

Problems and Solutions  
in  
Real and Complex Analysis,  
Integration,  
Functional Equations  
and  
Inequalities

by  
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## Preface

The purpose of this book is to supply a collection of problems in analysis. Please submit your solution to one of the email addresses below.

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### Prescribed books for problems.

1) Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume I: Introductory Level

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# Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	$n$ -dimensional Euclidean space
$\mathbb{C}^n$	space of column vectors with $n$ real components
$\mathcal{H}$	$n$ -dimensional complex linear space
$i$	space of column vectors with $n$ complex components
$\Re z$	Hilbert space
$\Im z$	$\sqrt{-1}$
$ z $	real part of the complex number $z$
$T \subset S$	imaginary part of the complex number $z$
$S \cap T$	modulus of complex number $z$
$S \cup T$	$ x + iy  = (x^2 + y^2)^{1/2}$ , $x, y \in \mathbb{R}$
$f(S)$	subset $T$ of set $S$
$f \circ g$	the intersection of the sets $S$ and $T$
$\mathbf{v}$	the union of the sets $S$ and $T$
$\mathbf{v}^T$	image of set $S$ under mapping $f$
$\mathbf{0}$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\ \cdot\ $	column vector in $\mathbb{C}^n$
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	transpose of $\mathbf{v}$ (row vector)
$\mathbf{x} \times \mathbf{y}$	zero (column) vector
$A, B, C$	norm
$\det(A)$	scalar product (inner product) in $\mathbb{C}^n$
$\text{tr}(A)$	vector product in $\mathbb{R}^3$
$\text{rank}(A)$	$m \times n$ matrices
$A^T$	determinant of a square matrix $A$
	trace of a square matrix $A$
	rank of matrix $A$
	transpose of matrix $A$

$\overline{A}$	conjugate of matrix $A$
$A^*$	conjugate transpose of matrix $A$
$A^\dagger$	conjugate transpose of matrix $A$ (notation used in physics)
$A^{-1}$	inverse of square matrix $A$ (if it exists)
$I_n$	$n \times n$ unit matrix
$I$	unit operator
$0_n$	$n \times n$ zero matrix
$AB$	matrix product of $m \times n$ matrix $A$ and $n \times p$ matrix $B$
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices $A$ and $B$
$[A, B] := AB - BA$	commutator for square matrices $A$ and $B$
$[A, B]_+ := AB + BA$	anticommutator for square matrices $A$ and $B$
$A \otimes B$	Kronecker product of matrices $A$ and $B$
$A \oplus B$	Direct sum of matrices $A$ and $B$
$\delta_{jk}$	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
$\lambda$	eigenvalue
$\epsilon$	real parameter
$t$	time variable
$\hat{H}$	Hamilton operator



# Chapter 1

## Sums and Products

---

### 1.1 Solved Problems

**Problem 1.** The *harmonic series* can be approximated by

$$\sum_{j=1}^n \frac{1}{j} \approx 0.5772 + \ln(n) + \frac{1}{2n}.$$

Calculate the left and right-hand side for  $n = 1$  and  $n = 10$ .

**Problem 2.** The *Bernoulli numbers*  $B_0, B_1, B_2, \dots$  are defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j \equiv B_0 + \frac{B_1}{1!} x + \frac{B_2}{2!} x^2 + \dots.$$

One finds  $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30$ . One has  $B_j = 0$  if  $j \geq 3$  and  $j$  odd. The Bernoulli numbers are utilized in the *Euler summation formula*

$$\sum_{j=1}^n f(j) = \int_1^n f(t) dt + \frac{1}{2}(f(n) + f(1)) + \sum_{k=1}^n \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(1)) + R_m(n)$$

where

$$|R_m(n)| \leq \frac{4}{(2\pi)^{2m}} \int_1^m |f^{(2m)}(t)| dt.$$

## 2 Problems and Solutions

Calculate the sum

$$\sum_{j=1}^n j^2$$

using  $f(t) = t^2$ .

**Problem 3.** Let  $z \in \mathbb{C}$  and  $|z| < 1$ . Calculate the sum

$$(1 - |z|^2)^{2s} \sum_{n=0}^{\infty} n \binom{2s + n - 1}{n} |z|^{2n}.$$

**Problem 4.** Let  $x \in \mathbb{R}$  and  $r \in \mathbb{N}$  with  $r \geq 1$ . Find the sum

$$\sum_{k=0}^{r-1} \exp(-2\pi i k x / r).$$

**Problem 5.** Let  $a_1, a_2, \dots, a_n \in [0, 1]$ . Show that there exists a real number  $x$  in the unit interval such that the average of the (unsigned) distances from  $x$  to the  $a_j$ 's is  $1/2$ .

**Problem 6.** Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be a differentiable function. Show that there exist  $m$  differentiable functions  $g_1, g_2, \dots, g_m$  defined on  $\mathbb{R}^m$  with the properties

$$f(\mathbf{x}) = f(\mathbf{0}) + \sum_{j=1}^m x_j g_j(\mathbf{x})$$

and

$$g_j(\mathbf{0}) = \frac{\partial f}{\partial x_j}(\mathbf{0})$$

where  $\mathbf{x} = (x_1, x_2, \dots, x_m)$ .

**Problem 7.** Let  $n$  be a positive integer and  $f(j) = j(j-1)(j-2)$  with  $j = 1, 2, \dots, n$ . Let  $a_j := f(j+1) - f(j)$ . By calculating the sum  $\sum_{j=1}^n a_j$  show that

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}.$$

**Problem 8.** Show that

$$\left( \sum_{j=1}^N \frac{1}{u - v_j} \right)^2 - \sum_{j=1}^N \frac{1}{(u - v_j)^2} \equiv 2 \sum_{\substack{i < j \\ j=2 \\ i=1}}^N \frac{1}{v_i - v_j} \left( \frac{1}{u - v_i} - \frac{1}{u - v_j} \right).$$

This identity plays a role in the *Bethe ansatz*.

**Problem 9.** Show that the series

$$f(\theta) = \sum_{j=0}^{\infty} \frac{\sin(3^j \theta)}{2^j}$$

is convergent. Is the series  $df/d\theta$  convergent?

**Problem 10.** Find the radius of convergence of the power series

$$\sum_{j=0}^{\infty} \binom{j+k}{j} z^j, \quad k > 0.$$

**Problem 11.** Find the radius of convergence of the power series

$$\sum_{j=1}^{\infty} \frac{j!}{j^j} z^{mj}, \quad m = 1, 2, \dots$$

**Problem 12.** Let  $(s_0, s_1, \dots, s_{n-1})^T \in \mathbb{R}^n$ , where  $n = 2^k$ . This vector in  $\mathbb{R}^n$  can be associated with a piecewise constant function  $f$  defined on  $[0, 1)$

$$f(x) = \sum_{j=0}^{2^k-1} s_j \Theta_{[j2^{-k}, (j+1)2^{-k})}(x)$$

where  $\Theta_{[j2^{-k}, (j+1)2^{-k})}(x)$  is the step function

$$\Theta_{[j2^{-k}, (j+1)2^{-k})}(x) := \begin{cases} 1 & x \in [j2^{-k}, (j+1)2^{-k}) \\ 0 & x \notin [j2^{-k}, (j+1)2^{-k}) \end{cases}$$

with the support  $[j2^{-k}, (j+1)2^{-k})$ . Let  $x_{j+1} = 4x_j(1 - x_j)$  with  $j = 0, 1, 2, \dots$  and  $x_0 = 1/3$ . Then

$$x_0 = \frac{1}{3}, \quad x_1 = \frac{8}{9}, \quad x_2 = \frac{32}{81}, \quad x_3 = \frac{6272}{6561}.$$

4 Problems and Solutions

Find  $f(x)$  for this data set and then calculate

$$\int_0^1 f(x)dx.$$

**Problem 13.** Let  $a_1, a_2, \dots, a_n$  be a finite sequence of numbers. Its *Cesáro sum* is defined as

$$\frac{s_1 + s_2 + \dots + s_n}{n}$$

where

$$s_k = a_1 + a_2 + \dots + a_k$$

for each  $k$ ,  $1 \leq k \leq n$ . Suppose that the Cesáro sum of the 99-term sequence  $a_1, a_2, \dots, a_{99}$  is 100. Find the Cesáro sum of the 100-term sequence  $1, a_1, a_2, \dots, a_{99}$ .

**Problem 14.** Each  $x \in [0, 1]$  can be written as

$$x = \sum_{j=1}^{\infty} \frac{\epsilon_j}{2^j}$$

with  $\epsilon_j = 0$  or  $\epsilon_j = 1$ . Define the function  $f : [0, 1] \rightarrow [0, 1)$  as

$$f(x) = \sum_{j=1}^{\infty} \frac{2\epsilon_j}{3^j}.$$

The function  $f$  is known as *Cantor function*. Let  $x = 1/8$ . Find  $f(x)$ .

**Problem 15.** Let  $s \geq 2$ . Simplify the sum

$$\sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{1}{(j_1 + j_2)^s}.$$

**Problem 16.** Consider

$$\frac{1+x}{1-x-x^2} = \sum_{j=0}^{\infty} c_j x^j.$$

Find  $c_j$ .

**Problem 17.** The *Cantor series approximation* is defined as follows. For arbitrary chosen integers  $n_1, n_2, \dots$  (equal or larger than 2), we can approximate any real number  $r_0$  as follows

$$\begin{aligned}x_j &= \text{integer part}(r_j), \quad j = 0, 1, 2, \dots \\r_{j+1} &= (r_j - x_j)n_j\end{aligned}$$

and

$$r_0 \approx x_0 + \sum_{j=1}^N \frac{x_j}{n_1 n_2 \cdots n_j}.$$

The approximation error is

$$E_N = \frac{1}{n_1 n_2 \cdots n_N}.$$

Apply this approximation to  $r_0 = 2/3$  and the golden mean number with  $n_j = 2$  for all  $j$  and  $N = 4$ .

**Problem 18.** Calculate the sum

$$S = \sqrt{2 + \sqrt{2}} - \sqrt{2} \sqrt{2 - \sqrt{2}} - \sqrt{2 - \sqrt{2}}.$$

**Problem 19.** Let  $n_1, n_2, n_3 \in \mathbb{Z}$ . Calculate

$$1 - e^{i\pi(n_1 + n_2 + n_3)}.$$

**Problem 20.** Show that  $7 + 2\sqrt{10}$  has a square root the form  $\sqrt{x} + \sqrt{y}$ .

**Problem 21.** Calculate

$$\sum_{j=0}^{\infty} \frac{j^3 + 1}{j!} x^j.$$

Hint. Write  $j^3 + 1$  in the form

$$a + bj + cj(j-1) + dj(j-1)(j-2).$$

**Problem 22.** Let  $n \in \mathbb{N}$ . Find the sum

$$\sum_{k=-(n-1)/2}^{(n-1)/2} k^2$$

where  $k$  runs in steps of 1.

**Problem 23.** Assume that the series

$$1, 14, 51, 124, 245, 426, \dots$$

is of the form

$$ak^3 + bk^2 + ck + d.$$

Find the integer coefficients  $a, b, c, d$ .

**Problem 24.** Find the sum

$$S_n = \frac{2}{1 \cdot 2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \dots + \frac{2}{n \cdot (n+1) \cdot (n+2)}.$$

Calculate  $S_n$  for  $n \rightarrow \infty$ . Hint. Find  $a, b, c$  for

$$\frac{2}{n \cdot (n+1) \cdot (n+2)} = \frac{a}{n} + \frac{b}{n+1} + \frac{c}{n+2}.$$

**Problem 25.** Let  $a > 0$ . Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{z - na} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2 a^2}.$$

**Problem 26.** Let  $x \neq y, x \neq z, y \neq z$ . Find the sum

$$\frac{1}{(x-y)(x-z)} + \frac{1}{(y-x)(y-z)} + \frac{1}{(z-x)(z-y)}.$$

**Problem 27.** Given the expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \quad -1 < x \leq 1.$$

Calculate  $\ln(11)$ .

**Problem 28.** Let  $n \in \mathbb{N}_0$  and  $p, a > 0$ . Show that

$$\sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} \exp(-2\pi kp/a) \equiv \frac{1}{(1 - \exp(-2\pi p/a))^{n+1}}.$$

Hint: Start with

$$f(n) := \sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} \exp(-2\pi kp/a)$$

and find  $f(n+1)$ .

**Problem 29.** Let  $(s_1, s_2) \in \mathbb{Z}^2$ . Let  $P(\phi_1, \phi_2)$  be a probability density and let  $\theta_1, \theta_2 \in \mathbb{R}$ . We can express the characteristic double sequence as

$$\chi(s_1, s_2) = \int_{\theta_1}^{\theta_1+2\pi} \int_{\theta_2}^{\theta_2+2\pi} \exp(i(s_1\phi_1 + s_2\phi_2))P(\phi_1, \phi_2)d\phi_1d\phi_2.$$

Find  $P(\phi_1, \phi_2)$  as a function of  $\chi(s_1, s_2)$ . Note that  $\chi(0, 0) = 1$ .

**Problem 30.** Show that

$$\frac{\pi}{4} = 4 \arctan(1/5) - \arctan(1/239).$$

Hint. Let  $\theta = \arctan(1/5)$ . Thus  $\tan(\theta) = 1/5$ . Applying the double angle formula for the tangent we have  $\tan(2\theta) = 5/12$  and  $\tan(4\theta) = 120/119$ . Finally apply that  $\tan(\pi/4) = 1$ .

**Problem 31.** Let  $n \in \mathbb{N}_0$ . Consider an infinite number of time variable  $\mathbf{t} = (t_1, t_2, t_3, \dots)$ . Consider the sum

$$p_n(x + t_1, t_2, t_3, \dots) = \sum_{\substack{k_0+k_1+2k_2+3k_3+\dots=n \\ k_0, k_1, k_2, k_3, \dots \geq 0}} \frac{x^{k_0} t_1^{k_1} t_2^{k_2} \dots}{k_0! k_1! k_2! \dots}.$$

Find  $p_0, p_1, p_2, p_3$ .

**Problem 32.** Let  $n$  be an integer with  $n \geq 1$ . Simplify the series

$$S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}.$$

**Problem 33.** Let  $N$  be a positive integer and  $a, b$  be positive integers or positive half-integers. Find

$$f(N, a, b) = \frac{1}{2}(1 + (-1)^{2a} + (-1)^{2b} + (-1)^{2a+2b+N}).$$

**Problem 34.** Let  $-1 < x < +1$  and  $n \geq 2$ .

(i) Show that

$$S_n(x) = \sum_{j=0}^{n-1} jx^j = \frac{nx^n}{x-1} + \frac{x(1-x^n)}{(1-x)^2}.$$

(ii) Show that

$$\lim_{n \rightarrow \infty} S_n(x) = \frac{x}{(1-x)^2}.$$

(iii) Let  $x = 1/2$ . Show that  $\lim_{n \rightarrow \infty} S_n(1/2) = 2$ .

**Problem 35.** Let  $x, y, z \in \mathbb{R}$  and  $x \neq y$ ,  $x \neq z$ ,  $y \neq z$ . Show that

$$\frac{x}{(z-x)(x-y)} + \frac{y}{(x-y)(y-z)} + \frac{z}{(y-z)(z-x)} = 0.$$

**Problem 36.** Let

$$x \neq 0, \quad x \neq \pm 2\pi, \quad x \neq \pm 4\pi, \dots$$

and  $n \in \mathbb{N}$ . Show that

$$\sin(x) + \sin(2x) + \dots + \sin(nx) = \frac{\cos(x/2) - \cos((n+1/2)x)}{2\sin(x/2)}.$$

For the values  $x = 0$ ,  $x = \pm 2\pi$ ,  $x = \pm 4\pi$  etc the sum is given by 0.

**Problem 37.** Let  $n = 0, 1, 2, \dots$ . The function

$$K_n(\theta) := \frac{1}{n+1} \frac{\sin^2\left(\left(\frac{n+1}{2}\right)\theta\right)}{\sin^2\left(\frac{1}{2}\theta\right)}$$

is called the *Fejér kernel*.

(i) Show that

$$K_n(\theta) := \sum_{j=-n}^{j=+n} \left(1 - \frac{|j|}{n+1}\right) e^{ij\theta}.$$

(ii) Show that  $K_n(\theta) \geq 0$ .

(iii) Show that for any continuous  $2\pi$  periodic function  $f$  on  $\mathbb{R}$  one has

$$\begin{aligned} K_n \star f(\theta) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta - \alpha) f(\alpha) d\alpha \\ &= \sum_{j=-n}^{j=+n} \left(1 - \frac{|j|}{n+1}\right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\alpha} f(\alpha) d\alpha\right) e^{ij\theta}. \end{aligned}$$



(iv) Show that  $K_n \star f(\theta) \rightarrow f(\theta)$  uniformly in  $\theta$  as  $n \rightarrow \infty$ .

**Problem 38.** Let  $\lambda > 0$ .

(i) Calculate

$$S_1(\lambda) = \sum_{j=0}^{\infty} \frac{j e^{-\lambda} \lambda^j}{j!}.$$

(ii) Calculate

$$S_2(\lambda) = \sum_{j=0}^{\infty} \frac{j(j-1)e^{-\lambda} \lambda^j}{j!}.$$

**Problem 39.** (i) Let  $\alpha > 0$ . Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k^2 + \alpha^2} = \frac{\pi}{2\alpha^2} \cdot \frac{\cosh(\alpha x)}{\sinh(\alpha x)} - \frac{1}{2\alpha^2}$$

where  $-\pi \leq x \leq \pi$ .

(ii) Let  $\alpha > 0$ . Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sin(kx)}{k^2 + \alpha^2} = -\frac{\pi}{2\alpha^2} \cdot \frac{\sinh(\alpha x)}{\sinh(\alpha \pi)}$$

for  $-\pi < x < \pi$ .

**Problem 40.** Let  $a > 0$  and  $b > 0$ . Show that the continued fraction

$$a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \frac{1}{a + \dots}}}}$$

can be written as

$$\frac{a}{2} + \sqrt{\frac{a^2}{4} + \frac{a}{b}}.$$

**Problem 41.** Let  $M, N \geq 1$ . Find the sum

$$\sum_{m=1}^M \sum_{n=1}^N \frac{(m+n)!}{(2m+n)!(m+2n)!}.$$

**Problem 42.** Study the sum

$$\sum_{\ell, m, n=1}^{\infty} \frac{\ell}{(\ell m + \ell n + mn + 1)^2} x^{\ell+m+n}.$$

**Problem 43.** Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \frac{3}{4}n + \frac{1}{8}} = 4\pi.$$

**Problem 44.** Let  $n \geq 1$ . Find the sum

$$\sum_{j=1}^n \frac{3}{j(j+3)}.$$

What happens for  $n \rightarrow \infty$ ?

**Problem 45.** Let  $F_1 = 1$ ,  $F_2 = 2$ ,  $F_{k+1} = F_{k-1} + F_k$  be the Fibonacci numbers. Calculate

$$\sum_{k=1}^{\infty} \frac{\log_2(F_{k+1})}{2^{k+1}}.$$

**Problem 46.** Let  $\mathbb{Z}$  be the set of integers. Simplify the sum

$$\sum_{k \in \mathbb{Z}} \frac{\sin(\pi(x-k)) \sin(\pi(y-k))}{\pi(x-k)\pi(y-k)}.$$

**Problem 47.** Let  $n \geq 1$ . One has

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k. \quad (1)$$

Show that

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n} = n2^{n-1}. \quad (2)$$

**Problem 48.** Let  $n > 2$  and  $A, h > 0$ . Find  $V_n$  given by

$$V_n = \frac{Ah}{n^3} \sum_{j=1}^{n-1} j^2.$$

Then find  $\lim_{n \rightarrow \infty} V_n$ . This concerns the volume of a *pyramide*,  $h$  is the height and  $A$  is the area of the base.

**Problem 49.** Let  $n \geq 1$  and  $c_1, \dots, c_n$  be constants. Find the minima of the function  $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \sum_{j=1}^n (x - c_j)^2.$$

**Problem 50.** Let  $x \in \mathbb{R}$ . The sequence of functions  $\{f_k(x)\}$  is defined by  $f_1(x) = \cos(x/2)$  and for  $k > 1$  by

$$f_k(x) = f_{k-1}(x) \cos(x/2^k).$$

Thus

$$f_k(x) = \cos(x/2) \cos(x/2^2) \cdots \cos(x/2^k).$$

Obviously, we have  $f_k(0) = 1$  for every  $k$ . Calculate  $\lim_{k \rightarrow \infty} f_k(x)$  as a function of  $x$  for  $x \neq 0$ .

### Programming Problems

**Problem 1.** Let  $m, n \geq 2$ . Calculate the finite sum

$$S_{n,m} = \sum_{j=1}^m \sum_{k=1}^n |j - k|.$$

Give a C++ implementation.

**Problem 2.** Let  $n \in \mathbb{N}$ . Calculate the finite sum

$$S_n = \sum_{j=-n}^n \sum_{k=-n}^n |j + k|.$$

**Problem 3.** Let  $m \geq 1$  and  $n \geq 1$ . Find the sum

$$\sum_{k=0}^{\min(m,n)} (n + m - 2k + 1).$$

## 1.2 Supplementary Problems

**Problem 1.** Show that

$$1 + 2 \tanh^2(\lambda) + 3 \tanh^4(\lambda) + \cdots + (n+1) \tanh^{2n}(\lambda) + \cdots \equiv \cosh^4(\lambda).$$

Note that  $\tanh(0) = 0$  and  $\cosh(0) = 1$ .

**Problem 2.** Show that

$$\sum_{j=2}^{\infty} \frac{1}{j(2j-1)} = 2 \ln(2) - 1.$$

**Problem 3.** (i) Show that

$$\frac{1}{4} = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots$$

(ii) Let  $n \geq 1$ . Show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

**Problem 4.** Let  $n, m \in \mathbb{N}$  and  $a, b, c, d \in \mathbb{R}$ . Show that

$$(a+b)^n (c+d)^m = \sum_{r=0}^n \sum_{s=0}^m \binom{n}{r} \binom{m}{s} a^r b^{n-r} c^s d^{m-s}.$$

**Problem 5.** Let  $\ell \geq 1$ . Study the *Gauss sum*

$$G(k, \ell) = \frac{1}{\sqrt{\ell}} \sum_{r=0}^{\ell-1} e^{2\pi i k r^2 / \ell}.$$

**Problem 6.** Show that if  $|x| < 1$  we have the expansion

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Let  $a > 0$ . Use this expansion to show that

$$\frac{1}{(a+x)^2} = \frac{1}{a^2} \left( 1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \cdots \right).$$

**Problem 7.** Apply mathematical induction to show that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n - 1) &= n^2 \\ 1^2 + 4^2 + 7^2 + \cdots + (3n - 2)^2 &= \frac{1}{2}n(6n^2 - 3n - 1) \\ 1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 &= n^2(2n^2 - 1). \end{aligned}$$

**Problem 8.** Show that

$$\ln(2) = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots$$

**Problem 9.** Let  $x > 0$ . Show that a complicated way to calculate  $1/x$  is given by

$$\frac{1}{x+1} + \frac{1!}{(x+1)(x+2)} + \frac{2!}{(x+1)(x+2)(x+3)} + \cdots$$

**Problem 10.** Let  $y \in \mathbb{R}$  and  $[y]$  be the integer part of  $y$ . Let  $\omega = (1 + \sqrt{5})/2$  be the golden mean number (which is an irrational number). We define

$$x_j := j + ((j+1)/\omega)(\lambda - 1), \quad j = 0, 1, \dots$$

where  $\lambda > 0$  and

$$\mu_k = x_k - x_{k-1}, \quad k = 1, 2, \dots$$

Show that the sequence  $\mu_1, \mu_2, \dots$  is non-periodic.

**Problem 11.** Let  $s > 0$  and

$$f(s) = \int_{t=0}^{\infty} \frac{\ln(1+st)}{1+t^2} dt.$$

Show that

$$f(s) = (1 - \ln(s))s + \frac{\pi s^2}{4} + \left( \frac{\ln(s)}{3} - \frac{1}{9} \right) s^3 + O(s^4).$$

**Problem 12.** Let  $n$  be an integer and  $n > 1$ . Show that

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

cannot be integer for all  $n > 1$ .

**Problem 13.** Let  $x \in \mathbb{R}$ . Show that the series

$$\frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \cdots$$

converges absolutely for all values of  $x$ , except for  $+1$  and  $-1$ .

**Problem 14.** Show that the series

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \quad \sum_{j=0}^{\infty} jz^j = \frac{z}{(1-z)^2}, \quad \sum_{j=0}^{\infty} j^2 z^j = \frac{z(1+z)}{(1-z)^3}$$

converge for all  $|z| < 1$ .

**Problem 15.** Let  $n$  be an integer with  $n \geq 2$ . Find

$$\sum_{1 \leq i < j \leq n} \left| \frac{1}{2^i} - \frac{1}{3^j} \right|.$$

**Problem 16.** Let  $\ell \geq 1$ . Show that

$$\sum_{j=1}^{\ell} \frac{1}{4j^2 - 1} = \frac{\ell}{2\ell + 1}$$

and

$$\sum_{j=2}^{\ell} \frac{1}{j(j^2 - 1)} = \frac{1}{4} \left( 1 - \frac{2}{\ell(\ell + 1)} \right).$$

**Problem 17.** Let  $0 \leq r < 1$ . Calculate the sum

$$\sum_{j=1}^{\infty} r^j \cos(j\phi)$$

utilizing the identity

$$\cos(j\phi) \equiv \frac{1}{2}(e^{ij\phi} + e^{-ij\phi}).$$

**Problem 18.** Calculate the sum

$$\sum_{j=1}^{\infty} \frac{1}{j^2 + 1}$$

with the knowledge that

$$\sum_{j=1}^{\infty} \frac{1}{j^2 + 1} \equiv \sum_{j=1}^{\infty} \frac{1}{j^2} - \sum_{j=1}^{\infty} \frac{1}{j^2(j^2 + 1)}, \quad \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

**Problem 19.** Show that

$$\begin{aligned} \cos(\pi/4) &= \frac{1}{2}\sqrt{2}, \quad \cos(\pi/8) = \frac{1}{2}\sqrt{2 + \sqrt{2}}, \quad \cos(\pi/16) = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}} \\ \sin(\pi/4) &= \frac{1}{2}\sqrt{2}, \quad \sin(\pi/8) = \frac{1}{2}\sqrt{2 - \sqrt{2}}, \quad \sin(\pi/16) = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2}}}. \end{aligned}$$

**Problem 20.** Let  $n \geq 0$  and  $x \neq 0$ . Find

$$S_n(x) = \sum_{k=0}^n kx^k.$$

Utilize that

$$\begin{aligned} S_{n+1}(x) &= S_n(x) + (n+1)x^{n+1} = \sum_{k=0}^n (k+1)x^{k+1} = \sum_{k=0}^n kx^{k+1} + \sum_{k=0}^n x^{k+1} \\ &= xS_n(x) + \sum_{k=0}^n x^{k+1} \end{aligned}$$

and a case study with  $x = 1$  and  $x \neq 1$ .

**Problem 21.** The *Fibonacci numbers* are given by  $x_{t+2} = x_{t+1} + x_t$  with  $t = 0, 1, \dots$  and  $x_0 = x_1 = 1$ . Consider

$$F = \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \dots \equiv \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \dots.$$

Show that the series converges. Is  $F < 4$ ?

**Problem 22.** Calculate the sum

$$S = \sum_{j=1}^{\infty} \frac{2n+1}{n^2(n+1)}.$$

**Problem 23.** Show that the *Cantor series*

$$c_1 + \frac{c_2}{2!} + \frac{c_3}{3!} + \frac{c_4}{4!} + \dots$$

with  $0 \leq c_n \leq n - 1$  is convergent.

**Problem 24.** Let  $N \geq 1$  be an integer and  $k = -4N, -4N + 1, \dots, 4N$ . We define

$$f(k) = \frac{1}{N} \sum_{j=1}^N \cos \left( \left( j - \frac{1}{2} \right) \frac{k\pi}{N} \right).$$

Show that  $f(0) = 1$ ,  $f(\pm 2N) = -1$ ,  $f(\pm 4N) = 1$  and 0 otherwise.

**Problem 25.** Show that

$$\sum_{k \in \mathbb{Z}} \frac{1}{(x - k)} = \pi \cot(\pi x).$$

Note that  $\cot(0) = \infty$  and  $\cot(\pi/2) = 0$ .

**Problem 26.** Let  $c > 0$ . Apply the *Poisson summation formula*

$$\sum_{n=-\infty}^{+\infty} f(n + a) = \sum_{k=-\infty}^{+\infty} \exp(2\pi i k a) \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i k x) dx$$

for the function

$$f(x) := \begin{cases} x \exp(-cx^2) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

**Problem 27.** Let  $k, m = 1, 2, \dots$ . Write a C++ program that implements the sum

$$E(k, m) = \frac{1}{2^{k+m-1}} \sum_{i=0}^{m-1} \binom{k+m-1}{i}$$

for a given  $k$  and  $m$ . The sum plays a role for the arithmetic triangle.

**Problem 28.** Let  $L \in \mathbb{N}_0$  and  $\kappa$  be a nonnegative real number.

(i) Show that

$$Z(\kappa) = \left( \sum_{n=0}^L \binom{L}{n} e^{-\kappa L n} \right)^2 = (1 + e^{-\kappa L})^{2L}.$$

(ii) Show that for  $\kappa > 0$  one has for large  $L$  that  $Z \approx 1 + 2Le^{-\kappa L}$ . Show that at  $\kappa = 0$  one has  $Z(\kappa = 0) = 2^{2L}$ .



**Problem 29.** Let  $n > 2$ . Show that

$$\sum_{j=2}^n \frac{1}{j} \approx \ln(n).$$

**Problem 30.** Let  $n \geq 1$  and  $\omega$  be the primitive  $n$ th root of 1. Show that

$$\sum_{j=0}^{n-1} \omega^j = 0.$$

**Problem 31.** Show that

$$e^x + e^{-x} + 2 \cos(x) = 4 \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}.$$

**Problem 32.** Show that

$$\left( \sum_{j=0}^{\infty} \frac{1}{j!} \right) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right) = 1$$

**Problem 33.** Let  $n \in \mathbb{N}$ . Applying the principle of mathematical induction show that

$$A(n) = \sum_{k=0}^n k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

Note that  $A(0) = 0$ .

**Problem 34.** Let  $N \geq 2$ . Find the sum

$$\sum_{\substack{j_1, j_2, j_3=1 \\ j_3 \geq j_2 \geq j_1}}^N (j_1 j_2 j_3 + (j_1 + j_2 + j_3)).$$

**Problem 35.** Let  $h, r > 0$  and fixed and  $m, n \in \mathbb{N}$ . Consider

$$\begin{aligned} S_{m,n} &= mnr \sin(\pi/n) \sqrt{\left(\frac{h}{m}\right)^2 + (r - r \cos(\pi/n))^2} \\ &= \pi r \frac{\sin(\pi/n)}{\pi/n} \sqrt{h^2 + \frac{1}{4} \pi^4 r^2 \frac{m^2}{n^4} \left(\frac{\sin(\pi/n)}{\pi/n}\right)^4}. \end{aligned}$$

- (i) Consider  $\lim_{m,n \rightarrow \infty} S_{m,n}$  with  $m = n^2$ .  
(ii) Consider  $\lim_{m,n \rightarrow \infty} S_{m,n}$  with  $m = n^3$ .  
(iii) Consider  $\lim_{m,n \rightarrow \infty} S_{m,n}$  with  $\lim_{m \rightarrow \infty, n \rightarrow \infty} (m^2/n^4) = 0$  and show that  $\lim_{m,n \rightarrow \infty} S_{m,n} = \pi r h$  which is a surface area of a half cylinder with radius  $r$  and length  $h$ .

**Problem 36.** (i) Let  $a > 0$ . Show that

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2 + a^2} = \frac{\pi}{2a} \frac{\cosh(a(\pi - x))}{\sinh(a\pi)} - \frac{1}{2a}.$$

(ii) Let  $a > 0$ . Show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2 + a^2} = \frac{\pi}{2a} \tanh\left(\frac{a\pi}{2}\right).$$

**Problem 37.** Let  $k \geq 2$ . Consider

$$f(k) = k! \left( \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^k}{k!} \right) \equiv k! \sum_{j=2}^k \frac{(-1)^j}{j!}.$$

Thus  $f(2) = 1$ . Find  $f(3)$ ,  $f(4)$ ,  $f(5)$ ,  $f(6)$ , ... . Which of these numbers is a prime number?

**Problem 38.** Let  $M$  be an integer with  $M \geq 2$  and  $\Im(\phi) < 0$ . Show that

$$S(k) = \sum_{m=1}^{M-1} \frac{e^{(2\pi i m k)/M}}{\sin^2(\pi(m + \phi)/M)} = -\frac{4kM e^{-2\pi i k \phi/M}}{1 - e^{-2\pi i \phi}} + \frac{M^2 e^{-2\pi i k \phi/M}}{\sin^2(\pi \phi)}.$$

Note that

$$\frac{1}{\sin^2(\pi(m + \phi)/M)} = -4 \sum_{p=0}^{\infty} p e^{-2\pi i(m + \phi)p/M}.$$

**Problem 39.** The  $q$ -exponential function is defined by

$$e_q^x := \sum_{j=0}^{\infty} \frac{x^j}{[j]!}$$

where

$$[j] := \frac{q^j - q^{-j}}{q - q^{-1}}.$$

Find  $e_q^x$  for  $x = 1$  and  $q = 1/2$ .

**Problem 40.** Let  $a, b, c \in \mathbb{R}$ . Factorize

$$(b - c)^3 + (c - a)^3 + (a - b)^3.$$

The following Maxima may be helpful

```
T: (b-c)^3 + (c-a)^3 + (a-b)^3;
T: expand(T);
T: ratsimp(T);
```

**Problem 41.** Let  $n \geq 0$ . Starting from

$$\sum_{j=0}^n c_j x^j = (1 + x)^n$$

show that

$$\begin{aligned} c_0 + c_1 + \cdots + c_n &= 2^n \\ c_0 - c_1 + c_2 - c_3 + \cdots + (-1)^n c_n &= 0 \\ c_1 + 2c_2 + 3c_3 + \cdots + nc_n &= n2^{n-1} \\ c_0^2 + c_1^2 + \cdots + c_n^2 &= \frac{(2n)!}{(n!)^2}. \end{aligned}$$

**Problem 42.** Let  $n \geq 1$ . Show that

$$\sum_{j=1}^n \ln(1 + 1/j) = \ln(n + 1).$$

**Problem 43.** Let  $x \in (-1, 1)$ . Show that

$$\sum_{j=0}^{\infty} jx^j = \frac{x}{(1-x)^2}.$$

**Problem 44.** (i) Assuming that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln(2).$$

Show that

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} \ln(2).$$

Note that the convergence of the first series is not absolute.

(ii) Show that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} - \frac{1}{2} \ln(n)$$

tends to a finite value as  $n \rightarrow \infty$ . Find this value.

**Problem 45.** Let  $\beta > 0$ . Show that

$$\sum_{n=0}^{\infty} e^{-\beta n} = \frac{1}{1 - e^{-\beta}}.$$

**Problem 46.** Let  $n$  be a positive integer and  $f(\theta_1, \dots, \theta_N)$  be a periodic function, i.e. periodic  $2\pi$  for each  $\theta_j$  ( $j = 1, \dots, N$ ). Show for large  $n$  we have

$$\begin{aligned} & \left( \prod_{k=1}^N \int_0^{2\pi} d\theta_k \right) \left( \sum_{j=1}^N \cos(\theta_j) \right)^n f(\theta_1, \dots, \theta_N) \\ & \approx N^n \left( \prod_{k=1}^N \int_{-\infty}^{\infty} d\theta_k e^{-n(\theta_k)^2/(2N)} \right) f(\theta_1, \dots, \theta_N). \end{aligned}$$

**Problem 47.** Let  $A, B$  be finite sets and  $n(A), n(B)$  the numbers of elements in  $A$  and  $B$ , respectively. Is

$$n(A) + n(B) = n(A \cup B) + n(A \cap B)?$$

Prove or disprove.

**Problem 48.** Let  $f : [0, 1] \rightarrow [0, 1]$  be a differentiable function, for example the logistic map  $f(x) = 4x(1-x)$ . Let  $f^{(k)}$  be the  $k$ -th iterate of the function  $f$ . Let

$$b_k := \sup_{0 \leq x \leq 1} \left| \frac{d}{dx} f^{(k)}(x) \right|.$$

Show that

$$\lim_{k \rightarrow \infty} (b_k)^{1/k}$$

exist. Apply the chain rule.

**Problem 49.** Let  $n \geq 2$ . Show that

$$\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}.$$

**Problem 50.** Is the series

$$S = \sum_{k=1}^{\infty} \frac{k}{1+2k^3}$$

convergent?

**Problem 51.** Give a non-trivial infinite sequence  $(x_0, x_1, x_2, \dots)$  such that

$$\sum_{j=0}^{\infty} \frac{|x_j|}{1+|x_j|}$$

is finite.

**Problem 52.** Let  $a > 0$ . Show that

$$\sum_{j=-\infty}^{\infty} (-1)^j \frac{1}{j^2 + a^2} = \frac{\pi}{a \sinh(\pi a)}.$$

**Problem 53.** Let  $n = 0, 1, 2, \dots$ . The *Fermat numbers* are given by

$$F_n = 2^{(2^n)} + 1.$$

Show that the Fermat numbers satisfy the recurrence relation

$$F_{n+1} = (F_n - 1)^2 + 1$$

with  $F_0 = 3$

**Problem 54.** Let  $N \geq 1$ . Consider the Hilbert space  $L_2([-1, 1])$ . The Chebyhev-Gauss-Lobatto points are given by

$$x_j = \cos(j\pi/N), \quad j = 0, 1, \dots, N.$$

Find the four points for  $N = 3$ .

**Problem 55.** Let  $|z| < 1$ . Show that

$$\sum_{j=1}^{\infty} jz^j = (1-z)^{-1} \sum_{j=1}^{\infty} z^j.$$

Show that

$$\sum_{j=1}^{\infty} j^2 z^j = (1+z)(1-z)^{-2} \sum_{j=1}^{\infty} z^j$$

**Problem 56.** Show that

$$\frac{1}{3} + \frac{2}{9} + \cdots + \frac{2^n}{3^{n+1}} + \cdots = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \frac{1}{1-2/3} = 1.$$

**Problem 57.** (i) Let  $a > 0$ . Show that

$$\sum_{n=0}^{\infty} \frac{1}{(a+n)(a+n+1)} = \frac{1}{a}.$$

(ii) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

(iii) Let  $n \geq 2$ . Show that

$$\sum_{j=0}^{n-1} \frac{1}{(j+1)(j+2)} = 1 - \frac{1}{n+1}$$

and thus show that

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)(j+2)} = 1.$$

**Problem 58.** Let  $n$  be a positive integer. Show that

$$\frac{\sin(2^n \alpha)}{2^n \sin(\alpha)} = \cos(\alpha) \cos(2\alpha) \cos(2^2 \alpha) \cdots \cos(2^{n-1} \alpha).$$

**Problem 59.** Let  $\Gamma(x)$  be the *gamma function*. Show that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(\alpha + 2k)\Gamma(\alpha + k)}{\Gamma(\alpha + k + n + 1)} = \delta_{n,0}.$$

**Problem 60.** Let  $a > 0$  and  $a \notin \mathbb{N}$ . Show that

$$\cos(ax) = \frac{2a \sin(ax)}{\pi} \left( \frac{1}{2a^2} - \frac{\cos(x)}{a^2 - 1^2} + \frac{\cos(2x)}{a^2 - 2^2} - \frac{\cos(3x)}{a^2 - 3^2} + \cdots \right)$$

for all  $x \in [-\pi, \pi]$ .

**Problem 61.** (i) Let  $n \in \mathbb{N}$ . Show that

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1).$$

(ii) Let  $n \geq 1$ . Show that

$$\sum_{k=0}^n k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

**Problem 62.** Let  $N \geq 1$ . Find the sums

$$v_k = \sum_{j=0}^{N-1} \left( \exp\left(\frac{2\pi}{N}jk\right) - 1 \right), \quad k = 0, 1, \dots, N-1.$$

**Problem 63.** (i) Let  $\ell$  be a positive integer and  $\phi \in \mathbb{R}$ . Show that

$$\sum_{m=-\ell}^{+\ell} e^{im\phi} = \frac{\sin((\ell+1/2)\phi)}{\sin(\phi/2)}.$$

(ii) The function

$$D_n(\theta) := \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$$

is called the *Dirichlet kernel*. Show that

$$D_n(\theta) = \sum_{k=-n}^{k=+n} e^{ik\theta}.$$

**Problem 64.** Let  $k$  be a positive integer and  $k \geq 2$ . Show that

$$1 + x - 2x^k \equiv (1-x) + 2x(1-x^{k-1}) \equiv (1-x) \left( 1 + 2 \sum_{j=1}^{k-1} x^j \right).$$

**Problem 65.** Let  $n \geq 2$ . Let  $S_n$  be the standard  $n$ -simplex embedded in  $\mathbb{R}^n$

$$S_n := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n x_j = 1, \text{ for } x_k \geq 0, \text{ for } k = 1, \dots, n \right\}.$$

We denote by  $S_{k,n-1}$  the  $k$ th face of the boundary of  $S_n$ . They are  $(n-1)$ -simplexes

$$S_{k,n-1} := \{ \mathbf{x} \in \mathbb{R}^n : x_k = 0, \mathbf{x} \in S_n \} \quad \text{for } k = 1, \dots, n.$$

Show that the boundary  $\partial S_n$  of  $S_n$  is the union

$$\partial S_n = \cup_{j=1}^n S_{j,n-1}$$

of the faces.

**Problem 66.** Show that

$$\prod_{j=1}^{\infty} \cos\left(\frac{x}{2^j}\right) = \frac{\sin(x)}{x}$$

for  $x \in \mathbb{R}$ .

**Problem 67.** The triple sum

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} ' \frac{(-1)^{k_1+k_2+k_3}}{\sqrt{k_1^2 + k_2^2 + k_3^2}}$$

plays a role in solid state physics (Madelung constant). Here ' indicates that the term  $(k_1, k_2, k_3) = (0, 0, 0)$  is omitted. Given a positive integer  $N$ . Write a C++ program that implements the sum

$$\sum_{k_1=-N}^N \sum_{k_2=-N}^N \sum_{k_3=-N}^N ' \frac{(-1)^{k_1+k_2+k_3}}{\sqrt{k_1^2 + k_2^2 + k_3^2}}.$$

Run the program for different  $N$  and compare with the exact value.

**Problem 68.** Find the sum

$$S_L = \sum_{n=1}^L \exp\left(i\pi \frac{2}{L} n^2\right)$$

for  $L = 1, 2, 3$ .



**Problem 69.** Show that the series

$$\sum_{j=1}^{\infty} \frac{\sin(jx)}{\sqrt{j}}$$

converges for all  $x \in \mathbb{R}$ .

**Problem 70.** Let  $k \geq 0$  and  $s > 0$ . Show that

$$\prod_{\ell=0}^k \frac{s + \ell + 1}{s + \ell} = \frac{s + k + 1}{s}.$$

**Problem 71.** Let  $n$  be a positive integer. Give a C++ implementation of the sum

$$\sum_{k=0}^n \sum_{\ell=0}^n \binom{k}{\ell}$$

using templates so that the `Verylong` class of `SymbolicC++` can be used.

**Problem 72.** Let  $n, m \geq 1$ . Implement the sum

$$S_{n,m} := \frac{(2m)!}{2^{2m} m!} \sum_{k=0}^m 2^k \binom{n}{k} \binom{m}{k}$$

using the `Verylong` class and `Rational` class of `SymbolicC++`.

**Problem 73.** Let  $n \in \mathbb{N}$  and  $m_1, m_2, \dots, m_n \in \mathbb{N}_0$ . Consider the function

$$f(m_1, m_2, \dots, m_n) := \begin{cases} \delta_{m_1,0} & \text{for } n = 1 \\ \delta_{m_2,0} \delta_{m_1,1} & \text{for } n = 2 \\ \delta_{m_n,0} \delta_{(m_1+\dots+m_n),n-1} \prod_{j=1}^{n-2} H(j - \sum_{k=1}^j m_{n-k}) & \text{for } n \geq 3 \end{cases}$$

where  $H$  is the step function

$$H(x) := \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Give an implementation of this function in `SymbolicC++` for a given  $n$  using the `Verylong` class.

# Chapter 2

## Maps

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### 2.1 Solved Problems

**Problem 1.** Newton's sequence takes the form of a difference equation

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}$$

where  $t = 0, 1, 2, \dots$  and  $x_0$  is the initial value at  $t = 0$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = x^2 - 1$$

and  $x_0 \neq 0$ . Find the fixed points of  $f$ . Find the fixed points of the difference equation. Let  $x_0 = 1/2$ . Find  $x_1, x_2, x_3$ . Find Newton's sequence for this function. Obtain the exact solution of the difference equation.

**Problem 2.** (i) Solve the nonlinear recurrence relation

$$x_{n+1} = x_n^2, \quad n = 0, 1, \dots$$

where  $x_0 = 2$ .

(ii) Solve the linear recurrence relation

$$x_{n+1} = x_n + x_{n-1} + x_{n-2}, \quad n = 2, 3, \dots$$

and the initial values  $x_0 = x_1 = x_2 = 1$ .

(iii) Solve the linear recurrence relation

$$x_{n+1} = 1 + \sum_{j=0}^{n-1} x_j, \quad x_0 = 1.$$

**Problem 3.** Let  $x > 0$  and  $p > 0$ . Consider the map

$$f(x) = xe^{p-x}.$$

- (i) Find the fixed points. Study the stability of the fixed points.  
 (ii) Show that  $f$  has a least one periodic point  $x^*$  with  $x^* \neq 0$  or  $p$ .

**Problem 4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  be a positive, continuously differentiable function, defined for all real numbers and whose derivative is always negative. Show that for any real number  $x_0$  (initial value) the sequence  $(x_k)$  obtained by *Newton's method*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

has always limit  $\infty$ .

**Problem 5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable map. Let  $f^{(n)}$  be the  $n$ -th iterate of  $f$ . Calculate the derivative of  $f^{(n)}$  at  $x_0$ .

**Problem 6.** (i) Solve the second order linear difference equation

$$x_{t+2} = x_{t+1} + x_t \quad t = 0, 1, 2, \dots$$

where  $x_0 = 0$  and  $x_1 = 1$ .

- (ii) Give the definition of the golden mean number and derive this number.  
 (iii) Calculate

$$\lim_{t \rightarrow \infty} \frac{x_{t+1}}{x_t}.$$

**Problem 7.** The recursion relation

$$F_{n+2} = F_n + F_{n+1}$$

with the initial values  $F_0 = F_1 = 1$  provides the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... A generalization of the Fibonacci sequence is the  $q$ -analogue of the sequence defined by the recursion relation

$$F_{n+2}(q) = F_n(q) + qF_{n+1}(q)$$

and the initial condition  $F_0(q) = 1$  and  $F_1(q) = q$ . Here,  $q$  is a real or complex number.

- (i) Give the first five terms of the sequence.
- (ii) Find a generating function of  $F_n(q)$ .
- (iii) Find an explicit expression for  $F_n(q)$ .

**Problem 8.** (i) Find the linear map  $f : \{0, 1\} \rightarrow \{-1, 1\}$  such that

$$f(0) = -1, \quad f(1) = 1. \quad (1)$$

(ii) Find a linear map  $g : \{-1, 1\} \rightarrow \{0, 1\}$  such that

$$g(-1) = 0, \quad g(1) = 1. \quad (2)$$

This is obviously the inverse map of  $f$ .

**Problem 9.** Consider the differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{1 - \cos(2\pi x)}{x}$$

where using L'Hospital  $f(0) = 0$ .

- (i) Find the zeros of  $f$ .
- (ii) Find the maxima and minima of  $f$ .

**Problem 10.** Given two manifolds  $M$  and  $N$ , a bijective map  $\phi$  from  $M$  to  $N$  is called a *diffeomorphism* if both  $\phi : M \rightarrow N$  and its inverse  $\phi^{-1}$  are differentiable. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by the analytic function

$$f(x) = 4x(1 - x)$$

and the analytic function  $g : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$g(x) = 1 - 2x^2.$$

(i) Can one find a diffeomorphism  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$g = \phi \circ f \circ \phi^{-1}?$$

(ii) Consider the diffeomorphism  $\psi : \mathbb{R} \rightarrow \mathbb{R}$

$$\psi(x) = \sinh(x).$$

Calculate  $\psi \circ g \circ \psi^{-1}$ .

**Problem 11.** Consider the polynomial  $p(x) = x^3 - 3x + 3$ . Show that for any positive integer  $N$ , there is an initial value  $x_0$  such that the sequence  $x_0, x_1, x_2, \dots$  obtained from *Newton's method*

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = \frac{2x_n^3 - 3}{3(x_n^2 - 1)}, \quad n = 0, 1, 2, \dots$$

has period  $N$ .

**Problem 12.** Consider the linear recursion equation for  $F(n)$  with  $n = 0, 1, 2, \dots$

$$(n + 1)F(n + 1) - nF(n - 1) + F(n) = 0$$

where  $F(0) = 0, F(1) = 1$ . We define

$$f(z) := \sum_{n=1}^{\infty} F(n)z^n$$

where  $z$  is an undeterminate. Find the differential equation for  $f$  with the initial condition. Solve the differential equation.

**Problem 13.** Let  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an analytic function. Consider the map

$$\mathbf{x}_j = \mathbf{f}(\mathbf{x}_{j-1}) = \dots = \mathbf{f}(\mathbf{x}_0).$$

To study the evolution of phase-space distributions, we can introduce the evolution operator  $U(\mathbf{x}', \mathbf{x}, j)$  such that any initial phase-space distribution  $\rho(\mathbf{x}, 0)$  evolves into

$$\rho(\mathbf{x}'; j) = \int_{\Omega} U(\mathbf{x}', \mathbf{x}; j) \rho(\mathbf{x}, 0) d\mathbf{x}$$

where  $\Omega$  is the phase space area. Find  $U(\mathbf{x}', \mathbf{x}; j)$ .

**Problem 14.** Find the solution of the recursion relation

$$a_{j+2} = \frac{2}{3}a_{j+1} + \frac{1}{3}a_j, \quad j = 1, 2, \dots$$

with  $a_1 = 5, a_2 = 1$ . Calculate  $\lim_{j \rightarrow \infty} a_j$ .

**Problem 15.** Solve the linear difference equation

$$x_{t+1} = 2x_t - t, \quad t = 0, 1, 2, \dots$$

with the initial value  $x_0 = 1$ .

**Problem 16.** Let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}$$

Find an invertible map  $f : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$  with

$$f(0, 0, 0) = 0, \quad f(0, 0, 1) = 1, \quad f(0, 1, 0) = 2, \quad f(0, 1, 1) = 3,$$

$$f(1, 0, 0) = 4, \quad f(1, 0, 1) = 5, \quad f(1, 1, 0) = 6, \quad f(1, 1, 1) = 7$$

etc. Find the inverse function.

**Problem 17.** Consider the second order difference equation

$$x_{t+2} = x_{t+1}x_t$$

with the initial values  $x_0 = a$ ,  $x_1 = b$ . Give the solution. What are the fixed points?

**Problem 18.** Let  $A$  be a set. Suppose that it is possible to define subsets  $A_1, A_2, \dots$  of  $A$  which have the properties that

(i) the sets are pairwise disjoint; that is  $A_i \cap A_j = \emptyset$  for all  $i, j = 1, 2, \dots$  and  $j \neq i$

(ii)  $A_1 \cup A_2 \cup \dots = A$ .

Then the family of sets  $\{A_1, \dots\}$  is called a *partition* of  $A$ .

Let  $S = \{1, 2, 3, \dots, 9\}$  and  $A = \{1, 4, 7\}$ ,  $B = \{2, 3, 5, 6\}$ ,  $C = \{7, 8, 9\}$ . Is  $\{A, B, C\}$  a partition of  $S$ ?

**Problem 19.** Let  $S$  be a finite set. Let  $\mathcal{P}(S)$  be the power set of  $S$ . Then  $\mathcal{P}(S)$  has  $2^{|S|}$  elements.

(i) Show that the composition

$$A \circ B := (A \cup B) \cap (\mathfrak{C}A \cup \mathfrak{C}B)$$

defines a group, where  $A, B \in \mathcal{P}(S)$ . Here  $\mathfrak{C}$  denotes the complement.

(ii) Show that the composition

$$A \circ B := (A \cap B) \cup (\mathfrak{C}A \cap \mathfrak{C}B)$$

defines a group, where  $A, B \in \mathcal{P}(S)$ . Here  $\mathfrak{C}$  denotes the complement.

**Problem 20.** Find a function  $f : [0, 1] \rightarrow (0, 1]$  that is one-to-one.

**Problem 21.** Let  $n \in \mathbb{N}$ . Consider the map

$$u_{t+1} = \frac{1}{2} \left( u_t + \frac{n}{u_t} \right), \quad t = 0, 1, 2, \dots$$

given the initial value  $u_0$  with  $u_0 > 0$ . Show that

$$\frac{u_{t+1} - \sqrt{n}}{u_{t+1} + \sqrt{n}} = \left( \frac{u_t - \sqrt{n}}{u_t + \sqrt{n}} \right)^2.$$

Show that  $u_t \rightarrow \sqrt{n}$  as  $t \rightarrow \infty$ . Show that  $\sqrt{n}$  is a fixed point.

**Problem 22.** Let  $0 \leq \alpha < \pi/4$ . Consider the transformation

$$X(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}} (x \cos(\alpha) + iy \sin(\alpha))$$

$$Y(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}} (-ix \sin(\alpha) + y \cos(\alpha)).$$

(i) Show that  $X^2 + Y^2 = x^2 + y^2$ .

(ii) Do the matrices

$$\frac{1}{\sqrt{\cos(2\alpha)}} \begin{pmatrix} \cos(\alpha) & i \sin(\alpha) \\ -i \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

form a group under matrix multiplication?

**Problem 23.** A smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called homogeneous of degree  $r$  if

$$f(\epsilon x_1, \dots, \epsilon x_n) = \epsilon^r f(x_1, \dots, x_n). \quad (1)$$

Show that (Euler's identity)

$$\sum_{j=1}^n \frac{\partial f}{\partial x_j} x_j = r f.$$

**Problem 24.** Find the invariance group of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \sin(x).$$

**Problem 25.** Consider the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \cos(\sin(x)) - \sin(\cos(x)).$$

Show that  $f$  admits at least one critical point. By calculating the second order derivative find out whether this critical point refers to a maxima or minima.

**Problem 26.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be analytic functions. We define the *star product*

$$f(x_1, x_2) \star g(x_1, x_2) := \lim_{x'_1 \rightarrow x_1, x'_2 \rightarrow x_2} \exp\left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x'_2} - \frac{\partial}{\partial x'_1} \frac{\partial}{\partial x_2}\right) f(x_1, x_2) g(x'_1, x'_2).$$

Let

$$f(x_1, x_2) = \sin(x_1 + x_2), \quad g(x_1, x_2) = \sin(x_1 - x_2).$$

Find the star product.

**Problem 27.** Let  $N_1, N_2$  be given positive integers. Let  $n_1 = 0, 1, \dots, N_1 - 1$ ,  $n_2 = 0, 1, \dots, N_2 - 1$ . There are  $N_1 \cdot N_2$  points. The points  $(n_1, n_2)$  are a subset of  $\mathbb{N}_0 \times \mathbb{N}_0$  and can be mapped one-to-one onto a subset of  $\mathbb{N}_0$

$$j(n_1, n_2) = n_1 N_2 + n_2$$

where  $j = 0, 1, \dots, (N_1 - 1)(N_2 - 1)$ . Find the inverse of this map.

**Problem 28.** Let  $a \in \mathbb{R}$ . Consider the transformation

$$\tilde{t}(t, x) = \frac{1}{a} e^{ax} \sinh(at), \quad \tilde{x}(t, x) = \frac{1}{a} (e^{ax} \cosh(at) - 1)$$

with  $\lim_{a \rightarrow 0} \tilde{t} = t$ ,  $\lim_{a \rightarrow 0} \tilde{x} = x$ . Find the inverse of the transformation.

**Problem 29.** Show that the map  $f : (0, 1) \rightarrow \mathbb{R}$

$$x \mapsto f(x) = \frac{x - 1/2}{x(x - 1)}$$

is bijective. Note that  $f(1/2) = 0$  and  $f(1/4) = 4/3$ .

**Problem 30.** Consider the map

$$f : \mathbb{R} \rightarrow \{(x_1, x_2) : x_1^2 + (x_2 - 1)^2 = 1 \wedge x_2 \neq 2\}$$

defined by

$$f(t) = \left( \frac{4t}{t^2 + 4}, \frac{2t^2}{t^2 + 4} \right).$$



Show that  $f$  is bijective. Show that  $f$  and  $f^{-1}$  are continuous.

**Problem 31.** Let  $N \geq 2$  and  $\delta := 1/(N + 1)$ . Solve the linear one-dimensional linear difference equation

$$x_{j+1} - 2x_j + x_{j-1} = -12\delta^4(j + 1)^2, \quad j = 0, 1, \dots, N - 1$$

with the boundary conditions  $x_{-1} = x_N = 0$ .

**Problem 32.** (i) Consider the analytic map  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x(1 - x).$$

Find the fixed points of  $f$ . Are the fixed points stable? Prove or disprove.

(ii) Let  $x_0 = 1/2$  and iterate

$$f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \dots$$

Does this sequence tend to a fixed point? Prove or disprove.

(iii) Let  $x_0 = 2$  and iterate

$$f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \dots$$

Does this sequence tend to a fixed point? Prove or disprove.

(iv) Find the critical points of  $f$ . Then find the extrema of  $f$ .

(v) Find the roots of  $f$ , i.e. solve  $f(x) = 0$ .

(vi) Find the minima of the function  $g(x) := |f(x)|$ .

**Problem 33.** Let  $c > 0$  and  $0 \leq v_1 < c$ ,  $0 \leq v_2 < c$ . We define the composition

$$v_1 \star v_2 := \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

Is the composition associative?

**Problem 34.** Solve the initial value problem of the system of first order difference equations

$$\begin{aligned} x_{1,t+1} &= -2x_{1,t} \\ x_{2,t+1} &= \frac{8}{9}x_{1,t} - x_{2,t}. \end{aligned}$$

The first difference equation is independent of  $x_{2,t}$  and we find the solution of the initial value problem

$$x_{1,t+1} = (-2)^t x_{1,0}.$$

**Problem 35.** Let  $m_A, m_B, \mathbf{R}_A, \mathbf{R}_B$  be the masses and centre-of mass coordinates of mass  $A$  and  $B$ , respectively. We set  $m = m_A + m_B$ . Find the inverse of the transformation

$$\mathbf{r}(\mathbf{R}_A, \mathbf{R}_B) = \mathbf{R}_A - \mathbf{R}_B, \quad \mathbf{R}(\mathbf{R}_A, \mathbf{R}_B) = \frac{1}{m}(m_A \mathbf{R}_A + m_B \mathbf{R}_B).$$

**Problem 36.** Consider the map  $\mathbf{f} : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$

$$f_1(n_1, n_2) = |n_1 - n_2|, \quad f_2(n_1, n_2) = n_1 + n_2.$$

Is the map invertible?

**Problem 37.** Let  $n, m \geq 0$ . Consider the differential operators

$$K_n := \sum_{j=0}^n u_j(x) \frac{d^j}{dx^j}, \quad L_m := \sum_{j=0}^m v_j(x) \frac{d^j}{dx^j}$$

where the  $u_j(x)$ 's and  $v_j(x)$ 's are smooth functions. If the two differential operators  $K_n$  and  $L_m$  commute, then there is a nonzero polynomial  $R(z, w)$  such that  $R(K_n, L_m) = 0$ . The curve  $\Gamma$  defined by  $R(z, w) = 0$  is called the spectral curve. If we consider the eigenvalue problem

$$K_n \psi = z \psi, \quad L_m \psi = w \psi$$

then  $(z, w) \in \Gamma$ .

(i) Let  $(\alpha \in \mathbb{R})$

$$K = \left( \frac{d^2}{dx^2} + x^3 + \alpha \right)^2 + 2x,$$

$$L = \left( \frac{d^2}{dx^2} + x^3 + \alpha \right)^3 + 3x \frac{d^2}{dx^2} + 3 \frac{d}{dx} + 3x(x^2 + \alpha).$$

Show that  $K$  and  $L$  commute with  $w^2 = z^3 - \alpha$ .

(ii) Show that

$$K = \left( \frac{d^3}{dx^3} + x^2 + \alpha \right)^2 + 2 \frac{d}{dx}$$

and

$$L = \left( \frac{d^3}{dx^3} + x^2 + \alpha \right)^3 + 3 \frac{d^4}{dx^4} + 3(x^2 + \alpha) \frac{d}{dx} + 3x$$

commute.

**Problem 38.** Let  $A, B, C$  be nonempty sets. Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are invertible maps. Then the composition of the functions

$g \circ f : A \rightarrow C$  is invertible and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ . Function composition is associative.

**Problem 39.** Consider the map  $f : (-1, 1) \rightarrow \mathbb{R}$

$$f(x) = \frac{x}{1 - |x|}.$$

Show that the map  $f^{-1} : \mathbb{R} \rightarrow (-1, 1)$  is given by

$$f^{-1}(x) = \frac{x}{1 + |x|}.$$

**Problem 40.** Let  $\tau \in \mathbb{R}$ . Show that the curves

$$\mathbf{x}(\tau) = \begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} = \begin{pmatrix} 1 \\ \tau \end{pmatrix}, \quad \mathbf{y}(\tau) = \begin{pmatrix} y_1(\tau) \\ y_2(\tau) \end{pmatrix} = \begin{pmatrix} \tau \\ \tau^2 \end{pmatrix}$$

are linearly independent.

**Problem 41.** Let  $r, k$  be positive integers with  $r \leq k$  and  $M = \{x_1, \dots, x_r\}$  and  $N = \{y_1, \dots, y_k\}$ . Then there

$$\frac{k!}{(k-r)!}$$

surjective maps  $f : M \rightarrow N$ . Find all the maps for  $r = k = 2$ .

**Problem 42.** Consider the first order difference equation

$$x_{\tau+1}x_\tau + 2x_\tau + x_{\tau+1} = 0, \quad \tau = 0, 1, 2, \dots$$

with  $x_0 \neq 0$ . Show that this difference equation can be linearized with  $x_\tau = 1/y_\tau$  where  $x_0 \neq 0$ . First show that  $x^* = 0$  and  $x^* = -3$  are solutions of the nonlinear difference equation.

## 2.2 Supplementary Problems

**Problem 1.** Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$  and  $\times$  be the *vector product*

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Hence the vector product is a map from  $\mathbb{R}^3 \times \mathbb{R}^3$  into  $\mathbb{R}^3$ . Show that the map is differentiable.

**Problem 2.** Study the map (*Legendre map*)  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = e^{x_1} \cos(x_2), \quad f_2(x_1, x_2) = -e^{x_1} \sin(x_2).$$

Find the functional determinant.

**Problem 3.** Consider the four symbols  $\{A, B, C, D\}$ . The *Rudin-Shapiro substitution* is

$$A \mapsto AC, \quad B \mapsto DC, \quad C \mapsto AB, \quad D \mapsto DB.$$

Find the sequence starting of with  $B$ .

**Problem 4.** Solve the difference equation

$$x_{t+1} = tx_t + t^2 \pmod{2}$$

with  $t = 0, 1, 2, \dots$  and  $x_0 = 1$ .

**Problem 5.** Study the map  $\mathbf{f} : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

$$f_1(x_1, x_2, x_3) = x_1 - x_2x_3, \quad f_2(x_1, x_2, x_3) = -x_2 + x_1x_3,$$

$$f_3(x_1, x_2, x_3) = x_3 - x_1x_2.$$

(i) Show that the fixed points are

$$(0, 0, 0), \quad (1, 0, 0), \quad (0, 0, 1), \quad (-1, 0, 0), \quad (0, 0, -1).$$

(ii) Show that  $(x_1 = 2, x_2 = 2, x_3 = 2)$  provides a periodic orbit.

(iii) Show that  $(x_1 = 1, x_2 = 2, x_3 = 1)$  provides an eventually periodic orbit.

(iv) Is the map invertible?

**Problem 6.** Consider the map  $\mathbf{f} : [0, 1]^3 \rightarrow [0, 1]^3$  given by

$$f_1(x_1, x_2, x_3) = 2x_1, \quad f_2(x_1, x_2, x_3) = \frac{1}{2}x_2, \quad f_3(x_1, x_2, x_3) = \frac{1}{2}x_3$$

for  $0 \leq x_1 \leq 1/2$  and

$$f_1(x_1, x_2, x_3) = 2x_1 - 1, \quad f_2(x_1, x_2, x_3) = \frac{1}{2}(x_2 + 1), \quad f_3(x_1, x_2, x_3) = \frac{1}{2}(x_3 + 1)$$

for  $1/2 < l \leq 1$ . Find the orbit for  $x_1 = x_2 = x_3 = 1/3$ .

**Problem 7.** Consider the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = \sinh(x_1) \sin(x_2), \quad f_2(x_1, x_2) = \cosh(x_1) \cos(x_2).$$

Find the fixed points of the map and study their stability.

**Problem 8.** Study the system of difference equations

$$\begin{aligned} x_{0,t+1} &= 2x_{0,t} - x_{1,t} - x_{2,t} - x_{3,t} - x_{4,t} \\ x_{1,t+1} &= -\frac{1}{2}x_{1,t} + x_{0,t} \\ x_{2,t+1} &= -\frac{1}{2}x_{2,t} + x_{0,t} \\ x_{3,t+1} &= -\frac{1}{2}x_{3,t} + x_{0,t} \\ x_{4,t+1} &= -\frac{1}{2}x_{4,t} + x_{0,t} \end{aligned}$$

with the initial conditions

$$x_{0,0} = 1, \quad x_{1,0} = x_{2,0} = x_{3,0} = x_{4,0} = 0.$$

**Problem 9.** Consider the difference equation

$$x_{t+1} = 2x_t - 2, \quad t = 0, 1, 2, \dots$$

where  $x_0 = 1$ . Find the general solution. Show that  $x^* = 2$  is a particular solution (the fixed point).

**Problem 10.** The *Fibonacci numbers* can be defined by

$$\begin{pmatrix} x_{t+2} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix}$$

with  $x_0 = x_1 = 1$ . Find the eigenvalues of the  $2 \times 2$  matrix. Study the sequence

$$\begin{pmatrix} x_{t+4} \\ x_{t+3} \\ x_{t+1} \\ x_t \end{pmatrix} = (A \otimes A) \begin{pmatrix} x_{t+3} \\ x_{t+2} \\ x_{t+1} \\ x_t \end{pmatrix}$$

with  $x_0 = x_1 = x_2 = x_3 = 1$  and  $\otimes$  denotes the Kronecker product.

**Problem 11.** Let  $x_0 \in \mathbb{R}^+$  and  $r \in \mathbb{R}^+$ . Show that

$$x_{j+1} = \frac{1}{2} \left( x_j + \frac{r}{x_j} \right), \quad j = 0, 1, 2, \dots$$

tends to  $\sqrt{r}$ .

**Problem 12.** Let  $A = (a_{jk})$  be a  $3 \times 3$  matrix with  $\det(A) \neq 0$ . Consider the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$f_1(x_1, x_2) = \frac{a_{11}x_1 + a_{12}x_2 + a_{13}}{a_{31}x_1 + a_{32}x_2 + a_{33}}, \quad f_2(x_1, x_2) = \frac{a_{21}x_1 + a_{22}x_2 + a_{23}}{a_{31}x_1 + a_{32}x_2 + a_{33}},$$

Find the inverse of the map.

**Problem 13.** Consider the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = x_1 + x_2, \quad f_2(x_1, x_2) = x_1x_2$$

or written as difference equation

$$x_{1,\tau+1} = x_{1,\tau} + x_{2,\tau}, \quad x_{2,\tau+1} = x_{1,\tau}x_{2,\tau}.$$

Find the fixed points of the map. Then solve the difference equation for  $x_{1,0} = 1/4$ ,  $x_{2,0} = 1/5$ .

**Problem 14.** Show that

$$\left( \frac{\partial F(x, y)}{\partial x} \right)^2 + \left( \frac{\partial F(x, y)}{\partial y} \right)^2$$

expressed in polar coordinates  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$  takes the form

$$\left( \frac{\partial F(x(r, \theta), y(r, \theta))}{\partial r} \right)^2 + \frac{1}{r} \left( \frac{\partial F(x(r, \theta), y(r, \theta))}{\partial \theta} \right)^2.$$

**Problem 15.** Let  $r > 1$  and  $\tau = 0, 1, 2, \dots$ . Study the system of difference equation

$$x_{1,\tau+1} = \frac{1}{r} e^{x_{2,\tau}}, \quad x_{2,\tau+1} = r e^{-x_{1,\tau}}$$

with the initial conditions  $x_{1,0} = x_{2,0} = 0$ . Are there fixed points?

**Problem 16.** Let  $m \in \mathbb{N}$ . Solve the recurrence relation

$$f(n) = 1 + f(n/2) \text{ for } n \geq 2, n = 2^m$$

with the initial condition  $f(1) = 1$ .

**Problem 17.** Study the initial value problem second order difference equation

$$x_{\tau+2} = x_{\tau+1} + x_{\tau} - x_{\tau+1}x_{\tau}$$

with  $x_0 = 1, x_1 = 1/2$ .

**Problem 18.** A set  $A$  is equivalent to the set  $B$  (one writes  $A \sim B$ ) if there is a mapping  $f : A \rightarrow B$  which is  $(1, 1)$  and onto. Show that  $\mathbb{R}^2 \sim \mathbb{R}$ . Utilize that

$$(\cdot a_1 a_2 a_3 \dots, \cdot b_1 b_2 b_3 \dots) \leftrightarrow \cdot a_1 b_1 a_2 b_2 \dots$$

defines a  $(1, 1)$  map between pairs of decimal expansions and single expressions of numbers in the interval  $(0, 1)$ .

**Problem 19.** Let  $\Omega = [0, 1)$ ,  $T(x) = 2x \pmod{1}$ , and  $\mu$  the Lebesgue measure. Show that  $T$  is ergodic.

**Problem 20.** (i) Let  $c > 0$ . Study the difference equation

$$x_{\tau+1} = \frac{x_{\tau}}{c + x_{\tau}}, \quad \tau = 0, 1, \dots$$

with  $x_0 > 0$ . First find the fixed points.

(ii) Study the difference equation

$$x_{\tau+1} = \frac{6x_{\tau}}{(1 + x_{\tau})^2}$$

with  $x_0 > 0$ . First find the fixed points.

**Problem 21.** Let  $x \in [0, 1]$  and  $[a]$  be the integer part of  $a$ . Show that the sequence  $x_t$  ( $t = 0, 1, \dots$ ) given by

$$x_0 = 0, \text{ and } x_{t+1} = x_t + \frac{1}{2^{t+1}} [2^{t+1}(x - x_t)]$$

converges to  $x$ .

**Problem 22.** The *Kustaanheimo-Stiefel transformation* is defined by the map from  $\mathbb{R}^4$  (coordinates  $u_1, u_2, u_3, u_4$ ) to  $\mathbb{R}^3$  (coordinates  $x_1, x_2, x_3$ )

$$\begin{aligned} x_1(u_1, u_2, u_3, u_4) &= 2(u_1 u_3 - u_2 u_4) \\ x_2(u_1, u_2, u_3, u_4) &= 2(u_1 u_4 + u_2 u_3) \\ x_3(u_1, u_2, u_3, u_4) &= u_1^2 + u_2^2 - u_3^2 - u_4^2 \end{aligned}$$

together with the constraint

$$u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4 = 0.$$

(i) Show that

$$r^2 = x_1^2 + x_2^2 + x_3^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

(ii) Show that

$$\Delta_3 = \frac{1}{4r} \Delta_4 - \frac{1}{4r^2} V^2$$

where

$$\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_4 = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}$$

and  $V$  is the vector field

$$V = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} - u_4 \frac{\partial}{\partial u_3} + u_3 \frac{\partial}{\partial u_4}$$

(iii) Consider the differential one form

$$\alpha = u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4.$$

Find  $d\alpha$ . Find  $L_V \alpha$ , where  $L_V(\cdot)$  denotes the Lie derivative.

(iv) Let  $g(x_1(u_1, u_2, u_3, u_4), x_2(u_1, u_2, u_3, u_4), x_3(u_1, u_2, u_3, u_4))$  be a smooth function. Show that  $L_V g = 0$ .

**Problem 23.** The Kustaanheimo-Stiefel map  $\mathbb{R}^4 \rightarrow \mathbb{R}^3 ((u_1, u_2, u_3, u_4) \rightarrow (x_1, x_2, x_3))$  is given by

$$x_1 = 2(u_1 u_3 - u_2 u_4), \quad x_2 = 2(u_1 u_4 + u_2 u_3), \quad x_3 = u_1^2 + u_2^2 - u_3^2 - u_4^2$$

together with the constraint

$$\alpha \equiv u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4 = 0.$$

Show that applying the Kustaanheimo-Stiefel map the Laplacian operator  $\Delta_3$  in  $\mathbb{R}^3$  can be written as

$$\Delta_3 = \frac{1}{4r} \Delta_4 - \frac{1}{4r^2} V^2$$

where

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2} = u_1^2 + u_2^2 + u_3^2 + u_4^2, \quad V = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} - u_4 \frac{\partial}{\partial u_3} + u_3 \frac{\partial}{\partial u_4}.$$



# Chapter 3

## Functions

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### 3.1 Solved Problems

**Problem 1.** Let  $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $j = 1, 2, \dots, n$  be real-valued functions with continuous second-order partial derivatives everywhere on  $\mathbb{R}^n$ . Suppose that there are constants  $c_{ij}$  such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all  $i$  and  $j$ ,  $1 \leq i \leq n$ . Prove that there is a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f_i + \partial g / \partial x_i$  is linear for all  $i = 1, 2, \dots, n$ . A linear function  $p : \mathbb{R}^n \rightarrow \mathbb{R}$  is of the form

$$p(\mathbf{x}) = a_0 + a_1x_1 + a_2x_2 + \dots + a_nx_n.$$

**Problem 2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable function and  $\mathbf{x} \in \mathbb{R}^n$ . Consider the invertible  $n \times n$  matrix  $R$  and the transformation  $\mathbf{x}' = R\mathbf{x}$ . The transformation operator  $\hat{P}_R$  associated with the invertible matrix  $R$  is defined by

$$\hat{P}_R f(\mathbf{x}) := f(R^{-1}\mathbf{x}).$$

Note that the operator acts upon the coordinates  $\mathbf{x}$  and not on the argument of  $f$ . This means  $\hat{P}_R \hat{P}_S f(\mathbf{x}) = f(S^{-1}R^{-1}\mathbf{x})$ , where  $S$  is another invertible  $n \times n$  matrix. Show that

$$\hat{P}_R \hat{P}_S f(\mathbf{x}) = \hat{P}_{RS} f(\mathbf{x}).$$

**Problem 3.** A ratio list is a finite list of positive numbers,  $(r_1, r_2, \dots, r_n)$ . An iterated function system realizing a ratio list  $(r_1, r_2, \dots, r_n)$  in a metric space  $S$  is a list  $(f_1, f_2, \dots, f_n)$ , where  $f_j : S \rightarrow S$  is a similarity with ratio  $r_j$ . A nonempty compact set  $K \subseteq S$  is an invariant set for the iterated function system  $(f_1, f_2, \dots, f_n)$  iff  $K = f_1(K) \cup f_2(K) \cup \dots \cup f_n(K)$ . The triadic Cantor set is an invariant set for an iterated function system realizing the ratio list  $(1/3, 1/3)$ . The Sierpinski gasket is an invariant set for an iterated system realizing the ratio list  $(1/2, 1/2, 1/2)$ . The dimension associated with a ratio list  $(r_1, r_2, \dots, r_n)$  is the positive number  $s$  such that  $r_1^s + r_2^s + \dots + r_n^s = 1$ . Let  $(r_1, r_2, \dots, r_n)$  be a ratio list. Suppose each  $r_j < 1$ . Show that there is a unique nonnegative number  $s$  satisfying

$$\sum_{j=1}^n r_j^s \equiv \sum_{j=1}^n e^{s \ln(r_j)} = 1.$$

The number  $s$  is 0 iff  $n = 1$ .

**Problem 4.** The *arc length* of the equilateral hyperbola

$$h(t) = \sqrt{t^2 - 1}, \quad t \geq 1$$

starting at  $t = 1$  is given by

$$L_h(x) = \int_1^x \sqrt{\frac{2t^2 - 1}{t^2 - 1}} dt$$

as a function of the terminal point  $t = x$ . The tangent line to the hyperbola at  $t = x$  is

$$T_h(t) = \sqrt{x^2 - 1} + \frac{x}{\sqrt{x^2 - 1}}(t - x)$$

whose intersection with the  $t$ -axis is  $t = 1/x$  ( $t \in (0, 1)$ ). The line

$$N_h(t) = -\frac{\sqrt{x^2 - 1}}{x}t$$

is perpendicular to  $T_h$  passing through the origin.

- (i) Find the point  $P_h$  where the lines  $T_h$  and  $N_h$  intersect.
- (ii) Calculate the distance from  $(x, h(x))$  to the common point  $P_h$ .

**Problem 5.** Given a differentiable function  $f$ . The *logarithmic derivative* of  $f$  is defined as

$$\frac{1}{f} \frac{df}{dx}.$$

When  $f$  is a real function of real variables and takes strictly positive values then the chain rule provides

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f} \frac{df}{dx}.$$

The logarithmic derivative has the following properties.

1. The logarithmic derivative of the product of functions is the sum of their logarithmic derivatives

$$\frac{1}{fg} \frac{d}{dx}(fg) = \frac{1}{f} \frac{df}{dx} + \frac{1}{g} \frac{dg}{dx}$$

2. The logarithmic derivative of the quotient of functions is the difference of their logarithmic derivatives

$$\frac{1}{f/g} \frac{d}{dx}(f/g) = \frac{1}{f} \frac{df}{dx} - \frac{1}{g} \frac{dg}{dx}$$

3. The logarithmic derivative of the  $\alpha$ -th power of a function  $f$  is  $\alpha$  times the logarithmic derivative of the function

$$\frac{1}{f^\alpha} \frac{d}{dx}(f^\alpha) = \alpha \frac{1}{f} \frac{df}{dx}$$

4. The logarithmic derivative of the exponential of a function equals the derivative of a function

$$\frac{1}{e^f} \frac{d}{dx} e^f = \frac{df}{dx}$$

(i) Find the logarithmic derivative of  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \cosh(x)$ .

(ii) Let  $f$  be a meromorphic function in the open and connected set  $D \subseteq \mathbb{C}$ . Let  $G \subseteq D$  be a region such that its closure  $\bar{G} \subseteq D$  and its boundary  $\partial G$  is a continuous curve not containing a zero or pole of  $f$ . Let  $N$  be the number of zeros of  $f$  lying inside  $G$  and  $P$  be the number of poles of  $f$  lying inside  $G$ . Then

$$N - P = \frac{1}{2\pi i} \int_{\partial G} \frac{1}{f(z)} \frac{df(z)}{dz} dz$$

where  $\partial G$  is an oriented boundary of  $G$ . Calculate the left and right-hand side for the function  $f(z) = 1/z$  and  $G = \{(x, y) : x^2 + y^2 \leq 1\}$ .

**Problem 6.** Let  $f$  be an analytic function. Let  $p \in \mathbb{N}$  and  $\alpha \in \mathbb{R}$ . Show that

$$\left(\frac{d}{dx} + \alpha x\right)^p f \equiv \exp\left(-\frac{\alpha x^2}{2}\right) \frac{d^p}{dx^p} \left(\exp\left(\frac{\alpha x^2}{2}\right) f\right).$$

**Problem 7.** Consider

$$K(t, s) = \begin{cases} 1 & \text{if } 0 \leq t \leq s \\ 0 & \text{if } s \leq t \leq 1 \end{cases}.$$

Show that this *kernel* satisfies the functional equation

$$K(cu + a, t) = K\left(u, \frac{t - a}{c}\right)$$

where  $c > 0$ .

**Problem 8.** Find a continuous differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which has no fixed points and no critical points.

**Problem 9.** Let  $a, b, p, q, r \in \mathbb{R}$ ,  $b > a$ ,  $a < p$ ,  $p < r$ ,  $r < q$ ,  $q < b$  and

$$p - a = r - p, \quad q - r = b - q.$$

In fuzzy logic the following membership function plays an important role

$$f(x; a, b, r, p, q) = \begin{cases} 0 & x \leq a \\ 2^{m-1}((x-a)/(r-a))^m & a < x \leq p \\ 1 - 2^{m-1}((r-x)/(r-a))^m & p < x \leq r \\ 1 - 2^{m-1}((x-r)/(b-r))^m & r < x \leq q \\ 2^{m-1}((b-x)/(b-r))^m & q < x < b \\ 0 & x \geq b \end{cases}$$

where  $m$  is the fuzzifier. In most cases  $m = 2$ . Where are the crossover points? What is the value at the centre?

**Problem 10.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the mapping of  $(x, y) \rightarrow (u, v)$  given by

$$u(x, y) = e^x \cos(y), \quad v(x, y) = e^x \sin(y).$$

What is the range of  $f$ ? Show that the Jacobian determinant is not zero at any point of  $\mathbb{R}^2$ . Thus every point of  $\mathbb{R}^2$  has a neighbourhood in which  $f$  is one-to-one. Nevertheless  $f$  is not one-to-one on  $\mathbb{R}^2$ . What are the images under  $f$  of lines parallel to the coordinate axes?

**Problem 11.** Let  $x, y \in \mathbb{R}$ . Calculate the square root

$$\sqrt{x^2 + y^2 - 2xy}.$$

**Problem 12.** A criterion for linearly independence of functions  $f_0, f_1, \dots, f_n \in C^n[a, b]$  is that the *Wronski determinant* is nonzero

$$\det \begin{pmatrix} f_0 & f_1 & \dots & f_n \\ f_0' & f_1' & \dots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \dots & f_n^{(n)} \end{pmatrix} \neq 0.$$

Here ' denotes derivative. Apply this criterion to the functions

$$f_0(x) = \cos(x), \quad f_1(x) = \sin(x).$$

**Problem 13.** For a sphere of radius  $r$  and mass density  $\rho$  the mass that must be concentrated at its centre is ( $\lambda \geq 0$ )

$$M(\lambda) = \frac{4\pi r \rho}{\lambda^2} (\cosh(\lambda r) - \sinh(\lambda r)/(\lambda r)).$$

Find  $\lim_{\lambda \rightarrow 0} M(\lambda)$ .

**Problem 14.** Let  $-1 < a < 1$ . Find the inverse of the transformation

$$\lambda(z) = \frac{z - a}{1 - az}.$$

**Problem 15.** Consider the Jacobi elliptic functions

$$\operatorname{sn}(x, k), \quad \operatorname{cn}(x, k), \quad \operatorname{dn}(x, k)$$

where  $x \in \mathbb{R}$  and  $k^2 \in [0, 1]$ . We have

$$\operatorname{sn}(x, 0) = \sin(x), \quad \operatorname{cn}(x, 0) = \cos(x), \quad \operatorname{dn}(x, 0) = 1$$

and

$$\operatorname{sn}(x, 1) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x), \quad \operatorname{cn}(x, 1) = \operatorname{dn}(x, 1) = \frac{2}{e^x + e^{-x}} \equiv \operatorname{sech}(x).$$

(i) Find an expression using Jacobi elliptic functions that interpolates between  $\sin(x)$  for  $k = 0$  and  $\sinh(x)$  for  $k = 1$ .

(ii) Find an expression using Jacobi elliptic functions that interpolates between  $\cos(x)$  for  $k = 0$  and  $\cosh(x)$  for  $k = 1$ .

(iii) Use this result to interpolate between the matrices

$$\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}.$$

**Problem 16.** Let  $f_1, f_2$  be differentiable functions and  $f_1(x) > 0, f_2(x) > 0$  for all  $x$ . Let  $f(x) = f_1(x)f_2(x)$ . Find

$$\frac{d}{dx}(\ln(f(x))).$$

**Problem 17.** Let  $x \in \mathbb{R}$ . The sequence of functions  $\{f_k(x)\}$  is defined by  $f_1(x) = \cos(x/2)$  and for  $k > 1$  by

$$f_k(x) = f_{k-1}(x) \cos(x/2^k).$$

Thus

$$f_k(x) = \cos(x/2) \cos(x/2^2) \cdots \cos(x/2^k).$$

Obviously, we have  $f_k(0) = 1$  for every  $k$ . Calculate  $\lim_{k \rightarrow \infty} f_k(x)$  as a function of  $x$  for  $x \neq 0$ .

**Problem 18.** (i) Consider the transformation in  $\mathbb{R}^3$

$$\begin{aligned} x_0(a, \theta_1) &= \cosh(a) \\ x_1(a, \theta_1) &= \sinh(a) \sin(\theta_1) \\ x_2(a, \theta_1) &= \sinh(a) \cos(\theta_1) \end{aligned}$$

where  $a \geq 0$  and  $0 \leq \theta_1 < 2\pi$ . Find

$$x_0^2 - x_1^2 - x_2^2.$$

(ii) Consider the transformation in  $\mathbb{R}^4$

$$\begin{aligned} x_0(a, \theta_1, \theta_2) &= \cosh(a) \\ x_1(a, \theta_1, \theta_2) &= \sinh(a) \sin(\theta_2) \sin(\theta_1) \\ x_2(a, \theta_1, \theta_2) &= \sinh(a) \sin(\theta_2) \cos(\theta_1) \\ x_3(a, \theta_1, \theta_2) &= \sinh(a) \cos(\theta_2) \end{aligned}$$

where  $a \geq 0, 0 \leq \theta_1 < 2\pi$  and  $0 \leq \theta_2 \leq \pi$ . Find

$$x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Extend the transformation to  $\mathbb{R}^n$ .

**Problem 19.** A fixed charge  $Q$  is located on the  $z$ -axis with coordinates  $\mathbf{r}_a = (0, 0, d/2)$ , where  $d$  is interfocal distance of the *prolate spheroidal*

coordinates

$$\begin{aligned} x(\eta, \xi, \phi) &= \frac{1}{2}d((1 - \eta^2)(\xi^2 - 1))^{1/2} \cos \phi \\ y(\eta, \xi, \phi) &= \frac{1}{2}d((1 - \eta^2)(\xi^2 - 1))^{1/2} \sin(\phi) \\ z(\eta, \xi, \phi) &= \frac{1}{2}d\eta\xi \end{aligned}$$

where  $-1 \leq \eta \leq +1$ ,  $1 \leq \xi \leq \infty$ ,  $0 \leq \phi \leq 2\pi$ . Express the *Coulomb potential*

$$V = \frac{Q}{|\mathbf{r} - \mathbf{r}_a|}$$

in prolate spheroidal coordinates.

**Problem 20.** *Toroidal coordinates* are defined by

$$\begin{aligned} x_1(\alpha, \beta, \phi) &= \frac{c \sinh(\alpha) \cos(\phi)}{\cosh(\alpha) - \cos(\beta)} \\ x_2(\alpha, \beta, \phi) &= \frac{c \sinh(\alpha) \sin(\phi)}{\cosh(\alpha) - \cos(\beta)} \\ x_3(\alpha, \beta, \phi) &= \frac{c \sin(\beta)}{\cosh(\alpha) - \cos(\beta)}. \end{aligned}$$

Use the L'Hospital rule to find  $x_1, x_2, x_3$  for  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$ .

**Problem 21.** Let  $c, \theta \in \mathbb{R}$  and

$$f(\theta) = c(e^{i\theta} + e^{-i\theta}).$$

Calculate  $\exp(f(\theta))$ .

**Problem 22.** A continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is called an *alternating function* if

$$f(x, y) = -f(y, x).$$

Give an example of an analytic alternating function. Find the minima and maxima of the function.

**Problem 23.** Consider the functions

$$\begin{aligned} j_1(x) &= \frac{1}{x^2}(\sin(x) - x \cos(x)) \\ j_2(x) &= \frac{1}{x^3}((3 - x^2) \sin(x) - 3x \cos(x)). \end{aligned}$$

Use the L'Hospital rule to find  $j_1(0)$  and  $j_2(0)$ .

**Problem 24.** Find the invariance group of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \sin(x).$$

**Problem 25.** Let  $a > 0$ . Consider the transformation

$$u(x, y) = \frac{a \sin(2ax)}{2(\sin^2(ax) + \sinh^2(ay))}, \quad v(x, y) = \frac{a \sinh(2ay)}{2(\sin^2(ax) + \sinh^2(ay))}.$$

Find the inverse transformation.

**Problem 26.** (i) Find an analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that has no fixed point and no critical point. Draw the function.

(ii) Find an analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that has no fixed point and exactly one critical point at  $x = 0$ . Draw the function.

**Problem 27.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an analytic function. Calculate the commutator

$$\left[ \cos(x) \frac{d}{dx}, \sin(x) \frac{d}{dx} \right] f.$$

**Problem 28.** (i) Consider the analytic function  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = \sinh(x_2), \quad f_2(x_1, x_2) = \sinh(x_1).$$

Show that this function admits the (only) fixed point  $(0, 0)$ . Find the functional matrix at the fixed point

$$\left( \begin{array}{cc} \partial f_1 / \partial x_1 & \partial f_1 / \partial x_2 \\ \partial f_2 / \partial x_1 & \partial f_2 / \partial x_2 \end{array} \right) \Big|_{(0,0)}.$$

(ii) Consider the analytic function  $\mathbf{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$g_1(x_1, x_2) = \sinh(x_1), \quad g_2(x_1, x_2) = -\sinh(x_2).$$

Show that this function admits the (only) fixed point  $(0, 0)$ . Find the functional matrix at the fixed point

$$\left( \begin{array}{cc} \partial g_1 / \partial x_1 & \partial g_1 / \partial x_2 \\ \partial g_2 / \partial x_1 & \partial g_2 / \partial x_2 \end{array} \right) \Big|_{(0,0)}.$$

(iii) Multiply the two matrices found in (i) and (ii).



(iv) Find the composite function  $\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\mathbf{h}(\mathbf{x}) = (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x})).$$

Show that this function also admits the fixed point  $(0, 0)$ . Find the functional matrix at this fixed point

$$\left( \begin{array}{cc} \partial h_1 / \partial x_1 & \partial h_1 / \partial x_2 \\ \partial h_2 / \partial x_1 & \partial h_2 / \partial x_2 \end{array} \right) \Big|_{(0,0)}.$$

Compare this matrix with the matrix found in (iii).

**Problem 29.** An approximation of  $e^{-x}$  ( $x \geq 0$ ) as a rational polynomial using a 3rd order Padé approximation is given by

$$e^{-x} \approx \frac{1 - x/2 + x^2/10 - x^3/120}{1 + x/2 + x^2/10 + x^3/120}.$$

Note that  $e^{-x} \geq 0$  for all  $x$ . For which value of  $x > 0$  does the right-hand side takes negative values?

**Problem 30.** Let  $x \in \mathbb{R}$ . Consider the function

$$f(x) = \frac{x - i}{x + i}.$$

Find  $f(0)$ ,  $f(1)$ ,  $f(-1)$  and  $f(x \rightarrow \infty)$ . Is there an inverse?

**Problem 31.** Consider the function

$$f(x) = x^3 - x^2.$$

Find an approximation of the derivative of  $f$  by using

$$f'(x) \approx \frac{f(x - 2h) - 8f(x - h) + 8f(x + h) - f(x + 2h)}{12h} - \frac{1}{30}h^4 f^{(5)}(\xi)$$

$$f'(x) \approx \frac{-3f(x - h) - 10f(x) + 18f(x + h) - 6f(x + 2h) + f(x + 3h)}{12h} + \frac{1}{20}h^4 f^{(5)}(\xi)$$

$$f'(x) \approx \frac{-25f(x) + 48f(x + h) - 36f(x + 2h) + 16f(x + 3h) - 3f(x + 4h)}{12h} - \frac{1}{5}h^4 f^{(5)}(\xi)$$

where  $x \leq \xi \leq x + h$ .

**Problem 32.** Let  $z \in \mathbb{C}$ . The *Airy functions*  $Ai(z)$  and  $Bi(z)$  are defined as sums of the power series

$$Ai(z) = c_1 f(z) - c_2 g(z)$$

$$Bi(z) = \sqrt{3}(c_1 f(z) + c_2 g(z))$$

where

$$f(z) = 1 + \frac{1}{3!}z^3 + \frac{1 \cdot 4}{6!}z^6 + \frac{1 \cdot 4 \cdot 7}{9!}z^9 + \dots$$

$$g(z) = z + \frac{2}{4!}z^4 + \frac{2 \cdot 5}{7!}z^7 + \frac{2 \cdot 5 \cdot 8}{10!}z^{10} + \dots$$

with

$$c_1 = \frac{1}{2\pi} \Gamma\left(\frac{1}{3}\right) 3^{-1/6}, \quad c_2 = \frac{1}{2\pi} \Gamma\left(\frac{2}{3}\right) 3^{1/6}.$$

Show that the radius of convergence of these infinite series is infinite.

**Problem 33.** Find the inverse of the function  $y = \tanh(x)$ . Note that  $y \in (-1, 1)$ .

**Problem 34.** Consider the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \sum_{j=0}^{\infty} \frac{x^j}{(j+1)!} \equiv \frac{e^x - 1}{x}.$$

Find the fixed points and critical points of the function. Note that  $f(0) = 1$ .

**Problem 35.** Find non-negative analytic functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 0.$$

**Problem 36.** The *Euler dilogarithm function*  $\text{Li}_2(x)$  is defined for  $x \in (0, 1)$  as

$$\text{Li}_2(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^2} = - \int_0^x \frac{\ln(1-t)}{t} dt.$$

It can be analytically continued to the complex plane with the branch cut from 1 to  $\infty$  along the real axis using the integral. Calculate  $\text{Li}_2(1/2)$ .

**Problem 37.** Let  $Ai$  be the Airy function and  $J_n, I_n$  are the Bessel and modified Bessel functions of order  $n$ . Show that

$$\frac{d^2 Ai}{dx^2} = x Ai(x)$$

$$Ai(x) = \frac{1}{3} x^{1/2} (I_{-1/3}(z) - I_{1/3}(z))$$

$$Ai(-x) = \frac{1}{3} x^{1/2} (J_{1/3}(z) + J_{-1/3}(z))$$

where  $z := 2x^{3/2}/3$ .

**Problem 38.** Find a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x) = f(-x), \quad f(x) = f(x + 2\pi), \quad f(0) = 0.$$

**Problem 39.** Let  $N$  be an integer and  $N \geq 2$ . The generalized hyperbolic function for a given  $N$  is defined as

$$f_j^{(N)}(x) := \sum_{k=0}^{\infty} \frac{x^{j+kN}}{(j+kN)!}, \quad j = 0, 1, \dots, N-1.$$

The functions  $f_j^{(N)}$  are analytic.

(i) Show that

$$f_j^{(N)}(x) = f_{j+N}^{(N)}(x)$$

i.e.  $f_j^{(N)}$  is periodic with respect to the index  $j$ .

(ii) Show that

$$\frac{d}{dx} f_j^{(N)}(x) = f_{j-1}^{(N)}(x).$$

(iii) Let  $\omega^N = 1$ , i.e.  $\omega$  is the  $N$ -th primitive root of unity. Show that

$$f_j^{(N)}(x) = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-jk} \exp(\omega^k x).$$

**Problem 40.** Let  $n = 2, 3, 4, \dots$ . Find

$$\frac{\Gamma(n/2)}{\Gamma(n)}$$

where  $\Gamma$  denotes the gamma function.

**Problem 41.** Consider the hyperspherical coordinates

$$\begin{aligned} x_1(\theta, \phi, \psi) &= \cos(\theta) \\ x_2(\theta, \phi, \psi) &= \sin(\theta) \cos(\phi) \\ x_3(\theta, \phi, \psi) &= \sin(\theta) \sin(\phi) \cos(\psi) \\ x_4(\theta, \phi, \psi) &= \sin(\theta) \sin(\phi) \sin(\psi) \end{aligned}$$

with  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ . Show that the *angular distance* can be calculated as

$$\cos(d_{jk}) =$$

$$\cos(\theta_j) \cos(\theta_k) + \sin(\theta_j) \sin(\theta_k) (\cos(\phi_j) \cos(\phi_k) + \sin(\phi_j) \sin(\phi_k) \cos(\psi_j - \psi_k)).$$

**Problem 42.** Consider the function

$$f(x) = 1 + x + 2x^2 + 3x^3.$$

Find the second order derivative of  $f$  at  $a = 1$  applying

$$f''(a) = \lim_{\epsilon \rightarrow 0} \frac{f(a + \epsilon) - 2f(a) + f(a - \epsilon)}{\epsilon^2}.$$

**Problem 43.** Let  $-1 < x < 1$  and

$$\sum_{j=0}^{\infty} a_j x^j = \frac{1}{\sqrt{1-x}}.$$

Find the expansion coefficients  $a_j$ .

**Problem 44.** Show that the function

$$f(x) = \cos(x) \cosh(x) + 1$$

has infinitely many roots, i.e. solutions of  $f(x) = 0$ . What happens for  $x$  large?

**Problem 45.** The Jacobi elliptic functions  $\text{sn}(x, k)$ ,  $\text{cn}(x, k)$ ,  $\text{dn}(x, k)$  with  $k \in [0, 1]$  and  $k^2 + k'^2 = 1$  have the properties

$$\text{sn}(x, 0) = \sin(x), \quad \text{cn}(x, 0) = \cos(x), \quad \text{dn}(x, 0) = 1$$

and

$$\text{sn}(x, 1) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{cn}(x, 1) = \frac{2}{e^x + e^{-x}} = \text{dn}(x, 1).$$

We define

$$\begin{aligned} u_1(x, y, k, k') &= \text{sn}(x, k) \text{dn}(y, k') \\ u_2(x, y, k, k') &= \text{cn}(x, k) \text{cn}(y, k') \\ u_3(x, y, k, k') &= \text{dn}(x, k) \text{sn}(y, k'). \end{aligned}$$

- (i) Find  $u_1(x, y, 0, 1)$ ,  $u_2(x, y, 0, 1)$ ,  $u_3(x, y, 0, 1)$  and calculate  $u_1^2(x, y, 0, 1) + u_2^2(x, y, 0, 1) + u_3^2(x, y, 0, 1)$ .  
(ii) Find  $u_1(x, y, 1, 0)$ ,  $u_2(x, y, 1, 0)$ ,  $u_3(x, y, 1, 0)$  and calculate  $u_1^2(x, y, 1, 0) + u_2^2(x, y, 1, 0) + u_3^2(x, y, 1, 0)$ .

**Problem 46.** Let  $m$  be an integer with  $m \geq 1$ . Then the *gamma function*  $\Gamma$  is given by

$$\Gamma(m) = (m - 1)!$$

Thus  $\Gamma(1) = \Gamma(2) = 1$ ,  $\Gamma(3) = 2$ ,  $\Gamma(4) = 6$ ,  $\Gamma(5) = 24$ . Furthermore

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2m - 1)}{2^m} \sqrt{\pi}.$$

Thus  $\Gamma(3/2) = \sqrt{\pi}/2$ ,  $\Gamma(5/2) = 3\sqrt{\pi}/4$ ,  $\Gamma(7/2) = 15\sqrt{\pi}/8$ ,  $\Gamma(9/2) = 105\sqrt{\pi}/16$ . Calculate

$$\frac{\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

for  $m = 5$ ,  $m = 10$ ,  $m = 20$ .

**Problem 47.** Let  $c_1 > 0$ ,  $c_2 > 0$ . Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^2 + \frac{c_1 x^2}{1 + c_2 x^2}.$$

Find the minima and maxima. Find the fixed points.

**Problem 48.** Let  $c > 0$ . Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f_c(x) = \frac{4}{x^2} \sin^2\left(\frac{cx}{2}\right).$$

Find  $f_c(0)$ . Show that  $f_c$  has a maximum at  $x = 0$ .

**Problem 49.** (i) Find the minima and maxima of the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \cosh(x) \cos(x).$$

Find the fixed points.

(ii) Find the minima and maxima of the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \sinh(x) \sin(x).$$

Find the fixed points.

**Problem 50.** Let  $S_1$  and  $S_2$  be two finite sets with  $n_1$  and  $n_2$  elements, respectively. The number of functions  $f : S_1 \rightarrow S_2$  is given by

$$n_2^{n_1}.$$

(i) Let  $n_1 = n_2 = 2$ . Find all possible functions.

(ii) Let  $n_1 = n_2 = n$ . Then there are  $n!$  one-to-one functions  $f : S_1 \rightarrow S_2$ . Let  $n = 3$ . Find all one-to-one functions.

**Problem 51.** Let  $\mu > 0$  and

$$\mathbf{R} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{R}' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

(i) Show that

$$\frac{\exp(-|\mathbf{R} - \mathbf{R}'|/\mu)}{|\mathbf{R} - \mathbf{R}'|} = \sum_{n=0}^{\infty} \varepsilon_n \cos(n(\phi - \phi')) \int_0^{\infty} \frac{1}{k^2 + 1/\mu} J_n(kr) J(kr') \exp(-\sqrt{k^2 + 1/\mu^2} |x_3 - x'_3|) k dk$$

where  $\varepsilon_n = 1$  for  $n = 0$  and  $\varepsilon_n = 2$  for  $n > 0$  and  $J_m(kr)$  is ordinary Bessel function of order  $m$ .

(ii) Consider the functions

$$f_{s,k,n}(\mathbf{R}) = \frac{\sqrt{k}}{2\pi} J_n(kr) e^{in\phi + isx_3}$$

where  $0 \leq k < \infty$ ,  $-\infty < s < \infty$ , and  $n = 0, \pm 1, \pm 2, \dots$ . Show that

$$\int_0^{2\pi} \int_{-\infty}^{+\infty} dx_3 \int_0^{\infty} f_{s,k,n}(\mathbf{R}) \bar{f}_{s',k',n'}(\mathbf{R}) r dr = \delta_{nn'} \delta(s - s') \delta(k - k').$$

**Problem 52.** Let  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  be analytic functions and

$$g_1(x) = \frac{1}{2}(f_1(x) + f_2(x) - |f_1(x) - f_2(x)|)$$

$$g_2(x) = \frac{1}{2}(f_1(x) + f_2(x) + |f_1(x) - f_2(x)|).$$

Let  $f_1(x) = \sin(x)$  and  $f_2(x) = \cos(x)$ . Find the maxima and minima of  $g_1$ . Find the maxima and minima of  $g_2$ .

**Problem 53.** Let the function  $f$  be continuous on a closed interval  $[a, b]$  and is continuous differentiable in the open interval  $(a, b)$ . Then there exists a point  $c$  in  $(a, b)$  such that (mean value theorem)

$$f(b) - f(a) = f'(c)(b - a).$$

Apply the theorem to the function

$$f(x) = 1 + 2x + 3x^2$$

and the interval  $[-1, 1]$ .

**Problem 54.** The plane  $\{(x_1, x_2, x_3) : x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1\}$  and the cylinder  $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 4\}$  intersect. Find the shortest distance from  $(0, 0, 0)$  to this curve.

**Problem 55.** Consider the plane  $\{(x_1, x_2, x_3) : 2x_1 + 3x_2 - x_3 = 5\}$  in the three dimensional Euclidean space  $\mathbb{E}^3$ . Find the shortest distance from  $(0, 0, 0)$  to this plane.

**Problem 56.** We denote by  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  the rational, real and complex fields, respectively. Let  $B_1$  and  $B_2$  be Banach spaces. A map  $f : B_1 \rightarrow B_2$  is said to be additive if and only if

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

for all  $x_1, x_2 \in B_1$ . Show that  $f$  is additive implies that  $f(qx) = qf(x)$  for all  $q \in \mathbb{Q}$ . Show that  $f$  is additive and continuous implies that  $f(rx) = rf(x)$  for all  $r \in \mathbb{R}$ .

**Problem 57.** Let  $a_1d_1 - b_1c_1 \neq 0$ ,  $a_2d_2 - b_2c_2 \neq 0$  and the functions  $f_1 : \mathbb{C} \rightarrow \mathbb{C}$

$$f_1(z) = \frac{a_1z + b_1}{c_1z + d_1}, \quad f_2(z) = \frac{a_2z + b_2}{c_2z + d_2}.$$

Find  $f_2 \circ f_1$ , i.e. the function composition.

**Problem 58.** Let  $x_1, x_2 \geq 0$ . Consider

$$f(x_1, x_2) = \frac{\sqrt{1+x_1}}{\sqrt{1+x_1}\sqrt{1+x_2}} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} F(j_1, j_2) x_1^{j_1} x_2^{j_2}.$$

Find the expansion coefficients.

**Problem 59.** Consider the function

$$f(x) = \frac{3}{x^3}(\sin(x) - x \cos(x)).$$

Find  $f(0)$  using the L'Hospital rule.

**Problem 60.** Let

$$f(x) = \frac{\sin(x)}{\sinh(x)}.$$

Find  $f(0)$ .

**Problem 61.** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^3.$$

Show that  $f$  is not convex. Show that  $f$  is convex if we restrict to the domain  $x \geq 0$ .

**Problem 62.** Consider the convex functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$ . Show that  $f + g$  is convex. Show that  $\max\{f, g\}$  is convex.

**Problem 63. Definition (Convex Set).** A subset  $C$  of  $\mathbb{R}^n$  is said to be *convex* if for any  $\mathbf{a}$  and  $\mathbf{b}$  in  $C$  and any  $\theta$  in  $\mathbb{R}$ ,  $0 \leq \theta \leq 1$ , the  $n$ -tuple  $\theta\mathbf{a} + (1 - \theta)\mathbf{b}$  also belongs to  $C$ . In other words, if  $\mathbf{a}$  and  $\mathbf{b}$  are in  $C$  then

$$\{\theta\mathbf{a} + (1 - \theta)\mathbf{b} : 0 \leq \theta \leq 1\} \subset C.$$

**Definition (Convex Polyhedron).** Let  $\mathbf{a}_1, \dots, \mathbf{a}_p$  be  $p$  points in  $\mathbb{R}^n$ . The  $n$ -tuple

$$\sum_{j=1}^p \theta_j \mathbf{a}_j, \quad \theta_j \geq 0, \quad j = 1, \dots, p, \quad \sum_{j=1}^p \theta_j = 1$$

is called a convex combination (or a convex sum) of  $\mathbf{a}_1, \dots, \mathbf{a}_p$ . If  $X \subset \mathbb{R}^n$  then the set of all (finite) convex combination of points of  $X$  is called the convex hull of  $X$  and is denoted by  $H(X)$ . If  $X$  is finite,  $X = \{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ , then  $H(X)$  is called the convex polyhedron spanned by  $\mathbf{a}_1, \dots, \mathbf{a}_p$  and is also denoted by  $H(\mathbf{a}_1, \dots, \mathbf{a}_p)$ .

**Definition.** Let  $S$  be a nonempty convex set in  $\mathbb{R}^n$ , where  $\mathbb{R}^n$  is the  $n$ -dimensional Euclidean space. The function  $f : S \rightarrow \mathbb{R}$  is said to be *convex* if

$$f(\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1 - \lambda)f(\mathbf{x}_2)$$

for each  $\mathbf{x}_1, \mathbf{x}_2 \in S$  and for each  $\lambda \in [0, 1]$ . The function is said to be strictly convex if the above inequality holds as a strict inequality for each distinct  $\mathbf{x}_1, \mathbf{x}_2 \in S$  and for each  $\lambda \in (0, 1)$ .

Let

$$\mathbf{a}_1 = (1, -1), \quad \mathbf{a}_2 = (2, 2), \quad \mathbf{a}_3 = (3, 1)$$

be points of  $\mathbb{R}^2$ . Find the convex polyhedron.

**Problem 64.** Show that the function  $\phi : [0, \infty) \rightarrow \mathbb{R}$  defined by

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \ln x & \text{if } x \neq 0 \end{cases} \quad (1)$$



is strictly convex, i.e.,

$$\phi(\alpha x + \beta y) \leq \alpha\phi(x) + \beta\phi(y) \tag{2}$$

if  $x, y \in [0, \infty)$ ,  $\alpha, \beta \geq 0$ ,  $\alpha + \beta = 1$  with equality only when  $x = y$  or  $\alpha = 0$  or  $\beta = 0$ . By induction we get

$$\phi\left(\sum_{i=1}^k \alpha_i x_i\right) \leq \sum_{i=1}^k \alpha_i \phi(x_i) \tag{3}$$

if

$$x_i \in [0, \infty), \quad \alpha_i \geq 0, \quad \sum_{i=1}^k \alpha_i = 1. \tag{4}$$

Equality holds only when all the  $x_i$ , corresponding to non-zero  $\alpha_i$ , are equal.

**Problem 65.** Let  $x \geq 0$  and

$$f(x) = x \ln(x) - x + 1. \tag{1}$$

(i) Show that

$$f(x) \geq 0. \tag{2}$$

(ii) Show that from L'Hospital rule we find that  $f(0) = 1$ .

(iii) Show that the function is convex.

**Problem 66.** Show that  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = |x| \tag{1}$$

is convex.

**Problem 67.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined as

$$f(x, y) = ax^2 + by^2 + 2cxy + d$$

where  $a, b, c, d \in \mathbb{R}$ . For what values of  $a, b, c, d$  is  $f$  concave?

**Problem 68.** Let  $r_1, r_2 \in \mathbb{R}$ . Consider the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = r_1 x_1 (1 - x_1 - x_2), \quad f_2(x_1, x_2) = r_2 x_1 x_2.$$

Find  $df_1, df_2, df_1 \wedge df_2$ .

**Problem 69.** Find  $x = \arctan(-1/\sqrt{3})$ .

**Problem 70.** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be analytic functions. Find

$$e^{f(x)} \frac{d^2}{dx^2} (e^{-f(x)} g(x)).$$

**Problem 71.** Consider the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = \sin(x_1^2 + x_2^2), \quad f_2(x_1, x_2) = \cos(x_1^2 + x_2^2).$$

Find the Jacobian matrix and the Jacobian determinant.

**Problem 72.** Consider the map  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = -\frac{1}{2}x^2 - x + \frac{1}{2}.$$

Then  $f(1) = -1$ ,  $f(-1) = 1$ . Thus we have a periodic orbit with period 2. Is the orbit attracting or repelling? We set  $x_1 = 1$ ,  $x_2 = -1$ . We have to test

$$|(df(x = x_1)/dx)(df(x = x_2)/dx)| < 1$$

for attracting and

$$|(df(x = x_1)/dx)(df(x = x_2)/dx)| > 1$$

for repelling.

**Problem 73.** Let  $\alpha \in \mathbb{R}$  and  $f(x) = e^{\alpha x}$ . Find

$$\frac{1}{h^2} (f(x+h) - 2f(x) + f(x-h)).$$

Then consider  $h \rightarrow 0$ .

**Problem 74.** Consider the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \cos(x)$ . The equation  $\cos(x) = x$  has one solution, i.e. we have one fixed point for  $f$ . Consider  $f(f(x)) = \cos(\cos(x))$ . Does the equation  $\cos(\cos(x)) = x$  admits other solutions besides the one of  $\cos(x) = x$ .

**Problem 75.** Let  $a, b \in \mathbb{N}$ . Consider the  $2 \times 2$  matrix

$$C = \begin{pmatrix} 1 & a \\ b & 1+ab \end{pmatrix}$$

with  $\det(C) = 1$ . Consider the map (generalized Arnold cat map)

$$\begin{pmatrix} x_{1,\tau+1} \\ x_{2,\tau+1} \end{pmatrix} = C \begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \end{pmatrix} \pmod{1}$$

where  $\tau = 0, 1, 2, \dots$ . Find the eigenvalues of  $C$  which relates to the one-dimensional Liapunov exponents.

**Problem 76.** Consider the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = 2x, \quad g(x) = 3x^2 - 1.$$

Find  $(f \circ g)(x) = f(g(x))$ ,  $(g \circ f)(x) = g(f(x))$  and the extrema of these functions.

**Problem 77.** Let  $a \in \mathbb{R}$ . Given the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\mathbf{f}(x_1, x_2) = (x_1 + x_2, 2x_1 + ax_2).$$

- (i) Find the fixed points of the map.
- (ii) Find the matrix  $D\mathbf{f}(x_1, x_2)$  and show that the matrix  $D\mathbf{f}(x_1, x_2)$  is invertible if and only if  $a \neq 2$ .

**Problem 78.** Let  $A, B$  be non-empty sets. If  $f : A \rightarrow B$  is (1,1) and onto (i.e. bijective) one can define the inverse function  $f^{-1} : B \rightarrow A$  as the unique function from  $B$  to  $A$  such that

$$(f \circ f^{-1})(y) = y \text{ for all } y \in B$$

$$(f^{-1} \circ f)(x) = x \text{ for all } x \in A.$$

The function  $f : (0, 1) \rightarrow \mathbb{R}$

$$x \mapsto f(x) = \frac{x - 1/2}{x(x - 1)}$$

is bijective. Find  $f^{-1}$ . Note that  $f(1/2) = 0$ ,  $f(1/4) = 4/3$ ,  $f(3/4) = -4/3$ .

**Problem 79.** Give non-trivial functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  which have no critical points and satisfy  $f(0) = 0$ . Trivial means  $f(x) = cx$ , where  $c$  is a constant.

### Programming Problems

**Problem 1.** Let  $f_1, f_2, f_3 : \mathbb{R}^3 \rightarrow \mathbb{R}$  be continuously differentiable function. Find the determinant of the  $3 \times 3$  matrix  $A = (a_{jk})$

$$a_{jk} := \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j}.$$

Apply computer algebra.

**Problem 2.** Consider the map  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) = x^2 + x + 1.$$

Find the minima and maxima of

$$g(x) = f(f(x)) - f(x)f(x).$$

Apply computer algebra.

**Problem 3.** Consider the polynomials

$$f(x) = x^2 + 2x + 1, \quad g(x) = x + 4;$$

Find  $(f \circ g)(x) = f(g(x))$ ,  $(g \circ f)(x) = g(f(x))$  and  $f(g(x)) - g(f(x))$ .

**Problem 4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an analytic function. The *Schwarzian derivative* is defined by

$$S(f(x)) := \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2$$

where  $'$  denotes the derivative with respect to  $x$ . Let  $a, b, c, d \in \mathbb{R}$  with  $ad - bc \neq 0$  and

$$g(x) = \frac{af(x) + b}{cf(x) + d}.$$

Show that  $S(g(x)) = S(f(x))$ . Apply computer algebra.

## 3.2 Supplementary Problems

**Problem 1.** Find

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}.$$

**Problem 2.** Consider the analytic function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x, y) = \frac{xe^x(e^y - 1) - ye^y(e^x - 1)}{xy(e^x - e^y)}.$$

(i) Show that  $f(0, 0) = \frac{1}{2}$  applying the L'Hospital rule.

(ii) Show that

$$f(x, x) = \frac{e^x - x - 1}{x^2}.$$

(iii) Show that

$$f(x, -x) = \frac{\tanh(x/2)}{x}.$$

**Problem 3.** Consider the map from  $\mathbb{R}^4$  to the vector space of  $2 \times 2$  matrices over  $\mathbb{R}$

$$f(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + x_4 & x_2 + x_3 \\ -x_2 + x_3 & x_1 - x_4 \end{pmatrix}.$$

Find the inverse of the map. Note that the determinant of the matrix is given by  $x_1^2 + x_2^2 - x_3^2 - x_4^2$ .

**Problem 4.** Let

$$D = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

Let  $\mathbf{v}$  be a normalized vector in  $\mathbb{R}^3$  with nonnegative entries and  $A$  be a  $3 \times 3$  matrix over  $\mathbb{R}$  with strictly positive entries. Show that the map  $f : D \rightarrow D$

$$f(\mathbf{v}) = \frac{A\mathbf{v}}{\|A\mathbf{v}\|}$$

has a fixed point, i.e. there is a normalized vector  $\mathbf{v}_0$  such that

$$\frac{A\mathbf{v}_0}{\|A\mathbf{v}_0\|} = \mathbf{v}_0.$$

**Problem 5.** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be analytic functions. Find

$$(e^{f(x)} \frac{d^2}{dx^2} e^{-f(x)})x.$$

**Problem 6.** Let  $\mathbb{Z}$  be the set of integers and  $\mathbb{N}_0$  be the set of natural numbers including 0. Consider the one-to-one map  $f : \mathbb{Z} \rightarrow \mathbb{N}_0$

$$f(j) = \begin{cases} 0 & \text{if } j = 0 \\ 2j - 1 & \text{if } j \text{ is positive} \\ -2j & \text{if } j \text{ is negative} \end{cases}.$$

Give a C++ implementation with the class `Verylong` of the function  $f$  and its inverse.

**Problem 7.** Let  $\{f_1, f_2, \dots, f_n\}$  be a set of convex functions from  $\mathbb{R}^n \rightarrow \mathbb{R}$ . Show that the nonnegative linear combination

$$f(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x}) + \dots + \alpha_n f_n(\mathbf{x}), \quad \alpha_1, \alpha_2, \dots, \alpha_n \geq 0$$

is convex.

**Problem 8.** Study the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = \sqrt{1 - \sin(x_2)}, \quad f_2(x_1, x_2) = \sqrt{1 + \sin(x_1)}.$$

First find the fixed points if there are any.

**Problem 9.** Let  $k \in \mathbb{N}$  and  $x \geq 0$ . Find the fixed points of the map

$$f_k(x) = \sqrt{k + x}$$

and study their stability.

**Problem 10.** Consider the continuous function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$f(x) = \begin{cases} 2x & x < 1 \\ \max\{4 - 2x, \frac{1}{4}\} & x \geq 1 \end{cases}$$

Find the fixed points and study their stability.

**Problem 11.** Let

$$D = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

Let  $\mathbf{v}$  be a normalized vector in  $\mathbb{R}^3$  with nonnegative entries and  $A$  be a  $3 \times 3$  matrix over  $\mathbb{R}$  with strictly positive entries. Show that the map  $f : D \rightarrow D$

$$f(\mathbf{v}) = \frac{A\mathbf{v}}{\|A\mathbf{v}\|}$$

has fixed point, i.e. there is a  $\mathbf{v}_0$  such that

$$\frac{A\mathbf{v}_0}{\|A\mathbf{v}_0\|} = \mathbf{v}_0.$$

**Problem 12.** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be two analytic functions and  $S$  be the Schwarzian derivative. Show that if  $Sf < 0$  and  $Sg < 0$ , then

$$S(f \circ g) < 0.$$

**Problem 13.** Let  $x, y \in \mathbb{R}^+$ . Consider the function  $f : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}$

$$f(x, y) = \frac{\sqrt{1+x}}{\sqrt{1+x} + \sqrt{1+y}}.$$

Then  $f(0, 0) = 1/2$  and  $f(x, y) + f(y, x) = 1$ . Set

$$\frac{\sqrt{1+x}}{\sqrt{1+x} + \sqrt{1+y}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F(m, n) x^m y^n.$$

Show that  $F(m, n) = -F(n, m)$  unless  $m = n = 0$  and  $F(m, m) = 0$  for  $m \geq 1$ . Show that

$$F(0, 0) = 1/2, \quad F(0, 1) = -F(1, 0) = -1/8, \quad F(0, 2) = -F(2, 0) = 1/16,$$

$$F(0, 3) = -F(3, 0) = -5/128, \quad F(1, 2) = -F(2, 1) = -1/128.$$

**Problem 14.** Consider the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = e^{x_1} \cos(x_2), \quad f_2(x_1, x_2) = e^{x_1} \sin(x_2).$$

Find the fixed points of the map.

**Problem 15.** Consider the function  $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} x_1 \cos(x_2) \sin(x_3) \\ x_1 \sin(x_2) \sin(x_3) \\ x_1 \cos(x_3) \end{pmatrix}.$$

Find the fixed points and study their stability.

**Problem 16.** Consider the function  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix}.$$

The Jacobian matrix is

$$\begin{pmatrix} \partial f_1 / \partial x_1 & \partial f_1 / \partial x_2 \\ \partial f_2 / \partial x_1 & \partial f_2 / \partial x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}$$

with determinant equal to  $4(x_1^2 + x_2^2)$ . Find the fixed points and study their stability.

**Problem 17.** Consider the differential operators

$$\begin{aligned} D_1 &= \frac{d^2}{dx^2} + \frac{c}{x^2} + \frac{x^2}{16} \\ D_2 &= \sinh(\alpha) \left( \frac{d^2}{dx^2} + \frac{c}{x^2} - \frac{x^2}{16} \right) + \cosh(\alpha) \left( -\frac{i}{2}x \frac{d}{dx} - \frac{i}{4} \right) \\ D_3 &= \cosh(\alpha) \left( \frac{d^2}{dx^2} + \frac{c}{x^2} - \frac{x^2}{16} \right) + \sinh(\alpha) \left( -\frac{i}{2}x \frac{d}{dx} - \frac{i}{4} \right). \end{aligned}$$

(i) Show the differential operator satisfy (Lie algebra  $su(1,1)$ )

$$[D_1, D_2] = -iD_3, \quad [D_2, D_3] = iD_1, \quad [D_3, D_1] = iD_2.$$

For the actual calculations let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function and one shows that  $[D_1, D_2]f(x) = -iD_3f(x)$  etc.

(ii) Show that

$$D_3^2 - D_1^2 - D_2^2 = -\frac{3}{16} - \frac{c}{4}.$$

**Problem 18.** Let

$$\mathbf{v}(\tau) = \begin{pmatrix} v_1(\tau) \\ v_2(\tau) \\ v_3(\tau) \end{pmatrix}, \quad \mathbf{w}(\tau) = \begin{pmatrix} w_1(\tau) \\ w_2(\tau) \\ w_3(\tau) \end{pmatrix}$$

where  $v_j(\tau), w_j(\tau)$  ( $j = 1, 2, 3$ ) are smooth functions of  $\tau$ . Calculate

$$\frac{d}{d\tau}(\mathbf{v} \times (\mathbf{v} \times \mathbf{w})).$$

**Problem 19.** Show that the  $(2, 2)$  Padé approximant of the cosine function is given by

$$\cos(x) \approx \frac{12 - 5x^2}{12 + x^2}.$$

**Problem 20.** Let  $f(x) = x^2$  and  $g(x) = x \ln(x)$ . Show that

$$\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0.$$

**Problem 21.** Let  $c > 0$ . Find

$$\lim_{x \rightarrow 0} \frac{\sinh(cx)}{x}, \quad \lim_{x \rightarrow 0} \frac{x}{\sinh(cx)}.$$



**Problem 22.** Show that the function  $f : (0, 1) \rightarrow \mathbb{R}$

$$f(x) = \frac{x - 1/2}{x(x - 1)}$$

is bijective.

**Problem 23.** Show that the function

$$f(x) = \frac{\pi}{\sin(\pi x)}$$

has poles at  $0, \pm 1, \pm 2, \dots$

**Problem 24.** For any real number  $x$  one defines  $\lfloor x \rfloor$  (floor of  $x$ ) as the largest integer less than or equal to  $x$ . One defines  $\lceil x \rceil$  (ceiling of  $x$ ) as the smallest integer greater than or equal to  $x$ . Find the minima and the maxima of the function

$$f(x) = 2x - \lfloor x \rfloor.$$

**Problem 25.** Let  $0 \leq t < 2$ . Show that

$$\ln \left( \frac{2+t}{2-t} \right) = \ln \left( \frac{1+t/2}{1-t/2} \right) = 2 \operatorname{arctanh}(t/2).$$

**Problem 26.** Consider the continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{1}{2} \left( \frac{x}{|x|+1} + 1 \right).$$

Find the fixed points and study their stability.

**Problem 27.** Consider the function  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_1(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2^2), \quad f_2(x_1, x_2) = x_1 x_2.$$

Find the fixed points and study their stability. Is there an inverse  $\mathbf{f}^{-1}$ .

**Problem 28.** Explain

$$\sin(\alpha/2) = \pm \sqrt{\frac{1 - \cos(\alpha)}{2}}, \quad \cos(\alpha/2) = \pm \sqrt{\frac{1 + \cos(\alpha)}{2}}.$$

**Problem 29.** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that the function is not analytic. Note that  $f(x) = f(-x)$  and

$$\frac{df}{dx} = \frac{2}{x^3}f(x), \quad \frac{d^2f}{dx^2} = \left(\frac{4}{x^6} - \frac{6}{x^4}\right)f(x)$$

Furthermore  $\lim_{x \rightarrow \pm\infty} f(x) = 1$ .

**Problem 30.** Let  $N > 0$ . Show that

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-N} e^{-(1/\epsilon)} = 0.$$

**Problem 31.** Consider the map  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$\begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 \cos(x_2) \\ x_1 \sin(x_2) \end{pmatrix}.$$

Find the fixed points and study their stability. Show that the functional matrix is given by

$$\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix} = \begin{pmatrix} \cos(x_2) & -x_1 \sin(x_2) \\ \sin(x_2) & x_1 \sin(x_2) \end{pmatrix}.$$

Is the map  $\mathbf{f}$  invertible?

**Problem 32.** Let  $a > 0$ . Show that

$$\lim_{x \rightarrow 0} \frac{\sinh(2ax)}{\sinh(x)} = 2a.$$

**Problem 33.** (i) Write down the first four terms of the Taylor expansion of  $\cos(x)$ . Then find  $(2, 2)[x]$  of the Padé approximant. Discuss the case  $x \rightarrow \infty$  for both functions.

(ii) Write down the first four terms of the Taylor expansion of  $\cosh(x)$ . Then find  $(2, 2)[2]$  of the Padé approximant. Discuss the case  $x \rightarrow \infty$  for both functions.

**Problem 34.** Consider the Hénon map ( $a > 0, b > 0$ )

$$x_{1,t+1} = a - bx_{2,t} + x_{1,t}^2, \quad x_{2,t+1} = x_{1,t}, \quad t = 0, 1, 2, \dots$$

Let  $|x_{1,0}| < R$ ,  $|x_{2,0}| < R$ , where  $R$  is the larger root of  $\lambda^2 - (|b|+1)\lambda - a = 0$ . Show that all points  $(x_{1,0}, x_{2,0})$  outside of this domain tend to  $\infty$  or  $-\infty$  for  $t \rightarrow \infty$  or  $t \rightarrow -\infty$ .

**Problem 35.** Consider the one-dimensional map ( $t = 0, 1, 2, \dots$ )

$$x_{t+1} = \frac{x_t + 2}{x_t + 1}, \quad x_0 \geq 0.$$

Find the fixed points of the map. Let  $x_0 = 0$ . Does  $\lim_{t \rightarrow \infty} x_t$  tend to a fixed point?

**Problem 36.** Consider the analytic functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \sinh(x), \quad g(x) = 2x^3.$$

Find  $f^{-1} \circ g \circ f$ ,  $g^{-1} \circ f \circ g$ .

**Problem 37.** Let  $R > r > 0$ . Consider the function  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  (battle of the sexes)

$$f_1(x_1, x_2) = (x_1 \quad 1 - x_1) \begin{pmatrix} R & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} x_2 \\ 1 - x_2 \end{pmatrix} = ((R+r)x_2 - r)x_1 + r(1 - x_2)$$

$$f_2(x_1, x_2) = (x_1 \quad 1 - x_1) \begin{pmatrix} r & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} x_2 \\ 1 - x_2 \end{pmatrix} = ((R+r)x_2 - R)x_2 + R(1 - x_2).$$

Find the fixed points and study their stability.

**Problem 38.** Let  $c_1, c_2 \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an analytic function. Show that

$$\exp\left(c_1 \frac{d}{dx}\right) \exp(c_2 x) f(x) \equiv \exp(c_2 x) \exp\left(c_1 \frac{d}{dx}\right) \exp(c_1 c_2) f(x).$$

**Problem 39.** Let  $n$  be a non-negative integer. Let  $k = 0, 1, \dots, n$ . Find the derivative

$$\frac{\partial^n}{\partial x^k \partial x^{n-k}} (x + y)^n.$$

Show that

$$\frac{\partial^n}{\partial x^k \partial x^{n-k}} (x + y)^n = n!.$$

**Problem 40.** The content ( $n$ -dimensional volume) bounded by a hypersphere of radius  $r$  is known to be

$$V_n = \frac{2r^n \pi^{n/2}}{n\Gamma(n/2)}$$

where  $\Gamma$  is the *gamma function*. Let  $r = 1$ . Show that

$$\lim_{n \rightarrow \infty} V_n = 0.$$

**Problem 41.** Let  $n \geq 1$ . Show that the vector space spanned by

$$x^n, yx^{n-1}, \dots, y^{n-1}x, y^n$$

is  $n + 1$  dimensional.

**Problem 42.** Let  $\epsilon, x \in \mathbb{R}$ . Show that

$$\lim_{\epsilon \rightarrow 0} \frac{\sinh(\epsilon x)}{\sinh(\epsilon)} = x, \quad \lim_{\epsilon \rightarrow 0} \frac{\sin(\epsilon x)}{\sin(\epsilon)} = x.$$

**Problem 43.** The *Hurwitz zeta-function*  $\zeta_H(s, a)$  is defined by

$$\zeta_H(s, a) := \sum_{k=0}^{\infty} (n+a)^{-s}, \quad 0 < a \leq 1, \quad \Re(s) > 1.$$

(i) Show that the Hurwitz zeta-function can be written in the form

$$\zeta_H(s, a) = a^{-s} + \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt$$

where  $\Gamma(s)$  is the Gamma function.

(ii) Show that

$$\begin{aligned} \frac{\partial}{\partial a} \zeta_H(s, a) &= -s \zeta_H(s+1, a) \\ \frac{\partial}{\partial s} \zeta_H(s, a) \Big|_{s=0} &= \ln(\Gamma(a)) - \frac{1}{2} \ln(2\pi). \end{aligned}$$

**Problem 44.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an analytic function. Show that (differential identity)

$$\frac{d^n}{dx^n} f(x) = x^{-n} \frac{d^n}{d\epsilon^n} f(\epsilon x) \Big|_{\epsilon=1}.$$

**Problem 45.** Find an analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(\infty) = 1, \quad f(-\infty) = 0$$

and

$$f(x_1) \leq f(x_2) \text{ whenever } x_1 \leq x_2.$$

Find the derivative of the function  $f$  and determine the maxima of the function  $df/dx$ .

**Problem 46.** Let

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that

$$\Theta(x_1 - x_3) - \Theta(x_2 - x_3) = \begin{cases} 1 & \text{for } x_1 > x_3 > x_2 \\ -1 & \text{for } x_2 > x_3 > x_1 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 47.** (i) Let

$$f(\theta) = \frac{\cos\left(\frac{\pi}{2} \cos(\theta)\right)}{\sin(\theta)}.$$

Find  $f(\theta)$  for  $\theta = n\pi$  ( $n \in \mathbb{Z}$ ) using the L'Hospital rule. Plot  $f(\theta)$  as a function of  $\theta$ .

(ii) Apply the L'Hospital rule to show that

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$$

(iii) Show that

$$\lim_{x \rightarrow 0} \frac{\sinh(x)}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{x}{\sinh(x)} = 1.$$

(iv) Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{x \cosh(x) - \sinh(x)}{2x \sinh^2(x)}.$$

Find

$$\lim_{x \rightarrow 0} f(x), \quad \lim_{x \rightarrow \infty} f(x).$$

**Problem 48.** Consider the functions

$$f(x) = \frac{x}{\sinh(x)}, \quad g(x) = \frac{\sinh(x)}{x}.$$

Find

$$\lim_{x \rightarrow 0} f(x), \quad \lim_{x \rightarrow 0} g(x).$$

**Problem 49.** Let  $x \in \mathbb{R}$ . Show that

$$\lim_{x \rightarrow \infty} \frac{1 - e^{-x}}{x} = 1.$$

**Problem 50.** (i) Show that

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0.$$

(ii) Show that

$$\lim_{x \rightarrow 1} \frac{x - 1}{\ln(x)} = 1.$$

(iii) Show that

$$\lim_{x \rightarrow 0} \frac{\cosh(x) - \cosh(2x)}{\sinh(x)} = 0.$$

(iv) Let  $x > 0$ . Show that

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1.$$

**Problem 51.** Let  $\alpha \in \mathbb{R}$ . Find

$$\lim_{\alpha \rightarrow 0} \frac{\sin(\alpha) \cos(\alpha)}{\alpha}.$$

**Problem 52.** Let  $k > 0$  and dimension 1/length. Find the extrema of the functions

$$\begin{aligned} f_1(x) &= \ln(\cosh(kx)) \\ f_2(x) &= \cosh(kx) + \alpha \sinh(kx), \quad -1 \leq \alpha \leq +1 \\ f_3(x) &= A \operatorname{sech}(kx) \equiv A \frac{1}{\cosh(kx)} \\ f_4(x) &= a \operatorname{sech}^2(kx) \equiv a \frac{\sinh^2(kx)}{\cosh^2(kx)}. \end{aligned}$$

**Problem 53.** Consider the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \cos(x) + \sin(x).$$

- (i) Find the critical points of  $f$  and the minima and maxima of the function.
- (ii) Find the roots of  $f$ , i.e. solve  $f(x) = 0$ .
- (iii) Find the fixed points of  $f$ , i.e. solve  $f(x) = x$ .
- (iv) Find the differential equation together with the initial conditions the function  $f$  satisfies.

**Problem 54.** Consider the polynomial  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^3 - \frac{3}{2}x^2 + x - \frac{3}{2}.$$

Show that there is a root, i.e. a solution of  $f(x) = 0$  in the interval  $[1, 2]$ . Apply the Newton method (say 3 steps) to find an approximate solution for the root. Apply the two initial condition  $x_0 = 1$  and  $x_0 = 2$ . After finding this root how would you proceed to find the other two roots?

**Problem 55.** A real valued function  $f$  defined on an open subset  $G$  of a Banach space  $E$  is said to be Fréchet differentiable at a point  $x \in G$  if there is  $f'(x) \in E^*$  such that

$$\lim_{v \rightarrow 0} \frac{|f(x+v) - f(x) - \langle f'(x), v \rangle|}{\|v\|} = 0.$$

Then  $f'(x)$  is called the Fréchet derivative of  $f$  at  $x$ . Let  $E = \mathbb{R}$ . Let  $f(x) = x^3 + 2x^2 + 3x + 4$ . Find the Fréchet derivative.

**Problem 56.** Consider the function

$$f(x) = \frac{\exp(-x^2/2) \int_0^x \exp(\tau^2/2) d\tau}{x}.$$

Show that

$$\lim_{x \rightarrow 0} f(x) = 1.$$

**Problem 57.** Consider the functions  $f : \mathbb{R} \rightarrow [-1, 1]$ ,  $g : \mathbb{R} \rightarrow [-1, 1]$

$$f(x) = \sin(x), \quad g(x) = \cos(x).$$

- (i) Find the minima and maxima of the function

$$h_1(x) = \max(f(x), g(x)).$$

Is the function  $h_1$  differentiable?

(ii) Find the minima and maxima of the function

$$h_2(x) = \min(f(x), g(x)).$$

Is the function  $h_2$  differentiable?

**Problem 58.** Let  $a \in \mathbb{R}$ . Show that

$$\lim_{x \rightarrow \infty} (1 + a/x)^x = e^a.$$

**Problem 59.** Let  $k \in \mathbb{R}$ ,  $x, y \in \{+1, -1\}$  such that  $xy = \pm 1$ . Show that

$$e^{kxy} = \cosh(k) + xy \sinh(k).$$

**Problem 60.** Consider the coordinate transformation

$$\begin{aligned} x_1(r, \alpha, \beta, \gamma) &= r \sin(\beta/2) \sin((\gamma - \alpha)/2) \\ x_2(r, \alpha, \beta, \gamma) &= r \sin(\beta/2) \cos((\gamma - \alpha)/2) \\ x_3(r, \alpha, \beta, \gamma) &= r \cos(\beta/2) \sin((\gamma + \alpha)/2) \\ x_4(r, \alpha, \beta, \gamma) &= r \cos(\beta/2) \sin((\gamma + \alpha)/2). \end{aligned}$$

Show that  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2$ .



## Chapter 4

# Polynomial

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### 4.1 Solved Problems

**Problem 1.** The *Chebyshev polynomials* are defined by

$$T_k(x) = \cos(k \arccos(x)), \quad k = 0, 1, \dots \quad x \in [-1, 1]$$

Thus the first six polynomials are

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 + 5x. \end{aligned}$$

Find  $1, x, x^2, x^3, x^4, x^5$  as functions of  $T_0, T_1, T_2, T_3, T_4, T_5$ .

**Problem 2.** Let  $L_n(x), H_n(x)$  be the *Laguerre polynomials* and *Hermite polynomials*, where  $n = 0, 1, \dots$ . Let

$$L_n^{(\alpha)}(x) := \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha})$$

be the associated Laguerre polynomials with  $\alpha > -1$  and  $n = 0, 1, \dots$ . The Laguerre polynomials are recovered by setting  $\alpha = 0$ . We have

$$H_{2n}(x) = (-4)^n n! L_n^{(-1/2)}(x^2) \quad (1)$$

and the following addition formula for the associated Laguerre polynomials  $L_n^\alpha(x)$

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^n L_{n-k}^{(\alpha)}(x) L_k^{(\beta)}(y). \quad (2)$$

- (i) Find a new sum rule by inserting (2) into (1).  
(ii) Consider the *sum rule*

$$\frac{1}{n!2^n} H_n(\sqrt{2}x) H_n(\sqrt{2}y) = \sum_{k=1}^n (-1)^k L_{n-k}^{(-1/2)}((x+y)^2) L_k^{(-1/2)}((x-y)^2). \quad (3)$$

Insert (1) into (3) to find a sum rule for Hermite polynomials.

**Problem 3.** The *Hermite polynomial* of degree  $n$  can be written as

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}.$$

Express  $x^n$  using the Hermite polynomials.

**Problem 4.** Given the differentiable function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with

$$f(x) = 4x(1-x).$$

- (i) Find the *fixed points* of  $f(x)$  and  $f(f(x))$ .  
(ii) Find the *critical points* of  $f(x)$  and  $f(f(x))$ .

**Problem 5.** Let  $P(z)$  be a polynomial of degree  $n \geq 2$  with distinct zeros  $\zeta_1, \dots, \zeta_n$ . Show that

$$\sum_{j=1}^n \frac{1}{P'(\zeta_j)} = 0$$

where  $'$  denotes the derivative, i.e.  $P'(\zeta) \equiv dP(z = \zeta)/dz$ .

**Problem 6.** Given a set of  $N$  real numbers  $x_1, x_2, \dots, x_N$ . It is often useful to express the sum of the  $j$  powers

$$s_j = x_1^j + x_2^j + \dots + x_N^j, \quad j = 0, 1, 2, \dots$$

in terms of the *elementary symmetric functions*

$$\begin{aligned}\sigma_1 &= \sum_{i=1}^N x_i \\ \sigma_2 &= \sum_{i<j}^N x_i x_j \\ \sigma_3 &= \sum_{i<j<k}^N x_i x_j x_k \\ &\vdots \\ \sigma_N &= x_1 x_2 \cdots x_N.\end{aligned}$$

Consider the special case with three numbers  $x_1, x_2, x_3$ . Then the elementary symmetric functions are given by

$$\sigma_1 = x_1 + x_2 + x_3, \quad \sigma_2 = x_1 x_2 + x_1 x_3 + x_2 x_3, \quad \sigma_3 = x_1 x_2 x_3.$$

We know that the elementary symmetric functions are the coefficients (up to sign) of the polynomial with the roots  $x_1, x_2, x_3$ . In other words the values of  $x_1, x_2, x_3$  each satisfy the polynomial equation

$$x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3 = 0. \tag{1}$$

Find a *recursion relation* for

$$s_j := x_1^j + x_2^j + x_3^j, \quad j = 0, 1, 2, \dots$$

and give the initial values  $s_0, s_1, s_2$ . Calculate  $s_3$  and  $s_4$ .

**Problem 7.** The *dominant tidal potential* at position  $(r, \phi, \lambda)$  due to the moon or sun is given by

$$U(\mathbf{r}) = \frac{GM^* r^2}{r^{*3}} P_2^0(\cos(\psi))$$

where  $M^*$  is the mass of the moon or sun located at  $(r^*, \phi^*, \lambda^*)$ . Moreover,  $\psi$  is the angle between mass  $M^*$  and the observation point at  $(r, \phi, \lambda)$ , where  $\phi$  is the latitude and  $\lambda$  is the longitude. By the *spherical cosine theorem* we have

$$\cos(\psi) = \sin(\phi) \sin(\phi^*) + \cos(\phi) \cos(\phi^*) \cos(\lambda - \lambda^*).$$

The *Legendre polynomials* are defined as

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

with  $n = 0, 1, 2, \dots$  and the *associated Legendre polynomials* are defined as

$$P_n^m(x) := (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) = \frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n$$

with  $P_n^0(x) = P_n(x)$  and  $P_n^m = 0$  if  $m > n$ .

(i) Show that  $U(\mathbf{r})$  can be written as

$$U(\mathbf{r}) = \frac{GM^*r^2}{r^{*3}} \left( P_2^0(\sin \phi) P_2^0(\sin \phi^*) + \frac{1}{3} P_2^1(\sin \phi) P_2^1(\sin \phi^*) \cos(\lambda - \lambda^*) + \frac{1}{12} P_2^2(\sin \phi) P_2^2(\sin \phi^*) \cos(2(\lambda - \lambda^*)) \right).$$

(ii) Give an interpretation (maxima and nodes) of the terms in the parenthesis.

**Problem 8.** Consider the cubic equation

$$y^3 + py + q = 0, \quad p, q \in \mathbb{R}, \quad pq \neq 0. \quad (1)$$

Show that applying the nonlinear transformation

$$y(z) := z - \frac{p}{3z}$$

equation (1) can be reduced to

$$z^6 + qz^3 - \frac{p^3}{27} = 0$$

and with  $u = z^3$  to a quadratic equation.

**Problem 9.** Find all integers  $c$  for which the cubic equation

$$x^3 - x + c = 0$$

has three integer roots.

**Problem 10.** Let  $\Phi$  be an endomorphism of the space  $\mathbb{C}^n[X]$  of polynomials of degree  $n$  with complex coefficients, which maps a polynomial  $p(X)$  to the polynomial  $p(X+1)$ . Let  $\Psi$  be an endomorphism of the space  $\mathbb{C}^n[X]$  which maps a polynomial  $p(X)$  to  $(1-X)^n p\left(\frac{X}{1-X}\right)$ , which of course is also a polynomial. Show that we have a *braid-like relation*

$$\Phi \circ \Psi \circ \Phi = \Psi \circ \Phi \circ \Psi.$$

**Problem 11.** Let  $a \in \mathbb{R}$ . Let  $r$  and  $s$  be the roots of the quadratic equation

$$x^2 + ax + \frac{a^2 - 1}{2} = 0.$$

Find  $r^3 + s^3$  in terms of  $a$ , and express it as a polynomial in  $a$  with rational coefficients.

**Problem 12.** Consider the polynomial  $p(x) = x^3 - x^2 + x - 2$ . Does there exist a nontrivial polynomial  $q(x)$  with real coefficients such that the degree of every term of the product  $p(x)q(x)$  is a multiple of 3? If so, find one. If not, show there is none.

**Problem 13.** (i) Let  $n$  be a positive integer. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. The *Bernstein polynomials* of degree  $n$  associated with the continuous function  $f$  are given by

$$B_n(f(x), x) := \frac{1}{(b-a)^n} \sum_{j=0}^n \binom{n}{j} (x-a)^j (b-x)^{n-j} f(x_j)$$

where

$$x_j = a + j \frac{b-a}{n}, \quad j = 0, 1, \dots, n.$$

Consider the function  $f : [0, 1] \rightarrow \mathbb{R}$  given by  $f(x) = \sin(4x)$ . Show that  $B_2(f, x)$  is not a “good approximation” for  $f$ . Consider  $x = \pi/8$ .

(ii) The *Bernstein basis polynomials* are defined as

$$B_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}, \quad j = 0, 1, 2, \dots, n, \quad x \in [0, 1].$$

Show that

$$\sum_{j=0}^n B_{n,j}(x) = 1.$$

Show that  $B_{n,j}$  satisfies the recursion relations

$$B_{n,j}(x) = (1-x)B_{n-1,j}(x) + xB_{n-1,j-1}(x), \quad j = 1, 2, \dots, n-1$$

$$B_{n,0}(x) = (1-x)B_{n-1,0}(x)$$

$$B_{n,n}(x) = xB_{n-1,n-1}(x).$$

**Problem 14.** A one-dimensional map  $f$  is called an *invariant* of a two-dimensional map  $g$  if

$$g(x, f(x)) = f(f(x)).$$

Let

$$f(x) = 2x^2 - 1.$$

Show that  $f$  is an invariant for

$$g(x, y) = y - 2x^2 + 2y^2 + d(1 + y - 2x^2).$$

**Problem 15.** Consider the functions  $f(z) = z^3$  and  $h(z) = z + 1/z$ . Find a function  $p$  such that

$$h(f(z)) = p(h(z)). \quad (1)$$

**Problem 16.** Consider the map  $f_c(z) = z^2 + c$ , where  $c \in \mathbb{C}$ . Find all complex  $c$ -values where the map  $f_c$  has a fixed point  $z^*$  with  $f'_c(z^*) = -1$ .

**Problem 17.** Let  $p(x, y)$  be a real polynomial. Show that if  $p(x, y) = 0$  for infinitely many  $(x, y)$  on the unit circle  $x^2 + y^2 = 1$ , then  $p(x, y) = 0$  on the unit circle.

**Problem 18.** Show that the equation

$$z^n = a \quad (1)$$

where  $n$  is a positive integer and  $a$  is any nonzero complex number, has exactly  $n$  roots. Hint. Set

$$a = \rho(\cos(\phi) + i \sin(\phi)). \quad (2)$$

**Problem 19.** Let  $A$  be a  $2 \times 2$  matrix over the real numbers  $\mathbb{R}$ . The *trace* is defined as

$$\text{tr}(A) = a_{11} + a_{22}.$$

It can be proved that the trace is the sum of the eigenvalues of  $A$ , i.e.

$$\text{tr}(A) = \lambda_1 + \lambda_2.$$

Thus we have

$$\text{tr}(A^2) = \lambda_1^2 + \lambda_2^2.$$

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find the eigenvalues using the two equations.

**Problem 20.** Find the zeros of the cubic polynomial

$$\alpha(x) := a_0 + a_1x + a_2x^2 + x^3 \quad (1)$$

over  $\mathbb{C}$ , where  $a_0, a_1, a_2 \in \mathbb{R}$  and  $a_0 \neq 0$ .

**Problem 21.** Show that the zeros of

$$z^3 - 6z^2 + 11z - 6 = 0 \quad (1)$$

are given by 1, 2, and 3.

**Problem 22.** Find the zeros of the quartic polynomial

$$\alpha(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + x^4 \quad (1)$$

over  $\mathbb{C}$  when  $a_0 \neq 0$ .

**Problem 23.** Find the zeros of

$$\alpha(x) = 35 - 16x - 4x^3 + x^4. \quad (1)$$

**Problem 24.** The variational equation of the *Lorenz model*

$$\frac{dX}{dt} = -\sigma X + \sigma Y \quad (1a)$$

$$\frac{dY}{dt} = -XZ + \tau X - Y \quad (1b)$$

$$\frac{dZ}{dt} = XY - bZ \quad (1c)$$

is given by

$$\begin{pmatrix} dx_0/dt \\ dy_0/dt \\ dz_0/dt \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (\tau - Z) & -1 & -X \\ Y & X & -b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}. \quad (2)$$

(i) Show that Lorenz model possess the steady-state solution  $X = Y = Z = 0$ , representing the state of no convection.

(ii) Show that with this basic solution, the characteristic equation of the variational matrix is

$$(\lambda + b)(\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - \tau)) = 0. \quad (3)$$

(iii) Show that this equation has three real roots when  $\tau > 0$ ; all are negative when  $\tau < 1$ , but one is positive when  $\tau > 1$ . The criterion for the onset of convection is therefore  $\tau = 1$ .

(iv) Show that when  $\tau > 1$ , system (1) possess two additional steady state solutions

$$X = Y = \pm\sqrt{b(\tau - 1)}, \quad Z = \tau - 1. \quad (4)$$

(v) Show that for either of these solutions, the characteristic equation of the matrix in (2) is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (\tau + \sigma)b\lambda + 2\sigma b(\tau - 1) = 0. \quad (5)$$

(vi) Show that this equation possesses one real negative root and two complex conjugate roots when  $\tau > 1$ . Show that the complex conjugate roots are pure imaginary if the product of the coefficients of  $\lambda^2$  and  $\lambda$  equals the constant term, or

$$\tau = \sigma(\sigma + b + 3)(\sigma - b - 1)^{-1}. \quad (6)$$

**Problem 25.** The variational equation of the *Lotka Volterra model*

$$\frac{du_1}{dt} = u_1 - u_1u_2, \quad \frac{du_2}{dt} = -u_2 + u_1u_2 \quad (1)$$

is given by

$$\begin{pmatrix} dv_1/dt \\ dv_2/dt \end{pmatrix} = \begin{pmatrix} 1 - u_2 & -u_1 \\ u_2 & -1 + u_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (2)$$

where  $u_1 > 0$  and  $u_2 > 0$ .

(i) Show that (1) possess the steady-state solution  $u_1 = u_2 = 1$ .

(ii) Show that with this basic solution, the characteristic equation of the matrix in (2) is

$$\lambda^2 + 1 = 0. \quad (3)$$

(iii) Find the solution of the characteristic equation and discuss.

**Problem 26.** Show the following. Let  $P(z)$  be a polynomial. Then either

1.  $P(z)$  has a fixed point  $q$  with  $P'(q) = 1$ ,
2.  $P(z)$  has a fixed point  $q$  with  $|P'(q)| > 1$ .

**Problem 27.** Let  $S_n$  be the *symmetric group*. The symmetric group  $S_n$  acts naturally on polynomials in  $n$  variables. For a polynomial  $p$  in  $n$  variables, define the symmetrized polynomial associated to  $p$ ,  $\text{Sym}_n(p)$ , by

$$\text{Sym}_n(p) := \sum_{\sigma \in S_n} \sigma(p).$$



Let  $n = 2$  and

$$p(x_1, x_2) = x_1^2 x_2 + 2x_2.$$

Find  $\text{Sym}_2(p)$ .

**Problem 28.** Consider the system of equations

$$\begin{aligned} z_1 + z_2 + \cdots + z_{n-1} + z_n &= 0 \\ z_1 z_2 + z_2 z_3 + \cdots + z_{n-1} z_n + z_n z_1 &= 0 \\ &\vdots \\ z_1 z_2 \cdots z_{n-1} + z_2 z_3 \cdots z_n + \cdots + z_{n-1} z_n \cdots z_{n-3} + z_n z_1 \cdots z_{n-2} &= 0 \\ z_1 z_2 \cdots z_n &= 1. \end{aligned}$$

Find the solutions for the case  $n = 2$  and  $n = 3$ . This system of equations arise as follows. Let  $p$  be a prime number. A vector  $\mathbf{x} = (x_0, x_1, \dots, x_{p-1}) \in \mathbb{C}^p$  viewed as a function  $\mathbb{Z}/p \rightarrow \mathbb{C}$  has discrete Fourier transform  $\hat{\mathbf{x}} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{p-1})$ , where

$$\hat{x}_j = \sum_{k=0}^{p-1} \omega^{jk} x_k, \quad \omega := e^{2\pi i/p}.$$

The vector  $\mathbf{x}$  is called equimodular if all its coordinates have the same absolute value, and  $\mathbf{x}$  is called bi-equimodular if both  $\mathbf{x}$  and  $\hat{\mathbf{x}}$  are equimodular. The question is: Which vectors are bi-equimodular?

**Problem 29.** Let  $\alpha \in \mathbb{R}$ . Find the roots of the characteristic equation

$$\lambda^6 - 2 \cos(3\alpha) \lambda^3 + 1 = 0.$$

**Problem 30.** Show that the  $n$ -th order polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

which goes exactly through  $n + 1$  data points is unique.

**Problem 31.** Let  $p : \mathbb{R} \rightarrow \mathbb{R}$  be a polynomial that satisfies

$$p(1 - x) + 2p(x) = 3x$$

for all  $x \in \mathbb{R}$ . Find the values of  $p(0)$  and  $p(1)$ . Give an example of a polynomial that satisfies this condition.

**Problem 32.** (i) Let  $x_1, x_2 \in \mathbb{R}$  and  $x_1 \neq x_2$ . Find the solutions of the system of equations

$$x_1 - \frac{1}{(x_1 - x_2)^2} = 0, \quad x_2 + \frac{1}{(x_1 - x_2)^2} = 0.$$

(ii) Let  $x_1, x_2, x_3 \in \mathbb{R}$  and  $x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$ . Find the solutions of the system of equations

$$\begin{aligned} x_1 - \frac{1}{(x_1 - x_2)^2} - \frac{1}{(x_1 - x_3)^2} &= 0 \\ x_2 + \frac{1}{(x_1 - x_2)^2} - \frac{1}{(x_2 - x_3)^2} &= 0 \\ x_3 + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} &= 0. \end{aligned}$$

**Problem 33.** Let  $z, w \in \mathbb{C}$ . Find the solution of the system of equations

$$|z|^2 + |w|^2 = 1, \quad z^2 + w^3 = 0.$$

**Problem 34.** Consider the two polynomials

$$p_1(x) = a_0 + a_1x + \cdots + a_nx^n, \quad p_2(x) = b_0 + b_1x + \cdots + b_mx^m$$

where  $n = \deg(p_1)$  and  $m = \deg(p_2)$ . Assume that  $n > m$ . Let  $r(x) = p_2(x)/p_1(x)$ . We expand  $r(x)$  in powers of  $1/x$ , i.e.

$$r(x) = \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots$$

From the coefficients  $c_1, c_2, \dots, c_{2n-1}$  we can form an  $n \times n$  *Hankel matrix*

$$H_n = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}.$$

The determinant of this matrix is proportional to the *resultant* of the two polynomials. If the resultant vanishes, then the two polynomials have a non-trivial greatest common divisor. Apply this theorem to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \quad p_2(x) = x^2 + 4x + 3.$$

**Problem 35.** Let  $p_1$  and  $p_2$  two polynomials with  $n = \text{degree}(p_1)$ ,  $m = \text{degree}(p_2)$  and  $n > m$ . We expand the rational function

$$r(x) = \frac{p_2(x)}{p_1(x)}$$

with respect to powers of  $1/x$ , i.e.,

$$r(x) = d_1x^{-1} + d_2x^{-2} + \dots$$

The coefficients  $d_1, d_2, \dots, d_{2n-1}$  are inserted into the  $n \times n$  Hankel matrix

$$H_n = \begin{pmatrix} d_1 & d_2 & \dots & d_n \\ d_2 & d_3 & \dots & d_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ d_n & d_{n+1} & \dots & d_{2n-1} \end{pmatrix}.$$

A *Hankel matrix* is a diagonal matrix in which all the elements are the same along any diagonal that slopes from northeast to southwest. If the determinant of this matrix is zero, then the two polynomials have a non-trivial common divisor. Apply this algorithm to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \quad p_2(x) = x^2 + 6x + 8$$

to test whether they have a common non-trivial divisor.

**Problem 36.** Let  $p$  be a polynomial with coefficients in  $\mathbb{R}$ . If the equation  $p(x) = 0$  has repeated roots, then  $p(x)$  and  $dp(x)/dx$  have a highest common factor. Apply this to the polynomial

$$p(x) = 32x^4 - 64x^3 + 24x^2 + 8x - 3.$$

**Problem 37.** Let  $p$  be a polynomial with real coefficients. The equation  $p(x) = 0$  cannot have more positive roots than there are changes of sign from  $+$  to  $-$  and from  $-$  to  $+$  in the coefficients of the polynomial  $p(x)$ , or more negative roots than there are changes of sign in  $p(-x)$  (Descartes rule of signs). Apply the rule to the polynomial

$$p(x) = x^6 + 7x^3 + x - 2 = 0.$$

**Problem 38.** The Liouville-Riemann definition for the fractional integral operator  $D_x^{-q}$  is given by

$$D_x^{-q}f(x) := \frac{1}{\Gamma(q)} \int_0^x (x-y)^{q-1} f(y) dy, \quad q > 0.$$

The fractional differential operator  $D_x^\nu$  for  $\nu > 0$  is given by the definition

$$D_x^\nu f(x) := \frac{d^n}{dx^n} (D_x^{\nu-n} f(x)), \quad \nu - n < 0.$$

Let  $f(x) = x^2$ . Find  $D_x^{-q} f(x)$ .

**Problem 39.** Let  $n \in \mathbb{N}$ . Show by induction that  $x^n - y^n$  is divisible without remainder by  $x - y$  for all values of  $n$ . We have

$$x^{n+1} - y^{n+1} \equiv x(x^n - y^n) + y^n(x - y).$$

**Problem 40.** Consider the polynomial

$$P_n(x) = \sum_{j=0}^n (-1)^j a_j x^{n-j}, \quad a_0 = 1.$$

The homogeneous product sums symmetric functions  $h_k(x_1, \dots, x_n)$  of the zeros of this polynomial are defined as follows

$$\prod_{j=1}^n (1 - x_j x) = (1 + h_1 x + h_2 x^2 + h_3 x^3 + \dots)^{-1}.$$

Show that the first few sums are given explicitly by

$$\begin{aligned} h_1(x_1, \dots, x_n) &= \sum_{j=1}^n x_j \\ h_2(x_1, \dots, x_n) &= \sum_{j=1}^n x_j^2 + \sum_{j < k}^n x_j x_k \\ h_3(x_1, \dots, x_n) &= \sum_{j=1}^n x_j^3 + \sum_{j \neq k}^n x_j x_k + \sum_{j < k < \ell}^n x_j x_k x_\ell. \end{aligned}$$

**Problem 41.** Consider the *quintic equation*

$$p(x) = x^5 - 5x^3 + 5x - 5 = 0.$$

Is  $p$  irreducible over the rational numbers? What is the Galois group of  $p$ ? Look for solutions of the form  $r + 1/r$ .

**Problem 42.** The *Bernoulli polynomials*  $B_n(x)$  ( $n = 0, 1, \dots$ ) can be defined recursively by

$$\frac{dB_n(x)}{dx} = nB_{n-1}, \quad n = 1, 2, \dots$$

with  $B_0(x) = 1$  and the condition

$$\int_0^1 B_n(x) dx = 0, \quad n \geq 1.$$

The *Bernoulli numbers*  $B_n$  are defined by  $B_n := B_n(x = 0)$ .

(i) Find the first four Bernoulli polynomials.

(ii) Show that

$$B_m(x) = \sum_{n=0}^m \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (x+k)^m.$$

(iii) Show that

$$B_n(1-x) = (-1)^n B_n(x).$$

**Problem 43.** Consider the operators

$$D_1 = x, \quad D_2 = \frac{d}{dx}, \quad D_3 = x \frac{d}{dx}$$

which apply to the function of the *Bargmann space*

$$B := \left\{ f(n) = \frac{x^n}{\sqrt{n!}}, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R} \right\}.$$

Show that

$$D_1 f(n) = \sqrt{n+1} f(n+1), \quad D_2 f(n) = \sqrt{n} f(n-1), \quad D_3 f(n) = n f(n).$$

Find the commutators  $[D_1, D_2]$ ,  $[D_1, D_3]$ ,  $[D_2, D_3]$ .

**Problem 44.** Express the polynomial

$$p(x) = 7x_1^2 + 6x_2^2 + 5x_3^2 - 4x_1x_2 - 4x_2x_3 + 14x_1 - 8x_2 + 10x_3 + 6$$

in matrix form

$$p(x) = (x_1 \quad x_2 \quad x_3) A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathbf{v}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + c$$

Find the eigenvalues and normalized eigenvectors of the  $3 \times 3$  matrix.

**Problem 45.** Find the polynomials generated by

$$\frac{dp_{j+1}(x)}{dx} = (j+1)p_j(x), \quad j = 0, 1, 2, \dots$$

with  $p_0(x) = 1$ . The constants of integration we set to 0.

**Problem 46.** The complete *Bell polynomials*  $B_j(x_1, x_2, \dots, x_j)$  are given by the exponential generating function

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \dots, x_n) \frac{t^n}{n!}. \quad (1)$$

Taking the  $n$ -th derivative with respect to  $t$  we obtain

$$\left. \frac{d^n}{dt^n} \exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) \right|_{t=0} = B_n(x_1, \dots, x_n) \quad (2)$$

with  $B_0 = 1$ . Find the first four Bell polynomials.

### Programming Problem

**Problem 1.** Consider the polynomials  $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f_2 : \mathbb{R} \rightarrow \mathbb{R}$

$$f_1(x) = x^2 + 2x + 1, \quad f_2(x) = 2x.$$

Apply Maxima to find

$$h_1(x) = f_1(f_2(x)), \quad h_2(x) = f_2(f_1(x))$$

and  $C = f_1 - f_2$ .

## 4.2 Supplementary Problems

**Problem 1.** Construct a polynomial

$$p(x) = x^2 + ax + b$$

that admits the roots

$$\frac{1}{6}(\sqrt{3} + 3), \quad -\frac{1}{6}(\sqrt{3} - 3)$$

and the fixed points

$$\frac{1}{6}(\sqrt{30} + 6), \quad -\frac{1}{6}(\sqrt{30} - 6).$$

**Problem 2.** Let  $a, b \in \mathbb{R}$ . Consider the *quartic equation*

$$(\lambda - 1)^2(\lambda - b^2)^2 - \lambda^2 a^2(1 - b^2)^2 = 0$$

Show that the roots are given by

$$\begin{aligned}\lambda_1(a, b) &= \frac{1}{2}(1 - a + (1 + a)b^2 + (T_+)^{1/2}) \\ \lambda_2(a, b) &= \frac{1}{2}(1 - a + (1 + a)b^2 - (T_+)^{1/2}) \\ \lambda_3(a, b) &= \frac{1}{2}(1 + a + (1 - a)b^2 + (T_-)^{1/2}) \\ \lambda_4(a, b) &= \frac{1}{2}(1 + a + (1 - a)b^2 - (T_-)^{1/2})\end{aligned}$$

where  $T_{\pm} = (1 + a^2)(1 - b^2)^2 \pm 2a(b^4 - 1)$ .

**Problem 3.** Let  $n$  be a positive integer. Show that  $x^n - y^2$  has  $x - y$  as a factor for all  $n$ .

**Problem 4.** Let  $s = 1/2, 1, 3/2, 2, \dots$  be the spin values. The *Brillouin function* is defined as

$$B_s(x) := \frac{2s + 1}{2s} \coth\left(\frac{2s + 1}{2s}x\right) - \frac{1}{2s} \coth\left(\frac{1}{2s}x\right)$$

Find  $B_s(x = 0)$  applying L'Hospital.

**Problem 5.** Let  $n \in \mathbb{N}_0$  and  $x, m \in \mathbb{R}$ . The *Sonine polynomials* are defined by

$$S_m^n(x) = \sum_{j=0}^n (-1)^j \frac{\Gamma(m + n + 1)x^j}{\Gamma(m + j + 1)(n - j)!j!}$$

where  $\Gamma$  denotes the gamma function. Show that

$$S_m^0(x) = 1, \quad S_m^1(x) = m + 1 - x.$$

Show that the Sonine polynomials satisfy

$$\int_0^\infty e^{-x} x^m S_m^j(x) S_m^k(x) dx = \frac{\Gamma(m + j + 1)}{j!} = \delta_{j,k}.$$

**Problem 6.** (i) Show that the cubic roots of unity  $z^3 = 1$  are

$$1, \quad w = -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \quad w^2 = -\frac{1}{2} - \frac{\sqrt{3}i}{2}. \quad (2)$$

(ii) Show that the zeros of  $z^3 + 1 = 0$  are given by

$$-1, \quad \frac{1}{2} + \frac{i}{2}\sqrt{3}, \quad \frac{1}{2} - \frac{i}{2}\sqrt{3}. \quad (1)$$

(iii) Show that the set

$$S = \left\{ \omega_1 = -\frac{1}{2} + \frac{i}{2}\sqrt{3}, \quad \omega_2 = -\frac{1}{2} - \frac{i}{2}\sqrt{3}, \quad \omega_3 = 1 \right\}$$

of the cubic roots of 1, forms an abelian group with respect to multiplication on the set of complex numbers  $\mathbb{C}$ , i.e. show that

$$\begin{aligned} \omega_1\omega_2 &= \omega_3, & \omega_2\omega_1 &= \omega_3, & \omega_1\omega_3 &= \omega_1, \\ \omega_3\omega_1 &= \omega_1, & \omega_2\omega_3 &= \omega_2, & \omega_3\omega_2 &= \omega_2. \end{aligned} \quad (1)$$

Obviously, the neutral element is  $\omega_3$ . From (1) we see that each element has an inverse (which is unique), i.e.

$$(\omega_1)^{-1} = \omega_2, \quad (\omega_2)^{-1} = \omega_1, \quad (\omega_3)^{-1} = \omega_3.$$

The associative law is true for all complex numbers.

**Problem 7.** Let  $a, b \in \mathbb{R}$  and  $z$  be denote a root of the quadratic equation

$$z^2 + az + b = 0. \quad (1)$$

Show that the sequence of powers of  $z$ ,  $\{z^n\}$ ,  $n \geq 2$ , satisfies the linear difference equation

$$w_n + aw_{n-1} + bw_{n-2} = 0, \quad n \geq 2. \quad (2)$$

Set  $w_n = z^n$ . Then  $w_{n-1} = z^{n-1}$ .

**Problem 8.** A real number is said to be *algebraic* if it is a zero of a polynomial

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$$

where the coefficients  $c_j$  are integers. The height of such polynomials is defined as the positive integer number

$$h = n + |c_n| + |c_{n-1}| + \cdots + |c_0|.$$

Show that there are only finitely many polynomials of height  $h$ . Show that the set of all algebraic real numbers is enumerable.



**Problem 9.** Let  $p_0 = 1, p_1, p_2, \dots$  be polynomials which satisfy the recursion relation

$$p_n = \frac{1}{2n} \sum_{k=1}^n 2^{2k} \left( \frac{n!}{(n-k)!} \right)^2 x_k p_{n-k}, \quad n \in \mathbb{N}.$$

Give a SymbolicC++ and Maxima implementation of this recursion relation. For example  $p_1 = 2x_1$ .

**Problem 10.** Show that every polynomial  $\alpha(x) \in C[x]$  of degree  $m \geq 1$  has precisely  $m$  zeros over  $\mathbb{C}$ , where any zero of multiplicity is to be counted as  $n$  of the  $m$  zeros.

**Problem 11.** Show that if  $r \in \mathbb{C}$  is a zero of any polynomial  $\alpha(x)$  with real coefficients, then  $\bar{r}$  is also a zero of  $\alpha(x)$ , where  $\bar{r}$  denotes the complex conjugate of  $r$ .

**Problem 12.** Let

$$P = \{f_i(x), i = 1, \dots, k\}, \quad \mathbf{x} = (x_1 \dots x_n) \quad (1)$$

be a set of multivariate polynomials in the ring  $Q[x_1 \dots x_n]$ , with a solution set

$$S = S(f_1 \dots f_k) = \{x \mid f_i(x) = 0, \forall i = 1, \dots, k\} \quad (2)$$

All polynomials

$$u(x) = \sum_j g_j(x) f_j(x) \quad (3)$$

for arbitrary polynomials  $g_j$  will vanish in all points of  $S$ . The set of all  $u$  establishes a polynomial ideal

$$I = I(f_1 \dots f_k) = \left\{ \sum_j g_j(x) f_j(x) \right\} \quad (4)$$

and classical algebra tells that  $S$  is invariant if we replace the set  $\{f_1, \dots, f_k\}$  by any other basis for the ideal  $I(f_1, \dots, f_k)$ .

The *Buchberger algorithm* allows one to transform the set of polynomials into a canonical basis of the same ideal, the Gröbner basis  $GB = GB(I)$ . For the purpose of the equation solving especially Gröbner bases computed under lexicographical term ordering are important. They allow one to determine the set  $S$  directly. If  $I$  has dimension zero ( $S$  is a finite set of isolated points,  $GB$  has in most cases the form

$$g_1(x_1, x_k) = x_1 + c_{1,m-1} x_k^{m-1} + c_{1,m-2} x_k^{m-2} + \dots + c_{1,0}$$

$$\begin{aligned}
g_2(x_2, x_k) &= x_2 + c_{2,m-1}x_k^{m-1} + c_{2,m-2}x_k^{m-2} + \cdots + c_{2,0} \\
&\quad \dots \\
g_{k-1}(x_{k-1}, x_k) &= x_{k-1} + c_{k-1,m-1}x_k^{m-1} + c_{k-1,m-2}x_k^{m-2} + \cdots + c_{k-1,0} \\
g_k(x_k) &= x_k^m + c_{k,m-1}x_k^{m-1} + c_{k,m-2}x_k^{m-2} + \cdots + c_{k,0}.
\end{aligned}$$

A basis in this form has the elimination property: the variable dependency has been reduced to a triangular form, just as with a Gaussian elimination in the linear case. The last polynomial is univariate in  $x_k$ . It can be solved with usual algebraic or numeric techniques; its zero  $\bar{x}_k$  then are propagated into the remaining polynomials, which then immediately allow one to determine the corresponding coordinates  $(\bar{x}_1, \dots, \bar{x}_{k-1})$ .

Consider the system

$$\{y^2 - 6y, xy, \quad 2x^2 - 3y - 6x + 18, \quad 6z - y + 2x\}. \quad (4)$$

(i) Show that that this system has for  $\{x, y, z\}$  the lexicographical Gröbner basis

$$\{g_1(x, z) = x - z^2 + 2z - 1, \quad g_2(y, z) = y - 2z^2 - 2z - 2, \quad g_3(z) = z^3 - 1\}. \quad (5)$$

(iii) Show that the roots of the third polynomial are given by

$$\left\{ z = 1, \quad z = \frac{1 - \sqrt{3}i}{2}, \quad z = \frac{\sqrt{3}i - 1}{2} \right\}. \quad (6)$$

If we propagate one of them into the basis, we generate univariate polynomials of degree one which can be solved immediately; e.g. Selecting  $\frac{1 - \sqrt{3}i}{2}$  for  $z$  the basis reduces to

$$\left\{ x = \frac{3\sqrt{3}i + 3}{2}, y, 0 \right\} \quad (7)$$

such that the final solution for this branch is

$$\left\{ x = \frac{3\sqrt{3}i + 3}{2}, \quad y = 0, \quad z = \frac{1 - \sqrt{3}i}{2} \right\}. \quad (8)$$

For zero dimensional problems the last polynomial will always be univariate. However, in degenerate cases the other polynomials can contain their leading variable in a higher degree, then containing more mixed terms with the following variables. And there can be additional polynomials with mixed leading terms (of lower degree) imposing some restrictions. But the variable dependency pattern will remain triangular (eventually with more than  $k$  rows). There is a special algorithm for decomposing such ideals using ideal quotients.

# Chapter 5

## Equations

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### 5.1 Solved Problems

**Problem 1.** Solve the equation

$$\frac{1}{3} = \frac{1}{x} + \frac{1}{4}.$$

**Problem 2.** Let  $x_1, x_2$  be positive real numbers. Consider the equation

$$x_2 \sin(\theta) = x_1 \cos(\theta).$$

Find  $\sin(\theta)$ .

**Problem 3.** Let  $L$  be a given positive real number. Solve the system of two coupled nonlinear equations

$$\begin{aligned} 1 &= x^2 L + 2x^2 y + Lx^2 y^2 \\ 0 &= -x^2 + 2x^2 y + Lx^2 y^2. \end{aligned}$$

**Problem 4.** Let  $c, y \in \mathbb{R}$ . Solve the quadratic equation

$$x = cy + yx - x^2$$

with respect to  $x$ .

**Problem 5.** Consider a triangle in the plane. Consider a point within the triangle. We draw lines from this point to the three vertices, thereby dividing the triangle into three triangles of area  $A_1, A_2, A_3$ . The sides of the triangle are designated by the same number as the opposite vertex. The areas are identified by the number of the adjacent side. The quantities  $L_j$  ( $j = 1, 2, 3$ )

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}$$

where  $A$  is the area of the original triangle, are defined to be the triangular coordinates. Show that

$$L_1 + L_2 + L_3 = 1.$$

The relationship between the Cartesian coordinates  $x, y$  which are the coordinates of the points in the elements, and the triangular coordinates  $L_1, L_2, L_3$  are

$$x = L_1x_1 + L_2x_2 + L_3x_3, \quad y = L_1y_1 + L_2y_2 + L_3y_3$$

where  $x_j, y_j$  ( $j = 1, 2, 3$ ) are the coordinates of the nodes. The triangular coordinates can be expressed in terms of the known locations of the vertices, i.e.

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}.$$

Find  $L_1, L_2, L_3$  as function of  $x_j, y_j, x, y$ .

**Problem 6.** Let  $x, y \neq 0$ . Find all solutions of the equation

$$\frac{4x^2y^2}{(x^2 + y^2)^2} = 1.$$

**Problem 7.** Solve the equation

$$71x^2 = 133 \pmod{11}.$$

**Problem 8.** Let  $x, y \in \mathbb{N}$ . Find all solutions of

$$16x + 7y = 601.$$

**Problem 9.** Let  $x, y \in \mathbb{Z}$ . Find all solutions of

$$18x + 12y = 4.$$

**Problem 10.** Consider the two hyperplane ( $n \geq 1$ )

$$x_1 + x_2 + \cdots + x_n = 2, \quad x_1 + x_2 + \cdots + x_n = -2.$$

The hyperplanes do not intersect. Find the shortest distance between the hyperplanes. First consider the cases  $n = 1$  and  $n = 2$  and then the general case. What happens if  $n \rightarrow \infty$ ?

**Problem 11.** Let  $d$  be a positive distance,  $v, V$  be velocities  $v \neq V$  and  $T_1, T_2$  time-intervals. Assume that

$$\frac{d}{V+v} = T_1, \quad \frac{d}{V-v} = T_2.$$

Find  $d/V$ .

**Problem 12.** Solve the quadratic equation

$$x^2 - ix - (1 + i) = 0.$$

**Problem 13.** Let  $z \in \mathbb{C}$ . Solve

$$\left( \frac{z+1}{z-1} \right) = i.$$

**Problem 14.** Solve the system of nonlinear equations

$$\begin{aligned} x^2 - (y-z)^2 &= a^2 \\ y^2 - (z-x)^2 &= b^2 \\ z^2 - (x-y)^2 &= c^2. \end{aligned}$$

**Problem 15.** Show that the quartic equation

$$x^4 + qx^2 + rx + s = 0$$

can be written as

$$(x^2 - ex + f)(x^2 + ex + g) = 0$$

where  $e^2$  is the root of the cubic equation

$$z^3 + 2qz^2 + (q^2 - 4s)z - r^2 = 0.$$

**Problem 16.** Let

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be elements of  $\mathbb{C}^2$ . Solve the equation  $\mathbf{z}^* \mathbf{w} = \mathbf{w}^* \mathbf{z}$ .

**Problem 17.** A special set of coordinates on  $S^n$  called spheroconical (or elliptic spherical) coordinates are defined as follows: For a given set of real numbers  $\alpha_1 < \alpha_2 < \dots < \alpha_{n+1}$  and nonzero  $x_1, \dots, x_{n+1}$  the coordinates  $\lambda_j$  ( $j = 1, \dots, n$ ) are the solutions of the equation

$$\sum_{j=1}^{n+1} \frac{x_j^2}{\lambda - \alpha_j}.$$

Find the solutions for  $n = 2$ .

**Problem 18.** Find all solutions of the system of equations

$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) \cos(\beta) \\ \cos(\alpha) \sin(\beta) \\ \sin(\alpha) \cos(\beta) \\ \sin(\alpha) \sin(\beta) \end{pmatrix}.$$

**Problem 19.** Let  $\phi \in [0, 2\pi)$ . Solve the *cubic equation*

$$4x^3 - 3x - \cos(\phi) = 0$$

over the real numbers.

**Problem 20.** Let  $\epsilon > 0$ . Find the solution of the coupled two non-linear equations

$$\epsilon x^2 + x - y - 1 = 0, \quad \epsilon y^2 + x - y - 1 = 0.$$

Study  $\epsilon \rightarrow 0$  for these solutions.

**Problem 21.** Consider the six  $3 \times 3$  matrices

$$X_{12} = \begin{pmatrix} a_{12} & b_{12} & 0 \\ c_{12} & d_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X'_{12} = \begin{pmatrix} a'_{12} & b'_{12} & 0 \\ c'_{12} & d'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{13} = \begin{pmatrix} a_{13} & 0 & b_{13} \\ 0 & 1 & 0 \\ c_{13} & 0 & d_{13} \end{pmatrix}, \quad X'_{13} = \begin{pmatrix} a'_{13} & 0 & b'_{13} \\ 0 & 1 & 0 \\ c'_{13} & 0 & d'_{13} \end{pmatrix},$$

$$X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{23} & b_{23} \\ 0 & c_{23} & d_{23} \end{pmatrix}, \quad X'_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a'_{23} & b'_{23} \\ 0 & c'_{23} & d'_{23} \end{pmatrix}.$$

Find the 9 conditions on the entries such that (local Yang-Baxter equation)

$$X_{12}X_{13}X_{23} = X'_{23}X'_{13}X'_{12}.$$

Find solutions of these 9 equations for the 24 unknowns  $a_{12}, \dots, d'_{23}$ .

**Problem 22.** Solve the equations

$$2 \arcsin(x) + \arcsin(2x) - \pi/2 = 0$$

$$2 \arcsin(x) - \arccos(3x) = 0$$

$$\arccos(x) - \arctan(x) = 0$$

$$\arccos(2x^2 - 4x - 2) - 2 \arcsin(x) = 0$$

$$2 \arctan(x) - \arctan\left(\frac{2x}{1-x^2}\right) = 0.$$

**Problem 23.** Let  $x \in \mathbb{R}$ . Find all values of  $x$  that satisfy

$$|5x + 7| = 3.$$

Then find the smallest and largest values.

**Problem 24.** Let  $\theta \in [0, 2\pi)$ . Can one find  $x \in \mathbb{R}$  such that

$$\sin(\theta) = \frac{e^{-x/2}}{\sqrt{1+e^{-x}}}$$

**Problem 25.** Let  $x < 1$ . Find a solution of the equation

$$f^2(x) - 2f(x) + x = 0.$$

**Problem 26.** What are the conditions on  $c_1 > 0$  and  $c_2 > 0$  such that the system of equations

$$x_1 + x_2 = c_1, \quad x_1x_2 = c_2$$

has real solutions?

**Problem 27.** Let  $m, n$  be positive integers. Find all solution of the system of equations

$$2mn + (m^2 + n^2) = (m^2 - n^2)^2, \quad m - n = 1.$$

**Problem 28.** Let  $x \in \mathbb{R}$ .

- (i) Find all solutions of  $2x = |x| + 1$ .  
 (ii) Find all solutions of  $2x = -|x| + 1$ .

**Problem 29.** Let  $x \in \mathbb{Z}$ . Solve the equation

$$x^2 - 2x + 2 = 0 \pmod{5}.$$

**Problem 30.** Let  $a > b > 0$ . Find the points of intersections of the two ellipses

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad \frac{x_1^2}{b^2} + \frac{x_2^2}{a^2} = 1.$$

**Problem 31.** Let  $r_1 \geq 0, r_2 \geq 0, r_3 \geq 0$ . Find the solutions of the system of equations

$$r_1 + r_2 + r_3 = 1, \quad r_1^2 + r_2^2 + r_3^2 = 1.$$

**Problem 32.** Consider the cubic equation

$$p(x) = x^3 - 6x^2 + 11x - 6 = 0.$$

Let  $x_1, x_2, x_3$  be the roots. Given the equations

$$x_1 + x_2 + x_3 = -a_1, \quad x_1x_2 + x_2x_3 + x_3x_1 = a_2, \quad x_1x_2x_3 = -a_3$$

Find  $a_1, a_2, a_3$ .

**Problem 33.** Let  $z \in \mathbb{C}$ . Find all solutions of

$$z + z^* + zz^* = 0.$$

Set  $z = re^{i\phi}$  with  $r \geq 0$  and  $\phi \in [0, 2\pi)$ . An alternative would be setting  $z = x + iy$  with  $x, y \in \mathbb{R}$ .

**Problem 34.** We want to find the positive root of  $f(x) = 0$  with

$$f(x) = x^3 - x^2 - x - 1.$$



We write  $x^3 - x^2 - x - 1 = 0$  as  $x = 1 + 1/x + 1/x^2$  and set

$$x_{t+1} = 1 + 1/x_t + 1/x_t^2, \quad t = 0, 1, \dots$$

with  $x_0 = 1$ . Do we find the positive root?

**Problem 35.** Find all  $2 \times 2$  invertible matrices  $S$  over  $\mathbb{R}$  with  $\det(S) = 1$  such that

$$S \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} S \quad S \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus we have to solve the three equations

$$s_{21} = 0, \quad s_{11} + s_{12} = s_{22}, \quad s_{11}s_{22} = 1.$$

**Problem 36.** Let  $z \in \mathbb{C}$ . Find all solutions of

$$z + z^* + zz^* = 0.$$

Set  $z = re^{i\phi}$  with  $r \geq 0$ .

**Problem 37.** Let  $a$  be a real constant. Solve the two equations

$$-a^3 + x_3 - 3ax_2 + 3a^2x_1 = 0, \quad x_2 - 2ax_1 + a^2 = 0$$

with respect to  $x_1, x_2, x_3$ .

### Programming Problems

**Problem 1.** Let  $\epsilon \in \mathbb{R}$ . Consider the quadratic equation

$$x^2 - \epsilon x + \epsilon - 1 = 0.$$

Find the roots  $x_1(\epsilon), x_2(\epsilon)$ . Then find the minimum of  $x_1^2 + x_2^2$  with respect to  $\epsilon$ .

A SymbolicC++ program to solve this problem is:

```
// quadratic.cpp
#include <iostream>
#include "symbolicc++.h"
using namespace std;
```

```

int main(void)
{
    Symbolic x("x"), eps("eps"), f = 0;
    Equations soln = solve((x^2)-eps*x+eps-1==0,x);
    cout << "Solutions: " << endl << soln << endl;
    Equations::iterator i;
    for(i=soln.begin();i!=soln.end();i++) f += (i->rhs^2);
    cout << "f(eps) = " << f << endl;
    Equations min = solve(df(f,eps)==0,eps);
    for(i=min.begin();i!=min.end();i++)
        if(double(df(f,eps,2)[*i]) > 0)
            cout << "Minimum at " << *i << endl;
    return 0;
}
/*
Solutions:
[ x == eps-1,
  x == 1 ]
f(eps) = eps^(2)-2*eps+2
Minimum at eps == 1
*/

```

## 5.2 Supplementary Problems

**Problem 1.** Let  $x, y \in \mathbb{R}$ . Find all solutions of

$$e^{-x-y} = e^{-x} + e^{-y}.$$

**Problem 2.** Solve the system of nonlinear equations

$$x_1x_2 = 1, \quad x_2x_3 = 2, \quad x_3x_1 = 3.$$

Extend to the general case

$$x_1x_2 = c_1, \quad x_2x_3 = c_2, \dots, x_{n-1}x_n = c_{n-1}, \quad x_nx_1 = c_n$$

where  $c_j$  ( $j = 1, \dots, n$ ) are positive constants.

**Problem 3.** Find the solution of the equation

$$e^{-x} - 1 = 2(\sqrt{2} + 1).$$

**Problem 4.** Let  $n$  be a positive integer. Solve the nonlinear equation

$$\frac{1 - ne^{-x}}{1 + ne^{-x}} = \frac{x}{2}$$

for  $n = 1, 2, 3, 4, 5$ .

**Problem 5.** Let  $c_1 > 0$  and  $c_2 > 0$ . What is the condition on  $c_1, c_2$  such that

$$x_1 + x_2 = c_1, \quad x_1 x_2 = c_2$$

has a real solution?

**Problem 6.** Let  $n$  be a positive integer. Solve

$$\frac{1 - ne^{-x}}{1 + ne^{-x}} = \frac{x}{2}$$

for  $n = 1, 2, 3, 4, 5$ .

**Problem 7.** Let  $x > 0$ . Show that solution of the equation

$$x(1 + x^{1/2}) = (1 - x^{1/2})$$

is given by

$$x = \frac{1}{9}((17 + 3\sqrt{33})^{1/3} + (17 - 3\sqrt{33})^{1/3} - 1)^2 \approx 0.2955977\dots$$

**Problem 8.** Let  $x_1, x_2, x_3 \in \mathbb{R}$ . Find solutions of the equation

$$x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 = x_1 x_2 x_3$$

with  $x_1 \neq 0, x_2 \neq 0, x_3 \neq 0$ . Note that

$$x_1 = x_2 = x_3 = 1/3, \quad x_1 = 1/3, x_2 = x_3 = -1/3, \quad x_1 = x_2 = -1/3, x_3 = 1/3, \quad x_1 = x_3 = -1/3, x_2 = 1/3$$

are solutions.

**Problem 9.** (i) Consider the function

$$f(z, \bar{z}) = z + \bar{z} + z\bar{z}.$$

Find all solutions of  $f(z, \bar{z}) = 0$ .

(ii) Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$ . Find all solutions of

$$A + A^* + AA^* = 0_n.$$

Find all  $2 \times 2$  matrices  $A$  such that

$$A + A^* + AA^* = I_2.$$

**Problem 10.** (i) Let  $\phi, \theta \in [0, 2\pi)$ . Consider the equation

$$e^{i\phi} \sin(\theta) = \frac{2\eta}{|\eta|^2 + 1}.$$

Let  $\eta = 0, 1, -1, i, -i$ . Find  $\phi$  and  $\theta$ .

(ii) Let  $\theta \in [0, 2\pi)$ . Consider the equation

$$\cos(\theta) = \frac{|\eta|^2 - 1}{|\eta|^2 + 1}.$$

Let  $\eta = 0, 1, -1, i, -i$ . Find  $\phi$  and  $\theta$ .

**Problem 11.** Show that roots of the polynomial  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^4 - x^3 - 10x^2 - x + 1$$

are given by

$$x_1 = \frac{1}{2}(\sqrt{5} - 3), \quad x_2 = \frac{1}{2}(-\sqrt{5} - 3), \quad x_3 = 2 + \sqrt{3}, \quad x_4 = 2 - \sqrt{3}.$$

**Problem 12.** Find the real root of the polynomial  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^3 + x - 1.$$

The real root lie in the interval  $[0, 1]$ . Note that  $f(0) = -1$  and  $f(1) = 1$ .

**Problem 13.** Find solution solutions of the equation (Scherk's surface)

$$\cos(x_2)e^{x_3} = \cos(x_1).$$

For example  $(x_1, x_2, x_3) = (0, 0, 0)$  is a solution and  $(x_1, x_2, x_3) = (\pi/3, \pi/3, 0)$  is a solution.

**Problem 14.** Let  $x > 0$ . Find all solutions of

$$x = \sqrt{1 + \sqrt{2 + x}}.$$

First show that  $x^4 - 2x^2 - x - 1 = 0$ .

**Problem 15.** Show that the system of equations

$$\begin{aligned}3x + y - z + u^2 &= 0 \\x - y + 2z + u &= 0 \\2x + 2y - 3z - 2u &= 0\end{aligned}$$

can be solved for  $x, y, u$  in terms of  $z$ ; for  $x, z, u$  in terms of  $y$ ; for  $y, z, u$  in terms of  $x$ ; but not for  $x, y, z$  in terms of  $u$ .

**Problem 16.** Are there solutions of the equation ( $z \in \mathbb{C}$ )

$$\sin(z) = e^z \quad \text{and} \quad \cos(z) = e^z ?$$

**Problem 17.** (i) Show that  $\cos(\pi/7)$  is a real root of

$$8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0.$$

(ii) Show that

$$\cos(\pi/7) \cos(2\pi/7) \cos(4\pi/7) = -1/8.$$

**Problem 18.** Draw the curve in the plane given by

$$\sinh(x) = \exp(-2y).$$

Note that if  $y = 0$ , then  $\sinh(2x) = 1$  and therefore  $2x = \operatorname{arcsinh}(1)$ .

**Problem 19.** Solve the system of integral equations

$$\int_0^{2\pi} (f(\phi))^3 \cos(\phi) d\phi = 0, \quad \int_0^{2\pi} (f(\phi))^3 \sin(\phi) d\phi = 0.$$

**Problem 20.** Consider the two circles in the plane

$$(x_1 + 4)^2 + (x_2 + 5)^2 = 194, \quad (x_1 - 3)^2 + (x_2 - 2)^2 = 40.$$

Show that the two circles intersect and find the points of intersection. Show that  $x_1 + x_2 = 9$ .

## Chapter 6

# Normed Spaces

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Let  $\mathbb{C}$  be the field of complex numbers. Let  $V$  be a vector space (linear space) over  $\mathbb{C}$ . Then  $V$  is a normed linear space if for every  $f \in V$  there is a real number  $\|f\|$  such that ( $c \in \mathbb{C}$ )

$$(i) \|f\| \geq 0$$

$$(ii) \|f\| = 0 \text{ if and only if } f = 0$$

$$(iii) \|cf\| = |c| \|f\| \text{ for every } c \in \mathbb{C}$$

$$(iv) \|f + g\| \leq \|f\| + \|g\|.$$

A normed linear space  $V$  which does have the property that all Cauchy sequences are convergent is said to be complete. A complete normed linear space is called a Banach space.

## 6.1 Solved Problems

**Problem 1.** Consider the Banach space  $\mathbb{R}^4$  and the four vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ x \\ x^2/2 \\ x^3/3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ x \\ x^2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2x \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

where  $x \in \mathbb{R}$ . Show that the vectors are linearly independent. Find the distances

$$\|\mathbf{v}_1 - \mathbf{v}_2\|, \quad \|\mathbf{v}_2 - \mathbf{v}_3\|, \quad \|\mathbf{v}_3 - \mathbf{v}_4\|, \quad \|\mathbf{v}_4 - \mathbf{v}_1\|.$$

**Problem 2.** Consider the Hilbert space  $\mathbb{C}^2$  and the vectors

$$\mathbf{v}_1 = \begin{pmatrix} \cos(i) \\ \sin(i) \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -\sin(i) \\ \cos(i) \end{pmatrix}.$$

Find the distance  $\|\mathbf{v}_1 - \mathbf{v}_2\|$ .

**Problem 3.** (i) Let  $a, b \in \mathbb{R}$ . Show that

$$d(a, b) := |\arctan(a) - \arctan(b)|$$

defines a distance in  $\mathbb{R}$ .

(ii) Show that  $x_n = \arctan(n)$ , ( $n \in \mathbb{N}$ ) is a *Cauchy sequence*. Is the metric space  $\{\mathbb{R}, d\}$  complete?

**Problem 4.** Let  $n \geq 1$ ,  $0 \leq a < b$  and  $p \in \mathbb{R}^n$ . Show that there exists a map  $k : C^\infty(\mathbb{R}^n, \mathbb{R})$  such that  $k(\mathbf{x}) = 0$  for  $\|\mathbf{x} - \mathbf{p}\| \geq b$ ,  $k(\mathbf{x}) = 1$  for  $\|\mathbf{x} - \mathbf{p}\| \leq a$ , and  $0 < k(\mathbf{x}) \leq 1$  for  $a < \|\mathbf{x} - \mathbf{p}\| < b$ .

**Problem 5.** Let  $\mathbf{x} \in \mathbb{R}^2$ . Is

$$\|\mathbf{x}\| := |x_1 x_2|^{1/2}$$

a norm on  $\mathbb{R}^2$ ?

**Problem 6.** Show that if a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$|f(x) - f(y)| \leq M(|x - y|)^a$$

for some fixed  $M > 0$  and  $a > 1$ , then  $f$  is a constant function, i.e.,  $f$  is identically equal to some real number  $b$  for all  $x \in \mathbb{R}$ .

**Problem 7.** Let  $f, g$  be continuously differentiable functions on the interval  $[0, 1]$ . One defines

$$\langle f, g \rangle = \int_0^1 \left( f(x)g(x) + \frac{df}{dx} \frac{dg}{dx} \right) dx.$$

Show that this satisfies the properties of an inner product. Calculate  $\langle f, g \rangle$  for  $f(x) = \sin(x)$ ,  $g(x) = \cos(x)$ . Extend it to

$$\langle f, g \rangle = \int_0^1 \left( \sum_{j=0}^n \frac{d^j f(x)}{dx^j} \frac{d^j g(x)}{dx^j} \right) dx.$$

**Problem 8.** The  $p$ -norm of a vector  $(x_1, \dots, x_n) \in \mathbb{R}^n$  is defined as

$$\|\mathbf{x}\|_p := \left( \sum_{j=1}^n |x_j|^p \right)^{1/p}$$

where  $p \in \mathbb{R}^+$ . Find the norm for  $p \rightarrow \infty$ .

**Problem 9.** Let  $\mathbf{x} \in \mathbb{R}^n$  and  $\|\mathbf{x}\|$  be the Euclidean norm of  $\mathbf{x}$ . If  $B \subset \mathbb{R}^n$  is a nonempty compact set and  $\mathbf{x} \in \mathbb{R}^n$ , then we define the distance

$$\text{dist}(\mathbf{x}, B) := \min\{\|\mathbf{x} - \mathbf{b}\| : \mathbf{b} \in B\}.$$

If  $A, B \subset \mathbb{R}^n$  are nonempty compact sets then we define the distance

$$\text{dist}(A, B) := \max\{\text{dist}(\mathbf{a}, B) : \mathbf{a} \in A\}.$$

Show that

$$\text{dist}(A, B) = 0 \Leftrightarrow A \subset B$$

and  $\text{dist}(A, B) < \epsilon$  means that  $A \subset N_\epsilon(B)$  where

$$N_\epsilon(B) := \{\mathbf{x} : \text{dist}(\mathbf{x}, B) < \epsilon\}$$

is the  $\epsilon$ -neighborhood of  $B$ .

**Problem 10.** Let  $\mathbf{v}$  and  $\mathbf{w}$  be two normalized column vectors in  $\mathbb{C}^n$ . Does

$$D(\mathbf{v}, \mathbf{w}) := 2 \arccos \left( \sqrt{\frac{(\mathbf{v}^* \mathbf{w})(\mathbf{w}^* \mathbf{v})}{(\mathbf{v}^* \mathbf{v})(\mathbf{w}^* \mathbf{w})}} \right)$$

provide a distance measure between  $\mathbf{v}$  and  $\mathbf{w}$ .



**Problem 11.** Let  $z_1, z_2 \in \mathbb{C}$ . One can define a distance measure via

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}$$

Let  $z_1 = e^{i\phi_1}$  and  $z_2 = e^{i\phi_2}$ . Find  $\rho(z_1, z_2)$ .

**Problem 12.** The *chordal distance* between two complex numbers  $z_1, z_2$  is defined as

$$d(z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \sqrt{1 + |z_2|^2}}.$$

Let  $z_1 = e^{i\pi/2}$  and  $z_2 = e^{-i\pi/2}$ . Find  $d(z_1, z_2)$ .

**Problem 13.** Find  $n \times n$  matrices  $A, B$  such that

$$\|[A, B] - I_n\| \rightarrow \min$$

where  $\|\cdot\|$  denotes the norm and  $[,]$  denotes the commutator.

## 6.2 Supplementary Problems

**Problem 1.** Let  $x, y \in \mathbb{R}$ . Is

$$d(x, y) = \frac{|x - y|}{2 + |x - y|}$$

a metric on  $\mathbb{R}$ ? Note that  $d(x, x) = 0$  and  $d(x, y) \geq 0$ .

**Problem 2.** Let  $\mathbb{R}$  be the set of real numbers and  $x, y \in \mathbb{R}$ . Show that

$$d(x, y) = |x - y|$$

provides a distance function. Let  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ . Show that

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\left( \sum_{j=1}^n (x_j - y_j)^2 \right)}$$

provides a distance function. Of course one always assume the positive square root.

**Problem 3.** Consider the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \exp(-x^2), \quad g(x) = \exp(-|x|).$$

Find the distance  $d(f, g)$  between  $f$  and  $g$  given by

$$d(f, g) = \int_{\mathbb{R}} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

## Chapter 7

# Complex Numbers and Complex Functions

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### 7.1 Solved Problems

**Problem 1.** Let  $z \in \mathbb{C}$ . Solve the nonlinear equation

$$z^3 = z|z|^2.$$

**Problem 2.** (i) Find the complex numbers  $z$  satisfying  $z^2 = \bar{z}$ .  
(ii) Find the complex numbers  $z$  satisfying  $z^3 = \bar{z}$ .

**Problem 3.** Solve the cubic equation  $z^3 = -1$ . Do the solutions form a group under multiplication? If not, what numbers have to be added to form a group. Find them by multiplication of the solutions of the cubic equation.

**Problem 4.** Find the square root of  $z = 4 \exp(i\pi/4)$ .

**Problem 5.** Let  $\phi \in [0, 2\pi)$ . Consider the complex number  $z = 1 - e^{i\phi}$ . Find the condition on  $\phi$  such that  $|z| = 1$ .

**Problem 6.** Let  $x, y \in \mathbb{R}$  and  $z = x + iy$ . Find the real and imaginary part of  $i/(\pi z)$ .

**Problem 7.** Let  $\phi \in [0, 2\pi)$  and  $\theta \in (-\pi, \pi)$ . Can any complex number be represented by

$$z = e^{i\phi} \tan\left(\frac{1}{2}\theta\right)?$$

**Problem 8.** Find

$$zd\bar{z} - \bar{z}dz, \quad (zd\bar{z} - \bar{z}dz) \otimes (zd\bar{z} - \bar{z}dz).$$

**Problem 9.** Let  $x, c \in \mathbb{R}$  and  $c \neq 0$ . Find the real and imaginary part of the function

$$f_c(x) = \frac{x - ic}{x + ic}.$$

**Problem 10.** Let  $x, y \in \mathbb{R}$ . Solve

$$\frac{x - iy - i}{x + iy + i} = \frac{xy - i}{xy + i}.$$

**Problem 11.** Find  $\sqrt{i}$ .

**Problem 12.** (i) Calculate

$$p(\alpha, \beta, \theta, \phi) = |\cos(\alpha) \cos(\beta) \sin(\theta)e^{i\phi} + \sin(\alpha) \sin(\beta) \cos(\theta)|^2.$$

(ii) Show that  $p(\alpha, \beta, \theta, \phi) \leq 1$ .

(iii) Simplify the result from (i) for  $\theta = \pi/4$  and  $\phi = 0$ .

**Problem 13.** Let  $z_1, z_2 \in \mathbb{C}$ . Consider the distance measures

$$d_1(z_1, z_2) := |z_1 - z_2|, \quad d_2(z_1, z_2) := \frac{|z_1 - z_2|}{1 + |z_1 - z_2|},$$

$$d_3(z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}.$$

Here  $d_3$  is the chordal distance. Let  $z_1 = e^{i\phi}$  and  $z_2 = e^{i\phi_2}$ . Find  $d_1$ ,  $d_2$ ,  $d_3$ .

**Problem 14.** Let  $f(z_1)$  and  $g(z_2)$  be a pair of analytic functions of  $z_1$  and  $z_2$ , respectively. We define

$$f(z_1) \circ g(z_2) := \frac{1}{2\pi} \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta}) d\theta.$$

Let

$$f(z_1) = \sum_{k=0}^{\infty} a_k z_1^k, \quad g(z_2) = \sum_{k=0}^{\infty} b_k z_2^k$$

for  $|z_j| < R$ ,  $j = 1, 2$ . Find  $f(z_1) \circ g(z_2)$ .

**Problem 15.** Let  $z = x + iy$ , where  $x, y \in \mathbb{R}$ . Find

$$\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}.$$

**Problem 16.** Study the behaviour (stability) of the fixed points of the complex map  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $f(z) = z^2$ .

**Problem 17.** Let  $z = x + iy$  ( $x, y \in \mathbb{R}$ ) be a nonzero complex number. We define the *principal argument* by  $z = |z| \exp(i \arg(z))$ , where  $\arg(z) \in (-\pi, \pi]$  and we define the imaginary remainder  $\text{Imr}(z)$  and the imaginary quotient  $\text{Imq}(z)$  by

$$\Im(z) = \text{Imr}(z) + 2\pi \text{Imq}(z)$$

where  $\text{Imr}(z) \in (-\pi, \pi]$  and  $\text{Imq}(z) \in \mathbb{Z}$ . Show that

$$\ln(e^z) = \Re(z) + i \text{Imr}(z)$$

and in particular

$$\ln(e^z) = z \quad \text{iff} \quad \Im(z) \in (-\pi, \pi].$$

**Problem 18.** Let  $z \in \mathbb{C}$  and consider the analytic map

$$f(z) = \exp(z).$$

Find the solutions of the equation

$$z = f(z).$$

This are the *fixed points* of  $f$ . We set  $z = x + iy$  ( $x, y \in \mathbb{R}$ ). Then

$$x + iy = \exp(x + iy) \equiv e^x e^{iy} = e^x (\cos(y) + i \sin(y)).$$

Thus we have to solve

$$e^x \cos(y) - x = 0, \quad e^x \sin(y) - y = 0.$$

**Problem 19.** Let  $A$  be an  $n \times n$  matrix. Suppose  $f$  is an analytic function inside on a closed contour  $\Gamma$  which encircles  $\lambda(A)$ , where  $\lambda(A)$  denotes the eigenvalues of  $A$ . We define  $f(A)$  to be the  $n \times n$  matrix

$$f(A) = \frac{1}{2\pi i} \oint_{\Gamma} f(z)(zI_n - A)^{-1} dz.$$

This is a matrix version of the *Cauchy integral theorem*. The integral is defined on an element-by-element basis  $f(A) = (f_{jk})$ , where

$$f_{jk} = \frac{1}{2\pi i} \oint_{\Gamma} f(z) \mathbf{e}_j^T (zI_n - A)^{-1} \mathbf{e}_k dz.$$

Let  $f(z) = z^2$  and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate  $f(A)$ .

**Problem 20.** Let  $x_1, y_1, x_2, y_2 \in \mathbb{R}$ . Consider the two complex numbers

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2.$$

In *polar form* we have

$$z_1 = r_1 e^{i\phi_1} = r_1 (\cos(\phi_1) + i \sin(\phi_1)), \quad z_2 = r_2 e^{i\phi_2} = r_2 (\cos(\phi_2) + i \sin(\phi_2))$$

where  $r_1^2 = x_1^2 + y_1^2$ ,  $r_2^2 = x_2^2 + y_2^2$ . Now

$$z = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Find  $z$  in polar form  $z = r e^{i\phi}$ .

**Problem 21.** Let  $a, b, \phi$  be fixed real numbers. Consider the function

$$w(z) = z + az^2 + be^{i\phi} z^3.$$

Write it as real and imaginary part, with  $w = u + iv$  and  $z = x + iy$ .

**Problem 22.** Consider the  $z$ -transform

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad x(n) = \frac{1}{2\pi i} \oint x(z) z^{n-1} dz.$$

Let

$$S(N) := \sum_{n=1}^N x(n).$$

Then

$$S(N) = \sum_{n=1}^N x(n) = \frac{1}{2\pi i} \oint x(z) \sum_{n=1}^N z^{n-1} dz.$$

It follows that (geometric series)

$$S(N) = \frac{1}{2\pi i} \oint \frac{x(z)(z^N - 1)}{z - 1} dz.$$

Apply this expression and the *residue theorem* to calculate

$$S(N) = \sum_{n=1}^N n^3.$$

**Problem 23.** (i) Consider the complex number  $z = e^{i\phi}$ . Let  $n \in \mathbb{N}$ . Find  $\sqrt[n]{z}$ .

(ii) Let  $z = re^{i\phi}$  and  $w = x + iy$  ( $x, y \in \mathbb{R}$ ). Find  $z^w$ .

**Problem 24.** Consider the complex numbers  $z_1 = 0.4 + 0.3i$ ,  $z_2 = 5 + 2i$ . Calculate  $z_1^{z_2}$ . Hint: Set  $z_1 = r_1 e^{i\phi_1}$ .

**Problem 25.** Consider the complex numbers

$$z_1 = x_1 + iy_1 = 1 + 4i, \quad z_2 = x_2 + iy_2 = 3 - 2i.$$

Calculate  $\log_{z_2}(z_1)$ .

**Problem 26.** Let  $A > 0$  and  $B \geq 0$ . Consider the quadratic conformal map in the complex  $w$ -plane of the unit disc in the complex  $z$ -plane

$$w(z) = Az + Bz^2.$$

Let  $\theta$  be the polar angle in the complex  $z$ -plane.

(i) Show that with the notation  $w := u + iv$ ,  $z := x + iy$  (with  $u, v, x, y$  real) one has the parametric equation of the boundary

$$u(\theta) = A \cos(\theta) + B \cos(2\theta), \quad v(\theta) = A \sin(\theta) + B \sin(2\theta).$$

(ii) Let  $C := B/A$ . Show that for  $C = 0$  (i.e.  $B = 0$ ) one obtains a circular disc. Show that for  $C = 1/4$  the curvature vanishes at  $\theta = \pi$ . Show that for  $C = 1/2$  the derivative  $dw/dz$  vanishes at the boundary, i.e. at  $\theta = \pi$ .

**Problem 27.** (i) Solve the equation

$$\left(\frac{z + \frac{i}{2}}{z - \frac{i}{2}}\right)^4 = 1.$$

(ii) Solve the system of equations

$$\left(\frac{z_1 + \frac{i}{2}}{z_1 - \frac{i}{2}}\right)^4 = \frac{z_1 - z_2 + i}{z_1 - z_2 - i}, \quad \left(\frac{z_2 + \frac{i}{2}}{z_2 - \frac{i}{2}}\right)^4 = \frac{z_2 - z_1 + i}{z_2 - z_1 - i}.$$

These equations play a role for the Bethe ansatz for spin systems.

**Problem 28.** Let  $z$  be a complex number such that  $|z| < \infty$  and  $\Re(z) > 0$ . Consider

$$Z(z) = \int_0^\infty \exp(-zx - x^2) dx.$$

Show that

$$Z(z) = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{1}{2} + \frac{1}{2}k)}{k!} z^k.$$

**Problem 29.** Let  $z \neq 0$ . Show that the function

$$f(z) = \frac{\ln(z)}{z-1}$$

is analytic near  $z = 1$  and admits the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^n \quad \text{for } |z-1| < 1.$$

**Problem 30.** Is the function  $f : \mathbb{C} \rightarrow \mathbb{C}$

$$f(z) = z + |z|$$

continuous? Find the fixed points of  $f$ .

**Problem 31.** Let  $x_1, x_2, y_1, y_2 \in \mathbb{R}$  and  $z_1 = x_1 + iy_1$ ,  $z_2 = x_2 + iy_2$  with  $z_1 \bar{z}_1 \neq 0$ ,  $z_2 \bar{z}_2 \neq 0$ . Find the conditions such that

$$\begin{pmatrix} \bar{z}_1 & \bar{z}_2 \end{pmatrix} \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} = 0.$$



**Problem 32.** Let  $z = \rho e^{i\theta}$  and  $\zeta = R e^{i\omega}$  with  $R > 0$  and  $\rho > 0$ . Show that

$$\frac{\rho^2 - R^2}{\rho^2 - 2R\rho \cos(\omega - \theta) + R^2} = \Re \left( \frac{z + \zeta}{z - \zeta} \right).$$

## 7.2 Supplementary Problems

**Problem 1.** Show that

$$\frac{1}{5}(-1 + 2i) = \frac{1}{\sqrt{5}} e^{i(\pi - \arctan(2))}.$$

**Problem 2.** Let  $z_1 = r_1 e^{i\phi_1}$ ,  $z_2 = r_2 e^{i\phi_2}$  with  $r_1, r_2 \geq 0$ . Show that

$$\Re \left( \frac{z_1 + z_2}{z_1 - z_2} \right) = \frac{r_1^2 + r_2^2}{|z_1 - z_2|^2}.$$

**Problem 3.** Let  $z = x + iy$  with  $x, y \in \mathbb{R}$ . Show that

$$\Re(z^2) = x^2 - y^2.$$

**Problem 4.** Let  $0 < r < 1$  and  $\phi \in [0, 2\pi)$ . Consider the map

$$f(z) = r e^{i\phi} + (1 - r e^{i\phi})z.$$

Find the fixed points. Find  $f(0)$ ,  $f(f(0))$ ,  $f(f(f(0)))$ .

**Problem 5.** Let  $x, y \in \mathbb{R}$ . Find real and imaginary part of

$$z = \sqrt{\frac{x - iy}{x + iy}} e^{i\phi}.$$

**Problem 6.** Let  $z \in \mathbb{C}$ . Show that

$$\sin(z) = z \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{k^2 \pi^2} \right).$$

**Problem 7.** Show that the *Bessel function*

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{2n+\nu}$$

is an entire function of  $z$  for  $\nu = 0, 1, \dots$ . Show that

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu \frac{1}{z} J_\nu(z).$$

**Problem 8.** (i) Show that the radius of convergence of the function

$$f(z_1, z_2) = \sum_{k=0}^{\infty} z_1^k z_2$$

is given by  $\{(z_1, z_2) : |z_1| < 1 \vee z_2 = 0\}$ .

(ii) Show that the radius of convergence of the function

$$f(z_1, z_2) = \sum_{k=0}^{\infty} (z_1 z_2)^k$$

is given by  $\{(z_1, z_2) : |z_1| \cdot |z_2| < 1\}$ .

(iii) Show that the radius of convergence of the function

$$f(z_1, z_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} z_1^{k_2} z_2^{k_1-k_2}$$

is given by  $\{(z_1, z_2) : |z_1| + |z_2| < 1\}$ .

**Problem 9.** (i) Show that the sum

$$\sum_{k=0}^{\infty} k! z^k$$

only converges for  $z = 0$ .

(ii) Show that the sum

$$\sum_{k=0}^{\infty} \frac{z^k}{k^2}$$

converges for  $|z| \leq 1$  and diverges for  $|z| > 1$ .

(iii) Show that the sum

$$\sum_{k=0}^{\infty} (z_1 z_2)^k$$

converges for  $\{(z_1, z_2) : |z_1| \cdot |z_2| < 1\}$ .

(iv) Show that the sum

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} z_1^{k_2} z_2^{k_1-k_2}$$

converges for  $\{(z_1, z_2) : |z_1| + |z_2| < 1\}$ . The sum appears at the expansion of  $1/(1 - (z_1 + z_2))$ .

**Problem 10.** Let  $x \in \mathbb{R}^+$  and  $f(x) = \sqrt{x}$ . Show that real function can be extended with a power series expansion around 1 into the complex domain

$$f(z) = \sqrt{1 + (z - 1)} = \sum_{k=0}^{\infty} \binom{1/2}{k} (z - 1)^k$$

and for  $z = re^{i\phi}$  with  $-\pi/2 < \phi < \pi/2$  we have  $f(z) = \sqrt{r}e^{i\phi/2}$ .

**Problem 11.** (i) Is the complex function

$$f(z) = \bar{z} = x_1 - ix_2$$

holomorph?

(ii) Is the complex function

$$f(z) = z^2 - \bar{z}^2 = 4ix_1x_2$$

holomorph?

**Problem 12.** (i) Show that if  $n$  is a positive integer then

$$(r(\sin(\phi) + i \cos(\phi)))^n \equiv r^n(\cos(n\phi) + i \sin(n\phi)). \quad (1)$$

*Hint.* Apply  $\exp(i\phi) \equiv \cos(\phi) + i \sin(\phi)$ .

(ii) Show that the  $n$ -th roots of unity are

$$\rho = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right), \quad \rho^2, \quad \rho^3, \dots, \rho^{n-1}, \quad \rho^n \equiv 1. \quad (1)$$

**Problem 13.** Let  $\epsilon \in \mathbb{R}$  and  $z \in \mathbb{C}$ . Consider the product

$$f(z) = \prod_{k=1}^{\infty} \frac{1 - e^{-\epsilon k}}{1 - e^{-\epsilon(k+z)}}.$$

Find  $f(0)$ . Show that

$$f(z) = (1 - e^{-\epsilon z})f(z - 1).$$

Show that

$$f(m) = \prod_{k=1}^m (1 - e^{-\epsilon k}), \quad m = 1, 2, \dots$$

Show that  $f$  is periodic with period  $2\pi i/\epsilon$ . Show that  $f$  has simple poles at  $z = -m$  ( $m = 1, 2, \dots$ ). Show that the residues are given by

$$\lim_{z \rightarrow -m} (z + m)f(z) = \frac{(-1)^{m+1} \exp(-\frac{1}{2}\epsilon m^2 + \frac{1}{2}\epsilon m)}{\epsilon f(m-1)}.$$

**Problem 14.** Show that

$$(1 + i)^{1/2} = \pm 2^{1/4} (\cos(\pi/8) + i \sin(\pi/8)).$$

**Problem 15.** Let  $n = 0, 1, \dots$  and  $x \in \mathbb{R}$ . Find the real and imaginary part of the functions

$$f_n(x) = \frac{1}{\sqrt{\pi}} \frac{(ix - 1)^n}{(ix + 1)^{n+1}}.$$

**Problem 16.** Show that

$$\frac{1}{(n-k)!} = \frac{1}{2\pi i} \oint dt \frac{e^t}{t^{n-k+1}}$$

where the integration contour is a small circle around the origin in the complex plane.

**Problem 17.** We know that  $0 \leq |\sin(x)| \leq 1$  and  $0 \leq |\cos(x)| \leq 1$  for  $x \in \mathbb{R}$ . This is no longer true for  $\sin(z)$  and  $\cos(z)$  with  $z \in \mathbb{C}$ .

(i) Let  $a > 0$ . Show that

$$|\sin(az)| = \sqrt{\sinh^2(ay) + \sin^2(ax)}, \quad |\cos(az)| = \sqrt{\sinh^2(ay) + \cos^2(ax)}.$$

(ii) Find all solutions of  $|\sin(z)| = 2$  and  $|\cos(z)| = 2$ .

(iii) Find all solutions of  $\cos(z) = i$ .

(iv) Let  $n \in \mathbb{N}$ . Show that ( $x, y \in \mathbb{R}$ )

$$\begin{aligned} |\sin(n(x + iy))| &= \sqrt{\sin^2(nx) \cosh^2(ny) + \cos^2(nx) \sinh^2(ny)} \\ &> |\sinh^2(ny)| \rightarrow \infty \end{aligned}$$

as  $n \rightarrow \infty$  for any  $y \neq 0$ .

# Chapter 8

## Integration

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### 8.1 Solved Problems

**Problem 1.** The time-average of a continuous function  $f$  is

$$\langle f \rangle := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) dt.$$

Find the time-average of the functions

$$f_1(t) = \cos(\omega t) \sin(\omega t), \quad f_2(t) = \cos^2(\omega t), \quad f_3(t) = \sin^2(\omega t).$$

**Problem 2.** Let  $j, k = 1, 2, \dots$ . Consider the function

$$f(j, k) = \int_0^1 x^{j-1} (1-x)^{k-1} dx, \quad j, k = 1, 2, \dots$$

Thus  $f(1, 1) = 1$ . Is

$$f(j-1, k+1) = \frac{k}{j-1} f(j, k), \quad j \geq 2$$

and

$$f(j+1, k) = f(j, k) - f(j, k+1)?$$

Prove or disprove.

**Problem 3.** Let

$$x(\tau) = \begin{cases} 1 & \text{for } 0 \leq \tau \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(\tau) = \begin{cases} 1 & \text{for } 0 \leq \tau \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau.$$

**Problem 4.** Let  $T > 0$  and  $\omega = 2\pi/T$ . Let  $m, n \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{R}$ . Calculate

$$I(\alpha, \beta) = \frac{1}{T} \int_0^T c_m \sin(m\omega t + \phi_m - \alpha) c_n \sin(n\omega t + \phi_n - \beta) dt.$$

**Problem 5.** A *cubic B-spline* with uniform knot spacing, centered at the origin, is given by

$$B(x) = \begin{cases} \frac{1}{6}(2 - |x|)^3 & \text{if } 1 \leq |x| < 2 \\ \frac{1}{6}((2 - |x|)^3 - 4(1 - |x|)^3) & \text{if } 0 \leq |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the integral

$$\int_{-\infty}^{\infty} B(x) dx.$$

**Problem 6.** Find the normalization constant  $K$  from the condition

$$1 = 4K \int_0^{2\pi} \sin^2(\omega/2) d\omega \int_0^\pi \sin(\theta) d\theta \int_{-\pi}^\pi d\phi.$$

**Problem 7.** Let  $\epsilon \in \mathbb{R}$ . Calculate the integral

$$f(\epsilon) = \int_0^{2\pi} \exp(\epsilon e^{i\phi}) d\phi$$

by finding an ordinary differential equation for  $f$  together with the initial conditions. Obviously  $f(0) = 2\pi$ .

**Problem 8.** Let  $b > a$ . Find the mean and variance of random variable  $x$  with uniform probability density function  $p$

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

**Problem 9.** Consider a one-dimensional lattice (chain) with lattice constant  $a$ . Let  $k$  be the sum over the first Brillouin zone we have

$$\frac{1}{N} \sum_{k \in 1.BZ} F(\epsilon(k)) \rightarrow \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} F(\epsilon(k)) dk = G$$

where

$$\epsilon(k) = \epsilon_0 - 2\epsilon_1 \cos(ka).$$

Using the identity

$$\int_{-\infty}^{\infty} \delta(E - \epsilon(k)) F(E) dE \equiv F(\epsilon(k))$$

we can write

$$G = \frac{a}{2\pi} \int_{-\infty}^{\infty} F(E) \left( \int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k)) dk \right) dE.$$

Calculate

$$g(E) = \int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k)) dk$$

where  $g(E)$  is called the density of states.

**Problem 10.** Let

$$i_1(t) = I^2 \sin^2(\omega t), \quad i_2(t) = I^2 \sin^2(\omega t + \phi).$$

Calculate

$$\langle i_1(t) i_2(t) \rangle := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T i_1(t) i_2(t) dt$$

and

$$\langle i_1(t) i_2(t) \rangle - \langle i_1(t) \rangle \langle i_2(t) \rangle.$$

**Problem 11.** Let  $\alpha > 0$ . Find

$$\int_0^{\infty} \exp(-\alpha x) dx = \frac{1}{\alpha}.$$



**Problem 12.** Let  $f(t)$  be a continuous function. Show that

$$\int_0^x d\zeta \int_0^\zeta f(t)dt = \int_0^x (x-t)f(t)dt.$$

**Problem 13.** (i) Calculate

$$\int_{-\infty}^{\infty} \operatorname{sech}(t)\tanh(t) \cos(t + t_0)dt.$$

(ii) Calculate

$$\int_{-\infty}^{\infty} \operatorname{sech}^2(t)\tanh^2(t)dt.$$

**Problem 14.** Let  $\lambda > 0$ . Calculate

$$\int_{-\infty}^{\infty} \frac{\sin(\lambda t)}{\lambda \sinh(t)} dt.$$

**Problem 15.** Calculate

$$\int_{-\infty}^{\infty} \exp(-x^2 + 2ixy)dx.$$

**Problem 16.** Calculate the integral

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{e^t}{(1 + e^t)^2} \cos(t + \lambda)dt$$

using the residue technique.

**Problem 17.** Let  $m$  be a non-negative integer. Find

$$\int_0^\pi d\theta |\cos^{2m+1}(\theta)|.$$

**Problem 18.** Let  $\epsilon > 0$ . Calculate

$$\int_0^\infty \frac{k^2 dk}{e^{\epsilon k^2} - 1}.$$

**Problem 19.** The mother *Haar wavelet* is given by

$$f(t) = \begin{cases} -1 & \text{for } 0 \leq t < 1/2 \\ +1 & \text{for } 1/2 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt.$$

**Problem 20.** The *Poisson wavelet* is given by

$$f(t) = \left( t \frac{d}{dt} + 1 \right) P(t)$$

where

$$P(t) = \frac{1}{\pi} \frac{1}{1+t^2}.$$

Find the Fourier transform of  $f$ .

**Problem 21.** Let  $m \in \mathbb{Z}$ . Calculate

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} e^{im\phi} 2 \cos(\phi) d\phi.$$

**Problem 22.** Let  $0 < \alpha < 1$ . Find the integral

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx.$$

**Problem 23.** The linear one-dimensional *diffusion equation* is given by

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad -\infty < x < \infty$$

where  $u(x, t)$  denotes the concentration at time  $t$  and position  $x \in \mathbb{R}$ .  $D$  is the diffusion constant which is assumed to be independent of  $x$  and  $t$ . Given the initial condition  $c(x, 0) = f(x)$ ,  $x \in \mathbb{R}$  the solution of the one-dimensional diffusion equation is given by

$$u(x, t) = \int_{-\infty}^{\infty} G(x, t|x', 0) f(x') dx'$$

where

$$G(x, t|x', t') = \frac{1}{\sqrt{4\pi D(t-t')}} \exp\left(-\frac{(x-x')^2}{4D(t-t')}\right).$$

Here  $G(x, t|x', t')$  is called the fundamental solution of the diffusion equation obtained for the initial data  $\delta(x-x')$  at  $t=t'$ , where  $\delta$  denotes the Dirac delta function.

(i) Let  $u(x, 0) = f(x) = \exp(-x^2/(2\sigma))$ . Find  $u(x, t)$ .

(ii) Let  $u(x, 0) = f(x) = \exp(-|x|/\sigma)$ . Find  $u(x, t)$ .

**Problem 24.** Let  $\omega_0 > 0$  be a fixed frequency and  $t$  the time. Calculate

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|\omega t|} e^{-i\omega t} dt.$$

**Problem 25.** Let  $\omega_0 > 0$  be a fixed frequency and  $t$  the time. Calculate

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\omega_0 t|} e^{-i\omega t} dt.$$

**Problem 26.** The sum

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \exp\left(2 \cos\left(\frac{k\pi}{n+1}\right)\right)$$

can be cast into the integral

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \exp(2 \cos(\pi x)) dx. \tag{1}$$

Calculate this integral.

**Problem 27.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an analytic function. The *Dirichlet integral identity* is given by

$$\int_0^u \int_0^{u-w_2} f(u-w_1-w_2) w_1^{\mu_1-1} w_2^{\mu_2-1} dw_1 dw_2 = \frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2)} \int_0^u f(u-w) w^{\mu_1+\mu_2-1} dw.$$

Let  $f(x) = e^{-x}$ . Calculate the left and right-hand side of this identity.

**Problem 28.** Let  $r_1 > 0, r_2 > 0$ . Find the integral

$$I(r_1, r_2) = \int_{-1}^1 \frac{dx}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2x}}.$$

**Problem 29.** Show that

$$e^{a^2} = \int_{\mathbb{R}} dx \exp\left(-\frac{x^2}{2} + \sqrt{2}ax\right).$$

**Problem 30.** Let  $u \geq 0$  and

$$\rho(u) = \frac{1}{2} \exp(-\sqrt{u}).$$

Find

$$\rho_n = \int_0^{-\infty} u^n \rho(u) du.$$

**Problem 31.** Calculate the integral

$$I = \int \int_D \sqrt{x^2 + 4y^2} dy dx$$

where  $D$  is the domain bounded by the positive  $x$ -axis, the positive  $y$ -axis and the parabola  $y^2 = 1 - x$ .

**Problem 32.** Calculate the definite integral

$$\int_0^1 \sin(x^2) dx.$$

**Problem 33.** (i) Consider the wavelet ( $\omega_0 > 0$ )

$$\psi_0(t) = (e^{i\omega_0 t} - e^{-\omega_0^2/2})e^{-t^2/2}.$$

Show that

$$\int_{-\infty}^{\infty} \psi_0(t) dt = 0.$$

Hint: Use ( $a > 0$ )

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/4a}.$$

(ii) We define

$$\psi_n(t) := \frac{d}{dt}\psi_{n-1}(t), \quad n = 1, 2, \dots$$

Show that

$$\int_{-\infty}^{\infty} t^k \psi_n(t) dt = 0, \quad 0 \leq k \leq n.$$

**Problem 34.** Let  $b > a$ . Consider the integral

$$\int_{y=a}^{y=b} f(y) dy$$

where  $f$  is a continuous function in  $[a, b]$ . Apply the transformation

$$y(x) = \frac{1}{2}((b-a)x + a + b)$$

so that the integration range is between  $-1$  and  $+1$ . Then the Gauss quadrature can be applied which extends over the interval  $[-1, +1]$ .

**Problem 35.** Let  $\epsilon > 0$ . Find  $f$  of the equation

$$\exp(-\epsilon t) = 1 - \int_0^t f(s) ds.$$

**Problem 36.** (i) Find the area of the set

$$S_2 := \{ (x_1, x_2) : 1 \geq x_1 \geq x_2 \geq 0 \}.$$

(ii) Find the volume of the set

$$S_3 := \{ (x_1, x_2, x_3) : 1 \geq x_1 \geq x_2 \geq x_3 \geq 0 \}.$$

Extend the  $n$ -dimensions.

**Problem 37.** Let  $\omega_1, \omega_2$  be real and positive. Find

$$J(\omega_1, \omega_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2 + i\omega_1 x + i\omega_2 x^2\right) dx.$$

**Problem 38.** Let  $e$  be the eccentricity of an ellipse, i.e.  $\sqrt{1-e^2} = b/a$ . Let  $n \geq 1$ . Calculate the integral

$$\int_0^{2\pi} \frac{dx}{\left(\frac{1}{e} - \cos(x)\right)^n}$$

by making the substitution  $z = \exp(ix)$ . Apply the residue theorem.

Hint. We have

$$\int_0^{2\pi} \frac{dx}{\left(\frac{1}{e} - \cos(x)\right)^n} = \int_{|z|=1} \frac{dz}{\left(\frac{1}{e} - \frac{1}{2}\left(z + \frac{1}{z}\right)\right)^n iz} = \int_{|z|=1} \frac{i(-1)^{n+1} 2^n z^{n-1} dz}{\left(z^2 - \frac{2}{e}z + 1\right)^n}.$$

**Problem 39.** Let  $q^2 > 0$  and  $(p - q)^2 > 0$ . Find

$$\int_0^\infty \exp(-xq^2) dx, \quad \int_0^\infty \exp(-x(p - q)^2) dx.$$

**Problem 40.** Find  $\alpha > 0$  such that

$$\int_{-\infty}^{+\infty} \exp(-\alpha|x|) dx = 1.$$

Afterwards calculate

$$\int_{-\infty}^\infty x \exp(-\alpha|x|) dx, \quad \int_{-\infty}^\infty x^2 \exp(-\alpha|x|) dx.$$

**Problem 41.** Let  $c > 0$ . Show that

$$\exp\left(\frac{1}{2}cy^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dx \exp\left(-\frac{1}{2}x^2 + \sqrt{c}yx\right).$$

**Problem 42.** The Gauss invariant for two given closed loops  $C_\alpha$  and  $C_\beta$  in  $\mathbb{R}^3$  parametrized by  $\mathbf{r}_\alpha(s)$ ,  $\mathbf{r}_\beta$  is defined by

$$G(C_\alpha, C_\beta) := \frac{1}{4\pi} \oint_{C_\alpha} ds \oint_{C_\beta} ds' \frac{d\mathbf{r}_\alpha(s)}{ds} \times \frac{d\mathbf{r}_\beta(s')}{ds'} \frac{\mathbf{r}_\alpha(s) - \mathbf{r}_\beta(s')}{|\mathbf{r}_\alpha(s) - \mathbf{r}_\beta(s')|^3}$$

where  $\times$  denotes the vector product. Find  $G(C_\alpha, C_\beta)$  for the two curves

$$C_\alpha : x_1^2 + x_3^2 = 1, \quad C_\beta : (x_1 - 1)^2 + x_2^2 = 1.$$

**Problem 43.** Let  $x \in \mathbb{R}$ . We define  $[x]$  as the integer part of  $x$  and  $\{x\} := x - [x]$ . Calculate the integrals

$$\int_0^{7/2} [x] dx, \quad \int_0^{7/2} \{x\} dx.$$

**Problem 44.** Let  $x \in \mathbb{R}$ . Consider the integral

$$f(x) = \int_x^\infty \frac{e^{iy}}{y} dy.$$

- (i) Show that  $f(-|x|) = f(|x|) - i\pi$ .  
 (ii) Show that for large  $x$

$$f(x) = e^{ix} \left( \frac{i}{x} + \frac{1}{x^2} + \dots \right).$$

**Problem 45.** Let  $n = 0, 1, 2, \dots$ . We define

$$a_n := \int_0^{\pi/2} \cos^{2n}(x) dx, \quad b_n := \int_0^{\pi/2} x^2 \cos^{2n}(x) dx.$$

Then  $a_0 = \pi/2$  and  $b_0 = \pi^3/24$ . Show that using integration by parts

$$a_n = (2n - 1)(a_{n-1} - a_n).$$

Show that for  $n \geq 1$  we have

$$a_n = (2n - 1)nb_{n-1} - 2n^2b_n.$$

Show that ( $n \geq 1$ )

$$0 \leq \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2} = 2 \frac{b_n}{a_n} \leq \frac{\pi^2}{4(n+1)}.$$

**Problem 46.** Calculate

$$I = \int_0^{10} (x - \text{int}(x)) dx$$

where  $\text{int}(x)$  defines the largest integer less than  $x$ .

**Problem 47.** Let  $0 < a < b < 1$ . Find

$$\int_a^b \frac{\ln(1-x)}{x} dx.$$

**Problem 48.** Show that

$$\int_0^{\pi/2} \cos^{10}(x) \cos^8(2x) \cos^6(4x) \cos^4(6x) \cos^2(8x) dx =$$

$$\frac{5166673\pi}{536870912} + \frac{2966549762512816}{98120709987525225}.$$

**Problem 49.** Let  $z \in \mathbb{C}$ . Let  $\ln(1+z)$  be the branch of the logarithm defined on  $\mathbb{C} \setminus (-\infty, -1]$ . Calculate

$$I_n(r) = Pv \int_{|z|=r} z^{n-1} \ln(1+z) dz, \quad r > 0, \quad n \in \mathbb{Z}$$

where  $Pv$  is the principal value.

**Problem 50.** Let  $b > a$  and  $a, b$  be finite. Consider the integral

$$\int_a^b f(x) dx.$$

(i) Apply the transformation

$$x = \frac{1}{2}(a+b+(b-a)\tanh(y)) \Leftrightarrow y = \tanh^{-1}\left(\frac{2x-a-b}{b-a}\right)$$

to the integral.

(ii) Apply the transformation

$$x = \frac{1}{2}\left(a+b+(b-a)\tanh\left(\frac{\pi}{2}\sinh(y)\right)\right)$$

with

$$\frac{dx}{dy} = \frac{(b-a)\pi \cosh(y)/4}{\cosh^2(\pi \sinh(y)/2)}$$

to the integral.

**Problem 51.** Calculate the integral

$$\int_0^\pi \frac{1 + \sin(x)}{1 + \cos(x)} dx$$

utilizing the transformation  $t = \tan(x/2)$  with  $-\pi < x < \pi$ . From this transformation it follows that

$$x = 2 \arctan(t), \quad dx = \frac{2}{1+t^2} dt$$

and  $x = 0 \rightarrow t = 0$ ,  $x = \pi/2 \rightarrow t = 1$ .



**Problem 52.** Simplify the integral

$$I = \int_0^1 \frac{\cos(x)}{\sqrt{x}} dx$$

for numerical calculation.

**Problem 53.** Let  $a > 0$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions with  $f(-x) = f(x)$  and  $g(-x) = -g(x)$ . Show that

$$\int_{-a}^a f(x)g(x)dx = 0.$$

**Problem 54.** Show that

$$\int_0^\infty e^{-x} x^2 dx = 2!, \quad \int_0^\infty e^{-x} x^3 dx = 3!.$$

Let  $n = 4, 5, \dots$  Show that

$$\int_0^\infty e^{-x} x^n dx = n!.$$

**Problem 55.** Let  $k_1 > 0$ ,  $k_2 > 0$ ,  $k_3 > 0$ . Find the integral

$$\int_0^\infty \sin(k_1 r) \sin(k_2 r) \sin(k_3 r) \frac{dr}{r}.$$

**Problem 56.** Calculate

$$F(s, t) = \int_{\mathbb{R}} e^{-ixt - |x-s|} dx.$$

**Problem 57.** Let  $x \in (0, 1)$ . Calculate

$$P \int_0^1 \frac{y^m}{(y-x)} dy$$

for  $m = 0$ ,  $m = 1$ ,  $m = 2$ ,  $m = 3$ .

**Problem 58.** Let  $a, b > 0$  and  $b > |a|$ . Show that

$$\int_{\mathbb{R}} \frac{dx}{x^2 + 2ax + b^2} = \frac{\pi}{\sqrt{b^2 - a^2}}.$$

**Problem 59.** Let  $a, b > 0$ . Show that

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \left( \frac{b}{a} \right).$$

**Problem 60.** Let  $x > 0$ . Show that

$$\int_0^{\infty} e^{-tx} dt = \frac{1}{x}.$$

**Problem 61.**  $a, b \in \mathbb{R}$ . Let

$$f(x) = a \exp(bx).$$

Find the conditions on  $a$  and  $b$  such that

$$\int_0^1 f(x) dx = 1, \quad \int_0^1 x f(x) dx = \frac{1}{2}.$$

**Problem 62.** *Dawson's integral* is given by

$$f(x) = \int_0^x e^{t^2 - x^2} dt, \quad x \geq 0.$$

(i) Show that for all complex values  $z$  the function  $f$  satisfies the linear differential equation

$$\frac{df(z)}{dz} + zf(z) = 1.$$

(ii) Let  $j = 1, 2, \dots$ . Show that

$$f^{(j+1)}(z) + 2zf^{(j)} + 2jf^{(j-1)}(z) = 0, \quad j = 1, 2, \dots$$

where  $f^{(j)}$  indicates the  $j$ th derivative.

**Problem 63.** Let  $a, b, c \in \mathbb{R}$  and  $a + b \cos(\theta) + c \sin(\theta) \neq 0$ . Show that

$$\int_{-\pi}^{+\pi} \frac{b \sin(\theta) - c \cos(\theta)}{a + b \cos(\theta) + c \sin(\theta)} d\theta = 0.$$

**Problem 64.** Let  $a > 0$ . Show that

$$\int_0^{\infty} \frac{\cos(ax)}{1+x^2} dx = \frac{\pi}{2} e^{-a}.$$

**Problem 65.** Show that

$$\int_0^\infty \frac{\ln(\cosh(x))}{\cosh^3(x)} dx = \frac{\pi}{4}(\ln(2) - 1/2).$$

**Problem 66.** Let  $m, n = 0, 1, 2, \dots$ . Find the integral

$$f_{mn}(t) = \int_0^t (t - \tau)^m \tau^n d\tau.$$

**Problem 67.** Let  $a, b > 0$ . Find the integral

$$\int_0^\infty \frac{\cos(at) - \cos(bt)}{t} dt.$$

**Problem 68.** Let  $q^2 > 0$ . Calculate the integral

$$\int_0^\infty \exp(-q^2 x) dx.$$

**Problem 69.** Let  $x > 0$ . Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is integrable over  $[0, x]$  for all  $x > 0$  and  $\lim_{x \rightarrow \infty} f(x) = a$ . Show that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x f(s) ds = a.$$

Utilize

$$\left| \frac{1}{x} \int_0^x f(s) ds - a \right| = \left| \frac{1}{x} \int_0^x (f(s) - a) ds \right| \leq \left| \frac{1}{x} \int_0^x |f(s) - a| ds \right|.$$

**Problem 70.** Consider the function  $f : [0, 1] \rightarrow [0, 1]$

$$f(x) = \begin{cases} 1/x - \text{int}(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where  $\text{int}(y)$  denotes the integer part of  $y$ . The function is integrable.

(i) Let  $k \geq 1$  be a positive integer. Find

$$\int_{1/(k+1)}^{1/k} f(x) dx.$$

(ii) Find

$$\int_{1/k}^1 f(x) dx.$$

(iii) Find

$$\int_0^1 f(x) dx.$$

**Problem 71.** Show that

$$\int_0^{\pi/2} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)} + \sqrt{\cos(x)}} dx = \frac{\pi}{4}.$$

**Problem 72.** Calculate

$$\int_0^3 \left(x - [x] + \frac{1}{2}\right) dx$$

where  $[x]$  denotes the greatest integer less than or equal to  $x$ .**Hilbert Transform****Problem 73.** The *Hilbert transform*  $H$  and its inverse is given by

$$g(y) = H(f(x)) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{f(x)}{x-y} dx$$

$$f(x) = H^{-1}(g(y)) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{g(y)}{y-x} dy$$

where the *Cauchy principal value* is defined by

$$P \int_{\mathbb{R}} f(x) dx := \lim_{R \rightarrow \infty} \int_{-R}^{+R} f(x) dx.$$

The Hilbert transform relates parts of the function in the same domain. Let  $k > 0$ . Find the Hilbert transform of  $f(x) = \cos(kx)$ .**Problem 74.** Consider the Hilbert transform

$$H[f] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(x')}{x' - x} dx'.$$

Find  $H^2$ .

**Problem 75.** The *Hilbert transform* acts on a one-dimensional function  $g(s)$  by a convolution with the kernel  $1/(\pi s)$ . The singularity at  $s = 0$  is handled in the Cauchy principal value sense. The Fourier transform of the Hilbert kernel is  $-i \operatorname{sgn} \sigma$ . Thus the Hilbert transform of  $g$  is

$$Hg(s) = \int_{-\infty}^{\infty} \frac{g(s-s')}{\pi s'} ds' = \int_{-\infty}^{\infty} (-i \operatorname{sgn} \sigma) G(\sigma) e^{i2\pi s \sigma} d\sigma$$

where  $G(\sigma)$  is the Fourier transform of  $g$ , i.e.

$$G(\sigma) = \int_{-\infty}^{\infty} g(s) e^{-i2\pi \sigma s} ds.$$

Suppose  $g$  is a function whose support is strictly less than radius  $R$ , i.e.  $g(s) = 0$  for all  $|s| > R - \epsilon$  for some small positive  $\epsilon$ . Find an inversion formula.

**Problem 76.** The *Hilbert transform* of a function  $f \in L_2(\mathbb{R})$  is defined as

$$H(f)(y) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(x)}{x-y} dx$$

where *PV* stands for *principal value*. Calculate the Hilbert transform of  $f(x) = 1/(1+x^4)$ .

**Problem 77.** The *Radon transform* for a function  $f(x, y)$  is given by the integral transform

$$P(r, \theta) = Rf(x, y) = \int_{-\infty}^{+\infty} f(r \cos(\theta) - s \sin(\theta), r \sin(\theta) + s \cos(\theta)) ds.$$

The function  $P(r, \theta)$  describes the values of points on projections. Show that the inverse Radon transform can be given by

$$f(x, y) = R^{-1}P(r, \theta) = -\frac{1}{2\pi} BHDP(r, \theta)$$

where  $D$  is the partial differential operator  $Dg(r, \theta) = \partial g / \partial r$  with respect to  $r$ ,  $H$  is the Hilbert transform operator

$$Hg(r, \theta) = -\frac{1}{\pi} \int \frac{g(u, \theta)}{r-u} du$$

and  $B$  is the backprojection operator

$$Bg(r, \theta) = \int_0^\pi g(x \cos(\theta) + y \sin(\theta), \theta) d\theta.$$

**Problem 78.** The *Hilbert transform* of a function  $f \in L_2(\mathbb{R})$  is defined as

$$H(f)(y) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(x)}{x-y} dx$$

where *PV* stands for *principal value*. Calculate the Hilbert transform of

$$f(x) = \exp(-x^2/2).$$

**Problem 79.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{x}{\sinh(x)}.$$

Then  $f(0) = 1$ . Let  $c > 0$ . Find

$$\int_0^c f(x) dx.$$

**Problem 80.** Let  $p_1, p_2, p_3 > 0$ . Calculate

$$f(p_1, p_2, p_3) = \frac{4}{\pi} \int_0^{\infty} \sin(p_1 r) \sin(p_2 r) \sin(p_3 r) \frac{dr}{r}.$$

**Problem 81.** Find the constant  $K$  (normalization) from the condition

$$1 = 4K \int_0^{2\pi} \sin^2(\omega/2) d\omega \int_0^{\pi} \sin(\theta) d\theta \int_{-\pi}^{\pi} d\phi.$$

**Problem 82.** Consider

$$I = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d\alpha \int_0^{\pi} \sin(\beta) d\beta \int_{-\pi}^{\pi} F(\alpha, \beta, \gamma) d\gamma.$$

- (i) Find  $I$  for  $F(\alpha, \beta, \gamma) = 1$ .
- (ii) Find  $I$  for  $F(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$ .
- (iii) Find  $I$  for  $F(\alpha, \beta, \gamma) = \alpha\beta\gamma$ .

**Problem 83.** Consider the circle around  $(0, 0, 0)$  in the  $x_1 - x_2$  plane

$$\mathbf{r}_1(t) = \begin{pmatrix} x_{1,1}(t) \\ x_{1,2}(t) \\ x_{1,3}(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}, \quad t \in [0, 2\pi]$$

and the circle around  $(0, 1, 0)$  in the  $x_2 - x_3$  plane

$$\mathbf{r}_2(s) = \begin{pmatrix} x_{2,1}(s) \\ x_{2,2}(s) \\ x_{2,3}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + \cos(s) \\ \sin(s) \end{pmatrix}.$$

Then the derivatives are

$$\frac{d\mathbf{r}_1(t)}{dt} = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix}, \quad \frac{d\mathbf{r}_2(s)}{ds} = \begin{pmatrix} 0 \\ -\sin(s) \\ \cos(s) \end{pmatrix}.$$

Calculate (Gauss formula)

$$\frac{1}{4\pi} \oint \oint dt ds \left( \frac{d\mathbf{r}_1(t)}{dt} \times \frac{d\mathbf{r}_2(s)}{ds} \right) \cdot \frac{\mathbf{r}_1(t) - \mathbf{r}_2(s)}{|\mathbf{r}_1(t) - \mathbf{r}_2(s)|^3}$$

where  $\times$  denotes the vector product,  $\cdot$  denotes the scalar product and contour integrations run from 0 to  $2\pi$ .

**Problem 84.** Show that

$$\int_0^\infty e^{-x} x^2 dx = 2!, \quad \int_0^\infty e^{-x} x^3 dx = 3!.$$

**Problem 85.** (i) Find the area of the set

$$S_2 := \{ (x_1, x_2) : 1 \geq x_1 \geq x_2 \geq 0 \}.$$

(ii) Find the volume of the set

$$S_3 := \{ (x_1, x_2, x_3) : 1 \geq x_1 \geq x_2 \geq x_3 \geq 0 \}.$$

Extend the  $n$ -dimensions.

**Problem 86.** Let  $n$  be a positive integer. Find

$$\int_0^\pi \cos^2(nx) dx.$$

**Problem 87.** Let  $n \geq 0$ . The Legendre polynomial of degree  $n$  is defined as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

with  $P_0(x) = 1$ . Let  $n \geq 1$ . Show that

$$\int_0^1 x^k P_n(x) dx = 0$$

for  $k = 0, 1, \dots, n-1$

**Problem 88.** Let  $k = 0, 1, \dots$ . Calculate

$$c_k = \frac{2}{\pi} \int_0^\pi \arcsin(\cos(\theta)) \cos(k\theta) d\theta.$$

**Problem 89.** Find  $\alpha > 0$  such that

$$\int_0^\infty 2\alpha x e^{-2\alpha x^3/3} dx = 1.$$

**Problem 90.** The complete elliptic integral of first kind  $K(m)$  can be defined by

$$K(m) = \int_0^1 ((1-t^2)(1-mt^2))^{-1/2} dt, \quad |m| < 1.$$

The *beta function* can be defined by

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad \Re(a) > 0, \quad \Re(b) > 0.$$

Show that  $2\sqrt{2}K(1/2) = B(1/4, 1/2)$ .

**Problem 91.** Consider the function  $f : [0, 1] \rightarrow [0, 1]$ ,  $f(x) = \sqrt{x}$  with

$$I = \int_0^1 \sqrt{x} dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}.$$

Find a approximation (upper bound) of the integral in the sense of Lebesgue by partitioning the ordinate interval into the intervals  $[0, 1/4]$ ,  $(1/4, 1/2]$ ,  $(1/2, 3/4]$ ,  $(3/4, 1]$  with  $a_0 = 0$ ,  $a_1 = 1/4$ ,  $a_2 = 1/2$ ,  $a_3 = 3/4$ ,  $a_4 = 1$ .

**Problem 92.** Let  $a > 0$ . Calculate

$$\int \frac{dx}{(a^2 - x^2)^{3/2}}.$$



Set  $x = a \sin(\theta)$ . Then  $dx = a \cos(\theta)d\theta$ .

**Double Integrals**

**Problem 93.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function and  $f(x_1, x_2) = f(x_2, x_1)$  for all  $x_1, x_2 \in \mathbb{R}$ . Let  $b, a \in \mathbb{R}$  and  $b > a$ . Calculate

$$\int_a^b \int_a^b f(x_1, x_2) \sin(x_1 - x_2) dx_1 dx_2.$$

**Problem 94.** Let  $f$  be a continuous function.

(i) Show that the double integral

$$\int_0^x d\xi \int_0^\xi f(t) dt$$

can be expressed by a single integral.

(ii) Show that ( $n \geq 2$ )

$$\int_0^x d\xi_1 \int_0^{\xi_1} d\xi_2 \cdots \int_0^{\xi_{n-1}} f(\xi_n) d\xi_n$$

can be expressed by a single integral.

**Problem 95.** Calculate the integral

$$I = \int_0^{\pi/2} \sin^{2n}(\theta) \cos^{2n+1}(\theta) d\theta$$

by considering the double integral

$$\int \int_D (r \sin(\theta))^{2n} (r \cos(\theta))^{2n+1} e^{-r^2} r dr d\theta$$

where  $D$  is the first quadrant.

**Problem 96.** Find the integral

$$I(\ell) = \int_0^{2\pi} \left( \int_0^{|\cos(\theta)|^{(\ell/2)}} dp \right) d\theta.$$

**Problem 97.** Evaluate the quadruple integral

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 dx' \int_0^1 dy' \left( \frac{1}{((x-x')^2 + (y-y')^2)^{1/2}} - \frac{1}{((x-x')^2 + (y-y')^2 + 1)^{1/2}} \right).$$

**Problem 98.** Let  $z \in (0, 1]$  and  $x \in (0, 1]$ . Find the double integral

$$f(z) = 1 - \int_z^1 dx \int_{z/x}^1 dy.$$

**Problem 99.** (i) Let  $r = \sqrt{x_1^2 + \cdots + x_n^2}$ . Show that the integral

$$\int \int_{r \geq 1} \cdots \int \frac{dx_1 \cdots dx_n}{r^\alpha}$$

is finite for  $\alpha > n$  and infinite for  $\alpha \leq n$ .

(ii) Let  $r = \sqrt{x_1^2 + \cdots + x_n^2}$ . Show that the integral

$$\int \int_{r \leq 1} \cdots \int \frac{dx_1 \cdots dx_n}{r^\alpha}$$

is finite for  $\alpha < n$  and infinite for  $\alpha \geq n$ .

**Problem 100.** Consider a three-dimensional probability distribution  $f(x_1, x_2, x_3)$  such that for all  $j$

$$f_x(x_j) = \frac{1}{2\sqrt{\pi}} \left( 1 + 2x_j^2 - \frac{-1 + 2x_j^2}{\sqrt{2}} \right) e^{-x_j^2}$$

where  $f_x(x)$  is the probability density associated with an individual variable. This means

$$f_x(x_1) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2, x_3) dx_2 dx_3$$

etc. Is it possible that the probability density associated with the sum of these variables  $s = x_1 + x_2 + x_3$  is given by

$$f_s(s) = \frac{1}{2\sqrt{\pi}} \left( 1 + 2s^2 + \frac{-1 + 2s^2}{\sqrt{2}} \right) e^{-s^2}$$

provided that  $f(x_1, x_2, x_3)$  is non-negative?

**Problem 101.** Consider a one-dimensional chain of length  $N$  with open end boundary conditions. The counting is from left to right starting at 0. The canonical partition function  $Z(\beta)$  ( $\beta > 0$ ) is given by the multiple integral

$$Z_N(\beta) = \int_{-1}^1 ds_0 \int_{-1}^1 ds_1 \cdots \int_{-1}^1 ds_{N-1} e^{\beta|s_0-s_1|} e^{\beta|s_1-s_2|} \cdots e^{\beta|s_{N-2}-s_{N-1}|}.$$

Show that there is a coordinate transformation which decouples the sites. Find  $Z_2(\beta)$  and  $Z_3(\beta)$ .

**Problem 102.** Let  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an analytic function. Consider the map

$$\mathbf{x}_j = \mathbf{f}(\mathbf{x}_{j-1}) = \cdots = \mathbf{f}(\mathbf{x}_0).$$

To study the evolution of phase-space distributions, we can introduce the evolution operator  $U(\mathbf{x}', \mathbf{x}, j)$  such that any initial phase-space distribution  $\rho(\mathbf{x}, 0)$  evolves into

$$\rho(\mathbf{x}'; j) = \int_{\Omega} U(\mathbf{x}', \mathbf{x}; j) \rho(\mathbf{x}, 0) d\mathbf{x}$$

where  $\Omega$  is the phase space area. Find  $U(\mathbf{x}', \mathbf{x}; j)$ .

**Problem 103.** Let

$$S := \{ (x, y) \in \mathbb{R}^2 : x, y \geq 0, 0 \leq x_1^2 + x_2^2 \leq 1 \}.$$

Let  $m, n$  be nonnegative integers. Find the integral

$$\int_S x^{2m+1} y^{2n+1} dx dy.$$

**Problem 104.** Calculate

$$\int \int_A dx dy$$

where

$$A = \{ (x, y) : x, y \geq 0, x + y \leq 1, x \geq 1/3 \}.$$

**Problem 105.** Calculate the integral

$$I = \int_1^\infty dx \int_1^\infty dy \frac{\sqrt{x^2-1} \sqrt{y^2-1}}{(x+y)^6}$$

using the substitution  $x = \cosh(\alpha)$ ,  $y = \cosh(\beta)$ .

**Problem 106.** Find the integral

$$I = \int_1^\infty dx \int_1^\infty dy \frac{\sqrt{x^2-1}\sqrt{y^2-1}}{(x+y)^6}.$$

Hint. Use the substitutions  $x = \cosh(\alpha)$ ,  $y = \cosh(\beta)$  and show that the integral can be written as

$$I = \int_0^\infty d\alpha \int_0^\infty d\beta \frac{\sinh^2(\alpha)\sinh^2(\beta)}{(\cosh(\alpha) + \cosh(\beta))^6}.$$

**Problem 107.** Calculate the double integral

$$\frac{1}{2\pi^2} \int_0^\pi \int_0^\pi d\alpha d\alpha' \ln(2 - \cos(\alpha) - \cos(\alpha'))$$

utilizing the identity

$$\cos(\alpha) + \cos(\alpha') \equiv 2 \cos\left(\frac{\alpha + \alpha'}{2}\right) \cos\left(\frac{\alpha - \alpha'}{2}\right)$$

the transformation  $x = (\alpha + \alpha')/2$ ,  $y = (\alpha - \alpha')/2$  and the integral

$$\int_0^\pi \ln(1 + \sin(x)) dx = -\pi \ln(2) + G$$

where  $G$  is the Catalan constant.

**Problem 108.** Let  $a > 0$  and  $R \geq 0$ . Find

$$I_a(R) = \int_0^R \int_0^R \frac{dx dy}{(x-y)^2 + a^2} \equiv \int_0^R \int_0^R \frac{dx dy}{x^2 + y^2 - 2xy + a^2}.$$

**Problem 109.** Calculate the integral

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{dx_1 dx_2 dx_3 dx_4 dx_5 dx_6}{1 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6}.$$

**Problem 110.** Let  $n_1, n_2 \in \mathbb{Z}$ . Consider a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  with a period  $2\pi$  for both  $x_1$  and  $x_2$

$$f(x_1, x_2) = \sum_{n_1, n_2 = -\infty}^{\infty} c_{n_1, n_2} e^{i(n_1 x_1 + n_2 x_2)}$$

and

$$\int_0^{2\pi} \int_0^{2\pi} f(x_1, x_2) dx_1 dx_2 = 0.$$

Then

$$c_{n_1, n_2} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x_1, x_2) e^{-i(n_1 x_1 + n_2 x_2)} dx_1 dx_2$$

(i) Let

$$f(x_1, x_2) = \sin(x_1) + \cos(x_2).$$

Find  $c_{n_1, n_2}$ .

(ii) Let

$$f(x_1, x_2) = \sin(x_1) \cos(x_2).$$

Find  $c_{n_1, n_2}$ .

**Problem 111.** Find a non-negative analytic function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) dx dy = 1$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} x^2 dx dy = \frac{1}{2}, \quad \int_{\mathbb{R}} \int_{\mathbb{R}} y^2 dx dy = \frac{1}{2}.$$

**Problem 112.** Consider the compact set

$$S := \{ (x, y) : y \geq x^2, y \leq \sqrt{x}, x, y \geq 0 \}.$$

Thus  $S \subset [0, 1] \times [0, 1]$ . Find

$$I(S) = \int_S d\mu = \int_S dx dy.$$

**Problem 113.** Let  $a > 0$ . Consider the compact set (*lemniscate*)

$$S := \{ (x, y) : (x^2 + y^2)^2 \leq 2a^2 xy \}.$$

Find

$$I(S) = \int_S d\mu = \int_S dx dy.$$

Introduce *polar coordinates*  $x(r, \phi) = r \cos(\phi)$ ,  $y(r, \phi) = r \sin(\phi)$ . Thus we have

$$x^2 + y^2 = r^2, \quad xy = r^2 \cos(\phi) \sin(\phi) \equiv \frac{1}{2} \sin(2\phi).$$

**Problem 114.** Calculate the integrals

$$\int_0^1 dx_2 \left( \int_0^{1-x_2} dx_1 \right) dx_2, \quad \int_0^1 dx_3 \left( \int_0^{1-x_3} dx_2 \left( \int_0^{1-x_3-x_2} dx_1 \right) \right).$$

Extend to  $n$ -dimensions. Give an interpretation of the result.

**Problem 115.** (i) Is the subset of  $\mathbb{R}^2$

$$S_2 = \{ (x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1 \}$$

convex?

(ii) Calculate

$$A = \int_{S_2} dx_1 dx_2.$$

(iii) The area of a triangle in the plane  $\mathbb{R}^2$  with vertices at  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  is given by

$$A = \pm \frac{1}{2} \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

where the sign is chosen so that the area is nonnegative. Let  $(x_1, y_1) = (0, 0)$ ,  $(x_2, y_2) = (1, 0)$ ,  $(x_3, y_3) = (0, 1)$ . Find  $A$ . Compare to (ii).

(iv) Is the subset of  $\mathbb{R}^3$

$$S_3 = \{ (x_1, x_2, x_3) : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \}$$

convex?

(v) Calculate

$$V = \int_{S_3} dx_1 dx_2 dx_3.$$

**Problem 116.** Consider

$$I = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d\alpha \int_0^{\pi} \sin(\beta) d\beta \int_{-\pi}^{\pi} F(\alpha, \beta, \gamma) d\gamma.$$

Find  $I$  for  $F(\alpha, \beta, \gamma) = 1$ ,  $F(\alpha, \gamma, \beta) = \alpha + \beta + \gamma$ ,  $F(\alpha, \beta, \gamma) = \alpha\beta\gamma$ .

**Problem 117.** Let  $\gamma > 0$ . A random variable  $X$  is said to be Lorentzian with parameters  $(\alpha, \gamma)$  if its probability density is given by

$$f_X(x) = \frac{1}{\pi} \frac{\gamma}{(x - \alpha)^2 + \gamma^2}, \quad x \in \mathbb{R}.$$

Let  $X, Y$  be two independent Lorentzian random variables with parameters  $(\alpha, \gamma)$  and  $(\beta, \delta)$ , respectively. Let  $\epsilon \geq 0$ . Show that

- (a)  $\epsilon X$  is distributed Lorentzian  $(\epsilon\alpha, \epsilon\gamma)$
- (b)  $\epsilon + X$  is distributed Lorentzian  $(\epsilon + \alpha, \gamma)$
- (c)  $-X$  is distributed Lorentzian  $(-\alpha, \gamma)$
- (d)  $X + Y$  is distributed Lorentzian  $(\alpha + \beta, \gamma + \delta)$
- (e)  $X^{-1}$  is distributed Lorentzian  $\left(\frac{\alpha}{\alpha^2 + \gamma^2}, \frac{\gamma}{\alpha^2 + \gamma^2}\right)$

**Problem 118.** A random variable  $X$  is said to be Lorentzian if its probability density  $p_X$  is a the form

$$p_X(x) = \frac{1}{\pi} \frac{\gamma}{(x - \alpha)^2 + \gamma^2}$$

where  $\gamma > 0$ . We say that  $X$  is  $(\alpha, \gamma)$  to indicate that the random variable is Lorentzian with the probability density given above. Let  $X, Y$  be two independent random variables with  $(\alpha, \gamma)$  and  $(\beta, \delta)$ . Let  $\epsilon$  be a nonnegative real number. Show that

$$\begin{aligned} \epsilon X & \text{ is } (\epsilon\alpha, \epsilon\gamma) \\ \epsilon + X & \text{ is } (\epsilon + \alpha, \gamma) \\ -X & \text{ is } (-\alpha, \gamma) \\ X + Y & \text{ is } (\alpha + \beta, \gamma + \delta) \\ X^{-1} & \text{ is } \left(\frac{\alpha}{\alpha^2 + \gamma^2}, \frac{\gamma}{\alpha^2 + \gamma^2}\right). \end{aligned}$$

**Problem 119.** Consider the set

$$M = \{ (x_1, x_2, x_3) : x_1^2 + x_2^2 \leq 1 \text{ and } x_1^2 + x_3^2 \leq 1 \}$$

which is subset of  $\mathbb{R}^3$ . Obviously  $M$  is compact and measurable

$$I(M) = \int_M d\mu = \int_{\mathbb{R}^3} \chi_M d\mu = \int_{\mathbb{R}^3} \chi_M dx_1 dx_2 dx_3$$

where  $\chi_M$  is the indicator function. Find  $I(M)$ .

**Problem 120.** Given a smooth Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^n \frac{p_j^2}{2} + U(\mathbf{q})$$

with  $n$  degrees of freedom ( $\mathbf{p} = (p_1, \dots, p_n)$ ,  $\mathbf{q} = (q_1, \dots, q_n)$ ). Let  $V(E)$  be the classical phase space volume at energy  $E$  of a smooth Hamilton function is given by

$$V(E) = \int_{\mathbb{R}^{2n}} \Theta(E - H(\mathbf{p}, \mathbf{q})) d^n \mathbf{p} d^n \mathbf{q}$$

where  $\Theta$  is the step function. Assume that  $U(\epsilon \mathbf{q}) = \epsilon^n U(\mathbf{q})$ .

(i) Consider the transformation

$$\mathbf{p} = E^{1/2} \mathbf{p}', \quad \mathbf{q} = E^{1/n} \mathbf{q}'$$

with the inverse transformation

$$\mathbf{p}' = E^{-1/2} \mathbf{p}, \quad \mathbf{q}' = E^{-1/n} \mathbf{q}.$$

Find  $d^n \mathbf{p}' d^n \mathbf{q}'$  and  $H(\mathbf{p}', \mathbf{q}')$ .

(ii) Calculate  $V(E)$  with the assumption that  $E > 0$ . Find the asymptotic behaviour.

**Problem 121.** Let  $a > 0$ . Find the area of the surface in  $\mathbb{R}^3$  given by the intersection of a hyperbolic paraboloid  $x_3 = x_1 x_2 / a$  and the cylinder  $x_1^2 + x_2^2 = a^2$ . We have the parameter representation

$$x_1(r, \phi) = r \cos(\phi), \quad x_2(r, \phi) = r \sin(\phi), \quad x_3(r, \phi) = \frac{x_1 x_2}{a} = \frac{1}{a} r^2 \cos(\phi) \sin(\phi).$$

Note that  $\sin(\phi) \cos(\phi) \equiv \frac{1}{2} \sin(2\phi)$ .

## 8.2 Programming Problems

**Problem 1.** Calculate the integral

$$\int_0^1 |\cos(2\pi x)| dx$$

using the random number generator described in problem 7, chapter 10, page 250, *Problems and Solutions in Scientific Computing*. Compare to the exact result by solving the integral.



### 8.3 Supplementary Problems

**Problem 1.** Let  $\alpha > 0$ . Show that

$$\int_{-\infty}^{\infty} \operatorname{sech}(\alpha t) dt = \frac{2}{\alpha}.$$

**Problem 2.** Consider the function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$

$$\phi(x) := \begin{cases} 1 & \text{for } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Find

$$\psi(x) := \phi(2x) - \phi(2x - 1).$$

Draw the function. Calculate

$$\int_{-\infty}^{+\infty} \psi(x) dx.$$

**Problem 3.** Show that

$$e^{-x^2/2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2 + ixy\right) dy.$$

**Problem 4.** Let  $\ell > 0$  and  $r_0 > 0$ . Find the integral

$$\int_0^{r_0} \frac{r^3}{\sqrt{1 + r^2/\ell^2}} dr.$$

**Problem 5.** Let  $0 \leq r < 1$ . Consider the Hilbert space  $L_2[0, 2\pi]$  and  $f(\theta) \in L_2[0, 2\pi]$ . Show that

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta + \frac{1}{\pi} \int_0^{2\pi} \sum_{j=1}^{\infty} r^j f(\theta) \cos(j(\phi - \theta)) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{1 - r^2}{1 - 2r \cos(\phi - \theta) + r^2} d\theta. \end{aligned}$$

**Problem 6.** Show that

$$\int_0^{\infty} e^{-kx} \sin(ky) \sin(k\gamma) \frac{dk}{k} = \frac{1}{4} \ln \left( \frac{x^2 + (y - \gamma)^2}{x^2 + (y + \gamma)^2} \right).$$

Note that

$$f(x) = \int_0^\infty e^{-kx} \frac{dk}{k} \Rightarrow \frac{df}{dx} = -\frac{1}{x}.$$

**Problem 7.** Show that

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^3} dx = \frac{x(x^2 - a^2)}{8a^2(x^2 + a^2)^3} + \frac{1}{8a^2} \arctan(x/a).$$

**Problem 8.** Show that

$$\frac{3}{\pi} \int_0^{\pi/6} \ln(2 \cos(x)) dx = 0.338314\dots$$

**Problem 9.** The content ( $n$ -dimensional volume) bounded by a hypersphere of radius  $r$  is known to be

$$V_n = \frac{2r^n \pi^{n/2}}{n\Gamma(n/2)}$$

where  $\Gamma$  is the gamma function. Let  $r = 1$ . Show that

$$\lim_{n \rightarrow \infty} V_n = 0.$$

**Problem 10.** Let  $k = 0, 1, 2, \dots$  and

$$y_k := \int_0^1 \frac{x^k}{1+x+x^2} dx.$$

Show that

$$y_{k+2} + y_{k+1} + y_k = \frac{1}{k+1}.$$

Note that

$$y_0 = \int_0^1 \frac{1}{1+x+x^2} dx = \frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) \Big|_{x=0}^{x=1}$$

$$y_1 = \int_0^1 \frac{x}{1+x+x^2} dx = \frac{1}{2} \ln(1+x+x^2) \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{1}{1+x+x^2} dx = \frac{1}{2} \ln(1+x+x^2) \Big|_0^1 - \frac{1}{2} y_0.$$

**Problem 11.** Let  $c \in \mathbb{C}$  with  $|c| < \infty$  and  $\Re(c) > 0$ . Consider the integral

$$I(c) = \int_0^\infty \exp(-cx - x^2) dx.$$

Show that

$$I(c) = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(1/2 + k/2)}{k!} c^k.$$

**Problem 12.** Find  $\alpha > 0$  such that

$$\int_0^{\infty} 2\alpha x^2 e^{-2\alpha x^3/3} dx = 1.$$

**Problem 13.** Let  $\mathbf{x} = (x_1 \ x_2 \ \cdots \ x_n)$ ,  $\mathbf{y} = (y_1 \ y_2 \ \cdots \ y_n)$  and  $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$ . Show that

$$\exp(a\mathbf{y}^2) = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \exp(-\mathbf{x}^2 + 2a^{1/2} \mathbf{x} \cdot \mathbf{y}) dx_1 \dots dx_n.$$

**Problem 14.** Let  $b > a$ . Show that

$$\int_a^b \sqrt{\frac{y-a}{b-y}} \left(1 - \frac{\gamma}{y}\right) \frac{dy}{y-x} = \pi \left(1 - \frac{\gamma}{x} \sqrt{\frac{a}{b}}\right).$$

**Problem 15.** Let  $c > 0$  and  $k \in \mathbb{R}$ . Show that

$$\int_0^{\infty} \exp(-cs^2) \cos(2ks) ds = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{c}} \exp(-k^2/c).$$

This integral plays a role in optics.

**Problem 16.** Let  $x > 0$ . Show that

$$2 \int_0^{\infty} e^{-s^2} ds = \int_{x^2}^{\infty} \frac{e^u}{\sqrt{u}}$$

**Problem 17.** Let  $x \geq 0$ . Consider the function  $f$  defined by

$$f(x) = \int_0^x \frac{\ln(1+y)}{y} dy.$$

Show that for small  $X$  we can write

$$f(x) = x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \cdots$$

Show that

$$f(x) = \frac{\pi^2}{6} + \frac{1}{2} \ln^2(x) - f\left(\frac{1}{x}\right).$$

Give a value of  $f(1)$ .

**Problem 18.** Calculating

$$\int \cos^3(x) \sin^3(x) dx$$

student Alice tells you the result is

$$\frac{1}{4} \sin^4(x) - \frac{1}{6} \sin^6(x) + C_1$$

and student Bob tells that

$$\frac{1}{6} \cos^6(x) - \frac{1}{4} \cos^4(x) + C_2$$

is the result.  $C_1, C_2$  are the constants of integration. Discuss.

**Problem 19.** Let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be an analytic function,  $g_j : \mathbb{R} \rightarrow \mathbb{R}$  ( $j = 1, \dots, n$ ) are analytic functions and  $c_1, \dots, c_n$  are constants. Consider

$$I(\epsilon) = \int_{c_1}^{g_1(\epsilon)} dx_1 \int_{c_2}^{g_2(\epsilon)} dx_2 \cdots \int_{c_n}^{g_n(\epsilon)} f(x_1, x_2, \dots, x_n, \epsilon).$$

Show that

$$\begin{aligned} \frac{dI(\epsilon)}{d\epsilon} &= \int_{c_1}^{g_1(\epsilon)} dx_1 \int_{c_2}^{g_2(\epsilon)} dx_2 \cdots \int_{c_n}^{g_n(\epsilon)} \frac{\partial f(x_1, x_2, \dots, x_n, \epsilon)}{\partial \epsilon} \\ &+ \frac{dg_1}{d\epsilon} \int_{c_2}^{g_2(\epsilon)} dx_2 \int_{c_3}^{g_3(\epsilon)} \cdots \int_{c_n}^{g_n(\epsilon)} f(x_1, x_2, \dots, x_n, \epsilon) \\ &+ \frac{dg_2}{d\epsilon} \int_{c_1}^{g_1(\epsilon)} dx_1 \int_{c_3}^{g_3(\epsilon)} dx_3 \cdots \int_{c_n}^{g_n(\epsilon)} f(x_1, x_2, \dots, x_n, \epsilon) \\ &+ \cdots \\ &+ \frac{dg_n}{d\epsilon} \int_{c_1}^{g_1(\epsilon)} dx_1 \int_{c_2}^{g_2(\epsilon)} dx_2 \cdots \int_{c_{n-1}}^{g_{n-1}(\epsilon)} dx_{n-1} f(x_1, x_2, \dots, x_n, \epsilon). \end{aligned}$$

**Problem 20.** Show that

$$\int_0^\infty \frac{t dt}{e^{2\pi t} - 1} = \frac{1}{4\pi^2} \int_0^\infty \frac{\tau d\tau}{e^\tau - 1} = \frac{1}{4\pi^2} \frac{\pi^2}{6} = \frac{1}{24}.$$

**Problem 21.** Let  $k \in \mathbb{N}$ . Show that

$$\int_0^{2\pi} \cos^k(\theta) d\theta = 2(1 + (-1)^k) \pi \frac{(k-1)!!}{k!!}.$$

**Problem 22.** Let  $k \in \mathbb{N}$ . Show that

$$\int_0^{2\pi} \cos^k(\theta) d\theta = 2(1 + (-1)^k) \pi \frac{(k-1)!!}{k!!}.$$

**Problem 23.** Show that

$$\int_0^1 \sqrt{1 - \sqrt{x}} dx = \frac{8}{15}.$$

Hint. Set  $x = \sin^4(y)$ . Then  $dx = 4 \sin^3(y) \cos(y) dy$ .

**Problem 24.** Let  $a > 0$ . Show that

$$\int \frac{x^2 dx}{(a^2 - x^2)^{3/2}} = \frac{x}{\sqrt{a^2 - x^2}} - \arcsin(x/a) + C$$

**Problem 25.** Show that

$$2 \int_0^\infty \frac{\sin(2px) \sin(qx)}{x} dx = \ln \frac{|2p + q|}{|2p - q|}.$$

**Problem 26.** For a  $\lambda/2$  antenna we obtain the expression

$$E(r, \theta) = -\frac{\omega I_0 \sin \theta}{4\pi \epsilon_0 c^2 r} \int_{-\lambda/4}^{\lambda/4} \cos(k\ell) \sin(\omega(t - c^{-1}(r - \ell \cos \theta))) d\ell.$$

Calculate  $E(r, \theta)$ .

**Problem 27.** Let  $\Re(a) > 0$  and  $n = 0, 1, \dots$ . Show that

$$\begin{aligned} \int_{-\infty}^{\infty} x^n e^{-ax^2 + px} dx &= \frac{\partial^n}{\partial p^n} \int_{-\infty}^{\infty} e^{-ax^2 + px} dx \\ &= \frac{\partial^n}{\partial p^n} \sqrt{\frac{\pi}{a}} e^{p^2/(4a)}. \end{aligned}$$

**Problem 28.** Let  $a \neq 0$ . Show that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x} dx = \pi \operatorname{sgn}(a).$$

**Problem 29.** Let  $c > 1$ . Show that

$$\int_1^c \sin(x^2) dx = \frac{1}{2} \int_1^{c^2} \frac{\sin(\tau)}{\sqrt{\tau}} d\tau.$$

**Problem 30.** Show that

$$\int_0^1 x(1+x^2)^{1/2} dx = \frac{1}{3}(2^{3/2} - 1)$$

setting  $u = 1 + x^2$  and hence  $du = 2x dx$ .

**Problem 31.** Let  $(B_\tau, \tau \geq 0)$  be the linear Brownian motion starting from 0. Show that

$$\Gamma_+ := \int_0^1 ds 1_{(B_s > 0)}$$

follows the arcsine distribution, i.e.

$$P(\Gamma_+ \in d\tau) = \frac{1}{\pi} \frac{d\tau}{\sqrt{\tau(1-\tau)}}.$$

**Problem 32.** Let  $a > 0$ . Show that

$$\int_{\mathbb{R}} \frac{\sin^2(ax/2)}{x^2} dx = \frac{1}{2}\pi a.$$

**Problem 33.** Let  $k = 1, 2, \dots$ . Study the integrals

$$L_k(bcc) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi (\cos(x_1) \cos(x_2) \cos(x_3))^k dx_1 dx_2 dx_3$$

$$L_k(sc) = \frac{1}{\pi^3} \frac{1}{3^k} \int_0^\pi \int_0^\pi \int_0^\pi (\cos(x_1) + \cos(x_2) + \cos(x_3))^k dx_1 dx_2 dx_3$$

$$L_k(fcc) = \frac{1}{\pi^3} \frac{1}{3^k} \int_0^\pi \int_0^\pi \int_0^\pi (\cos(x_1) \cos(x_2) + \cos(x_2) \cos(x_3) + \cos(x_3) \cos(x_1)) dx_1 dx_2 dx_3.$$

They play a role in solid state physics for the body-centred cubic lattice, simple cubic lattice, face-center cubic lattice.

**Problem 34.** Draw the functions

$$\begin{aligned} f_1(x) &= \cos(2\pi x) \\ f_2(x) &= \cos(2\pi(\cos(2\pi x))) \\ f_3(x) &= \cos(2\pi(\cos(2\pi(\cos(2\pi x))))). \end{aligned}$$

Extend it to  $f_n(x)$ . Find the integral

$$\int_0^{2\pi} f_n(x) dx.$$

**Problem 35.** Show that (*Fresnel's integral*)

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

**Problem 36.** Show that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{sech}(\alpha - \beta) e^{i\omega\beta} d\beta = \operatorname{sech}\left(\frac{1}{2}\pi\omega\right) e^{i\omega\alpha}.$$

**Problem 37.** Let  $\sigma > 0$ . Show that

$$\frac{1}{4\sigma^2} \int_0^{\infty} x^{5/2} \exp\left(-\frac{x^2}{8\sigma^2}\right) dx = \left(\frac{32}{10}\right)^{1/4} \Gamma(3/4)\sigma^{3/2}.$$

**Problem 38.** Let  $\alpha > 0$ . Show that

$$\int_0^{\infty} x e^{-\alpha x^2} J_0(\beta x) dx = \frac{1}{2\alpha} e^{-\beta^2/(4\alpha)}.$$

**Problem 39.** Show that

$$\int_0^{\infty} \frac{dx}{\cosh^3(x)} = \frac{\pi}{4}.$$

## Chapter 9

# Functional Equations

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A functional equation is any equation that specifies a function in implicit form. An example is the Gamma function  $\Gamma(z)$

$$\Gamma(z + 1) = z\Gamma(z)$$

with  $\Gamma(1) = \Gamma(2) = 1$ . The trigonometric function  $\sin(x)$  and  $\cos(x)$  satisfy a system of functional equations. So functional equations are equations where the unknowns are functions. In most cases it is assumed that the functions are continuous.

### 9.1 Solved Problems

**Problem 1.** To solve a number of nonlinear functional equations it is helpful to have the solution of the linear functional equation

$$f(x + y) = f(x) + f(y) \tag{1}$$

where we assume that the function  $f$  is continuous.

**Problem 2.** Find the solution of the functional equations

$$f(x + y) = f(x)f(y) \tag{1}$$

where we assume that  $f$  is continuous.



**Problem 3.** Find the solution of the functional equation

$$f(xy) = f(x) + f(y) \tag{1}$$

where we assume that the function  $f$  is continuous.

**Problem 4.** Find the solution of the functional equation

$$f(xy) = f(x)f(y). \tag{1}$$

We assume that the function  $f$  is continuous.

**Problem 5.** Find the solution of the *Jensen equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \tag{1}$$

where we assume that  $f$  is continuous.

**Problem 6.** Show that the functional equation

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$

admits the solutions  $f(x) = \tan(cx)$ , where  $c$  is a constant.

**Problem 7.** Show that the functional equation

$$f(x+y) = \frac{f(x) + f(y)}{1 + \frac{f(x)f(y)}{C^2}} \tag{1}$$

admits the solution

$$f(x) = C \tanh(cx). \tag{2}$$

**Problem 8.** Show that the functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x) + f(y)}$$

admits the solution

$$f(x) = \frac{c}{x}.$$

**Problem 9.** Show that the functional equation

$$f(x+y) = \frac{f(x) + f(y) - 1}{2f(x) + 2f(y) - 2f(x)f(y) - 1}$$

admits the solution

$$f(x) = \frac{1}{1 + \tan(cx)}.$$

**Problem 10.** Show that the functional equation

$$f(x + y) = f(x)f(y) + \sqrt{f(x)^2 - 1}\sqrt{f(y)^2 - 1}$$

admits the solution

$$f(x) = \cosh(cx).$$

**Problem 11.** Show that the functional equation

$$f(x + y + axy) = f(x)f(y)$$

admits the solution

$$f(x) = (1 + ax)^c.$$

**Problem 12.** Show that the functional equation

$$f(x + y) - f(x - y) = 4\sqrt{f(x)f(y)}$$

admits the solution

$$f(x) = cx^2.$$

**Problem 13.** Solve the functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

assuming that  $f$  is a continuous function.

**Problem 14.** Show that the trigonometric functions  $f(x) = \cos(x)$  and  $g(x) = \sin(x)$  satisfy the system of functional equations

$$\begin{aligned} g(x + y) &= g(x)f(y) + f(x)g(y) \\ f(x + y) &= f(x)f(y) - g(x)g(y) \\ g(x - y) &= g(x)f(y) - g(y)f(x) \\ f(x - y) &= f(x)f(y) + g(x)g(y). \end{aligned}$$

**Problem 15.** Show that the Jacobi elliptic functions satisfy the system of functional equations

$$f(x \pm y) = \frac{f(x)g(y)h(y) \pm f(y)g(x)h(x)}{1 - k^2 f(x)^2 f(y)^2}$$

$$g(x \pm y) = \frac{g(x)g(y) \mp f(x)f(y)h(x)h(y)}{1 - k^2 f(x)^2 f(y)^2}$$

$$h(x \pm y) = \frac{h(x)h(y) \mp k^2 f(x)f(y)g(x)g(y)}{1 - k^2 f(x)^2 f(y)^2}.$$

**Problem 16.** Let  $a$  be a positive integer with  $a \geq 2$ . Let  $1 \leq x \leq a$ . Consider the equation

$$g(x-1) - 2g(x) + g(x+1) = -\lambda g(x).$$

Show that

$$g_j(x) = \sin(j\pi x/(a+1)), \quad \lambda_j(x) = 2(1 - \cos(j\pi/(a+1)))$$

satisfy this equation.

**Problem 17.** Let  $a, c, \epsilon$  be positive constants. Solve the functional equation

$$g((\theta + c) \pmod{1}) = ag(\theta) + \epsilon \sin(2\pi\theta).$$

## 9.2 Supplementary Problems

**Problem 1.** Show that the functional equation

$$f(x+y) = \frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)}$$

admits the solution

$$f(x) = \frac{cx}{1 - cx}.$$

**Problem 2.** Consider the analytic function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \arctan(x).$$

Show that

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right).$$

What happens at  $xy = 1$ ?

**Problem 3.** Consider the functional equation

$$g(x) = \alpha g(g(x/\alpha))$$

with  $g(0) = 0$  and  $g'(x=0) = 1$ . Show that

$$g(x) = \frac{x}{1 - cx}$$

is a solution with  $c$  an arbitrary constant.

**Problem 4.** We know that

$$\tan(\alpha + \beta) \equiv \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.$$

Show that

$$\begin{aligned} \tan(\alpha + \beta + \gamma) = \\ \frac{\tan(\alpha) + \tan(\beta) + \tan(\gamma) - \tan(\alpha)\tan(\beta)\tan(\gamma)}{1 - \tan(\alpha)\tan(\beta) - \tan(\alpha)\tan(\gamma) - \tan(\beta)\tan(\gamma)}. \end{aligned}$$

# Chapter 10

## Inequalities

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### 10.1 Solved Problems

**Problem 1.** Let  $a, b \in \mathbb{R}$ . Show that

$$2ab \leq a^2 + b^2.$$

**Problem 2.** Let  $a, b \in \mathbb{R}$ . Show that

$$|a + b| \leq |a| + |b|.$$

**Problem 3.** Let  $a, b \in \mathbb{R}^+$ . Show that

$$\frac{1}{2}(a + b) \geq \sqrt{ab}.$$

**Problem 4.** Let  $x \geq 0$  and  $0 < p < 1$ . Show that

$$\frac{1}{p}(1 - x^p) \geq 1 - x.$$

**Problem 5.** Let  $x \in (0, 1)$ . Show that

$$x(1 - x) < x.$$

**Problem 6.** Let  $n \in \mathbb{N}$ .

- (i) Show that  $n < 2^n$ .
- (ii) Show that  $n^2 < 4^n$ .
- (iii) Show that if  $n \geq 4$  then  $2^n < n!$ .

**Problem 7.** Let  $x, y$  be two nonnegative real numbers. Show that

$$xy \leq \left(\frac{x+y}{2}\right)^2 \equiv \frac{1}{4}(x^2 + y^2 + 2xy).$$

**Problem 8.** Let  $a, b, c, d$  be nonnegative real numbers. Show that

$$(abcd)^{1/4} \leq \frac{1}{4}(a + b + c + d).$$

**Problem 9.** Let  $x_j$  ( $j = 1, \dots, n$ ) be nonnegative real numbers. Show that

$$x_1 x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n.$$

**Problem 10.** Let  $a, b, c, d$  be nonnegative real numbers. Show that

$$a^4 + b^4 + c^4 + d^4 \geq 4abcd.$$

**Problem 11.** Let  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Show that (*Bernoulli inequality*)

$$(1 + x)^n \geq 1 + nx.$$

**Problem 12.** Let  $x \geq 0$ . Show that

$$1 - e^{-x} \geq \frac{x}{1+x}.$$

**Problem 13.** Let

$$e_n := 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

where  $n = 1, 2, \dots$ . Let  $m > n$ . Show that

$$|e_m - e_n| = e_m - e_n \leq \frac{2}{(n+1)!}.$$

**Problem 14.** Does the inequality

$$1 + 2 \cos(\theta) - \cos(2\theta) \leq 2$$

hold for all  $\theta \in [0, 2\pi)$ ?

**Problem 15.** Let  $x > 0$ . Show that

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}.$$

**Problem 16.** Let  $x > 0$ . Show that

$$1 - \frac{1}{x} \leq \ln(x) \leq x - 1$$

with equality iff  $x = 1$ .

**Problem 17.** Show that if a twice differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has a second order derivative which is non-negative (positive) everywhere, then the function is convex (strictly convex).

**Problem 18.** Let  $a_1, a_2, \dots, a_n$  be positive numbers and  $b_1, b_2, \dots, b_n$  be nonnegative numbers such that

$$\sum_{j=1}^n b_j > 0.$$

Show that (*log-sum inequality*)

$$\sum_{j=1}^n \left( a_j \log \frac{a_j}{b_j} \right) \geq \left( \sum_{j=1}^n a_j \right) \log \frac{\left( \sum_{j=1}^n a_j \right)}{\left( \sum_{j=1}^n b_j \right)}$$

with the conventions based on continuity arguments

$$0 \cdot \log 0 = 0, \quad 0 \cdot \log \frac{p}{0} = \infty, \quad p > 0.$$

Show that equality holds if and only if  $a_j/b_j = \text{constant}$  for all  $j = 1, 2, \dots, n$ .

**Problem 19.** Let  $x_j$  ( $j = 1, 2, \dots, n$ ) be positive real numbers. Show that

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{\sum_{k=1}^n x_k}{n}.$$

This is the arithmetic-geometric mean inequality.

**Problem 20.** Let  $n$  be a positive integer and  $a, b \geq c/2 > 0$ . Show that

$$|a^{-n} - b^{-n}| \leq 4nc^{-n-1}|a - b|.$$

**Problem 21.** Let  $X \subset \mathbb{R}$  be an interval. A function  $\psi : X \rightarrow \mathbb{R}$  is *convex* if for all  $x_1, x_2 \in X$  and numbers  $\alpha_1, \alpha_2 \geq 0$  with  $\alpha_1 + \alpha_2 = 1$ ,

$$\psi(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 \psi(x_1) + \alpha_2 \psi(x_2). \quad (1)$$

This means that every chord of the graph of  $\psi$  lies above the graph. Let  $\psi : X \rightarrow \mathbb{R}$  be convex, let  $x_1, x_2, \dots, x_n \in X$ , and let  $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$  satisfy  $\sum_{j=1}^n \alpha_j = 1$ . Show that (Jensen's inequality)

$$\psi\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i \psi(x_i). \quad (2)$$

**Problem 22.** Consider the differentiable function  $f : [0, \infty) \rightarrow \mathbb{R}$

$$f(x) = \frac{x}{1+x}.$$

Let  $a, b \in \mathbb{R}^+$ . Show that

$$f(|a+b|) \leq f(|a|+|b|). \quad (1)$$

**Problem 23.** Let  $a, b \in \mathbb{R}$ . Show that

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

**Problem 24.** Show that  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = |x|$  is convex.

**Problem 25.** Let  $n \in \mathbb{N}$  and  $n \geq 2$ . Show that

$$e^n \ln(n+1) < e^{n+1} \ln(n).$$

**Problem 26.** Show that the function  $f : (0, \infty) \rightarrow \mathbb{R}$

$$f(x) = x \ln(x)$$



is convex.

**Problem 27.** Let  $x, y \in \mathbb{R}$ . Show that

$$\frac{x+y}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

**Problem 28.** Let  $A, B$  be  $n \times n$  matrices over  $\mathbb{C}$ . Show that

$$\|AB\| \leq \|A\| \cdot \|B\|.$$

**Problem 29.** Let  $A, B$  be  $n \times n$  matrices over  $\mathbb{C}$ . Show that

$$\|A+B\| \leq \|A\| + \|B\|.$$

**Problem 30.** Let  $\mathcal{H}$  be a Hilbert space. Let  $\psi, \phi \in \mathcal{H}$ . Assume that  $\psi \neq 0, \phi \neq 0$ . Show that

$$|\langle \phi, \psi \rangle| \leq \sqrt{\langle \phi, \phi \rangle} \sqrt{\langle \psi, \psi \rangle} \tag{1}$$

where  $\langle , \rangle$  denotes the scalar product in the Hilbert space.

**Problem 31.** Let  $n \geq 2$ . Let  $x_1, x_2, \dots, x_n$  be given positive real number with

$$x_1 < x_2 < \dots < x_n.$$

Let  $\lambda_1, \dots, \lambda_n \geq 0$  and  $\sum_{j=1}^n \lambda_j = 1$ . Show that

$$\left( \sum_{j=1}^n \lambda_j x_j \right) \left( \sum_{j=1}^n \lambda_j x_j^{-1} \right) \leq A^2 G^{-2}$$

where

$$A = \frac{1}{2}(x_1 + x_n), \quad G = (x_1 x_n)^{1/2}.$$

**Problem 32.** Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  (column vectors) and  $\epsilon > 0$ . Show that

$$2\mathbf{x}^T \mathbf{y} \leq \epsilon \mathbf{x}^T \mathbf{x} + \frac{1}{\epsilon} \mathbf{y}^T \mathbf{y}.$$

**Problem 33.** Let  $a, b \in \mathbb{R}$  and  $\epsilon > 0$ . Show that

$$2ab \leq \epsilon a^2 + \epsilon^{-1} b^2.$$

**Problem 34.** Show that the analytic function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(\alpha, \beta) = \sin(\alpha) \sin(\beta) \cos(\alpha - \beta)$$

is bounded between  $-1/8$  and  $1$ .

**Problem 35.** Let  $n$  be an integer and  $n \geq 2$ . Show that

$$e^n \ln(n+1) < e^{n+1} \ln(n).$$

**Problem 36.** Let  $a, b, c$  be positive numbers. Assume that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Show that

$$x + y + z \leq \sqrt{a^2 + b^2 + c^2}.$$

Use the Lagrange multiplier method.

**Problem 37.** Let  $a, b, c, d$  be positive numbers and  $s := a + b + c + d$ . Show that

$$\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} + \frac{s}{s-d} \geq \frac{16}{3}.$$

**Problem 38.** Let  $x \geq 0$  and  $0 < p < 1$ . Show that

$$\frac{1}{p}(1-x^p) \geq 1-x.$$

**Problem 39.** Let  $b$  be a real number such that the elements of the infinite sequence  $(a_k)_{k=1}^{\infty}$  satisfy

$$a_{k+m} \leq a_k + a_m + b$$

for all  $k, m = 1, 2, \dots$ . Show that

$$a := \lim_{k \rightarrow \infty} \frac{a_k}{k}$$

exists and  $a_k \geq ka - b$  for all  $k$ .

**Problem 40.** Let  $n \in \mathbb{N}$  and  $n \geq 2$ . Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

**Problem 41.** Let  $a, b \geq 0$ . Show that

$$\frac{a+b}{1+a+b} \leq \frac{a+b+ab}{1+a+b+ab} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

**Problem 42.** Show that

$$\frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin^2(x-y)}{(x-y)^2} dx dy < \frac{1}{4}.$$

**Problem 43.** Let  $x_1, x_2, x_3 \in \mathbb{R}$ . Show that

$$x_1^2 + x_2^2 + x_3^2 \geq x_1x_2 + x_2x_3 + x_3x_1.$$

Why could it be useful to consider the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1?$$

**Problem 44.** Let  $x_1, x_2, \dots, x_n$  be positive numbers. Assume that the numbers satisfy

$$\sum_{j=1}^n x_j = 1, \quad \sum_{j=1}^n x_j^2 = b^2.$$

Show that

$$\max\{x_j : 1 \leq j \leq n\} \leq \frac{1}{n}(1 + \sqrt{n-1}\sqrt{nb^2-1}).$$

**Problem 45.** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$ . Show that

$$|f(x_2) - f(x_1)| \leq 4|x_2 - x_1|$$

for all  $x_1$  and  $x_2$  in  $[-2, 2]$ . Hint. Apply

$$x_1^2 - x_2^2 \equiv (x_1 + x_2)(x_1 - x_2).$$

**Problem 46.** Let  $n \in \mathbb{N}$ . Show that

$$\frac{n}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n - 1} \leq n.$$

**Problem 47.** Let

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_n, \quad 0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$$

and  $r \geq 1$ . Show that (*Chebyshev inequality*)

$$\left( \frac{1}{n} \sum_{j=1}^n a_j^r \right)^{1/r} \left( \frac{1}{n} \sum_{j=1}^n b_j^r \right)^{1/r} \leq \left( \frac{1}{n} \sum_{j=1}^n (a_j b_j)^r \right)^{1/r}.$$

**Problem 48.** Let  $a, x, y \in \mathbb{R}$ . Show that

$$|\sqrt{a^2 + x^2} - \sqrt{a^2 + y^2}| \leq |x - y|.$$

**Problem 49.** Let  $b > a > 0$ . Show that

$$\left(1 - \frac{a}{b}\right) < \ln\left(\frac{b}{a}\right) < \frac{b}{a} - 1.$$

**Problem 50.** Show that

$$\left| \int_0^1 \frac{\cos(nx)}{x+1} \right| \leq \ln(2)$$

for  $n \in \mathbb{N}$ .

**Problem 51.** Let  $n \geq 2$ . Show that

$$2^n < \binom{2n}{n} < 2^{2n}.$$

**Problem 52.** Consider the two manifolds

$$x_1^2 + x_2^2 = 1, \quad y_1^2 + y_2^2 = 1.$$

Show that

$$|x_1 y_1 + x_2 y_2| \leq 1.$$

Hint. Set

$$x_1(t) = \cos(t), \quad x_2(t) = \sin(t), \quad y_1(\tau) = \cos(\tau), \quad y_2(\tau) = \sin(\tau).$$

**Problem 53.** Let  $n \in \mathbb{N}$  and  $x > 0$ . Show that

$$x^{n+1} + \frac{1}{x^{n+1}} > x^n + \frac{1}{x^n}.$$

**Problem 54.** Find all  $x \in \mathbb{R}$  and  $x > 0$  such that

$$|\sqrt{x} - \sqrt{2}| < 1.$$

**Problem 55.** Let  $j, k$  be positive integers. Find all pairs  $(j, k)$  such that the following four inequalities are satisfied

$$j + k < 10, \quad j + k \geq 6, \quad \frac{j}{k} > 1, \quad \frac{j}{k} < 2.$$

**Problem 56.** Let  $n$  be a positive integer. Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

**Problem 57.** Let  $z = x + iy$  with  $x, y \in \mathbb{R}$ . Show that

$$\frac{1}{\sqrt{2}}(|x| + |y|) \leq |z| \leq |x| + |y|.$$

**Matrices**

**Problem 58.** Let  $A, B$  be  $n \times n$  matrices over  $\mathbb{C}$ . Show that

$$|\operatorname{tr}(AB^*)|^2 \leq \operatorname{tr}(AA^*)\operatorname{tr}(BB^*).$$

**Problem 59.** Let  $A$  and  $B$  be two  $n \times n$  matrices over  $\mathbb{R}$ . It can be shown that

$$\begin{aligned} \operatorname{tr}e^{A+B} &\leq \operatorname{tr}(e^A e^B) \leq \frac{1}{2} \operatorname{tr}(e^{2A} + e^{2B}) \\ \operatorname{tr}e^{A+B} &\leq \operatorname{tr}(e^A e^B) \leq (\operatorname{tr}e^{pA})^{1/p} (\operatorname{tr}e^{qB})^{1/q} \end{aligned}$$

where  $p > 1, q > 1$  with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Is

$$(\operatorname{tr}e^{pA})^{1/p} (\operatorname{tr}e^{qB})^{1/q} \leq \frac{1}{2} \operatorname{tr}(e^{2A} + e^{2B})?$$

Prove or disprove.

**Problem 60.** Let  $A, B$  be hermitian matrices. Then

$$\operatorname{tr}(e^{A+B}) \leq \operatorname{tr}(e^A e^B).$$

Assume that

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate the left and right-hand side of the inequality. Does equality hold?

**Problem 61.** Let  $A, B$  be positive definite matrices. Show that

$$\operatorname{tr}(AB)^{2^{p+1}} \leq \operatorname{tr}(A^2 B^2)^{2^p}, \quad p \text{ a non-negative integer}$$

**Problem 62.** Let  $A$  be an  $n \times n$  matrix with  $\|A\| < 1$ .

(i) Show that  $(I_n + A)^{-1}$  exists.

(ii) Show that

$$\|(I_n + A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

**Problem 63.** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{R}$ . Assume that  $a_{jj} \geq 1$  for all  $j$  and

$$\sum_{j \neq k}^n a_{jk}^2 < 1.$$

Show that  $A$  is invertible.

**Problem 64.** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$ . Let  $I_n$  be the  $n \times n$  identity matrix. Assume that

$$\sum_{k=1}^n |a_{jk}| < 1$$

for each  $j$ . Show that  $I_n - A$  is invertible.

**Problem 65.** Let  $A$  be an  $n \times n$  positive definite matrix over  $\mathbb{R}$ . Let  $B$  be an  $n \times n$  positive semidefinite matrix over  $\mathbb{R}$ . Show that

$$\det(A + B) \geq \det(A).$$

**Problem 66.** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{R}$ . Show that there exists nonnull vectors  $\mathbf{x}_1, \mathbf{x}_2$  in  $\mathbb{R}^n$  such that

$$\frac{\mathbf{x}_1^T A \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \leq \frac{\mathbf{x}_2^T A \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2}$$

for every nonnull vector  $\mathbf{x}$  in  $\mathbb{R}^n$ .

**Problem 67.** Let  $A$  be an  $n \times n$  skew-symmetric matrix over  $\mathbb{R}$ . Show that

$$\det(I_n + A) \geq 1$$

with equality holding if and only if  $A = 0$ .

**Problem 68.** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{R}$ . Assume that  $a_{jj} \geq 1$  for  $j = 1, 2, \dots, n$  and

$$\sum_{j \neq k}^n a_{jk}^2 < 1.$$

Show that  $A$  is invertible.

**Problem 69.** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$ . Assume that

$$|a_{jj}| > \sum_{k \neq j}^n |a_{jk}|$$

for all  $j = 1, 2, \dots, n$ . Show that  $A$  is invertible.

**Problem 70.** Let  $A, B$  be  $n \times n$  positive definite matrices. Show that

$$\operatorname{tr}(A \ln(A)) - \operatorname{tr}(A \ln(B)) \geq \operatorname{tr}(A - B).$$

**Problem 71.** Let  $\mathbf{v}$  be a normalized (column) vector in  $\mathbb{C}^n$  and let  $A$  be an  $n \times n$  hermitian matrix. Is

$$\mathbf{v}^* e^A \mathbf{v} \geq e^{\mathbf{v}^* A \mathbf{v}}$$

for all normalized  $\mathbf{v}$ ? Prove or disprove.



## 10.2 Supplementary Problems

**Problem 1.** Let  $n \in \mathbb{N}$  and  $n \geq 2$ . Show that

$$\frac{1}{\sqrt{n-1}} > 2\sqrt{n} - 2\sqrt{n-1} > \frac{1}{\sqrt{n}}.$$

Use this result to show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

diverges.

**Problem 2.** Let  $b > a$ . Show that

$$\frac{b-a}{1+b^2} < \arctan(b) - \arctan(a) < \frac{b-a}{1+a^2}.$$

Note that  $\arctan(-x) = -\arctan(x)$ .

**Problem 3.** Let  $n$  be a positive integer. Is

$${}^{n+1}\sqrt{(n+1)!} > \sqrt[n]{n!}$$

for all  $n$ ?

**Problem 4.** Let  $a, b, c \in \mathbb{C}$  with  $|a| = |b| = |c| = 1$ . Suppose that  $\Im(a) \geq 0$ ,  $\Im(b) \geq 0$ ,  $\Im(c) \leq 0$ . Show that

$$\left| \frac{-1 + ab + bc + ca}{4} \right| \leq \frac{1}{\sqrt{2}}.$$

**Problem 5.** Let  $n$  be a positive integer with  $n \geq 2$ . Show that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} < 1.$$

**Problem 6.** Let  $n \geq 2$  and  $x_1, x_2, \dots, x_n$  be positive numbers. Assume that the numbers satisfy

$$\sum_{j=1}^n x_j = 1, \quad \sum_{j=1}^n x_j^2 = b^2.$$

Show that

$$\max\{x_j : 1 \leq j \leq n\} \leq \frac{1}{n} \left( 1 + \sqrt{n-1} \sqrt{nb^2 - 1} \right).$$

**Problem 7.** Show that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

**Problem 8.** Let  $x \geq 0$ . Show that  $1 + x \leq 3^x$ .

**Problem 9.** Let  $A$  be an  $n \times n$  matrix. Is

$$|\det(I_n + A)| \leq \exp(\|A\|)?$$

Does it depend on the chosen norm?

**Problem 10.** Let  $z_j$  ( $j = 1, \dots, p$ ) be fixed complex numbers. Is

$$\prod_{j=1}^p |z - z_j| > \prod_{j=1}^p (|z_j| - |z|)?$$

**Problem 11.** Let  $x \in (0, 1)$ . Show that

$$x > \ln(1 + x) > x - \frac{x^2}{2}.$$

**Problem 12.** Let  $x, y \in \mathbb{R}$ . Show that

$$\frac{|x + y|}{1 + |x + y|} \leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}.$$

**Problem 13.** Let  $x_1 > 0$ ,  $x_2 > 0$ ,  $\alpha_1 > 0$ ,  $\alpha_2 > 0$  and  $\alpha_1 + \alpha_2 = 1$ . Show that

$$x_1^{\alpha_1} x_2^{\alpha_2} \leq \alpha_1 x_1 + \alpha_2 x_2.$$

Apply  $\ln(x)$ .



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