x}, \mathbf{y} \rangle.$$

**Problem 9.** Let $|0\rangle$, $|1\rangle$ be an orthonormal basis in the Hilbert space $\mathbb{C}^2$. Let

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

where $\theta, \phi \in \mathbb{R}$.

(i) Find $\langle\psi|\psi\rangle$.
(ii) Find the probability $|\langle 0|\psi\rangle|^2$. Discuss $|\langle 0|\psi\rangle|^2$ as a function of $\theta$.
(iii) Assume that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the $2 \times 2$ matrix $|\psi\rangle\langle\psi|$ and calculate the eigenvalues.

**Problem 10.** Consider the Hilbert space $\mathbb{R}^2$. Show that the vectors

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

are linearly independent. Find

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Show that these four vectors form a basis in $\mathbb{R}^4$. Consider the $4 \times 4$ matrix $Q$ which is constructed from the four vectors given above, i.e. the columns of the $4 \times 4$ matrix are the four vectors. Find $Q^T$. Is $Q$ invertible? If so find the inverse $Q^{-1}$. What is the use of the matrix $Q$?

**Problem 11.** Consider the Hilbert space $\mathbb{R}^4$. Let $A$ be a symmetric $4 \times 4$ matrix over $\mathbb{R}$. Assume that the eigenvalues are given by $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$ and $\lambda_4 = 3$ with the corresponding normalized eigenfunctions

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Find the matrix $A$ by means of the *spectral theorem*.

**Problem 12.** Show that the $2 \times 2$ matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ & -1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $M_2(\mathbb{C})$.

**Problem 13.** Consider the Hilbert space $\mathcal{H}$ of the $2 \times 2$ matrices over the complex numbers with the scalar product

$$\langle A, B \rangle := \mathrm{tr}(AB^*), \qquad A, B \in \mathcal{H}.$$

Show that the rescaled Pauli matrices $\mu_j := \frac{1}{\sqrt{2}}\sigma_j$, $j = 1, 2, 3$

$$\mu_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the rescaled $2 \times 2$ identity matrix

$$\mu_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $\mathcal{H}$.

**Problem 14.** Let $A$, $B$ be two $n \times n$ matrices over $\mathbb{C}$. We introduce the scalar product

$$\langle A, B \rangle := \frac{\mathrm{tr}(AB^*)}{\mathrm{tr} I_n} = \frac{1}{n}\mathrm{tr}(AB^*).$$

This provides us with a Hilbert space.
The Lie group $SU(N)$ is defined by the complex $n \times n$ matrices $U$

$$SU(N) := \{\, U \,:\, U^*U = UU^* = I_n \,,\, \det(U) = 1 \,\}.$$

The dimension is $N^2 - 1$. The Lie algebra $su(N)$ is defined by the $n \times n$ matrices $X$

$$su(N) := \{\, X \,:\, X^* = -X \,,\, \mathrm{tr} X = 0 \,\}.$$

(i) Let $U \in SU(N)$. Calculate $\langle U, U \rangle$.
(ii) Let $A$ be an arbitrary complex $n \times n$ matrix. Let $U \in SU(N)$. Calculate $\langle UA, UA \rangle$.
(iii) Consider the Lie algebra $su(2)$. Provide a basis. The elements of the basis should be orthogonal to each other with respect to the scalar product given above. Calculate the commutators of these matrices.

**Problem 15.** Let $\hat{H} = \omega S_1$ be a Hamilton operator, where

$$S_1 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\omega$ is the frequency.
(i) Find $\exp(-i\hat{H}t/\hbar)\psi(0)$, where $\psi(0) = (1, 1, 1)^T/\sqrt{3}$.

(ii) Calculate the time evolution of

$$S_3 := \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

using the Heisenberg equation of motion. The matrices $S_x$, $S_y$, $S_z$ are the spin-1 matrices, where

$$S_2 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

**Problem 16.** Consider the linear operator

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in the Hilbert space $\mathbb{R}^3$. Find

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

using the method of the Lagrange multiplier.

**Problem 17.** Consider the Hilbert space $\mathbb{R}^4$. Show that the *Bell basis*

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

forms an orthonormal basis in this Hilbert space.

**Problem 18.** Consider the Hilbert space $\mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{x}$ is considered as a column vector. Find the matrix $\mathbf{x}\mathbf{x}^T$. Show that at least one eigenvalue is equal to 0.

**Problem 19.** (i) Consider the Hilbert space $\mathbb{C}^4$. Show that the matrices

$$\Pi_1 = \frac{1}{2}(I_2 \otimes I_2 + \sigma_1 \otimes \sigma_1), \qquad \Pi_2 = \frac{1}{2}(I_2 \otimes I_2 - \sigma_1 \otimes \sigma_1)$$

are projection matrices in $\mathbb{C}^4$.

(ii) Find $\Pi_1\Pi_2$. Discuss.
(iii) Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ be the standard basis in $\mathbb{C}^4$. Calculate

$$\Pi_1\mathbf{e}_j, \qquad \Pi_2\mathbf{e}_j, \qquad j = 1,2,3,4$$

and show that we obtain 2 two-dimensional Hilbert spaces under these projections.

**Problem 20.** Consider the $3 \times 3$ matrix

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) The matrix $A$ can be considered as an element of the Hilbert space of the $3 \times 3$ matrices with the scalar product $\langle A, B\rangle := \mathrm{tr}(AB^T)$. Find the norm of $A$ with respect to this Hilbert space.
(ii) On the other hand $A$ can be considered as a linear operator in the Hilbert space $\mathbb{R}^3$. Find die norm

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{R}^3.$$

(iii) Find the eigenvalues of $A$ and $AA^T$. Compare the result with (i) and (ii).

**Problem 21.** Consider the Hilbert space $\mathbb{R}^3$. Find the spectrum (eigenvalues and normalized eigenvectors) of matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Find $\|A\| := \sup_{\mathbf{x}=1} \|A\mathbf{x}\|$, where $\|.\|$ denotes the norm and $\mathbf{x} \in \mathbb{R}^3$.

**Problem 22.** Find the spectrum (eigenvalues and normalized eigenvectors) of the $3 \times 3$ matrix

$$A = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

Find $\|A\|$, where $\|.\|$ denotes the norms

$$\|A\|_1 := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

$$\|A\|_2 := \sqrt{\operatorname{tr}(AA^*)}.$$

Compare the norms with the eigenvalues. Find $\exp(A)$.

**Problem 23.** Consider the Hilbert space $M_4(\mathbb{C})$ of all $4 \times 4$ matrices over $\mathbb{C}$ with the scalar product $\langle A, B \rangle := \operatorname{tr}(AB^*)$, where $A, B \in M_4(\mathbb{C})$. The $\gamma$-matrices are given by

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We define the $4 \times 4$ matrices

$$\sigma_{jk} := \frac{i}{2}[\gamma_j, \gamma_k], \qquad j < k$$

where $j = 1, 2, 3$, $k = 2, 3, 4$ and $[\,,\,]$ denotes the commutator.
(i) Calculate $\sigma_{12}$, $\sigma_{13}$, $\sigma_{14}$, $\sigma_{23}$, $\sigma_{24}$, $\sigma_{34}$.
(ii) Do the 16 matrices

$$I_4,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_5\gamma_1,\ \gamma_5\gamma_2,\ \gamma_5\gamma_3,\ \gamma_5\gamma_4,\ \sigma_{12},\ \sigma_{13},\ \sigma_{14},\ \sigma_{23},\ \sigma_{24},\ \sigma_{34}$$

form a basis in the Hilbert space $M_4(\mathbb{C})$? If so is the basis orthogonal?

**Problem 24.** Find the spectrum (eigenvalues and normalized eigenvectors) of matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find $\|A\|$, where $\|.\|$ denotes the norm.

**Problem 25.** Let $A$ and $B$ be two arbitrary matrices. Give the definition of the Kronecker product. Let $\mathbf{u}_j$ $(j = 1, 2, \ldots, m)$ be an orthonormal basis in the Hilbert space $\mathbb{R}^m$. Let $\mathbf{v}_k$ $(k = 1, 2, \ldots, n)$ be an orthonormal basis in the Hilbert space $\mathbb{R}^n$. Show that $\mathbf{u}_j \otimes \mathbf{v}_k$ $(j = 1, 2, \ldots, m)$, $(k = 1, 2, \ldots, n)$ is an orthonormal basis in $\mathbb{R}^{m+n}$.

**Problem 26.** Show that the $2 \times 2$ matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $M^2(\mathbb{C})$.

**Problem 27.** Show that the $2 \times 2$ matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis in the Hilbert space $M^2(\mathbb{R})$. Apply the *Gram-Schmidt technique* to obtain an orthonormal basis.

**Problem 28.** Consider the $3 \times 3$ matrices over the real numbers

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) The matrix $A$ can be considered as an element of the Hilbert space of the $3 \times 3$ matrices over the real nunbers with the scalar product

$$\langle B, C \rangle := \operatorname{tr}(BC^T).$$

Find the norm of $A$ with respect to this Hilbert space.
(ii) On the other hand the matrix $A$ can be considered as a linear operator in the Hilbert space $\mathbb{R}^3$. Find the norm

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{R}^3.$$

(iii) Find the eigenvalues of $A$ and $A^T A$. Compare the result with (i) and (ii).

**Problem 29.** Consider the Hilbert space $\mathbb{C}^2$. The Pauli spin matrices $\sigma_x$, $\sigma_y$, $\sigma_z$ act as linear operators in this Hilbert space. Let

$$\hat{H} = \hbar\omega\sigma_3$$

be a Hamilton operator, where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\omega$ is the frequency. Calculate the time evolution (intial value problem) of

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i.e.

$$i\hbar \frac{d\sigma_x}{dt} = [\sigma_1, \hat{H}](t).$$

The matrices $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli matrices, where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

**Problem 30.** Consider the Hilbert space $\mathbb{C}^4$. Consider the Hamilton operator

$$\hat{H} := \hbar\omega \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Find the time-evolution of the operator

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

using the Heisenberg equation of motion

$$i\hbar \frac{d\gamma_3}{dt} = [\gamma_3, \hat{H}](t).$$

**Problem 31.** Let $M$ be any $n \times n$ matrix. Let $\mathbf{x} = (x_1, x_2, \ldots)^T$. The linear operator $A$ is defined by

$$A\mathbf{x} = (w_1, w_2, \ldots)^T$$

where

$$w_j := \sum_{k=1}^{n} M_{jk} x_k, \qquad j = 1, 2, \ldots, n$$
$$w_j := x_j, \qquad j > n$$

and $\mathcal{D}(A) = \ell_2(\mathbf{N})$. Show that $A$ is self-adjoint if the $n \times n$ matrix $M$ is hermitian. Show that $A$ is unitary if $M$ is unitary.

**Problem 32.** Consider the Hilbert space $\mathbb{C}^n$. Let $\mathbf{u}_j$, $j = 1, 2, \ldots, n$, and $\mathbf{v}_j$, $j = 1, 2, \ldots, n$ be orthonormal bases in $\mathbb{C}^n$, where $\mathbf{u}_j$, $\mathbf{v}_j$ are considered as column vectors. Show that

$$U = \sum_{j=1}^{n} \mathbf{u}_j \mathbf{v}_j^*$$

is a unitary $n \times n$ matrix.

**Problem 33.** Consider the Hilbert space $\mathbb{R}^2$. Given the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}.$$

The three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are at 120 degrees of each other and are normalized, i.e. $\|\mathbf{u}_j\| = 1$ for $j = 1, 2, 3$. Every given two-dimensional vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

in many different ways. Given the vector $\mathbf{v}$ minimize

$$\frac{1}{2}(c_1^2 + c_2^2 + c_3^2)$$

subject to the two constraints

$$\mathbf{v} - c_1\mathbf{u}_1 - c_2\mathbf{u}_2 - c_3\mathbf{u}_3 = \mathbf{0}.$$

**Problem 34.** Let $A$, $H$ be $n \times n$ hermitian matrices, where $H$ plays the role of the Hamilton operator. The Heisenberg equations of motion is given by

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[H, A(t)].$$

with $A = A(t = 0) = A(0)$. Let $E_j$ $(j = 1, 2, \ldots, n^2)$ be an orthonormal basis in the Hilbert space $\mathcal{H}$ of the $n \times n$ matrices with scalar product

$$\langle X, Y \rangle := \operatorname{tr}(XY^*), \qquad X, Y \in \mathcal{H}.$$

Now $A(t)$ can be expanded using this orthonormal basis as

$$A(t) = \sum_{j=1}^{n^2} c_j(t) E_j$$

and $H$ can be expanded as

$$H = \sum_{j=1}^{n^2} h_j E_j.$$

Find the time evolution for the coefficients $c_j(t)$, i.e. $dc_j/dt$, where $j = 1, 2, \ldots, n^2$.

**Problem 35.** The sequence space consists of the set of all (bounded or unbounded) sequences of complex numbers

$$x = (x_1, x_2, \ldots)$$

Thus we have a vector space. Can we define a metric in this vector space which is not implied by a norm?

# Chapter 3

# Hilbert Space $L_2(\Omega)$

## 3.1 Solved Problems

**Problem 1.** A basis in the Hilbert space $L_2[0,1]$ is given by

$$B := \left\{ e^{2\pi i x n} : n \in \mathbb{Z} \right\}.$$

Let

$$f(x) = \begin{cases} 2x & 0 \le x < 1/2 \\ 2(1-x) & 1/2 \le x < 1 \end{cases}$$

Is $f \in L_2[0,1]$? Find the first two expansion coefficients of the Fourier expansion of $f$ with respect to the basis given above.

**Problem 2.** (i) Consider the Hilbert space $L_2[-1,1]$. Consider the sequence

$$f_n(x) = \begin{cases} -1 \text{ if } & -1 \le x \le -1/n \\ nx \text{ if } & -1/n \le x \le 1/n \\ +1 \text{ if } & 1/n \le x \le 1 \end{cases}$$

where $n = 1, 2, \ldots$. Show that $\{ f_n(x) \}$ is a sequence in $L_2[-1,1]$ that is a Cauchy sequence in the norm of $L_2[-1,1]$.
(ii) Show that $f_n(x)$ converges in the norm of $L_2[-1,1]$ to

$$\text{sgn}(x) = \begin{cases} -1 \text{ if } -1 \le x < 0 \\ +1 \text{ if } \ 0 < x \le 1 \end{cases}.$$

(iii) Use this sequence to show that the space $C[-1,1]$ is a subspace of $L_2[-1,1]$ that is not closed.

**Problem 3.** Let $f \in L_2(\mathbb{R})$. Give the definition of the Fourier transform. Let us call the transformed function $\hat{f}$. Is $\hat{f} \in L_2(\mathbb{R})$? What is preserved under the Fourier transform?

**Problem 4.** Consider the Hilbert space $L_2[a,b]$, where $a, b \in \mathbb{R}$ and $b > a$. Find the condition on $a$ and $b$ such that

$$\langle \cos(x), \sin(x) \rangle = 0$$

where $\langle \, , \, \rangle$ denotes the scalar product in $L_2[a,b]$.
Hint. Since $b > a$, we can write $b = x + \epsilon$, where $\epsilon > 0$.

**Problem 5.** Consider the Hilbert space $L_2[0,1]$. The *Legendre polynomials* are defined as

$$P_0(x) = 1, \qquad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1).$$

Show that the first first four elements are given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Normalize the four elements. Show that the four elements are pairwise orthonormal.

**Problem 6.** Let $R$ be a bounded region in $n$-dimensional space. Consider the eigenvalue problem

$$-\Delta u = \lambda u, \qquad u(q \in \partial R) = 0$$

where $\partial R$ denotes the boundary of $R$.
(i) Show that all eigenvalues are real and positive
(ii) Show that the eigenfunctions which belong to different eigenvalues are orthogonal.

**Problem 7.** Consider the inner product space

$$C[a,b] = \{ \, f(x) \, : \, f \text{ is continuous on } x \in [a,b] \, \}$$

with the inner product

$$\langle f, g \rangle := \int_a^b f(x) g^*(x) dx.$$

This implies a norm

$$\langle f, f \rangle = \int_a^b f(x) f^*(x) dx = \|f\|^2.$$

Show that $C[a,b]$ is incomplete. This means find a Cauchy sequence in the space $C[a,b]$ which converges to an element which is not in the space $C[a,b]$.

**Problem 8.** Consider the Hilbert space $L_2[-\pi,\pi]$. Given the function

$$f(x) = \begin{cases} 1 & 0 < x \le \pi \\ 0 & x = 0 \\ -1 & -\pi \le < 0 \end{cases}$$

Obviously $f \in L_2[-\pi,\pi]$. Find the Fourier expansion of $f$. The orthonormal basis $\mathcal{B}$ is given by

$$\mathcal{B} := \left\{ \phi_k(x) = \frac{1}{\sqrt{2\pi}} \exp(ikx) \;\; k \in \mathbb{Z} \right\}.$$

Find the approximation $a_0\phi_0(x) + a_1\phi_1(x) + a_{-1}\phi_{-1}(x)$, where $a_0$, $a_1$, $a_{-1}$ are the Fourier coefficients.

**Problem 9.** Consider the linear operator $A$ in the Hilbert space $L_2[0,1]$ defined by $Af(x) := xf(x)$. Find the matrix elements

$$\langle P_i, AP_j \rangle$$

for $i, j = 0, 1, 2, 3$, where $P_i$ are the (normalized) Legrende polynomials. Is the matrix $A_{ij}$ symmetric?

**Problem 10.** Consider the Hilbert space $L_2[0, 2\pi)$. Let

$$g(x) = \cos(x), \qquad f(x) = x.$$

Find the conditions on the coefficients of the polynomial

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that

$$\langle g(x), p(x) \rangle = 0, \qquad \langle f(x), p(x) \rangle = 0.$$

Solve the equations for $a_3$, $a_2$, $a_1$, $a_0$.

**Problem 11.** Let $b > a$. Consider the Hilbert space $L_2([a,b])$ and the functions

$$\phi_n(x) := \sin\left(n\pi \frac{x-a}{b-a}\right), \quad n = 1, 2, \ldots$$

which form an orthonormal basis in $L_2([a,b])$. Find

$$\langle \phi_m(x), x\phi_n(x) \rangle \equiv \int_a^b \phi_m(x) x \phi_n(x) dx, \quad m,n = 0,1,2,\ldots$$

**Problem 12.** Let $b > a$. Consider the Hilbert space $L_2([a,b])$ and the functions

$$\phi_n(x) := \cos\left(n\pi \frac{x-a}{b-a}\right), \quad n = 0,1,2,\ldots$$

which form an orthonormal basis in $L_2([a,b])$. Find

$$\langle \phi_m(x), x\phi_n(x) \rangle \equiv \int_a^b \phi_m(x) x \phi_n(x) dx, \quad m,n = 0,1,2,\ldots$$

**Problem 13.** Let $m,n \in \mathbb{N}$. Consider the Hilbert space $L_2([-1,1])$ the functions

$$f_n(x) = \frac{1}{1+nx^2}, \quad n = 1,2,\ldots$$

which are elements in this Hilbert space. Find

$$\|f_n(x) - f_m(x)\|.$$

**Problem 14.** Consider the Hilbert space $L_2(\mathbb{R})$. Give the definition and an example of an even function in $L_2(\mathbb{R})$. Give the definition and an example of an odd function in $L_2(\mathbb{R})$. Show that any function $f \in L_2(\mathbb{R})$ can be written as a combination of an even and an odd function.

**Problem 15.** The Chebyshev polynomials $T_n(x)$ of the 1-st kind are defined for $x \in [-1,1]$ and given by

$$T_n(x) = \cos(n \arccos x), \qquad n = 0,1,2,\ldots$$

The Chebyshev polynomials $U_n(x)$ of the 2-nd kind are defined for $x \in [-1,1]$ and given by

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}}, \qquad n = 0,1,2,\ldots.$$

Consider the Hilbert spaces

$$\mathcal{H}_1 = L_2\left([-1,1], \frac{dx}{\pi\sqrt{1-x^2}}\right), \qquad \mathcal{H}_2 = L_2\left([-1,1], \frac{2\sqrt{1-x^2}dx}{\pi}\right)$$

which bases are formed by the Chebyshev polynomials of the 1-st and 2-nd type

$$\Phi_n^{(1)}(x) = \sqrt{2}T_n(x), \quad n \geq 1, \quad \Phi_0^{(1)} = T_0(x) = 1$$
$$\Phi_n^{(2)}(x) = U_n(x), \qquad n \geq 0$$

Find a recursion relation for $\Phi_n^{(1)}$ and $\Phi_n^{(2)}$.

**Problem 16.** Consider the Hilbert space $L_2[-\pi, \pi]$. Obviously $\cos(x) \in L_2[-\pi, \pi]$. Find the norm $\|\cos(x)\|$. Find nontrivial functions $f, g \in L_2[-\pi, \pi]$ such that

$$\langle f(x), \cos(x) \rangle = 0, \qquad \langle g(x), \cos(x) \rangle = 0$$

and

$$\langle f(x), g(x) \rangle = 0.$$

**Problem 17.** Consider the Hilbert space $L_2[0, 1]$. Find a non-trivial polynomial $p$

$$p(x) = ax^3 + bx^2 + cx + d$$

such that

$$\langle p, 1 \rangle = 0, \qquad \langle p, x \rangle = 0, \qquad \langle p, x^2 \rangle = 0.$$

**Problem 18.** Consider the set of polynomials

$$\{ 1, x, x^2, \ldots, x^n, \ldots \}.$$

Use the Gram-Schmidt procedure and the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\omega(x)dx, \qquad \omega(x) > 0$$

to obtain the first four orthogonal polynomials when
(i) $a = -1$, $b = 1$, $\omega(x) = 1$ (Legrendre polynomials)
(ii) $a = -1$, $b = 1$, $\omega(x) = (1 - x^2)^{-1/2}$ (Chebyshev polynomials)
(iii) $a = 0$, $b = +\infty$, $\omega(x) = e^{-x}$ (Laguerre polynomials)
(iv) $a = -\infty$, $b = +\infty$, $\omega(x) = e^{-x^2}$ (Hermite polynomials)

**Problem 19.** Consider the function

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \cos(jx).$$

Is $f$ an element of $L_2[-\pi, \pi]$?

**Problem 20.** Consider the Hilbert space $L_2([0,1])$. The *shifted Legrendre polynomials* , defined on the interval $[0,1]$, are obtained from the Legrendre polynomial by the transformation $y = 2x - 1$. The shifted Legrendre polynomials are given by the recurrence formula

$$P_j(x) = \frac{(2j+1)(2x-1)}{j+1} P_j(x) - \frac{j}{j+1} P_{j-1}(x) \qquad j = 1, 2, \dots$$

and $P_0(x) = 1$, $P_1(x) = 2x - 1$. They are elements of the Hilbert space $L_2([0,1])$. A function $u$ in the Hilbert space $L_2([0,1])$ can be approximated in the form of a series with $n+1$ terms

$$u(x) = \sum_{j=0}^{n} c_j P_j(x)$$

where the coeffients $c_j \in \mathbb{R}$, $j = 0, 1, \dots, n$. Consider the Volterra integral equation of first kind

$$\lambda \int_0^x \frac{y(t)}{(x-t)^\alpha} dt = f(x), \qquad 0 \le t \le x \le 1$$

with $0 < \alpha < 1$ and $f \in L_2([0,1])$. Consider the ansatz

$$y_n(x) = a_0 x^\alpha + \sum_{j=0}^{n} c_j P_j(x).$$

to find an approximate solution to the *Volterra integral equation* of first kind ($\alpha = 1/2$)

$$\lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x)$$

where

$$f(x) = \frac{2}{105} \sqrt{x}(105 - 56x^2 + 48x^3).$$

**Problem 21.** The *Fock space* $\mathcal{F}$ is the Hilbert space of entire functions with inner product given by

$$\langle f|g\rangle := \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dxdy, \qquad z = x + iy$$

where $\mathbb{C}$ denotes the complex numbers. Therefore the growth of functions in the Hilbert space $\mathcal{F}$ is dominated by $\exp(|z|^2/2)$. Let $f, g \in \mathcal{F}$ with Taylor expansions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \qquad g(z) = \sum_{j=0}^{\infty} b_j z^j.$$

(i) Find $\langle f|g\rangle$ and $\|f\|^2$.
(ii) Consider the special that $f(z) = \sin(z)$ and $g(z) = \cos(z)$. Calculate $\langle f|g\rangle$.
(iii) Let

$$\mathcal{K}(z,w) := e^{z\overline{w}}, \qquad z, w \in \mathbb{C}.$$

Calculate $\langle f(z)|\mathcal{K}(z,w)\rangle$.

**Problem 22.** Consider the Hilbert space $L_2[0,\pi]$. Let $\| \; \|$ be the norm induced by the scalar product of $L_2[0,\pi]$. Find the constants $a$, $b$ such that

$$\|\sin(x) - (ax^2 + bx)\|$$

is a minimum.

**Problem 23.** Consider the Hilbert space $L_2(\mathbb{R})$. Let

$$f_n(x) = \frac{x}{1+nx^2}, \qquad n = 1, 2, \ldots .$$

(i) Find $\|f_n(x)\|$ and

$$\lim_{n\to\infty} \|f_n(x)\|.$$

(ii) Does the sequence $f_n(x)$ converge uniformely on the real line?

**Problem 24.** Let $n = 1, 2, \ldots$. We define the functions $f_n \in L_2[0,\infty)$ by

$$f_n(x) = \begin{cases} \sqrt{n} \text{ for } n \le x \le n + 1/n \\ 0 \qquad \text{otherwise} \end{cases}$$

(i) Calculate the norm $\|f_n - f_m\|$ implied by the scalar product. Does the sequence $\{ f_n \}$ converge in the $L_2[0,\infty)$ norm?
(ii) Show that $f_n(x)$ converges pointwise in the domain $[0,\infty)$ and find the limit. Does the sequence converge pointwise uniformly?
(iii) Show that $\{ f_n \}$ $(n = 1, 2, \ldots)$ is an orthonormal system. Is it a basis in the Hilbert space $L_2[0,\infty)$?

**Problem 25.** Consider the function $f \in L_2[0,1]$

$$f(x) = \begin{cases} x \quad \text{for } 0 \le x \le 1/2 \\ 1 - x \text{ for } 1/2 \le x \le 1 \end{cases}$$

A basis in the Hilbert space is given by

$$\mathcal{B} := \left\{ 1, \; \sqrt{2}\cos(\pi n x) \quad : \quad n = 1, 2, \ldots \right\}.$$

Find the Fourier expansion of $f$ with respect to this basis. From this expansion show that

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

**Problem 26.** A particle is enclosed in a rectangular box with impenetrable walls, inside which it can move freely. The Hilbert space is

$$L_2([0,a] \times [0,b] \times [0,c])$$

where $a, b, c > 0$. Find the eigenfunctions and the eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions?

**Problem 27.** Consider the Hilbert space $L_2[0,1]$. Find a non-trivial function

$$f(x) = ax^3 + bx^2 + cx + d.$$

such that

$$\langle f(x), x \rangle = 0, \qquad \langle f(x), x^2 \rangle = 0, \qquad \langle f(x), x^3 \rangle = 0$$

where $\langle\, , \rangle$ denotes the scalar product

**Problem 28.** Consider the Hilbert space $L_2[0,1]$. Find a non-trivial function $f$ such that

$$\langle f(x), x \rangle = 0, \qquad \langle f(x), x^2 \rangle = 0, \qquad \langle f(x), x^3 \rangle = 0$$

where $\langle\, , \rangle$ denotes the scalar product

**Problem 29.** Consider the Hilbert space $L_2[0,1]$ and the polynomials

$$1,\ x,\ x^2,\ x^3,\ x^4\,.$$

Apply the Gram-Schmidt orthogonalization process to these polynomials.

**Problem 30.** Consider the Hilbert space $L_2(\mathbb{T})$. Let $f \in L_2(\mathbb{T})$. Give an example of a bounded linear functional.

**Problem 31.** Consider the Hilbert space $L_2(\mathbb{R})$. Show that the Hilbert space is the direct sum of the Hilbert space $\mathcal{M}$ of even functions and the Hilbert space $\mathcal{N}$ of odd functions. Give an example of such functions in this Hilbert space.

**Problem 32.** Let $a > 0$. Consider the Hilbert space $L_2[0, a]$. Let

$$Af(x) := xf(x)$$

for $f \in L_2[0, a]$. Find the norm of the operator $A$. We define

$$\|A\| := \sup_{\|f\|=1} \|Af\|.$$

**Problem 33.** Consider the Hilbert space $L_2[0, 2\pi]$. Let

$$g(x) = \cos(x), \qquad f(x) = x.$$

Find the conditions on the coefficients of the polynomial

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that

$$\langle g(x), p(x)\rangle = 0, \qquad \langle f(x), p(x)\rangle = 0.$$

Solve the equations for $a_3$, $a_2$, $a_1$, $a_0$.

**Problem 34.** Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1 - \cos(2\pi x)}{x}.$$

Using L'Hospital rules we have $f(0) = 0$. Is $f \in L_2(\mathbb{R})$?

**Problem 35.** Consider the Hilbert space $L_2[-1, 1]$. The *Legendre polynomials* are given by

$$P_j(x) := \frac{1}{2^j j!} \frac{d^j}{dx^j}(x^2 - 1)^j.$$

Find the scalar product

$$\langle P_j(x), P_k(x)\rangle.$$

**Problem 36.** Consider the Hilbert space $\mathcal{H} = L_2(\mathbb{T})$. This is the vector space of $2\pi$-periodic functions. Then

$$u(x) = \frac{1}{\sqrt{2}}$$

is a constant function which is normalized, i.e. $\|u\| = 1$. Show that the projection operator $P_u$ defined by

$$P_uf := \langle u, f\rangle u$$

maps a function $f$ to its mean. This means

$$P_u f = \langle f \rangle, \qquad \langle f \rangle = \int_0^{2\pi} f(x)dx.$$

**Problem 37.** Consider the Hilbert space $L_2[-\pi, \pi]$ and the vector space of continiuous real-valued functions $C[-\pi, \pi]$ on the interval $[-\pi, \pi]$. Let $k > 0$ and

$$f_k(x) = \begin{cases} 0 \text{ if } -\pi \leq x \leq 0 \\ kx \text{ if } 0 \leq x \leq 1/k \\ 1 \text{ if } 1/k \leq x \leq \pi \end{cases}$$

The sequence of functions $f_k$ belongs to the vector space $C[-\pi, \pi]$.
(i) Show that $f_n \to \chi$ in the norm of the Hilbert space $L_2[-\pi, \pi]$, where

$$\chi(x) := \begin{cases} 0 \text{ if } -\pi \leq x \leq 0 \\ 1 \text{ if } \ 0 < x \leq \pi \end{cases}$$

so that the sequence $\{ f_k \}$ is a Cauchy sequence in the Hilbert space $L_2[-\pi, \pi]$.
(ii) Show that $\|\chi - g\| > 0$ for every $g \in C[-\pi, \pi]$. Conclude that $C[-\pi, \pi]$ is not a Hilbert space.

**Problem 38.** Let $\Omega$ be the unit disk. A Hilbert space of analytic functions can be defined by

$$\mathcal{H} := \left\{ f(z) \ \text{analytic}, \ \ |z| < 1 \ : \ \sup_{a<1} \int_{|z|=a} |f(z)|^2 ds < \infty \right\}$$

and the scalar product

$$\langle f, g \rangle := \lim_{a \to 1} \int_{|z|=a} \overline{f(z)} g(z) ds.$$

Let $c_n$ $(n = 0, 1, 2, \ldots)$ be the coefficients of the power-series expansion of the analytic function $f$. Find the norm of $f$.

**Problem 39.** Let $\mathbb{C}^n$ denote the complex Euclidean space. Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n$ then the scalar product (inner product) is given by

$$\mathbf{z} \cdot \mathbf{w} := \mathbf{z}\mathbf{w}^* = \mathbf{z}\overline{\mathbf{w}}^T$$

where $\overline{\mathbf{z}} = (\overline{z}_1, \ldots, \overline{z}_n)$. Let $E_n$ denote the set of entire functions in $\mathbb{C}^n$. Let $F_n$ denote the set of $f \in E_n$ such that

$$\|f\|^2 := \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(\mathbf{z})|^2 \exp(-|\mathbf{z}|^2) dV$$

is finite. Here $dV$ is the volume element (Lebesgue mesure)

$$dV = \prod_{j=1}^{n} dx_j dy_j = \prod_{j=1}^{n} r_j dr_j d\theta_j$$

with $z_j = r_j e^{i\theta_j}$. The norm follows from the scalar product of two functions $f, g \in F_n$

$$\langle f, g \rangle := \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(\mathbf{z})\overline{g(\mathbf{z})} \exp(-|\mathbf{z}|^2) dV.$$

Let

$$\mathbf{z}^m := z_1^{m_1} \cdots z_n^{m_n}$$

where the multiindex $m$ is defined by $m! = m_1! \cdots m_n!$ and $|m| = \sum_{j=1}^{n} m_j$. Find the scalar product

$$\langle \mathbf{z}^m, \mathbf{z}^p \rangle.$$

**Problem 40.** Let $\Psi$ be a complex-valued differentiable function of $\phi$ in the interval $[0, 2\pi]$ and $\Psi(0) = \Psi(2\pi)$, i.e. $\Psi$ is an element of the Hilbert space $L_2([0, 2\pi])$. Assume that (normalization condition)

$$\int_0^{2\pi} \Psi^*(\phi)\Psi(\phi) d\phi = 1.$$

Calculate

$$\Im \frac{\hbar}{i} \int_0^{2\pi} \Psi^*(\phi) \phi \frac{d}{d\phi} \Psi(\phi) d\phi$$

where $\Im$ denotes the imaginary part.

**Problem 41.** The *Legrendre polynomials* are defined on the interval $[-1, 1]$ and defined by the recurrence formula

$$L_j(x) = \frac{2j+1}{j+1} x L_j(x) - \frac{j}{j+1} L_{j-1}(x) \qquad j = 1, 2, \ldots$$

and $L_0(y) = 1$, $L_1(x) = x$. They are elements of the Hilbert space $L_2([-1, 1])$. Calculate the scalar product

$$\langle L_j(x), L_k(x) \rangle$$

for $j, k = 0, 1, \ldots$. Discuss.

**Problem 42.** Let $f_n : [-1, 1] \to [-1, 1]$ be defined by

$$f_n(x) = \begin{cases} 1 & \text{for } -1 \le x \le 0 \\ \sqrt{1 - nx} & \text{for } 0 \le x \le 1/n \\ 0 & \text{for } 1/n \le x \le 1 \end{cases}$$

Show that $f_n \in L_2[-1,1]$. Show that $f_n$ is a Cauchy sequence.

**Problem 43.** Consider the Hilbert space $L_2([-1,1])$. The *Chebyshev polynomials* are defined by

$$T_n(x) := \cos(n \cos^{-1} x), \quad n = 0, 1, 2, \ldots .$$

They are elements of the Hilbert space $L_2([-1,1])$. We have

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

Calculate the scalar products

$$\langle T_0, T_1 \rangle, \quad \langle T_1, T_2 \rangle, \quad \langle T_2, T_3 \rangle.$$

Calculate the integrals

$$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx$$

for $(m,n) = (0,1), (m,n) = (1,2), (m,n) = (2,3)$.

**Problem 44.** (i) Consider the Hilbert space $L_2[0,1]$ with the scalar product $\langle \cdot, \cdot \rangle$. Let $f : [0,1] \to [0,1]$

$$f(x) := \begin{cases} 2x & \text{if } x \in [0,1/2) \\ 2(1-x) & \text{if } x \in [1/2,1] \end{cases}$$

Thus $f \in L_2[0,1]$. Calculate the moments $\mu_k$, $k = 0, 1, 2, \ldots$ defined by

$$\mu_k := \langle f(x), x^k \rangle \equiv \int_0^1 f(x) x^k dx.$$

(ii) Show that

$$\sum_{k=0}^{\infty} |\mu_k|^2 < \pi \int_0^1 |f(x)|^2 dx.$$

**Problem 45.** Let $a, b \in \mathbb{R}$ and $-\infty < a < b < +\infty$. Let $f$ be a function in the class $C^1$ (i.e., the derivative $df/dt$ exists and is continuous) on the interval $[a,b]$. Thus $f$ is also an element of the Hilbert space $L_2([a,b])$. Show that

$$\lim_{\omega \to \infty} \int_a^b f(t) \sin(\omega t) dt = 0. \tag{1}$$

**Problem 46.** Consider the *Lie group*

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \ |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

The elements of this Lie group act as analytic automorphism of the disk

$$\Omega := \{\, |z| < 1 \,\}$$

under

$$z \to zg = \frac{\bar{\alpha}z + \beta}{\bar{\beta}z + \alpha}$$

where $(zg)h = z(gh)$. Let $n \geq 2$. We define

$$\mathcal{H}_n := \{\, f(z) \text{ analytic in } \Omega,\ \|f\|^2 = \int_{\Omega} |f(z)|^2 (1 - |z|^2)^{n-2} dxdy < \infty \,\}$$

and

$$U_n(g)f(z) := \frac{1}{(\bar{\beta}z + \alpha)^n} f((\bar{\alpha}z + \beta)/(\bar{\beta}z + \alpha)).$$

Then $\mathcal{H}_n$ is a Hilbert space, i.e. the analytic functions in

$$L_2(\Omega, (1 - |z|^2)^{n-2} dxdy)$$

form a closed subspace. $U_n$ is a representation, i.e.,

$$U_n(gh) = U_n(g)U_n(h)$$

and $U_n(e) = I$, where $e$ is the identity element in $SU(1,1)$ ($2 \times 2$ unit matrix).
Show that

$$\frac{1}{(1 - |z|^2)^2} dx \wedge dy$$

is invariant $z \to zg$.

**Problem 47.** Consider the problem of a particle in a one-dimensional box. The underlying Hilbert space is $L_2(-a, a)$. Solve the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

as follows: The formal solution is given by

$$\psi(t) = \exp(-i\hat{H}t/\hbar)\psi(0).$$

Expand $\psi(0)$ with respect to the eigenfunctions of the operator $\hat{H}$. The eigenfunctions form a basis of the Hilbert space. Then apply $\exp(-i\hat{H}t/\hbar)$. Calculate the probability

$$P = |\langle \phi, \psi(t) \rangle|^2$$

where

$$\phi(q) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right)$$

and

$$\psi(q,0) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right).$$

**Problem 48.** Let $f \in L_2(\mathbb{R}^n)$. Consider the following operators

$$\begin{aligned} T_{\mathbf{y}} f(\mathbf{x}) &= f(\mathbf{x} - \mathbf{y}), \qquad \text{translation operator} \\ M_{\mathbf{k}} f(\mathbf{x}) &= e^{i\mathbf{x}\cdot\mathbf{k}} f(\mathbf{x}), \qquad \text{modulation operator} \\ D_s f(\mathbf{x}) &= |s|^{-n/2} f(s^{-1}\mathbf{x}), \quad s \in \mathbb{R} \setminus \{0\} \quad \text{dilation operator} \end{aligned}$$

where $\mathbf{x} \cdot \mathbf{k} = k_1 x_1 + \cdots + x_n k_n$.
(i) Find $\|T_{\mathbf{y}} f\|$, $\|M_{\mathbf{k}}\|$, $\|D_s f\|$, where $\| \; \|$ denotes the norm in $L_2(\mathbb{R}^n)$.
(ii) Find the adjoint operators of these three operators.

**Problem 49.** Consider the vector space

$$H_1(a,b) := \{\, f(x) \in L_2(a,b) \; : \; f'(x) \in L_2(a,b) \,\}$$

with the norm $g \in H_1(a,b))$

$$\|g\|_1 := \sqrt{\|g\|_0^2 + \|\partial g/\partial x\|_0^2}.$$

Consider the Hilbert space $L_2(-\pi,\pi)$ and $f(x) = \sin(x)$. Find the norm $\|f\|_1$.

**Problem 50.** Let $f \in H_1(a,b)$. Then for $a \le x < y \le b$ we have

$$f(y) = f(x) + \int_x^y f'(s) ds.$$

(i) Show that $f \in C[a,b]$.
(ii) Show that

$$|f(y) - f(x)| \le \|f\|_1 \sqrt{|y-x|}.$$

**Problem 51.** Consider the Hilbert space $L_2[0,\infty)$. The Laguerre polynomials are defined by

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}), \qquad n = 0,1,2,\ldots.$$

The first five Laguerre polynomials are given by

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= 1 - x \\ L_2(x) &= 2 - 4x + x^2 \\ L_3(x) &= 6 - 18x + 9x^2 - x^3 \\ L_4(x) &= 24 - 96x + 72x^2 - 16x^3 + x^4. \end{aligned}$$

Show that the function

$$\phi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x)$$

form an orthonormal system in the Hilbert space $L_2[0, \infty)$.

**Problem 52.** Consider the Hilbert space $L_2[-\pi, \pi]$. A basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \ : \ k \in \mathbb{Z} \right\}.$$

Find the Fourier expansion of

$$f(x) = 1.$$

**Problem 53.** (i) Consider the functions

$$f(x) = \frac{1}{1+x^2}, \qquad g(x) = \frac{x}{1+x^2}.$$

Obviously $f, g \in L_2(\mathbb{R})$. Calculate the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx.$$

(ii) Let $\omega > 0$. Consider the functions

$$f(t) = \frac{\sin(\omega t)}{\omega t}, \qquad g(t) = \frac{1 - \cos(\omega t)}{\omega t}.$$

Obvioulsy $f(0) = 1$, $g(0) = 0$ and $f, g \in L_2(\mathbb{R})$. Calculate the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) g(t) dt.$$

**Problem 54.** Consider the Hilbert space $L_2[0,1]$. Let $\mathcal{P}^n$ be the $n+1$-dimensional real linear space of all polynomial of maximal degree $n$ in the variable $x$, i.e.

$$\mathcal{P}^n = \text{span}\{ 1, x, x^2, \ldots, x^n \}.$$

The linear space $\mathcal{P}^n$ can be spanned by various systems of basis functions. An important basis is formed by the *Bernstein polynomials*

$$\{B_0^n(x), B_1^n(x), \ldots, B_n^n(x)\}$$

of degree $n$ with

$$B_i^n(x) := x^i(1-x)^{n-i}, \qquad i = 0, 1, \ldots, n.$$

The Bernstein polynomials have a unique dual basis

$$\{ D_0^n(x), D_1^n(x), \ldots, D_n^n(x) \}$$

which consists of the $n+1$ dual basis functions

$$D_i^n(x) = \sum_{j=0}^{n} c_{ij} B_j^n(x).$$

The dual basis functions satisfy

$$\langle D_i^n(x), B_j^n(x) \rangle = \delta_{ij}.$$

(i) Calculate the scalar product

$$\langle B_i^m(x), B_j^n(x) \rangle.$$

(ii) Find the coefficients $c_{ij}$.

**Problem 55.** Consider Fourier series and analytic (harmonic) functions on the disc

$$\mathbb{D} := \{ z \in \mathbb{C} \, : \, |z| \leq 1 \}.$$

A Fourier series can be viewed as the boundary values of a Laurent series

$$\sum_{n=-\infty}^{\infty} c_n z^n.$$

Suppose we are given a function $f$ on $\mathbb{T}$. Find the harmonic extension $u$ of $f$ into $\mathbb{D}$. This means

$$\Delta u = 0 \quad \text{and} \quad u = f \quad \text{on} \quad \partial \mathbb{D} = \mathbb{T}$$

where $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$.

**Problem 56.** Consider the compact abelian Lie group $U(1)$

$$U(1) = \{ e^{2\pi i \theta} \, : \, 0 \leq \theta < 1 \}.$$

The Hilbert space $L_2(U(1))$ is the space $L_2([0,1])$ consisting of all measureable funcrions $f(\theta)$ with period 1 such that

$$\int_0^1 |f(\theta)|^2 d\theta < \infty.$$

Now the set of functions

$$\{ e^{2\pi i m\theta} \, : \, m \in \mathbb{Z} \}$$

form an orthonormal basis for the Hilbert space $L_2([0,1])$. Thus every $f \in L_2([0,1])$ can be expressed uniquely as

$$f(\theta) = \sum_{m=-\infty}^{+\infty} c_m e^{2\pi i m\theta}, \qquad c_m = \int_0^1 f(\theta) e^{-2\pi i m\theta} d\theta.$$

Calculate

$$\int_0^1 |f(\theta)|^2.$$

**Problem 57.** The Hilbert space $L_2(\mathbb{R})$ is the vector space of measurable functions defined almost everywhere on $\mathbb{R}$ such that $|f|^2$ is integrable. $H_1(\mathbb{R})$ is the vector space of functions with first derivatives in $L_2(\mathbb{R})$. Give two examples of such a function.

**Problem 58.** Consider the Hilbert space $L_2[-\pi,\pi]$. The set of functions

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{-inx} \right\}_{n\in\mathbb{Z}}$$

is an orthonormal basis for $L_2[-\pi,\pi]$. Let

$$K(x,t) = \frac{1}{\sqrt{2\pi}} e^{itx}.$$

For $t$ fixed find the Fourier expansion of this function.

**Problem 59.** Consider the vector space $C([0,1])$ of continouos functions. We define the *triangle function*

$$\Lambda(x) := \begin{cases} 2x & 0 \le x \le 1/2 \\ 2-2x & 1/2 < x \le 1 \end{cases}.$$

Let $\Lambda_0(x) := x$ and

$$\Lambda_n(x) := \Lambda(2^j x - k)$$

where $j = 0,1,2,\ldots$, $n = 2^j + k$ and $0 \le k < 2^j$. The functions

$$\{ 1, \, \Lambda_0, \, \Lambda_1, \, \ldots \}$$

are the *Schauder basis* for the vector space $C([0,1])$. Let $f \in C([0,1])$. Then

$$f(x) = a + bx + \sum_{n=1}^{\infty} c_n \Lambda_n(x).$$

(i) Find the Schauder coefficients $a$, $b$, $c_n$.
(ii) Consider $g : [0,1] \to [0,1]$

$$g(x) = 4x(1-x).$$

Find the Schauder coefficients for this function.

**Problem 60.** Let $s$ be a nonnegative integer. Let $x \in \mathbb{R}$ and $h_n$ ($n = 0, 1, 2, \ldots$ be

$$h_n(x) = \frac{(-1)^n}{2^{n/2}\sqrt{n!}\sqrt[4]{\pi}} \exp(x^2/2) \frac{d^n e^{-x^2}}{dx^n}.$$

Thus $h_n$ for an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$. Consider the sequence

$$f_s(x) = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta} h_n(x)$$

where $s = 0, 1, 2, \ldots$. Show that the sequence converges weakly but not strongly to 0.

**Problem 61.** Let $\mathbb{C}^{n\times N}$ be the vector space of all $n \times N$ complex matrices. Let $Z \in \mathbb{C}^{n\times N}$. Then $Z^* \equiv \bar{Z}^T$, where $^T$ denotes transpose. One defines a *Gaussian measure* $\mu$ on $\mathbb{C}^{n\times N}$ by

$$d\mu(Z) := \frac{1}{\pi^{nN}} \exp(-\mathrm{tr}(ZZ^*))dZ$$

where $dZ$ denotes the Lebesgue measure on $\mathbb{C}^{n\times N}$. The *Fock space* $\mathcal{F}(\mathbb{C}^{n\times N})$ consists of all entire functions on $\mathbb{C}^{n\times N}$ which are square integrable with respect to the Gaussian measure $d\mu(Z)$. With the scalar product

$$\langle f|g\rangle := \int_{\mathbb{C}^{n\times N}} f(Z)\overline{g(Z)}d\mu(Z), \qquad f, g \in \mathbb{F}(\mathbb{C}^{n\times N})$$

one has a Hilbert space. Show that this Hilbert space has a *reproducing kernel* $K$. This means a continuous function $K(Z, Z') : \mathbb{C}^{n\times N} \times \mathbb{C}^{n\times N} \to \mathbb{C}$ such that

$$f(Z) = \int_{\mathbb{C}^{n\times N}} K(Z, Z') f(Z') d\mu(Z')$$

for all $Z \in \mathbb{C}^{n\times N}$ and $f \in \mathcal{F}(\mathbb{C}^{n\times N})$.

**Problem 62.** Consider the Hilbert space $L_2[0, \infty)$ and the function $f \in L_2[0, \infty)$

$$f(x) = \exp(-u^{1/4}) \sin(u^{1/4}).$$

Find

$$\int_0^\infty f(x)x^n dx, \qquad n = 0, 1, 2, \ldots.$$

**Problem 63.** Consider the Hilbert space $L_2[0, 2\pi)$ with the scalar product

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{i\theta}) \overline{f_2(e^{i\theta})} d\theta.$$

(i) Let $f_1(z) = z$ and $f_2(z) = z^2$. Find $\langle f_1, f_2 \rangle$.
(ii) Let $f_1(z) = z^2$ and $f_2(z) = \sin(z)$. Find $\langle f_1, f_2 \rangle$.

**Problem 64.** Consider the Hilbert space $L_2(\mathbb{R}^2)$ with the basis

$$\psi_{mn}(x_1, x_2) = N H_m(x_1) H_n(x_2) e^{-(x_1^2 + x_2^2)/2}$$

where $m, n = 0, 1, \ldots$ and $N$ is the normalization factor. Consider the two-dimensional potential

$$V(x_1, x_2) = \frac{a}{4}(x_1^4 + x_2^4) + c x_1 x_2.$$

(i) Find all linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$V(T\mathbf{x}) = V(\mathbf{x}).$$

(ii) Show that these $2 \times 2$ matrices form a group. Is the group abelian.
(iii) Find the conjugacy classes and the irreducible representations.
(iv) Consider the Hilbert space $L_2(\mathbb{R}^2)$ with the orthogonal basis

$$\psi_{mn}(x_1, x_2) = H_m(x_1) e^{-x_1^2/2} H_n(x_2) e^{-x_2^2/2}$$

where $m, n = 0, 1, 2, \ldots$. Find the invariant subspaces from the projection operators of the irreducible representations.

**Problem 65.** Consider the Hilbert space $L_2[-\pi, \pi]$. A basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \; : \; k \in \mathbb{Z} \right\}.$$

Find the Fourier expansion of

$$f(x) = 1.$$

**Problem 66.** Consider the Hilbert space $L_2[0,1]$. Let $\mathcal{P}^n$ be the $n+1$-dimensional real linear space of all polynomial of maximal degree $n$ in the variable $x$, i.e.

$$\mathcal{P}^n = \text{span}\{\, 1, x, x^2, \dots, x^n \,\}.$$

The linear space $\mathcal{P}^n$ can be spanned by various systems of basis functions. An important basis is formed by the *Bernstein polynomials* $\{B_0^n(x), B_1^n(x), \dots, B_n^n(x)\}$ of degree $n$ with

$$B_i^n(x) := x^i(1-x)^{n-i}, \qquad i = 0, 1, \dots, n.$$

The Bernstein polynomials have a unique dual basis $\{D_0^(x), D_1^n(x), \dots, D_n^n(x)\,\}$ which consists of the $n+1$ dual basis functions

$$D_i^n(x) = \sum_{j=0}^{n} c_{ij} B_j^n(x).$$

The dual basis functions satisfy

$$\langle D_i^n(x), B_j^n(x) \rangle = \delta_{ij}.$$

(i) Find the scalar product

$$\langle B_i^m(x), B_j^n(x) \rangle.$$

(ii) Find the coefficients $c_{ij}$.

**Problem 67.** Let $V$ be a metric vector space. A *reproducing kernel Hilbert space* on $V$ is a Hilbert space $\mathcal{H}$ of functions on $V$ such that for each $x \in V$, the point evaluation functional

$$\delta_x(f) := f(x), \qquad f \in \mathcal{H}$$

is continouos. A reproducing kernel Hilbert space $\mathcal{H}$ possesses a unique reproducing kernel $K$ which is a function on $V \times V$ characterized by the properties that for all $f \in \mathcal{H}$ and $x \in V$, $K(x,\cdot) \in \mathcal{H}$ and

$$f(x) = \langle f, K(x,\cdot) \rangle_{\mathcal{H}}$$

where $\langle \cdot,\cdot \rangle_{\mathcal{H}}$ denotes the inner product on $\mathcal{H}$. The reproducing kernel $K$ uniquely determines the reproducing kernel Hilbert space $\mathcal{H}$. The reproducing kernel Hilbert space of a reproducing kernel $K$ is denoted by $\mathcal{H}_K$. The *Paley-Wiener space* is defined by

$$S := \{\, f \in C(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) \,:\, \text{supp}\hat{f} \subseteq [-\pi,\pi]^d \,\}$$

is a reproducing kernel Hilbert space. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is given by

$$\hat{f}(\mathbf{k}) := \frac{1}{(\sqrt{2\pi})^{2d}} \int_{\mathbb{R}^{2d}} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{k}} d\mathbf{x}, \qquad \mathbf{k} \in \mathbb{R}^d$$

where $\mathbf{x}\cdot\mathbf{k} = x_1k_1 + \cdots + x_dk_d$ is the inner product in $\mathbb{R}^d$. The norm on the vector space $S$ inherits from that in $L_2(\mathbb{R}^d)$. Show that the reproducing kernel for the Paley-Wiener space $S$ is the *sinc function*

$$\mathrm{sinc}(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^{d} \frac{\sin(\pi(x_j - y_j))}{\pi(x_j - y_j)}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

**Problem 68.** Can one construct an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$ starting from ($\sigma > 0$)

$$e^{-|x|/\sigma}, \quad xe^{-|x|/\sigma}, \quad x^2e^{-|x|/\sigma}, \quad x^3e^{-|x|/\sigma}, \ldots.$$

**Problem 69.** Consider the Hilbert space $L_2[-1,1]$. Normalize the function $f(x) = x$ in this Hilbert space.

**Problem 70.** Show that (Mehler's formula)

$$\exp(-(u^2+v^2-2uvz)/(1-z^2)) = (1-z^2)^{1/2}\exp(-(u^2+v^2))\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(u)H_n(v)$$

where $H_n$ are the Hermite polynomials.

**Problem 71.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $j, k = 1, 2, \ldots$. Consider the functions

$$f_j(x) = x^j e^{-|x|}, \qquad f_k(x) = x^k e^{-|x|}.$$

Find the scalar product

$$(f_j(x), f_k(x)) := \int_{-\infty}^{\infty} f_j(x)\bar{f}_k(x)dx = \int_{-\infty}^{\infty} f_j(x)f_k(x)dx.$$

Discuss.

**Problem 72.** Consider the Hilbert space $L_2(\mathbb{R})$ and the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2} + V(x)\right)u(x) = Eu(x)$$

where the potential $V$ is given by

$$V(x) = x^2 + \frac{ax^2}{1+bx^2}$$

where $b > 0$. Insert the ansatz

$$u(x) = e^{-x^2/2}v(x)$$

and find the differential equation for $v$. Discuss. Make a polynomial ansatz for $v$.

**Problem 73.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $g > 0$. Consider the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2} + x^2 + \frac{\lambda x^2}{1+gx^2}\right)u(x) = Eu(x).$$

Find a solution of the second order differential equation by making the ansatz

$$u(x) = A(1+gx^2)\exp(-x^2/2).$$

**Problem 74.** (i) Consider the Hilbert space $L_2[-1/2, 1/2]$. Show that the following sets

$$\mathcal{B}_1 := \{\, \phi_k(x) = \exp(2\pi ikx),\; k \in \mathbb{Z} \,\}$$
$$\mathcal{B}_2 := \{\, \psi_k(x) = \sqrt{2}\sin(2\pi kx),\; k \in \mathbb{N} \,\}$$

each form an orthonormal basis in this Hilbert space.
(ii) Expand the step function

$$f(x) = \begin{cases} -1 \text{ for } x \in [-1/2, 0] \\ \;\;1 \;\text{ for }\; x \in [0, 1/2] \end{cases}$$

with respect to the basis $\mathcal{B}_1$ and with respect to the basis $\mathcal{B}_2$. Show that the two expansions are equivalent. Recall that

$$2\sin(x)\sin(y) \equiv \cos(x-y) - \cos(x+y).$$

**Problem 75.** Consider the problem of a free particle in a one-dimensional box $[-a, a]$. The underlying Hilbert space is $L_2[-a, a]$. An orthonormal basis in $L_2[-a, a]$ is given by

$$\mathcal{B} = \{\, u_k^{(+)}(q),\, u_k^{(-)}(q) \;:\; k \in \mathbb{N} \,\}$$

where

$$u_k^{(+)} = \frac{1}{\sqrt{a}} \cos\left(\frac{(k-1/2)\pi q}{a}\right), \qquad u_k^{(-)} = \frac{1}{\sqrt{a}} \sin\left(\frac{k\pi q}{a}\right).$$

The formal solution of the initial value problem of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

is given by

$$\psi(t) = \exp(-i\hat{H}t/\hbar)\psi(0).$$

Let

$$\psi(q,0) = \frac{1}{\sqrt{a}}\sin(\pi q/a), \qquad \phi(q) = \frac{1}{\sqrt{a}}\sin(\pi q/a).$$

Find $\exp(-i\hat{H}t/\hbar)$ and $P = |\langle\phi, \psi(t)\rangle|^2$.

**Problem 76.** Let $n$ be a positive integer. Consider the Hilbert space $L_2[0,n]$ and the function

$$f(x) = e^{-x}.$$

Find $a, b \in \mathbb{R}$ such that

$$\|f(x) - (ax^2 + bx)\|$$

is a minimum. The norm in the Hilbert space $L_2[0,n]$ is induced by the scalar product.

**Problem 77.** Give a function $f \in L_2([0,\infty))$ such that

$$\int_0^\infty f(x)dx = 1, \quad \int_0^\infty xf(x)dx = 1.$$

**Problem 78.** Consider the Hilbert space $L_2[0,2\pi]$. The linear operator $Lf(x) := df(x)/dx$ acts on a dense subset of $L_2[0,2\pi]$. Show that this linear operator is not bounded.

**Problem 79.** Consider the Hilbert space $L_2(\mathbb{R}^3, d\mathbf{x})$ and let

$$S^2 = \{\, (x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 + x_3^2 = 1 \,\}.$$

In spherical coordinates this Hilbert space has the decomposition

$$L_2(\mathbb{R}^3, d\mathbf{x}) = L_2(\mathbb{R}^+, r^2dr) \otimes L_2(S^2, \sin(\theta)d\theta d\phi).$$

Let $\hat{I}$ be the identity operator in the Hilbert space $L_2(S^2, \sin(\theta)d\theta d\phi)$. Then the radial momentum operator

$$\hat{P}_r := -i\hbar\frac{1}{r}\left(\frac{\partial}{\partial r}\right)r$$

is identified with the closure of the operator $\hat{P}_r \otimes \hat{I}$ defined on $D(\hat{P}_r) \otimes L_2(S^2, \sin(\theta)d\theta d\phi)$ where

$$D(\hat{P}_r) = \left\{ f \in L_2(\mathbb{R}^+, r^2 dr) \,:\, f \in AC(\mathbb{R}^+),\, \frac{1}{r}\frac{d}{dr}rf(r) \in L_2(\mathbb{R}^+, r^2 dr) \,\, \lim_{r\to 0} r|f(r)| = 0 \right\}$$

and for each $f \in D(\hat{P}_r)$

$$\hat{P}_r f(r) = -i\hbar\frac{1}{r}\frac{d}{dr}(rf(r))$$

where $\hat{P}_r$ is maximal symmetric in $L_2(\mathbb{R}^+, r^2 dr)$. Show that $\hat{P}_r$ is not self-adjoint.

**Problem 80.** Consider the Hilbert space $L_2[-\pi, \pi]$ and the functions

$$f(x) = |\sin(x)| \qquad g(x) = |\cos(x)|.$$

Find the distance

$$\|f(x) - g(x)\|$$

in this Hilbert space.

**Problem 81.** Consider the Hilbert space $L_2(\mathbb{R})$. Show that the spectrum of the *position operator* $\hat{x}$ is the real line denoted by $\mathbb{R}$.

**Problem 82.** Consider the Hilbert space $L_2(\mathbb{R})$. Is

$$\phi_n(x) = \frac{1}{\sqrt{\pi(1+x^2)}}e^{2in\arctan(x)}, \quad n \in \mathbb{Z}$$

an orthonormal basis in $L_2(\mathbb{R})$?

**Problem 83.** Consider the Hilbert space $L_2[0, \infty)$. Show that the functions

$$\phi_n(x) = e^{-x/2}L_n(x), \quad n = 0, 1, 2, \ldots$$

form an orthonormal basis in $L_2[0, \infty)$, where $L_n$ are the Laguerre polynomials defined by

$$L_n(x) = \frac{x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{k}x^k.$$

**Problem 84.** Consider the Hilbert space $L_2(\mathbb{R})$. Show that the functions

$$\phi_n(x) = \frac{1}{2^{n/2}\sqrt{n!}(\pi)^{1/4}} H_n(x) e^{-x^2/2}, \quad n = 0, 1, 2, \ldots$$

form an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$, where $H_n$ are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \quad n = 0, 1, 2, \ldots$$

**Problem 85.** Let $b > a$ and $n = 1, 2, \ldots$. Consider the Hilbert space $L_2[a, b]$. Find

$$\int_a^b \sin^2\left(\frac{n\pi(x-a)}{b-a}\right).$$

The functions

$$\phi_n(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

form an orthonormal basis in the Hilbert space $L_2[a, b]$.

**Problem 86.** Consider the Hilbert space $L_2[0, \infty)$ and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) \cos(yx) dy, \qquad g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos(yx) dy.$$

Let $g : \mathbb{R}^+ \to \mathbb{R}$

$$g(y) = e^{-y}$$

Find $f(x)$.

**Problem 87.** Consider the Hilbert space $L_2[-1, 1]$. An orthonormal basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \ : \ |x| \leq \pi, \ k \in \mathbb{Z} \right\}.$$

Consider the function $f(x) = e^{iax}$ in this Hilbert space, where the constant $a$ is real but not an integer. Apply *Parseval's relation*

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} |\langle f, \phi_k \rangle|^2, \quad \phi_k(x) = \frac{1}{\sqrt{2}} e^{ikx}$$

to show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(a-k)^2} = \frac{\pi^2}{\sin^2(ax)}.$$

**Problem 88.** Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{x}{\sinh(x)}$$

with $f(0) = 1$. Show that

$$\frac{x}{\sinh(x)} = 1 + 2\sum_{j=1}^{\infty}(-1)^j \frac{x^2}{x^2 + (j\pi)^2}.$$

**Problem 89.** Consider the vector space

$$\mathcal{H}(\mathbb{C}^{n\times N}) := \left\{ f \,:\, \mathbb{C}^{n\times N} \to \mathbb{C} \,|\, f \,\text{holomorphic} \int_{\mathbb{C}^{n\times N}} |f(z)|^2 d\mu(z) < \infty \right\}$$

where $z = (z_{jk})$ with $z_{jk} = x_{jk} + iy_{jk}$ $(j = 1, \ldots, n; k = 1, \ldots, N)$ and

$$d\mu(z) = \frac{1}{\pi^{nN}} \exp(-\mathrm{tr}(zz^*)), \quad dz = \prod_{j=1}^{n}\prod_{k=1}^{N} dx_{jk}dy_{jk}.$$

Show that $\mathcal{H}(\mathbb{C}^{n\times N}$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^{n\times N}} f(z)\overline{g(z)}d\mu(z).$$

**Problem 90.** Let $\mu > 0$ and

$$\mathbf{R} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{R}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}.$$

(i) Show that

$$\frac{\exp(-|\mathbf{R} - \mathbf{R}'|/\mu}{|\mathbf{R} - \mathbf{R}'|} = \sum_{n=0}^{\infty} \epsilon_n \cos(n(\phi - \phi')) \int_0^{\infty} \frac{1}{k^2 + 1/\mu} J_n(kr)J(kr') \exp(-\sqrt{k^2 + 1/\mu^2}|x_3 - x_3'|)kdk$$

where $\varepsilon_n = 1$ for $n = 0$ and $\varepsilon_n = 2$ for $n > 0$ and $J_m(kr)$ is ordinary Bessel function of order $m$.
(ii) Consider the functions

$$f_{s,k,n}(\mathbf{R}) = \frac{\sqrt{k}}{2\pi} J_n(kr)e^{in\phi + isx_3}$$

where $0k\infty$, $-\infty s\infty$, and $n = 0, \pm 1, \pm 2, \ldots$. Show that

$$\int_0^{2\pi} \int_{-\infty}^{+\infty} dx_3 \int_0^{\infty} f_{s,k,n}(\mathbf{R}) \bar{f}_{s',k',n'}(\mathbf{R}) r dr = \delta_{nn'} \delta(s-s') \delta(k-k')$$

**Problem 91.** Let $a > 0$. Consider the Hilbert space $L_2([0,a])$ and the function $f_n \in L_2([0,a])$

$$f_n(x) = \frac{1}{\sqrt{a}} e^{2\pi i x n/a}, \quad n = 1, 2, \ldots$$

Find $\|f_n(x)\|$ and $\|\frac{df_n(x)}{dx}\|$.

**Problem 92.** Let $a > 0$. Consider the Hilbert space $L_2([0,a])$ and the linear bounded operator

$$Tf(x) = xf(x), \quad f \in L_2([0,a]).$$

Find $\|A\|$.

**Problem 93.** Let $a > 0$. Consider the Hilbert space $L_2([0,a])$ and the functions $f_n \in L_2([0,a])$

$$f_n(x) = \frac{1}{\sqrt{a}} e^{2\pi i x n/a}, \quad n = 1, 2, \ldots$$

Find $\|f_n(x)\|$ and $\|df_n(x)/dx\|$.

**Problem 94.** Consider the Hilbert space $L_2([0,2\pi])$. For any $2\pi$ periodic function $k(\tau)$ in $L_2([0,2\pi])$ we define

$$K(u) := \int_0^{2\pi} k(x-\tau) u(\tau) d\tau.$$

Find the eigenfunctions and eigenvalues of $K$. Note that the functions $\{e^{inx} : n \in \mathbb{Z}\}$ form an orthonormal basis in $L_2([0,2\pi])$.

**Problem 95.** Let $b > a$. Consider the Hilbert space $L_2([a,b])$ and the second order ordinary differential equations

$$\frac{d^2u}{dx^2} + \lambda u = 0$$

with the boundary conditions $u(a) = u(b) = 0$. Solve the differential equation with this boundary condition. Discuss.

## 3.2 Supplementary Problems

**Problem 96.** Let $0 \leq r < 1$. Consider the Hilbert space $L_2[0, 2\pi]$ and $f(\theta) \in L_2[0, 2\pi]$. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta + \frac{1}{\pi}\int_0^{2\pi}\sum_{j=1}^{\infty} r^j f(\theta)\cos(j(\phi-\theta))d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}d\theta.$$

**Problem 97.** Consider the Hilbert space $L_2[-\pi, \pi]$. Find the series

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

where $\theta \in [-\pi, \pi]$ and

$$c_0 = 1, \;\; c_1 = c_{-1} = 1, \;\; c_2 = c_{-2} = \frac{1}{2}, \ldots, c_n = c_{-n} = \frac{1}{n}.$$

**Problem 98.** Consider the real axis $\mathbb{R}$ and a cover

$$\mathbb{R} = \cup_{j=-\infty}^{j=\infty} I_j, \quad I_j = [\epsilon_j, \epsilon_{j+1}), \;\; \epsilon_j < \epsilon_{j+1}$$

with $d_j := \epsilon_{j+1} - \epsilon_j = |I_j|$. Let $f_j$ be a window function supported in the interval $[\epsilon_j - d_{j-1}/2, \epsilon_{j+1} + d_{j+1}/2)$ such that

$$\sum_{j=-\infty}^{j=\infty} f_j^2(x) = 1$$

and $f_j^2(x) = 1 - f_j^2(2\epsilon_{j+1} - x)$ for $x$ near $\epsilon_{j+1}$. Show that the functions

$$g_{j,k}(x) = \frac{2}{\sqrt{2d_j}} f_j(x)\sin\left(\frac{\pi}{2d_j}(2k+1)(x-\epsilon_j)\right), \quad k = 0, 1, 2, \ldots$$

form an orthonormal basis of the Hilbert space $L_2(\mathbb{R})$ subordinate to the partition $f_j$.

**Problem 99.** Let

$$S = L_{2,\mathrm{real}}(\mathbb{R}, dx/(1+|x|^3))$$

and $q \in S$. Consider the operator

$$L(q) := -\frac{d^2}{dx^2} + q.$$

Show that the operator $L(q)$ defines a selfadjoint operator in the Hilbert space $L_2(\mathbb{R}, dx)$.

**Problem 100.** Let $a > 0$. Consider the Hilbert space $L_2([0,a] \times [0,a] \times [0,a])$. Let $n_j \in \mathbb{Z}$ with $j = 1, 2, 3$. Show that the functions

$$\phi_{n_1,n_2,n_3}(\mathbf{x}) = \frac{1}{a^{3/2}} \exp(i(n_1x_1 + n_2x_2 + n_3x_3))$$

form an orthonormal basis in this Hilbert space.

**Problem 101.** Consider the Hilbert space $L_2[0, 2\pi]$, the curve in the plane expressed in polar coordinates

$$r(\theta) = 1 + r\cos(2\theta)$$

and

$$r(\theta) = \sum_{n=-\infty}^{+\infty} C_n \exp(in\theta), \quad C_n = \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\exp(-in\theta)d\theta.$$

Find the coefficents $C_n$.

**Problem 102.** Let $\ell_1 > 0$, $\ell_2 > 0$, $\ell_3 > 0$ with dimension length. Consider the Hilbert space $L_2([0,\ell_1] \times [0,\ell_2] \times [0,\ell_3])$. Let $n_1, n_2, n_3 \in \mathbb{N}_0$.
(i) Show that the functions

$$\begin{aligned} u(x_1,x_2,x_3) &= \cos(k_1x_1)\sin(k_2x_2)\sin(k_3x_3) \\ v(x_1,x_2,x_3) &= \sin(k_1x_1)\cos(k_2x_2)\sin(k_3x_3) \\ w(x_1,x_2,x_3) &= \sin(k_1x_1)\sin(k_2x_2)\cos(k_3x_3) \end{aligned}$$

in this Hilbert space, where

$$k_1 = \frac{n_1\pi}{\ell_1}, \quad k_2 = \frac{n_2\pi}{\ell_2}, \quad k_3 = \frac{n_3\pi}{\ell_3}.$$

(ii) Show that $u(x_1,x_2,x_3)$, $v(x_1,x_2,x_3)$, $w(x_1,x_2,x_3)$ are 0 at the boundaries of the box (rectangular cavity) $[0,\ell_1] \times [0,\ell_2] \times [0,\ell_3]$.
(iii) Show that the functions ($A, B, C$ are constants)

$$\begin{aligned} \widetilde{u}(x_1,x_2,x_3,t) &= Au(x_1,x_2,x_3)e^{i\omega t} \\ \widetilde{v}(x_1,x_2,x_3,t) &= Bv(x_1,x_2,x_3)e^{i\omega t} \\ \widetilde{w}(x_1,x_2,x_3,t) &= Cw(x_1,x_2,x_3)e^{i\omega t}. \end{aligned}$$

satisfy the wave equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

with (*dispersion relation*))

$$\omega_{\mathbf{k}} = c(k_1^2 + k_2^2 + k_3^2)$$

**Problem 103.** Let $\ell = 0, 1, 2, \ldots$. Show that

$$\sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta,\phi) Y_{\ell m}^*(\theta,\phi) = \frac{2\ell+1}{4\pi}.$$

**Problem 104.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $T$ be the linear operator of pointwise multiplication on $L_2(\mathbb{R})$ given by

$$(Tf)x = xf(x) \quad \text{for} \quad f \in L_2(\mathbb{R}).$$

Find the spectrum of $T$.

**Problem 105.** Let $Ai(x)$ be the *Airy function*, $dAi(x)/dx$ be the derivative and $a_n$ $(n = 1, 2, \ldots)$ be the zeros of the Airy functions. Show that the functions

$$\frac{Ai(x+a_n)}{dAi(x=a_n)/dx}, \quad n = 1, 2, \ldots$$

form an orthonormal basis in the Hilbert space $L_2([0,\infty))$.

**Problem 106.** Let $b > a$. Consider the Hilbert space $L_2([a,b])$. Do the functions (*Chebyshev polynomial*)

$$T_n(x) = \frac{(b-a)^n}{2^{2n-1}} \cos\left(n \arccos\left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)\right)$$

form an orthonormal basis in $L_2([a,b])$?

# Chapter 4

# Hilbert Space $\ell_2(\mathbb{N})$

**Problem 1.** Consider the Hilbert space $\ell_2(\mathbb{N})$. Let $\mathbf{x} = (x_1, x_2, \ldots)^T$ be an element of $\ell_2(\mathbb{N})$. We define the linear operator $A$ in $\ell_2(\mathbb{N})$ as

$$A\mathbf{x} = (x_2, x_3, \ldots)^T$$

i.e. $x_1$ is omitted and the $n+1$st coordinate replaces the $n$th for $n = 1, 2, \ldots$. Then for the domain we have $\mathcal{D}(A) = \ell_2(\mathbb{N})$. Find $A^*\mathbf{y}$ and the domain of $A^*$, where $\mathbf{y} = (y_1, y_, \ldots)$. Is $A$ unitary?

**Problem 2.** Consider the Hilbert space $\ell_2(\mathbb{N})$ and $\mathbf{x} = (x_1, x_2, \ldots)^T$. The linear bounded operator $A$ is defined by

$$A(x_1, x_2, x_3, \ldots, x_{2n}, x_{2n+1}, \ldots)^T = (x_2, x_4, x_1, x_6, x_3, x_8, x_5, \ldots, x_{2n+2}, x_{2n-1}, \ldots)^T.$$

Show that the operator $A$ is unitary. Show that the point spectrum of $A$ is empty and the continuous spectrum is the entire unit circle in the $\lambda$-plane.

**Problem 3.** Consider the Hilbert space $\ell_2(\mathbb{N})$. Suppose that $S$ and $T$ are the right and left shift linear operators on this sequence space, defined by

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \qquad T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Show that $T = S^*$.

**Problem 4.** Find the spectrum of the infinite dimensional matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ & & \ddots & & \ddots & \end{pmatrix}.$$

In other words

$$a_{ij} = \begin{cases} 1 \text{ if } i = j+1 \\ 1 \text{ if } i = j-1 \\ 0 \ \text{ otherwise} \end{cases}$$

**Problem 5.** Let $P_j$ $(j = 0, 1, 2, \ldots)$ be the *Legendre polynomials*

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots .$$

Calculate the infinite dimensional matrix $A = (A_{jk})$

$$A_{jk} = \int_{-1}^{+1} P_j(x) \frac{dP_k(x)}{dx} dx$$

where $j, k = 0, 1, \ldots$. Consider the matrix $A$ as a linear operator in the Hilbert space $\ell_2(\mathbb{N}_0)$. Is $\|A\| < \infty$?

**Problem 6.** Let $\mathbb{Z}$ be the set of integers. Consider the Hilbert space $\ell_2(\mathbb{Z}^2)$. Let $(m_1, m_2) \in \mathbb{Z}^2$. Let $f(m_1, m_2)$ be an element of $\ell_2(\mathbb{Z}^2)$. Consider the unitary operators

$$Uf(m_1, m_2) := e^{-2\pi i \alpha m_2} f(m_1{+}1, m_2), \quad Vf(m_1, m_2) := e^{-2\pi i \beta m_1} f(m_1, m_2{+}1).$$

They are the so-called magnetic translation operators with phase $\alpha$ and $\beta$, respectively. Find the spectrum of $U$ and $V$. Find the commutator $[U, V]$. The so-called *Harper operator* which is self-adjoint is defined by

$$\hat{H} := U + U^* + V + V^*.$$

Find the spectrum of $\hat{H}$. Consider the case $\alpha$, $\beta$ irrational and $\alpha$, $\beta$ rational.

**Problem 7.** The spectrum $\sigma(\hat{H})$ of a linear operator $\hat{H}$ is defined as the set of all $\lambda$ for which the *resolvent*

$$R(\lambda) = (\lambda I - \hat{H})^{-1}$$

does not exist. If the linear operator $\hat{H}$ is self-adjoint, the spectrum is a subset of the real axis. The Lebesgue decomposition theorem states that

$$\sigma = \sigma_{pp} \cup \sigma_{ac} \cup \sigma_{sing}$$

where $\sigma_{pp}$ is the countable union of points (the pure point spectrum), $\sigma_{ac}$ is absolutely continuous with respect to Lebesgue measure and $\sigma_{sing}$ is singular with respect to Lebesgue measure, i.e. it is supported on a set of measure zero. Consider the Hilbert space $\ell_2(\mathbb{Z})$ and the linear operator

$$\hat{H} = \cdots \otimes I_2 \otimes I_2 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_3 \otimes I_2 \otimes I_2 \otimes \cdots$$

where $\sigma_1$ is at position 0. Find the spectrum of this linear operator.

**Problem 8.** Let $M$ be any $n \times n$ matrix. Let $\mathbf{x} = (x_1, x_2, \ldots)^T$. The linear operator $A$ is defined by

$$A\mathbf{x} = (w_1, w_2, \ldots)^T$$

where

$$w_j = \sum_{k=1}^{n} M_{jk} x_k, \qquad j = 1, 2, \ldots, n$$
$$w_j = x_j, \qquad j > n$$

and $\mathcal{D}(A) = \ell_2(\mathbb{N})$. Show that $A$ is self-adjoint if the $n \times n$ matrix $M$ is hermitian. Show that $A$ is unitary if $M$ is unitary.

**Problem 9.** Let $\Omega$ be the unit disk. A Hilbert space of analytic functions can be defined by

$$\mathcal{H} := \left\{ f(z) \text{ analytic } |z| < 1 : \sup_{a<1} \int_{|z|=a} |f(z)|^2 ds < \infty \right\}$$

and the scalar product

$$\langle f, g \rangle := \lim_{a \to 1} \int_{|z|=a} \overline{f(z)} g(z) ds.$$

Let $c_n$ $(n = 0, 1, 2, \ldots)$ be the coefficients of the power-series expansion of the analytic function $f$. Find the norm of $f$.

**Problem 10.** Let $|n\rangle$ be the number states $(n = 0, 1, \ldots)$. Let $k = 0, 1, \ldots$. Define the operators

$$T_k := \sum_{n=0}^{\infty} |n\rangle\langle 2n + k|.$$

(i) Show that $T_k T_{k'}^\dagger = \delta_{kk'} I$.
(ii) Show that $T_k^\dagger T_k = P_k$ is a projection operator.
(iii) Show that $\sum_{k=0}^{\infty} P_k = I$.
(iv) Is the operator

$$\sum_{k=0}^{\infty} T_k \otimes T_k^\dagger$$

unitary?

# Chapter 5

# Fourier Transform

**Problem 1.** Consider the Hilbert space $L_2(\mathbb{R})$. Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad -1 \leq x \leq 0 \\ e^{-x} & \text{if} \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 2.** (i) Find the Fourier transform for

$$f_\alpha(x) = \frac{\alpha}{2}\exp(-\alpha|x|), \qquad \alpha > 0.$$

Discuss $\alpha$ large and $\alpha$ small.
(ii) Calculate

$$\int_{-\infty}^{\infty} f_\alpha(x)dx.$$

**Problem 3.** Find the Fourier transform of the *hat function*

$$f(t) = \begin{cases} 1 - |t| & \text{for} \quad -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 4.** Let $f \in L_2(\mathbb{R})$ and $f \in L_1(\mathbb{R}$. Assume that $f(x) = f(-x)$. Can we conclude that $\hat{f}(k) = \hat{f}(-k)$?

**Problem 5.** Consider the Hilbert space $L_2(\mathbb{R})$. Find the Fourier transform of

$$f(x) = e^{-a|x|}, \qquad a > 0.$$

**Problem 6.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $a > 0$. Define

$$f_a(x) = \begin{cases} \frac{1}{2a} & |x| < a \\ 0 & |x| > a \end{cases}$$

Calculate

$$\int_{\mathbb{R}} f_a(x) dx$$

and the Fourier transform of $f_a$. Discuss the result in dependence of $a$.

**Problem 7.** Consider the Hilbert space $L_2(\mathbb{R})$. Let

$$\hat{\psi}(\omega) = \begin{cases} 1 & \text{if} \quad 1/2 \le |\omega| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\phi}(\omega) = e^{-\alpha|\omega|}, \qquad \alpha > 0.$$

(i) Calculate the inverse Fourier transform of $\hat{\psi}(\omega)$ and $\hat{\phi}(\omega)$, i.e.

$$\psi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \hat{\psi}(\omega)$$
$$\phi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \hat{\phi}(\omega).$$

(ii) Calculate the scalar product $\langle \psi(t) | \phi(t) \rangle$ by utilizing the identity

$$2\pi \langle \psi(t) | \phi(t) \rangle = \langle \hat{\psi}(\omega) | \hat{\phi}(\omega) \rangle.$$

**Problem 8.** Consider the Hilbert space $L_2(\mathbb{R})$ and the function $f \in L_2(\mathbb{R})$

$$f(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| \ge 1 \end{cases}$$

Calculate $f * f$ and verify the *convolution theorem*

$$\widehat{f * f} = \hat{f}\hat{f}.$$

**Problem 9.** Let

$$\hat{f}(\omega) = \begin{cases} (1 - \omega^2) & \text{for } |\omega| \le 1 \\ 0 & \text{for } |\omega| > 1 \end{cases}$$

Find $f(t)$.

**Problem 10.** Let $a > 0$. Find the Fourier transform of the function $f_a : \mathbb{R} \to \mathbb{R}$

$$f_a(x) = \begin{cases} x/a^2 + 1/a & \text{for} \quad -a \le x \le 0 \\ -x/a^2 + 1/a & \text{for} \quad 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

**Problem 11.** Let $a > 0$. Find the Fourier transform of

$$f_a(t) = \frac{1}{\sqrt{a}} e^{-a|t|}.$$

Discuss the cases $a$ large and $a$ small. Is $f_a \in L_2(\mathbb{R})$.

**Problem 12.** Show that the Fourier transform of the rectangular window of size $N$

$$w_n = \begin{cases} 1 & \text{for} \quad 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

is

$$W(e^{i\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-i\omega(N-1)/2}.$$

**Problem 13.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $T > 0$. Consider the function in $L_2(\mathbb{R})$

$$f(t) = \begin{cases} A\cos(\Omega t) & \text{for} \quad -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

where $A$ is a positive constant. Calculate the Fourier transform.

**Problem 14.** Let $\sigma > 0$. Show that the Fourier transform of the *Gaussian function*

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

is again a Gaussian function

$$\hat{g}_\sigma(k) = e^{-\sigma^2 k^2/2}.$$

We have $\int_{-\infty}^{\infty} g_\sigma(x) dx = 1$. Is

$$\int_{-\infty}^{\infty} \hat{g}_{\sigma_k}(k) dk = 1\,?$$

**Problem 15.** Show that the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \operatorname{sech}(\pi x)$$

is an element of $L_2(\mathbb{R})$ and $L_1(\mathbb{R})$. Find the Fourier transform of the function.

**Problem 16.** Let $a > 0$. Find the Fourier transform of

$$\sqrt{2\pi} f_a(x) + \frac{\sin(ax)}{ax}$$

where $f_a$ is the function with 1 for $|x| \le a$ and 0 otherwise.

**Problem 17.** Consider the Hermite-Gauss functions

$$f_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi} x) \exp(-\pi x^2), \qquad n = 0, 1, 2, \ldots$$

where $H_n$ is the $n$th Hermite polynomial. They for an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$. Do the Fourier transform of the functions form an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$.

**Problem 18.** The Hilbert transform $h(t)$ of the function $f(t)$ is the principal value of the convolution of $f(t)$ with the *kernel function* $k(t) = 1/(\pi t)$

$$h(t) = \int_{-\infty}^{\infty} f(s) k(t-s) ds = \int_{-\infty}^{\infty} f(s) \frac{1}{t-s} ds.$$

Let

$$G(\omega) = \int_{-\infty}^{\infty} g(t) \exp(i\omega t) dt$$

be the Fourier transform of $g$. Show that the Hilbert transform can be written as

$$H(\omega) = F(\omega) K(\omega) = -i \operatorname{sgn}(\omega) F(\omega).$$

**Problem 19.** Let $\omega_0 > 0$ be a fixed frequency and $t$ the time. Calculate

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\omega_0 t|} e^{-i\omega t} dt.$$

# Chapter 6

# Wavelets

**Problem 1.** Consider the Hilbert space $L_2[0,1]$ and the function $f(x) = x^2$ in this Hilbert space. Project the function $f$ onto the subspace of $L_2[0,1]$ spanned by the functions $\phi(x)$, $\psi(x)$, $\psi(2x)$, $\psi(2x-1)$, where

$$\phi(x) := \begin{cases} 1 & \text{for} \quad 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(x) := \begin{cases} 1 & \text{for} \quad 0 \leq x < 1/2 \\ -1 & \text{for} \quad 1/2 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

This is related to the Haar wavelet expansion of $f$. The function $\phi$ is called the father wavelet and $\psi$ is called the mother wavelet.

**Problem 2.** Consider the function $H \in L_2(\mathbb{R})$

$$H(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$H_{mn}(x) := 2^{-m/2} H(2^{-m}x - n)$$

where $m, n \in \mathbb{Z}$. Draw a picture of $H_{11}$, $H_{21}$, $H_{12}$, $H_{22}$. Show that

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \delta_{mk}\delta_{nl}, \qquad k, l \in \mathbb{Z}$$

where $\langle\,.\,\rangle$ denotes the scalar product in $L_2(\mathbb{R})$ Expand the function

$$f(x) = \exp(-|x|)$$

with respect to $H_{mn}$. The functions $H_{mn}$ form an orthonormal basis in $L_2(\mathbb{R})$.

**Problem 3.** Consider the Hilbert space $L_2[0,1]$ and the *Haar scaling function* (father wavelet)

$$\phi(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $n$ be a positive integer. We define

$$g_k(x) := \sqrt{n}\phi(nx - k), \qquad k = 0, 1, \ldots, n-1.$$

(i) Show that the set of functions $\{ g_0, g_1, \ldots, g_{n-1} \}$ is an orthonormal set in the Hilbert space $L_2[0,1]$.
(ii) Let $f$ be a continuous function on the unit interval $[0,1]$. Thus $f \in L_2[0,1]$. Form the projection $f_n$ on the subspace $S_n$ of the Hilbert space $L_2[0,1]$ spanned by $\{ g_0, g_1, \ldots g_{n-1} \}$, i.e.

$$f_n = \sum_{k=0}^{n-1} \langle f, g_k \rangle g_k \,.$$

Show that $f_n(x) \to f(x)$ pointwise in $x$ as $n \to \infty$.

**Problem 4.** The *continuous wavelet transform*

$$Wf(a,b) = \frac{1}{a} \int_{-\infty}^{+\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \qquad (a, b \in \mathbb{R},\ a > 0)$$

decomposes the function $f \in L_2(\mathbb{R})$ hierarchically in terms of elementary components $\psi((t-b)/a)$. They are obtained from a single *analyzing wavelet* $\psi$ applying *dilations* and *translations*. Here $\bar{\psi}$ denotes the complex conjugate of $\psi$ and $a$ is the scale and $b$ the shift parameter. The function $\psi$ has to be chosen so that it is well localized both in physical and Fourier space. The signal $f(t)$ can be uniquely recovered by the *inverse wavelet transform*

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} Wf(a,b) \psi\left(\frac{t-b}{a}\right) \frac{da}{a} db$$

if $\psi(t)$ (respectively its Fourier transform $\hat{\psi}(\omega)$ satisfies the *admissibility condition*

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty.$$

Consider the analytic function

$$\psi(t) = te^{-t^2/2}.$$

Does $\psi$ satisfies the admissibility condition?

**Problem 5.** Consider the function $H \in L_2(\mathbb{R})$

$$H(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$H_{mn}(x) := 2^{-m/2} H(2^{-m}x - n)$$

where $m, n \in \mathbb{Z}$. Draw a picture of $H_{11}$, $H_{21}$, $H_{12}$, $H_{22}$. Show that

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \delta_{mk}\delta_{nl}, \qquad k, l \in \mathbb{Z}$$

where $\langle\, .\, \rangle$ denotes the scalar product in $L_2(\mathbb{R})$. Expand the function

$$f(x) = \exp(-|x|)$$

with respect to $H_{mn}$. The functions $H_{mn}$ form an orthonormal basis in $L_2(\mathbb{R})$.

**Problem 6.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $\phi \in L_2(\mathbb{R})$ and assume that $\phi$ satisfies

$$\int_{\mathbb{R}} \phi(t)\overline{\phi(t-k)}dt = \delta_{0,k}$$

i.e. the integral equals 1 for $k = 0$ and vanishes for $k = 1, 2, \ldots$. Show that for any fixed integer $j$ the functions

$$\phi_{jk}(t) := 2^{j/2}\phi(2^j t - k), \qquad k = 0, \pm 1, \pm 2, \ldots$$

form an orthonormal set.

**Problem 7.** Consider the function $\phi : \mathbb{R} \to \mathbb{R}$

$$\psi(x) := \begin{cases} 1 & \text{for} \quad x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Find $\psi(x) := \phi(2x) - \phi(2x - 1)$. Calculate

$$\int_{-\infty}^{\infty} \psi(x)dx.$$

**Problem 8.** Consider the Littlewood-Paley orthonormal basis of wavelets. The mother wavelet of this set is

$$L(x) := \frac{1}{\pi x}(\sin(2\pi x) - \sin(\pi x)).$$

Show that

$$L_{mn}(x) = \frac{1}{2^{m/2}} L(2^{-m}x - n), \qquad m, n \in \mathbb{Z}$$

generates an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$. Apply the rule of L'Hospital to find $L(0)$.

**Problem 9.** (i) Consider the Hilbert space $L_2(\mathbb{R})$ and $\phi \in L_2(\mathbb{R})$. The basic scaling function (father wavelet) satisfies a scaling relation of the form

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x - k).$$

Show that the *Hilbert transform* of $\phi$

$$H(\phi)(y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\phi(x)}{x - y} dx$$

is a solution of the same scaling relation. Note that the scaling function $\phi$ may have compact support, the Hilbert transform has support on the real line and decays as $y^{-1}$.
(ii) Show that the Hilbert transform of the related mother wavelet $\psi$ is also noncompact and decays like $y^{-p-1}$ where

$$\int_{\mathbb{R}} x^m \psi(x) dx = 0$$

for $m = 0, 1, \ldots, p - 1$.

**Problem 10.** Is

$$f(x) = e^{-x^2/2} \cos(x)$$

a mother wavelet for the Hilbert space $L_2(\mathbb{R})$.

# Chapter 7

# Linear Operators

**Problem 1.** Show that an isometric operator need not be a unitary operator.

**Problem 2.** Consider the Hilbert space $L_2[0,1]$. Show that the linear operator $T : L_2[0,1] \to L_2[0,1]$ defined by

$$Tf(x) = xf(x)$$

is a bounded self-adjoint linear operator without eigenvalues.

**Problem 3.** Show that if two bounded self-adjoint linear operators $S$ and $T$ on a Hilbert space $\mathcal{H}$ are positive semi-definite and commute ($ST = TS$), then their product $ST$ is positive semi-definite. We have to show that $\langle STf, f\rangle \geq 0$ for all $f \in \mathcal{H}$.

**Problem 4.** Let $a > 0$. Consider the Hilbert space $L_2[-a,a]$. Consider the Hamilton operator

$$\hat{H} = \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

where

$$V(x) = \begin{cases} 0 & \text{for} \quad |x| \leq a \\ \infty & \text{otherwise} \end{cases}$$

Solve the Schrödinger equation, where the initial function $\psi(t=0) = \phi(x)$ is given by

$$\phi(x) = \begin{cases} x/a^2 + 1/a & \text{for } -a \le x \le 0 \\ -x/a^2 + 1/a & \text{for } \ 0 \le x \le a \end{cases}$$

Normalize $\phi$. Calculate the probability to find the particle in the state

$$\chi(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right)$$

after time $t$. A basis in the Hilbert space $L_2[-a,a]$ is given by

$$\left\{ \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{a}\right), \ \frac{1}{\sqrt{a}} \cos\left(\frac{(n-1/2)\pi x}{a}\right) \quad n = 1, 2, \dots \right\}.$$

**Problem 5.** Show that in one dimensional problems the energy spectrum of the bound state is always non-degenerate. Hint. Suppose that the oppsite is true. Let $u_1$ and $u_2$ be two linearly independent eigenfunctions with the same energy eigenvalues $E$, i.e.

$$\frac{d^2u_1}{dx^2} + \frac{2m}{\hbar^2}(E-V)u_1 = 0, \qquad \frac{d^2u_2}{dx^2} + \frac{2m}{\hbar^2}(E-V)u_2 = 0.$$

**Problem 6.** A particle is enclosed in a rectangular box with imprenetrable walls, inside which it can move freely. The Hilbert space is $L_2([0,a] \times [0,b] \times [0,c])$. Find the eigenfunctions and eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions.

**Problem 7.** Conside the Hilbert space $L_2[0,1]$ and the linear operator $T : L_2[0,1] \to L_2[0,1]$ defined by

$$(Tf)(x) := xf(x).$$

Show that $T$ is self-adjoint and positive definite. Find its positive square root.

**Problem 8.** Consider the Hilbert space $\ell_2(\mathbb{N})$ and the linear operator $T$ defined by

$$T : (x_1, x_2, x_3, \dots) \mapsto (0, 0, x_3, x_4, \dots).$$

Is $T$ bounded? Is $T$ self-adjoint? If so is $T$ positive?

**Problem 9.** In classical mechanics we have

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \qquad \mathbf{T} = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{T}$ is the torque, $\mathbf{F} = -\nabla V$ ($V$ potential depending only on $\mathbf{r}$) and

$$\frac{d\mathbf{L}}{dt} = \mathbf{T}\,.$$

In quantum mechanics with $\mathbf{p} \to -i\hbar\nabla$, $\mathbf{r} \to \mathbf{r}$ and wave function $\psi$ we have

$$\mathbf{L} = -i\hbar \int_{\mathbb{R}^3} d^3\mathbf{x}\,\psi^*(\mathbf{r}\times\nabla)\psi$$

and

$$\mathbf{T} = -\int_{\mathbb{R}^3} d^3\mathbf{x}\,\psi^*(\mathbf{r}\times\nabla V)\psi$$

since $\mathbf{F} = -\nabla V$. $\psi$ and $\psi^*$ obey the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*\,.$$

Show that

$$\frac{d\mathbf{L}}{dt} = \mathbf{T}.$$

**Problem 10.** Let $\hat{H}$ be a bounded self-adjoint Hamilton operator with normalized eigenfunctions $\phi_j$ ($j \in I$) which form an orthonormal basis in the underlying Hilbert space. We can write

$$\psi(t) = \sum_{j\in I} c_j e^{-iE_j t/\hbar}\phi_j$$

where $E_j$ are the eigenvalues of $\hat{H}$. Find $P(t) = \langle\psi(t=0)|\psi(t)\rangle$.

**Problem 11.** Consider the Hamilton operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + D(1-e^{-\alpha x})^2 + eEx\cos(\omega t)$$

where $\alpha > 0$. Find the quantum Liouville equation for this Hamilton operator.

**Problem 12.** Consider the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2m}\Delta + V(x)\right)\psi$$

Find the coupled system of partial differential equations for

$$\rho := \psi^* \psi, \qquad v := \Im\left(\frac{\nabla\psi}{\psi}\right).$$

**Problem 13.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $f \in L_2(\mathbb{R})$ and $\theta \in \mathbb{R}$. We define the operator $U(\theta)$ as

$$U(\theta)f(x) := e^{i\theta/2} f(xe^{i\theta}).$$

Is the operator $U(\theta)$ unitary?

**Problem 14.** Let $a > 0$. Consider the Hilbert space $L_2([0,a])$ and the linear bounded operator $A$ defined by

$$Af(x) := xf(x), \quad f \in L_2([0,a])$$

Find $\|A\|$.

**Problem 15.** Consider the Hilbert space $L_2(\mathbb{R})$. Let $k \in \mathbb{Z}$. For $k = 0$ we define $s_0 = 0$, for $k \geq 1$ we define

$$s_k := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

and for $k < 0$ we define $s_k = -s_{-k}$. Let $\epsilon > 0$. Define the indicator functions $W_k$ as

$$W_k(x) := \begin{cases} 1 & \text{for} \quad s_k < x/\epsilon \leq s_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

Let $u \in L_2(\mathbb{R})$. Define the linear operator $O$ as

$$(Ou)(x) := g(x)u(x)$$

where

$$g(x) = -\frac{x}{\epsilon} + \sum_{k\in\mathbb{Z}} \left(\frac{s_k + s_{k+1}}{2}\right) W_k(x).$$

(i) Show that $O$ is a bounded self-adjoint operator for any $\epsilon > 0$.
(ii) Show that the norm of $O$

$$\|O\| = \sup_{\|u\|=1} \|Ou\|$$

is given by $1/2$.

**Problem 16.** Consider the *momentum operator*

$$\hat{p} = -i\frac{d}{dx}$$

defined on $C_0^\infty[0,\infty)$. Show that the operator has no self-adjoint extensions on the Hilbert space $L_2[0,\infty)$.

**Problem 17.** Consider the Hilbert space $L_2(\mathbb{R})$ and the linear operator

$$\hat{p} = -i\frac{d}{dx} \quad \text{on} \quad D = \{f \,:\, f, \frac{df}{dx} \in L_2(\mathbb{R})\}.$$

Show that the spectrum is the whole real axis.

# Chapter 8

# Generalized Functions

## 8.1 Solved Problems

**Problem 1.** Consider the function $H : \mathbb{R} \to \mathbb{R}$

$$H(x) := \begin{cases} 1 & 0 \le x \le 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the derivative of $H$ in the sense of generalized functions. Obviously $H$ can be considered as a regular functional

$$\int_{\mathbb{R}} H(x)\phi(x)dx.$$

Find the Fourier transform of $H$. Draw a picture of the Fourier transform.

**Problem 2.** Let $C^m[a,b]$ be the vector space of $m$-times differentiable functions and the $m$-th derivative is continuous over the interval $[a,b]$ ($b > a$). We define an inner product (scalar product) of such two functions $f$ and $g$ as

$$\langle f, g \rangle_m := \int_a^b \left( fg + \frac{df}{dx}\frac{dg}{dx} + \cdots + \frac{d^m f}{dx^m}\frac{d^m g}{dx^m} \right) dx.$$

Given (Legendre polynomials)

$$f(x) = \frac{1}{2}(3x^2 - 1), \qquad g(x) = \frac{1}{2}(5x^3 - 3x)$$

and the interval $[-1,1]$, i.e. $a=-1$ and $b=1$. Show that $f$ and $g$ are orthogonal with respect to the inner product $\langle f,g\rangle_0$. Are they orthogonal with respect to $\langle f,g\rangle_1$?

**Problem 3.** Let $P$ be the parity operator, i.e.

$$P\mathbf{r} := -\mathbf{r}.$$

Obviously, $P = P^{-1}$. We define

$$O_P u(\mathbf{r}) := u(P^{-1}\mathbf{r}) \equiv u(-\mathbf{r}).$$

The vector $\mathbf{r}$ can be expressed in spherical coordinates as

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

where

$$0 \le \phi < 2\pi \quad 0 \le \theta < \pi.$$

(i) Calculate $P(r,\theta,\phi)$.
(ii) Let

$$Y_{lm}(\theta,\phi) = \frac{(-1)^{l+m}}{2^l l!}\left(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2}(\sin\theta)^m \frac{d^{l+m}}{d(\cos\theta)^{l+m}}(\sin\theta)^{2l}e^{im\phi}$$

be the *spherical harmonics*. Find

$$O_P Y_{lm}.$$

**Problem 4.** In the Hilbert space $\mathcal{H} = \ell_2(\mathbb{N}_0)$ Bose annihilation and creation operators denoted by $b$ and $b^\dagger$ are defined as follows: They have a common domain

$$\mathcal{D}(b) = \mathcal{D}(b^\dagger) = \left\{ \xi = (x_0,x_1,x_2,\dots)^T : \sum_{j=0}^{\infty} j|x_j|^2 < \infty \right\}.$$

Then $b\eta$ is given by

$$b(x_0,x_1,x_2,\dots)^T = (x_1, \sqrt{2}x_2, \sqrt{3}x_3, \dots)^T$$

and $b^\dagger\eta$ is given by

$$b^\dagger(x_0,x_1,x_2,\dots) = (0, x_0, \sqrt{2}x_1, \sqrt{3}x_2, \dots).$$

The infinite dimensional vectors

$$u_n = (0,0,\dots,0,1,0,\dots)^T$$

where the 1 is at the $n$ position ($n = 0, 1, 2, \ldots$) form the standard basis in $\mathcal{H} = \ell_2(\mathbb{N}_0)$. Is

$$\xi = (1, 1/2, 1/3, \ldots, 1/n, \ldots)$$

an element of $\mathcal{D}(a)$?

**Problem 5.** Given a function (signal) $f(\mathbf{t}) = f(t_1, t_2, \ldots, t_n) \in L_2(\mathbb{R}^n)$ of $n$ real variables $\mathbf{t} = (t_1, t_2, \ldots, t_n)$. We define the *symplectic tomogram* associated with the square integrable function $f$

$$w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{k=1}^{n} \frac{1}{2\pi|\nu_k|} \left| \int_{\mathbb{R}^n} dt_1 dt_2 \cdots dt_n f(\mathbf{t}) \exp\left( \sum_{j=1}^{n} \left( \frac{i\mu_j}{2\nu_j} t_j^2 - \frac{iX_j}{\nu_j} t_j \right) \right) \right|^2$$

where ($\nu_j \neq 0$ for $j = 1, 2, \ldots, n$)

$$\mathbf{X} = (X_1, X_2, \ldots, X_n), \quad \boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n), \quad \boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n).$$

(i) Prove the equality

$$\int_{\mathbb{R}^n} w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) d\mathbf{X} = \int_{\mathbb{R}^n} |f(\mathbf{t})|^2 d\mathbf{t} \tag{1}$$

for the special case $n = 1$. The tomogram is the probabilty distribution function of the random variable $\mathbf{X}$. This probability distribution function depends on $2n$ extra real parameters $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$.
(ii) The map of the function $f(\mathbf{t})$ onto the tomogram $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is invertible. The square integrable function $f(\mathbf{t})$ can be associated to the density matrix

$$\rho_f(\mathbf{t}, \mathbf{t}') = f(\mathbf{t}) f(\mathbf{t}').$$

This density matrix can be mapped onto the *Ville-Wigner function*

$$W(\mathbf{q}, \mathbf{p}) = \int_{\mathbb{R}^n} \rho_f \left( \mathbf{q} + \frac{\mathbf{u}}{2}, \mathbf{q} - \frac{\mathbf{u}}{2} \right) e^{-i\mathbf{p} \cdot \mathbf{u}} d\mathbf{u}.$$

Show that this map is invertible.
(iii) How is the tomogram $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ related to the Ville-Wigner function?
(iv) Show that the Ville-Wigner function can be reconstructed from the function $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$.
(v) Show that the density matrix $f(\mathbf{t}) f^*(\mathbf{t}')$ can be found from $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$.

**Problem 6.** Starting with the set of polynomials $\{ 1, x, x^2, \ldots, x^n, \ldots \}$ use the Gram-Schmidt procedure the scalar product (inner product)

$$\langle f, g \rangle = \int_{-1}^{1} (f(x)g(x) + f'(x)g'(x)) dx$$

to find the first five orthogonal polynomials, where $f'$ denotes derivative.

**Problem 7.** Describe the one-dimensional scattering of a particle incident on a Dirac delta function, i.e.

$$U(q) = U_0 \delta(q)$$

where $u_0 > 0$. Find the transmission and reflection coefficient.

**Problem 8.** (i) Give the definition of the current density, transmission coefficient, and reflection coefficient.
(ii) Calculate the transmission and the reflection coefficients of a particle having total energy $E$, at the potential barrier given by

$$V(x) = a\delta(x), \qquad a > 0$$

**Problem 9.** Show that

$$\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} = \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$$

in the sense of generalized functions
Hint. Expand the $2\pi$ periodic function

$$f(x) = \frac{1}{2} - \frac{x}{2\pi}$$

into a Fourier series.

**Problem 10.** (i) Give the definition of a generalized function.
(ii) Calculate the first and second derivative in the sense of generalized function of

$$f(x) = \begin{cases} 0 & x < 0 \\ 4x(1-x) & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

(iii) Calculate the Fourier transform of $f(x) = 1$ in the sense of generalized functions.

**Problem 11.** Consider the generalized function

$$f(x) = |\cos(x)|.$$

Find the derivative in the sense of generalized functions.

**Problem 12.** Consider the generalized function

$$f(x) := \begin{cases} \cos(x) & \text{for} \quad x \in [0, 2\pi) \\ 0 & \text{otherwise} \end{cases}$$

Find the first and second derivative of $f$ in the sense of generalized functions.

**Problem 13.** Find the derivative of $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |x|$$

in the sense of generalized functions.

**Problem 14.** Find the first three derivatives of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-|x|}$$

in the sense of generalized functions.

**Problem 15.** The *Sobolev space* of order $m$, denoted by $H^m(\Omega)$, is defined to be the space consisting of those functions in the Hilbert space $L_2(\Omega)$ that, together with all their weak partial derivatives up to and including those of order $m$, belong to the Hilbert space $L_2(\Omega)$, i.e.

$$H^m(\Omega) := \{\, u \,:\, D^\alpha u \in L_2(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| \le m \,\}.$$

We consider real-valued functions only, and make $H^m(\Omega)$ an inner product space by introducing the Sobolev inner product $\langle \cdot, \cdot \rangle_{H^m}$ defined by

$$\langle u, v \rangle_{H^m} := \int_\Omega \sum_{|\alpha| \le m} (D^\alpha u)(D^\alpha v) dx \quad \text{for} \quad u, v \in H^m(\Omega).$$

This inner product generates the Sobolev norm $\| \cdot \|_{H^m}$ defined by

$$\|u\|^2_{H^m} = \langle u|u \rangle_{H^m} = \int_\Omega \sum_{|\alpha| \le m} (D^\alpha u)^2 dx.$$

Thus $H^0(\Omega) = L_2(\Omega)$. We can write

$$\langle u, v \rangle = \sum_{|\alpha| \le m} \langle D^\alpha, D^\alpha v \rangle_{L_2(\Omega)}.$$

In other words the Sobolev inner product $\langle u, v \rangle_{H^m(\Omega)}$ is equal to the sum of the $L_2(\Omega)$ inner products of $D^\alpha u$ and $D^\alpha v$ over all $\alpha$ such that $|\alpha| \le m$.

(i) Consider the domain $\Omega = (0,2)$ and the function

$$u(x) = \begin{cases} x^2 & 0 < x \leq 1 \\ 2x^2 - 2x + 1 & 1 < x < 2. \end{cases}$$

Obviously $u \in L_2(\Omega)$. Find the Sobolev space to which $u$ belongs.
(ii) Find the norm of $u$.

**Problem 16.** Let $c > 0$. Consider the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + c\delta^{(n)}(x)\psi = E\psi$$

where $\delta^{(n)}$ $(n = 0, 1, 2, \ldots)$ denotes the $n$-th derivative of the delta function. Derive the joining conditions on the wave function $\psi$.

**Problem 17.** The *Morlet wavelet* consists of a plane wave modulated by a Gaussian, i.e.

$$\psi(\eta) = \frac{1}{\pi^{1/4}} e^{i\omega\eta} e^{-\eta^2/2}$$

where $\omega$ is the dimensionless frequency. Show that if $\omega = 6$ the admissibility condition is satisfied.

**Problem 18.** Let

$$f_0(x) = \exp(-x^2/2).$$

We define the mother wavelets $f_n$ as

$$f_n(x) = -\frac{d}{dx} f_{n-1}(x), \qquad n = 1, 2, \ldots$$

Show that the family of $f_n$'s obey the Hermite recursion relation

$$f_n(x) = x f_{n-1}(x) - (n-1) f_{n-2}(x), \qquad n = 2, 3, \ldots.$$

**Problem 19.** Let $a > 0$. Show that

$$\sum_{m=-\infty}^{\infty} \exp(i2\pi m(x+q)/a) \equiv a \sum_{k=-\infty}^{\infty} \delta(x + q - ka).$$

**Problem 20.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $x_0$ a root of $f$, i.e. $f(x_0) = 0$. Show that

$$\delta'(f(x)) = \frac{1}{(f'(x_0))^2} \left( \delta'(x - x_0) + \frac{f''(x_0)}{f'(x_0)} \delta(x - x_0) \right)$$

**Problem 21.** Show that the sum

$$\frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(x)P_\ell(y)$$

of Legendre polynomials $P_\ell$ is given by the Dirac delta function $\delta(y-x)$ for $-1 \leq x \leq +1$ and $-1 \leq y \leq +1$.

**Problem 22.** Show that

$$\delta(x-x') = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}(\cos(nx)\cos(nx') + \sin(nx)\sin(nx')).$$

**Miscellaneous**

**Problem 23.** Let

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

be the Schrödinger equation, where

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + U(r), \qquad \Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

and $\mathbf{r} = (x_1, x_2, x_3)$. Let

$$\rho(\mathbf{r}, t) := \bar{\psi}(\mathbf{r}, t)\psi(\mathbf{r}, t)$$

Find $\mathbf{j}$ such that

$$\text{div}\mathbf{j} + \frac{\partial\rho}{\partial t} = 0.$$

**Problem 24.** A particle is enclosed in a rectangular box with impenetrable walls, inside which in can move freely. The Hilbert space is

$$L_2([0,a] \times [0,b] \times [0,c])$$

where $a, b, c > 0$. Find the eigenfunctions and eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions?

**Problem 25.** Show that in one-dimensional problems the energy spectrum of the bound states is always non-degenerate. Hint. Suppose that the opposite is true. Let $u_1$, $u_2$ be two linearly independent eigenfunctions with the same energy eigenvalue $E$, i.e.

$$\frac{d^2u_1}{dx^2} + \frac{2m}{\hbar^2}(E - V)u_1 = 0$$

$$\frac{d^2u_2}{dx^2} + \frac{2m}{\hbar^2}(E - V)u_2 = 0.$$

**Problem 26.** Derive the Heisenberg uncertainty relation.

**Problem 27.** Give the standard postulates in quantum mechanics and discuss the problematic.

**Problem 28.** Show that in one-dimensional problems the energy spectrum of the bound states is always non-degenerate.

Hint. Suppose that the opposite is true.

Let $u_1$ and $u_2$ be two linearly independent eigenfunctions with the same energy eigenvalues $E$.

$$\frac{d^2u_1}{dx^2} + \frac{2m}{\hbar^2}(E-V)u_1 = 0$$

$$\frac{d^2u_2}{dx^2} + \frac{2m}{\hbar^2}(E-V)u_2 = 0$$

**Problem 29.** Let $a > 0$ and let $f_a : \mathbb{R} \to \mathbb{R}$ be given by

$$f_a(x) = \begin{cases} x/a^2 + 1/a & \text{for } -a \le x \le 0 \\ -x/a^2 + 1/a & \text{for } \ 0 \le x \le a \end{cases}$$

The function $f_a$ generates regular functional. Find the derivative of $f_a$ in the sense of generalized functions.

**Problem 30.** Consider a one-dimensional lattice (chain) with lattice constant $a$. Let $k$ be the sum over the first Brioullin zone we have

$$\frac{1}{N}\sum_{k\in 1.BZ} F(\epsilon(k)) \to \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} F(\epsilon(k))dk = G$$

where

$$\epsilon(k) = \epsilon_0 - 2\epsilon_1\cos(ka).$$

Using the identity

$$\int_{-\infty}^{\infty}\delta(E-\epsilon(k))F(E)dE \equiv F(\epsilon(k))$$

we can write

$$G = \frac{a}{2\pi}\int_{-\infty}^{\infty} F(E)\left(\int_{-\pi/a}^{\pi/a}\delta(E-\epsilon(k))dk\right)dE.$$

Calculate

$$g(E) = \int_{-\pi/a}^{\pi/a}\delta(E-\epsilon(k))dk$$

where $g(E)$ is called the density of states.

**Problem 31.** Let $\epsilon > 0$. Consider the Schrödinger eigenvalue equation

$$\left(-\frac{d^2}{dx^2} + 2\epsilon\delta(x)\right)u(x,\epsilon) = E(\epsilon)u(x,\epsilon)$$

with the boundary conditions $u(\pm 1, \epsilon) = 0$. Here $\epsilon$ is the coupling constant and determines the penetrability of the potential barrier. Find the eigenfunctions and the eigenvalues.

**Problem 32.** Show that in the sense of generalized functions

$$\delta(x) = \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{-|x|/\epsilon}$$

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to \infty} \epsilon \frac{\sin^2(\epsilon x)}{(\epsilon x)^2}$$

$$\delta(x) = \frac{1}{4} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(1 + \frac{|x|}{\epsilon}\right) e^{-|x|/\epsilon}.$$

**Problem 33.** Give two interpretations of the series of derivatives of $\delta$ functions

$$f(k) = 2\pi \sum_{n=0}^{\infty} c_n (-1)^n \delta^{(n)}(k). \qquad (1)$$

**Problem 34.** Show that

$$H(x-a) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \exp(iu(\tau - x)).$$

**Problem 35.** Let $a > 0$. Show that

$$\delta(x^2 - a^2) = \frac{1}{2a}(\delta(x-a) + \delta(x+a)).$$

**Problem 36.** (i) Show that the Fourier transform in the sense of generalized function of the *Dirac comb*

$$\sum_{n \in \mathbb{Z}} \delta(x-n)$$

is again a Dirac comb.
(ii) Find the Fourier transform in the sense of generalized functions of

$$1 + \sqrt{2\pi}\delta(x).$$

**Problem 37.** (i) Consider the nonlinear differential equation

$$3u\frac{du}{dx} = 2\frac{du}{dx}\frac{d^2u}{dx^2} + u\frac{d^3u}{dx^3}.$$

Show that $u(x) = e^{-|x|}$ is a solution in the sense of generalized function.
(ii) Consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + 3u\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}.$$

Show that $u(x,t) = c\exp(-|x-ct|)$ (*peakon*) is a solution in the sense of generalized functions.

**Problem 38.** Let $f$ be a differentiable function with a simple zero at $x = a$ such that $f(x=a) = 0$ and $df(x=a)/dx \neq 0$. Let $g$ be a differentiable function with a simple zero at $x = b \neq a$ such that $g(x=b) = 0$ and $dg(x=b)/dx \neq 0$. Show that

$$\delta(f(x)g(x)) = \frac{1}{|f'(a)g(a)|}\delta(x-a) + \frac{1}{|f(b)g'(b)|}\delta(x-b)$$

where $'$ denotes differentiation.

**Problem 39.** Consider the non-relativistic hydrogen atom, where $a_0$ is the Bohr radius and $a = a_0/Z$. The Schrödinger-Coulomb Green function $G(\mathbf{r}_1, \mathbf{r}_2; E)$ corresponding to the energy variable $E$ is the solution of the partial differential equation

$$\left(-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{amr_1} - E\right) G(\mathbf{r}_1, \mathbf{r}_2; E) = \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

with the appropiate boundary conditions. Show that expanding $G$ in terms of spherical harmonics $Y_{\ell m}$

$$G(\mathbf{r}_1, \mathbf{r}_2; E) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell(r_1, r_2; E) Y_{\ell m}(\theta_1, \phi_1) Y_{\ell m}^*(\theta_2, \phi_2)$$

we find for the radial part $g_\ell$ of the Schrödinger-Coulomb Green function

$$\left(\frac{1}{r_1^2}\frac{d}{dr_1}\left(r_1^2\frac{d}{dr_1}\right) - \frac{\ell(\ell+1)}{r_1^2} + \frac{2}{ar_1} - \frac{1}{\nu^2 a^2}\right) g_\ell(r_1, r_2; \nu) = -\frac{2m}{\hbar^2}\frac{\delta(r_1 - r_2)}{r_1 r_2}$$

where $\nu^2 a^2 := -\hbar^2/(2mE)$.
Hint. Utilize the identity

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\delta(r_1 - r_2)}{r_1 r_2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta_1, \phi_1) Y_{\ell m}^*(\theta_2, \phi_2)$$

**Problem 40.** Show that (distributional identity on $L_1(\mathbb{R})$)

$$\frac{1}{\pi^2}\int_{\mathbb{R}} \frac{1}{(t-x)(s-x)} dx = \delta(t-s)$$

where the integral is evaluated in the principal value sense.

**Problem 41.** Let $c > 0$. Show that an integral representation of the delta function is given by

$$\delta(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tx}dt$$

where the path of the $t$-integration can be closed to the right or left.

**Problem 42.** Show that

$$\delta\left(t - \frac{x}{c}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t(t-x/c)}d\omega.$$

**Problem 43.** Show that in the sense of generalized functions

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(ikx)dx, \qquad \theta(x) = \int_0^{\infty} \delta(\lambda - x)d\lambda.$$

**Problem 44.** Let $\epsilon > 0$. Show that

$$f_\epsilon(x-a) = \frac{1}{\sqrt{\pi\epsilon}}\exp\left(-\frac{(x-a)^2}{\epsilon}\right)$$

tends to $\delta(x-a)$ in the sense of generalized function if $\epsilon \to 0_+$.

**Problem 45.** Let $J_0$ be the Bessel functions. Show that

$$\delta(x)\delta(x) = \frac{1}{2\pi}\int_0^{\infty} kJ_0(k\sqrt{(x^2+y^2)})dk$$

in the sense of generalized functions.

**Problem 46.** Let $\alpha \in [0,1)$. Show that

$$\int_0^{\infty} x^{\alpha-1}P\left(\frac{1}{1-x^2}\right) = \frac{\pi}{2}\cot(\pi\alpha/2).$$

**Problem 47.** Show that

$$\delta(\mathbf{x}-\mathbf{x}') = \lim_{\alpha\to\beta}\left(\frac{2\pi}{\beta-\alpha}\right)^{3/2}\exp\left(-\frac{\alpha\beta}{2(\beta-\alpha)}(\mathbf{x}-\mathbf{x}')^2\right).$$

**Problem 48.** Let $p \in [0,1]$ and

$$\rho(x) = \frac{1}{2}pe^{-|x|} + (1-p)\delta(x).$$

Then $\rho(x) \geq 0$. Show that in the sense of generalized function

$$\int_{\mathbb{R}} \rho(x)dx = 1.$$

**Problem 49.** The two-dimensional Dirac comb function is defined by

$$C(x_1, x_1) := \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \delta(x_1 - m)\delta(x_2 - n).$$

Find the Fourier transform of $C$ in the sense of generalized functions.

**Problem 50.** Let $T > 0$. Consider the sequence of functions

$$f_n(t) = \frac{1}{n!}\frac{n}{T}\left(\frac{nt}{T}\right)^n \exp(-nt/T)$$

where $n = 1, 2, \ldots$. Find $f_n(t)$ for $n \to \infty$ in the sense of generalized functions. Find the Laplace transform of $f_n(t)$.

**Problem 51.** What charge distribution $\rho(r)$ does the spherical symmetric potential

$$V(r) = \frac{e^{-\mu r}}{r}$$

give? For $r \neq 0$ *Poisson's equation* in spherical coordinates is given by

$$\Delta V(\mathbf{r}) = \frac{1}{r}\frac{d^2}{dr^2}(rV(\mathbf{r})) + R(\theta, \phi)V(\mathbf{r}) = -4\pi\rho(\mathbf{r})$$

where $R(\theta, \phi)$ is the differential operator depending on the angles $\theta$, $\phi$.

**Problem 52.** Let $a_1, a_2, \ldots, a_n \neq 0$. Show that

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 d\epsilon_1 \cdots \int_0^1 d\epsilon_n \frac{\delta(1 - \sum_{j=1}^n \epsilon_j)}{(\sum_{j=1}^n \epsilon_j a_j)^n}.$$

**Problem 53.** Prove the identity in the sense of generalized function

$$f(y)\frac{\partial}{\partial y}\delta(x-y) \equiv -f(x)\frac{\partial}{\partial x}\delta(x-y) + \frac{df(y)}{dy}\delta(x-y).$$

**Problem 54.** Let $\phi \in S(\mathbb{R})$. Show that in the sense of generalized functions

$$\lim_{\zeta \to \infty} \frac{\sin(\zeta x)}{x} \phi(x) dx = \pi \phi(0).$$

**Problem 55.** Show that

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left( -\frac{(x-x_0)^2}{4Dt} \right)$$

satisfies the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

with the initial condition

$$u(x,0) = \delta(x - x_0).$$

**Problem 56.** Let $A$, $B$ be $n \times n$ matrices. Show that

$$e^{A+B} = \int_0^\infty d\alpha_1 e^{\alpha_1 A} \delta(1-\alpha_1) + \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 e^{\alpha_1 A} B e^{\alpha_2 A} \delta(1 - \alpha_1 - \alpha_2)$$
$$+ \int_0^\infty \int_0^\infty \int_0^\infty e^{\alpha_1 A} B e^{\alpha_2 A} B e^{\alpha_3 A} \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) + \cdots$$

**Problem 57.** Consider the Hilbert space $L_2([-1,1])$. The *Legendre polynomials* are given by

$$P_0 = 1, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

where $n = 1, 2, \ldots$. They satisfy

$$\int_{-1}^{+1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{m,n}, \quad n, m = 0, 1, 2, \ldots$$

Let

$$\delta(x) = \sum_{j'=0}^\infty d_{j'} P_{j'}(x).$$

Find the expansion cofficients $d_{j'}$.

## 8.2 Supplementary Problems

**Problem 1.** Show that the 2-dimensional complex $\delta$-function can be written as ($\alpha \in \mathbb{C}$)

$$\delta^{(2)}(z) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha \exp(\alpha^* z - z^*\alpha) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha \exp(i(\alpha^* z + z^*\alpha)).$$

**Problem 2.** Show that

$$\delta(x - x') = \frac{1}{\pi}\left(1 + 2\sum_{k=1}^{\infty}\cos(kx)\cos(kx')\right).$$

**Problem 3.** Consider the Hilbert space $L_2([0,\infty))$. The *Laguerre polynomials* are defined as

$$L_n(x) = e^x \frac{d^n}{dx^n}(e^{-x}x^n), \quad n = 0, 1, 2, \ldots$$

For the Hilbert space $L_2([0,\infty))$ we have the basis

$$B = \{\, e^{-x/2}L_n(x) \,:\, n = 0, 1, 2, \ldots \}.$$

Let $a \in \mathbb{R}$. Show that

$$\delta(x - a) = e^{-(x+a)/2}\sum_{k=0}^{\infty} L_k(x)L_k(a).$$

**Problem 4.** Show that

$$\delta(\cos(\theta_1) - \cos(\theta_2))\delta(\phi_1 - \phi_2) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1,\phi_1)Y^*_{\ell,m}(\theta_2,\phi_2).$$

**Problem 5.** Let $H_k$ ($k = 0, 1, 2, \ldots$) be the Hermite polynomials. Show that

$$\delta(x - a) = \frac{e^{-(x^2+a^2)/2}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{H_k(x)H_k(a)}{2^k k!}$$

**Problem 6.** Show that

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) = \pi\delta(z)$$

where

$$\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

**Problem 7.** Let $a \in \mathbb{R}$. Show that

$$\delta(x-a) = \int_{-\infty}^{+\infty} Ai(s-x)Ai(s-a)ds$$

where $Ai$ is the Airy function.

**Problem 8.** Let $x \in \mathbb{R}$ and $\alpha > 0$. Show that

$$\sum_{n=-\infty}^{\infty} \delta(x-n\alpha) \equiv \frac{1}{\alpha}\sum_{n=-\infty}^{\infty} \exp(2\pi i n x/\alpha).$$

**Problem 9.** Find the first and second derivative of the function

$$f(x) := \begin{cases} x\exp(-cx^2) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

in the sense of generalized function. Note that the function is continuous.

**Problem 10.** Show that

$$\int_0^{\infty} \cos(\omega t)dt = \pi\delta(\omega).$$

**Problem 11.** Let $r^2 = x_1^2 + x_2^2$ and $r'^2 = x_1'^2 + x_2'^2$.
(i) Show that

$$\delta(x_1, x_2) = \frac{1}{2\pi r}\delta(r)$$

(ii) Show that

$$\delta(x_1 - x_1', x_2 - x_2') = \frac{1}{r}\delta(r-r')\delta(\theta-\theta'), \quad r' > 0$$

where we applied polar coordinates.

**Problem 12.** Let $r^2 = x_1^2 + x_2^2 + x_3^2$. Show that

$$\delta(x_1 - x_1', x_2 - x_2', x_3 - x_3') = \frac{1}{r\sin(\theta)}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$$

where we applied spherical coordinates.

**Problem 13.** Consider the linear partial differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

Show that $u(x,t) = f(x - ct)$ is a solution for any one-dimensional generalized function.

**Problem 14.** Let $p > 0$. Show that

$$\frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{i(m+1/2)\phi}\cos(p|m+1/2|x) = \frac{1}{2p}\left(\delta(x-\phi/p)+\delta(x+\phi/p)\right).$$

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