Problems and Solutions in Real and Complex Analysis, Integration, Functional Equations and Inequalities

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Preface

The purpose of this book is to supply a collection of problems in analysis. Please submit your solution to one of th email addresses below.

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Prescribed books for problems.

1) Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume I: Introductory Level

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2) Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume II: Advanced Level

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Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\lvert z\rvert$	modulus of complex number z
	$\lvert x+iy\rvert = (x^2+y^2)^{1/2},\ \ x, y \in \mathbb{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{v}$	column vector in $\mathbb{C}^n$
$\mathbf{v}^T$	transpose of $\mathbf{v}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\lVert\,.\,\rVert$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^*\mathbf{y}$	scalar product (inner product) in $\mathbb{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbb{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of matrix A
A^T	transpose of matrix A

$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

Chapter 1

Sums and Products

1.1 Solved Problems

Problem 1. The *harmonic series* can be approximated by

$$\sum_{j=1}^{n} \frac{1}{j} \approx 0.5772 + \ln(n) + \frac{1}{2n}.$$

Calculate the left and rigt-hand side for $n = 1$ and $n = 10$.

Problem 2. The *Bernoulli numbers* B_0, B_1, B_2, $\ldots$ are defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j \equiv B_0 + \frac{B_1}{1!} x + \frac{B_2}{2!} x^2 + \cdots.$$

One finds $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$. One has $B_j = 0$ if $j \geq 3$ and j odd. The Bernoulli numbers are utilized in the *Euler summation formula*

$$\sum_{j=1}^{n} f(j) = \int_1^n f(t)dt + \frac{1}{2}(f(n)+f(1)) + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(1)) + R_m(n)$$

where

$$|R_m(n)| \leq \frac{4}{(2\pi)^{2m}} \int_1^m |f^{(2m)}(t)| dt.$$

Calculate the sum

$$\sum_{j=1}^{n} j^2$$

using $f(t) = t^2$.

Problem 3. Let $z \in \mathbb{C}$ and $|z| < 1$. Calculate the sum

$$(1-|z|^2)^{2s} \sum_{n=0}^{\infty} n \binom{2s+n-1}{n} |z|^{2n}.$$

Problem 4. Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$ with $r \geq 1$. Find the sum

$$\sum_{k=0}^{r-1} \exp(-2\pi ikx/r).$$

Problem 5. Let $a_1, a_2, \ldots, a_n \in [0,1]$. Show that there exists a real number x in the unit interval such that the average of the (unsigned) distances from x to the a_j's is $1/2$.

Problem 6. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function. Show that there exist m differentiable functions $g_1,\ g_2,\ \ldots,\ g_m$ defined on $\mathbb{R}^m$ with the properties

$$f(\mathbf{x}) = f(\mathbf{0}) + \sum_{j=1}^{m} x_j g_j(\mathbf{x})$$

and

$$g_j(\mathbf{0}) = \frac{\partial f}{\partial x_j}(\mathbf{0})$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$.

Problem 7. Let $x \in \mathbb{R}$. The sequence of functions $\{ f_k(x) \}$ is defined by $f_1(x) = \cos(x/2)$ and for $k > 1$ by

$$f_k(x) = f_{k-1}(x)\cos(x/2^k).$$

Thus

$$f_k(x) = \cos(x/2)\cos(x/2^2)\cdots\cos(x/2^k).$$

Obviously, we have $f_k(0) = 1$ for every k. Calculate $\lim_{k\to\infty} f_k(x)$ as a function of x for $x \neq 0$.

Problem 8. Let n be a positive integer and $f(j) = j(j-1)(j-2)$ with $j = 1, 2, \ldots, n$. Let $a_j := f(j+1) - f(j)$. By calculating the sum $\sum_{j=1}^{n} a_j$ show that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 9. Show that

$$\left(\sum_{j=1}^{N} \frac{1}{u - v_j} \right)^2 - \sum_{j=1}^{N} \frac{1}{(u - v_j)^2} \equiv 2 \sum_{\substack{i<j \\ j=2 \\ i=1}}^{N} \frac{1}{v_i - v_j} \left(\frac{1}{u - v_i} - \frac{1}{u - v_j} \right).$$

This identity plays a role in the *Bethe ansatz*.

Problem 10. Show that the series

$$f(\theta) = \sum_{j=0}^{\infty} \frac{\sin(3^j \theta)}{2^j}$$

is convergent. Is the series $df/d\theta$ convergent?

Problem 11. Find the radius of convergence of the power series

$$\sum_{j=0}^{\infty} \binom{j+k}{j} z^j, \qquad k > 0.$$

Problem 12. Find the radius of convergence of the power series

$$\sum_{j=1}^{\infty} \frac{j!}{j^j} z^{mj}, \qquad m = 1, 2, \ldots.$$

Problem 13. Let $(s_0, s_1, ..., s_{n-1})^T \in \mathbb{R}^n$, where $n = 2^k$. This vector in $\mathbb{R}^n$ can be associated with a piecewise constant function f defined on $[0, 1)$

$$f(x) = \sum_{j=0}^{2^k - 1} s_j \Theta_{[j2^{-k}, (j+1)2^{-k})}(x)$$

where $\Theta_{[j2^{-k}, (j+1)2^{-k})}(x)$ is the step function

$$\Theta_{[j2^{-k}, (j+1)2^{-k})}(x) := \begin{cases} 1 \; x \in [j2^{-k}, (j+1)2^{-k}) \\ 0 \; x \notin [j2^{-k}, (j+1)2^{-k}) \end{cases}$$

with the support $[j2^{-k}, (j+1)2^{-k})$. Let $x_{j+1} = 4x_j(1-x_j)$ with $j = 0, 1, 2, \ldots$ and $x_0 = 1/3$. Then

$$x_0 = \frac{1}{3}, \quad x_1 = \frac{8}{9}, \quad x_2 = \frac{32}{81}, \quad x_3 = \frac{6272}{6561}.$$

Find $f(x)$ for this data set and then calculate

$$\int_0^1 f(x)dx.$$

Problem 14. Let $a_1, a_2, \ldots, a_n$ be a finite sequence of numbers. Its *Cesáro sum* is defined as

$$\frac{s_1 + s_2 + \cdots + s_n}{n}$$

where

$$s_k = a_1 + a_2 + \cdots + a_k$$

for each k, $1 \leq k \leq n$. Suppose that the Cesáro sum of the 99-term sequence $a_1, a_2, \ldots, a_{99}$ is 100. Find the Cesáro sum of the 100-term sequence 1, $a_1, a_2, \ldots, a_{99}$.

Problem 15. Each $x \in [0,1]$ can be written as

$$x = \sum_{j=1}^{\infty} \frac{\epsilon_j}{2^j}$$

with $\epsilon_j = 0$ or $\epsilon_j = 1$. Define the function $f : [0,1] \to [0,1)$ as

$$f(x) = \sum_{j=1}^{\infty} \frac{2\epsilon_j}{3^j}.$$

The function f is known as *Cantor function*. Let $x = 1/8$. Find $f(x)$.

Problem 16. Let $s \geq 2$. Simplify the sum

$$\sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{1}{(j_1 + j_2)^s}.$$

Problem 17. Consider

$$\frac{1+x}{1-x-x^2} = \sum_{j=0}^{\infty} c_j x^j.$$

Find c_j.

Problem 18. The *Cantor series approximation* is defined as follows. For arbitrary chosen integers $n_1, n_2, \ldots$ (equal or larger than 2), we can approximate any real number r_0 as follows

$$\begin{aligned} x_j &= \text{integer part}(r_j), \quad j = 0, 1, 2, \ldots \\ r_{j+1} &= (r_j - x_j)n_j \end{aligned}$$

and

$$r_0 \approx x_0 + \sum_{j=1}^{N} \frac{x_j}{n_1 n_2 \cdots n_j}.$$

The approximation error is

$$E_N = \frac{1}{n_1 n_2 \cdots n_N}.$$

Apply this approximation to $r_0 = 2/3$ and the golden mean number with $n_j = 2$ for all j and $N = 4$.

Problem 19. Calculate the sum

$$S = \sqrt{2+\sqrt{2}} - \sqrt{2}\sqrt{2-\sqrt{2}} - \sqrt{2-\sqrt{2}}.$$

Problem 20. Let $n_1, n_2, n_3 \in \mathbb{Z}$. Calculate

$$1 - e^{i\pi(n_1+n_2+n_3)}.$$

Problem 21. Show that $7 + 2\sqrt{10}$ has a square root the form $\sqrt{x} + \sqrt{y}$.

Problem 22. Calculate

$$\sum_{j=0}^{\infty} \frac{j^3+1}{j!} x^j.$$

Hint. Write $j^3 + 1$ in the form

$$a + bj + cj(j-1) + dj(j-1)(j-2).$$

Problem 23. Let $n \in \mathbb{N}$. Find the sum

$$\sum_{k=-(n-1)/2}^{(n-1)/2} k^2$$

where k runs in steps of 1.

Problem 24. Assume that the series

$$1,\ 14,\ 51,\ 124,\ 245,\ 426,\ \ldots$$

is of the form

$$ak^3 + bk^2 + ck + d.$$

Find the integer coefficents a, b, c, d.

Problem 25. Find the sum

$$S_n = \frac{2}{1 \cdot 2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \cdots + \frac{2}{n \cdot (n+1) \cdot (n+2)}.$$

Calculate S_n for $n \to \infty$. Hint. Find a, b, c for

$$\frac{2}{n \cdot (n+1) \cdot (n+2)} = \frac{a}{n} + \frac{b}{n+1} + \frac{c}{n+2}.$$

Problem 26. Let $a > 0$. Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{z - na} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2 a^2}.$$

Problem 27. Let $x \neq y$, $x \neq z$, $y \neq z$. Find the sum

$$\frac{1}{(x-y)(x-z)} + \frac{1}{(y-x)(y-z)} + \frac{1}{(z-x)(z-y)}.$$

Problem 28. The sum

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} {}' \frac{(-1)^{k_1+k_2+k_3}}{\sqrt{k_1^2 + k_2^2 + k_3^2}}$$

plays a role in solid state physics. Here $'$ indicates that the term $(k_1, k_2, k_3) = (0, 0, 0)$ is omitted. Given a positive integer N. Write a C++ program that implements the sum

$$\sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} \sum_{k_3=-N}^{N} {}' \frac{(-1)^{k_1+k_2+k_3}}{\sqrt{k_1^2 + k_2^2 + k_3^2}}.$$

Run the program for different N and compare with the exact value.

Problem 29. Find the sum

$$S_L = \sum_{n=1}^{L} \exp\left(i\pi \frac{2}{L} n^2 \right)$$

for $L = 1, 2, 3$.

Problem 30. Given the expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \leq 1.$$

Calculate $\ln(11)$.

Problem 31. Let $n \in \mathbb{N}_0$ and $p, a > 0$. Show that

$$\sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} \exp(-2\pi kp/a) \equiv \frac{1}{(1-\exp(-2\pi p/a))^{n+1}}.$$

Hint: Start with

$$f(n) := \sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} \exp(-2\pi kp/a)$$

and find $f(n+1)$.

Problem 32. Let $(s_1, s_2) \in \mathbb{Z}^2$. Let $P(\phi_1, \phi_2)$ be a probability density and let $\theta_1, \theta_2 \in \mathbb{R}$. We can express the characteristic double sequence as

$$\chi(s_1, s_2) = \int_{\theta_1}^{\theta_1+2\pi} \int_{\theta_2}^{\theta_2+2\pi} \exp(i(s_1\phi_1 + s_2\phi_2)) P(\phi_1, \phi_2) d\phi_1 d\phi_2.$$

Find $P(\phi_1, \phi_2)$ as a function of $\chi(s_1, s_2)$. Note that $\chi(0,0) = 1$.

Problem 33. Show that

$$\frac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239).$$

Hint. Let $\theta = \arctan(1/5)$. Thus $\tan(\theta) = 1/5$. Applying the double angle formula for the tangent we have $\tan(2\theta) = 5/12$ and $\tan(4\theta) = 120/119$. Finally apply that $\tan(\pi/4) = 1$.

Problem 34. Let $n \geq 2$. Let S_n be the standard n-simplex embedded in $\mathbb{R}^n$

$$S_n := \left\{ \mathbf{x} \in \mathbb{R}^n \,:\, \sum_{j=1}^{n} x_j = 1, \text{ for } x_k \geq 0, \text{ for } k = 1, \ldots, n \right\}.$$

We denote by $S_{k,n-1}$ the kth face of the boundary of S_n. They are $(n-1)$-simplexes

$$S_{k,n-1} := \{ \mathbf{x} \in \mathbb{R}^n \,:\, x_k = 0, \ \mathbf{x} \in S_n \} \quad \text{for } k = 1, \ldots, n.$$

Show that the boundary ∂S_n of S_n is the union

$$\partial S_n = \cup_{j=1}^{n} S_{j,n-1}$$

of the faces.

Problem 35. A triple sum related to the Madelung constant is given by

$$\sum_{i,j,k=-\infty}^{\infty} \frac{(-1)^{i+j+k}}{\sqrt{i^2+j^2+k^2}}$$

where the prime indicates that $(0,0,0)$ is excluded from the summation. Write a C++ program that finds

$$\sum_{i=-100}^{100} \sum_{j=-100}^{100} \sum_{k=-100}^{100} \frac{(-1)^{i+j+k}}{\sqrt{i^2+j^2+k^2}}.$$

Problem 36. Let $n \in \mathbb{N}_0$. Consider an infinite number of time variable $\mathbf{t} = (t_1, t_2, t_3, \ldots)$. Consider the sum

$$p_n(x + t_1, t_2, t_3, \ldots) = \sum_{\substack{k_0+k_1+2k_2+3k_3+\cdots=n \\ k_0,k_1,k_2,k_3,\ldots\geq 0}} \frac{x^{k_0} t_1^{k_1} t_2^{k_2} \cdots}{k_0! k_1! k_2! \cdots}.$$

Find p_0, p_1, p_2, p_3.

Problem 37. Let n be an integer with $n \geq 1$. Simplify the series

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

Problem 38. Let $n \geq 1$. Show that

$$\sum_{j=1}^{n} \ln(1+1/j) = \ln(n+1).$$

Problem 39. Let $x \in (-1, 1)$. Show that

$$\sum_{j=0}^{\infty} j x^j = \frac{x}{1-x)^2}.$$

Problem 40. Let M be an integer with $M \geq 2$ and $\Im(\phi) < 0$. Show that

$$S(k) = \sum_{m=1}^{M-1} \frac{e^{(2\pi i m k)/M}}{\sin^2(\pi(m+\phi)/M)} = -\frac{4kMe^{-2\pi i k\phi/M}}{1-e^{-2\pi i\phi}} + \frac{M^2 e^{-2\pi i k\phi/M}}{\sin^2(\pi\phi)}.$$

Note that

$$\frac{1}{\sin^2(\pi(m+\phi)/M)} = -4\sum_{p=0}^{\infty} p e^{-2\pi i(m+\phi)p/M}.$$

Problem 41. Let N be a positive integer and a, b be positive integers or positive half-integers. Find

$$f(N, a, b) = \frac{1}{2}(1 + (-1)^{2a} + (-1)^{2b} + (-1)^{2a+2b+N}).$$

Problem 42. Let ℓ be a positive integer and $\phi \in \mathbb{R}$. Show that

$$\sum_{m=-\ell}^{+\ell} e^{im\phi} = \frac{\sin((\ell+1/2)\phi)}{\sin(\phi/2)}.$$

Problem 43. Let n be a positive integer. Show that

$$\frac{\sin(2^n\alpha)}{2^n\sin(\alpha)} = \cos(\alpha)\cos(2\alpha)\cos(2^2\alpha)\cdots\cos(2^{n-1}\alpha).$$

Problem 44. Let $-1 < x < +1$ and $n \geq 2$.
(i) Show that

$$S_n(x) = \sum_{j=0}^{n-1} j x^j = \frac{nx^n}{x-1} + \frac{x(1-x^n)}{(1-x)^2}.$$

(ii) Show that

$$\lim_{n\to\infty} S_n(x) = \frac{x}{(1-x)^2}.$$

(iii) Let $x = 1/2$. Show that $\lim_{n\to\infty} S_n(1/2) = 2$.

Problem 45. Let $x, y, z \in \mathbb{R}$ and $x \neq y$, $x \neq z$, $y \neq z$. Show that

$$\frac{x}{(z-x)(x-y)} + \frac{y}{(x-y)(y-z)} + \frac{z}{(y-z)(z-x)} = 0.$$

Problem 46. Let

$$x \neq 0, \quad x \neq \pm 2\pi, \quad x \neq \pm 4\pi, \ldots$$

and $n \in \mathbb{N}$. Show that

$$\sin(x) + \sin(2x) + \cdots + \sin(nx) = \frac{\cos(x/2) - \cos((n+1/2)x)}{2\sin(x/2)}.$$

For the values $x = 0$, $x = \pm 2\pi$, $x = \pm 4\pi$ etc the sum is given by 0.

Problem 47. Let $n = 0, 1, 2, \ldots$. The function

$$K_n(\theta) := \frac{1}{n+1} \frac{\sin^2\left(\left(\frac{n+1}{2}\right)\theta\right)}{\sin^2\left(\frac{1}{2}\theta\right)}$$

is called the *Fejér kernel*.
(i) Show that

$$K_n(\theta) := \sum_{j=-n}^{j=+n} \left(1 - \frac{|j|}{n+1}\right) e^{ij\theta}.$$

(ii) Show that $K_n(\theta) \geq 0$.
(iii) Show that for any continuous 2π periodic function f on $\mathbb{R}$ one has

$$\begin{aligned} K_n \star f(\theta) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\theta-\alpha) f(\alpha) d\alpha \\ &= \sum_{j=-n}^{j=+n} \left(1 - \frac{|j|}{n+1}\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\alpha} f(\alpha) d\alpha\right) e^{ij\theta}. \end{aligned}$$

(iv) Show that $K_n \star f(\theta) \to f(\theta)$ uniformely in θ as $n \to \infty$.

Problem 48. The function

$$D_n(\theta) := \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$$

is called the *Dirichlet kernel*. Show that

$$D_n(\theta) = \sum_{k=-n}^{k=+n} e^{ikx}.$$

Problem 49. Let $\lambda > 0$.
(i) Calculate

$$S_1(\lambda) = \sum_{j=0}^{\infty} \frac{je^{-\lambda}\lambda^j}{j!}.$$

(ii) Calculate

$$S_2(\lambda) = \sum_{j=0}^{\infty} \frac{j(j-1)e^{-\lambda}\lambda^j}{j!}.$$

Problem 50. (i) Let $\alpha > 0$. Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k^2+\alpha^2} = \frac{\pi}{2\alpha^2} \cdot \frac{\cosh(\alpha x)}{\sinh(\alpha x)} - \frac{1}{2\alpha^2}$$

where $-\pi \le x \le \pi$.
(ii) Let $\alpha > 0$. Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sin(kx)}{k^2+\alpha^2} = -\frac{\pi}{2\alpha^2} \cdot \frac{\sinh(\alpha x)}{\sinh(\alpha \pi)}$$

for $-\pi < x < \pi$.

Problem 51. Let $a > 0$ and $b > 0$. Show that the continued fraction

$$a + \frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots$$

can be written as

$$\frac{a}{2} + \sqrt{\frac{a^2}{4} + \frac{a}{b}}.$$

Problem 52. Let $M, N \ge 1$. Find the sum

$$\sum_{m=1}^{M}\sum_{n=1}^{N} \frac{(m+n)!}{(2m+n)!(m+2n)!}.$$

Problem 53. Let $\Gamma(x)$ be the gamma function. Show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\alpha+2k)\Gamma(\alpha+k)}{\Gamma(\alpha+k+n+1)}=\delta_{n,0}.$$

Problem 54. Study the sum

$$\sum_{\ell,m,n=1}^{\infty}\frac{\ell}{(\ell m+\ell n+mn+1)^2}x^{\ell+m+n}.$$

Problem 55. Show that

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+\frac{3}{4}n+\frac{1}{8}}=4\pi.$$

Problem 56. Let $n\geq 1$. Find the sum

$$\sum_{j=1}^{n}\frac{3}{j(j+3)}.$$

What happens for $n\to\infty$?

Problem 57. Let $F_1=1$, $F_2=2$, $F_{k+1}=F_{k-1}+F_k$ be the Fibonacci numbers. Calculate

$$\sum_{k=1}^{\infty}\frac{\log_2(F_{k+1})}{2^{k+1}}.$$

Problem 58. Let $\mathbb{Z}$ be the set of integers. Simplify the sum

$$\sum_{k\in\mathbb{Z}}\frac{\sin(\pi(x-k))\sin(\pi(y-k))}{\pi(x-k)\pi(y-k)}.$$

Problem 59. Let n be a positive integer. Give a C++ implementation of sum

$$\sum_{k=0}^{n}\sum_{\ell=0}^{n}\binom{k}{\ell}$$

using templates so that the `Verylong` class of SymbolicC++ can be used.

Problem 60. Let $n \geq 1$. One has

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1}$$

Show that

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n} = n 2^{n-1}. \tag{2}$$

Problem 61. Let $n > 2$ and $A, h > 0$. Find V_n given by

$$V_n = \frac{Ah}{n^3} \sum_{j=1}^{n-1} j^2.$$

Then find $\lim_{n\to\infty} V_n$.

Problem 62. Let $n \geq 1$ and $c_1, \ldots, c_n$ be constants. Find the minima of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{j=1}^{n} (x - c_j)^2.$$

Problem 63. Let $n \geq 1$. Show that

$$\sum_{k=0}^{n} k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

Programming Problems

Problem 64. Let $m, n \geq 2$. Calculate the finite sum

$$S_{n,m} = \sum_{j=1}^{m} \sum_{k=1}^{n} |j - k|.$$

Give a C++ implementation.

Problem 65. Let $n \in \mathbb{N}$. Calculate the finite sum

$$S_n = \sum_{j=-n}^{n} \sum_{k=-n}^{n} |j + k|.$$

Problem 66. Let $m \geq 1$ and $n \geq 1$. Find the sum

$$\sum_{k=0}^{\min(m,n)} (n+m-2k+1).$$

1.2 Supplementary Problems

Problem 1. Show that

$$1 + 2\tanh^2(\lambda) + 3\tanh^4(\lambda) + \cdots + (n+1)\tanh^{2n}(\lambda) + \cdots \equiv \cosh^4(\lambda).$$

Note that $\tanh(0) = 0$ and $\cosh(0) = 1$.

Problem 2. Show that

$$\sum_{j=2}^{\infty} \frac{1}{j(2j-1)} = 2\ln(2) - 1.$$

Problem 3. Let $n, m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{R}$. Show that

$$(a+b)^n (c+d)^m = \sum_{r=0}^{n} \sum_{s=0}^{m} \binom{n}{r} \binom{m}{s} a^r b^{n-r} c^s d^{m-s}.$$

Problem 4. Let $\ell \geq 1$. Study the *Gauss sum*

$$G(k, \ell) = \frac{1}{\sqrt{\ell}} \sum_{r=0}^{\ell-1} e^{2\pi i k r^2/\ell}.$$

Problem 5. Show that if $|x| < 1$ we have the expansion

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Let $a > 0$. Use this expansion to show that

$$\frac{1}{(a+x)^2} = \frac{1}{a^2}\left(1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \cdots\right).$$

Problem 6. Apply mathematical induction to show that

$$\begin{aligned}
1 + 3 + 5 + \cdots + (2n-1) &= n^2 \\
1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 &= \frac{1}{2}n(6n^2 - 3n - 1) \\
1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 &= n^2(2n^2 - 1).
\end{aligned}$$

Problem 7. Show that

$$\ln(2) = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$$

Problem 8. Let $x > 0$. Show that a complicated way to calculate $1/x$ is given by

$$\frac{1}{x+1} + \frac{1!}{(x+1)(x+2)} + \frac{2!}{(x+1)(x+2)(x+3)} + \cdots$$

Problem 9. Let $n, m \geq 1$. Implement the sum

$$S_{n,m} := \frac{(2m)!}{2^{2m}m!}\sum_{k=0}^{m} 2^k \binom{n}{k}\binom{m}{k}$$

using the `Verylong` class and `Rational` class of SymbolicC++.

Problem 10. Let $y \in \mathbb{R}$ and $\lfloor y \rfloor$ be the integer part of y. Let $\omega = (1+\sqrt{5})/2$ be the golden mean number (which is an irrational number). We define

$$x_j := j + ((j+1)/\omega)(\lambda - 1), \qquad j = 0, 1, \ldots$$

where $\lambda > 0$ and

$$\mu_k = x_k - x_{k-1}, \qquad k = 1, 2, \ldots$$

Show that the sequence μ_1, μ_2, ... is non-periodic.

Problem 11. Let $s > 0$ and

$$f(s) = \int_{t=0}^{\infty} \frac{\ln(1+st)}{1+t^2} dt.$$

Show that

$$f(s) = (1 - \ln(s))s + \frac{\pi s^2}{4} + \left(\frac{\ln(s)}{3} - \frac{1}{9}\right) s^3 + O(s^4).$$

Problem 12. Let n be an integer and $n > 1$. Show that

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

cannot be integer for all $n > 1$.

Problem 13. Let $x \in \mathbb{R}$. Show that the series

$$\frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \cdots$$

converges absolutely for all values of x, except for $+1$ and -1.

Problem 14. Show that the series

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \quad \sum_{j=0}^{\infty} j z^j = \frac{z}{(1-z)^2}, \quad \sum_{j=0}^{\infty} j^2 z^j = \frac{z(1+z)}{(1-z)^3}$$

converge for all $|z| < 1$.

Problem 15. Let n be an integer with $n \geq 2$. Find

$$\sum_{1 \leq i < j \leq n} \left| \frac{1}{2^i} - \frac{1}{3^j} \right|.$$

Problem 16. Let $\ell \geq 1$. Show that

$$\sum_{j=1}^{\ell} \frac{1}{4j^2-1} = \frac{\ell}{2\ell+1}$$

and

$$\sum_{j=2}^{\ell} \frac{1}{j(j^2-1)} = \frac{1}{4}\left(1 - \frac{2}{\ell(\ell+1)}\right).$$

Problem 17. Let $0 \leq r < 1$. Calculate the sum

$$\sum_{j=1}^{\infty} r^j \cos(j\phi)$$

utilizing the identity

$$\cos(j\phi) \equiv \frac{1}{2}(e^{ij\phi} + e^{ij\phi}).$$

Problem 18. Calculate the sum

$$\sum_{j=1}^{\infty} \frac{1}{j^2+1}$$

with the knowledge that

$$\sum_{j=1}^{\infty} \frac{1}{j^2+1} \equiv \sum_{j=1}^{\infty} \frac{1}{j^2} - \sum_{j=1}^{\infty} \frac{1}{j^2(j^2+1)}, \quad \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

Problem 19. Show that

$$\cos(\pi/4) = \frac{1}{2}\sqrt{2}, \ \cos(\pi/8) = \frac{1}{2}\sqrt{2+\sqrt{2}}, \ \cos(\pi/16) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\sin(\pi/4) = \frac{1}{2}\sqrt{2}, \ \sin(\pi/8) = \frac{1}{2}\sqrt{2-\sqrt{2}}, \ \sin(\pi/16) = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}.$$

Problem 20. Let $n \geq 0$ and $x \neq 0$. Find

$$S_n(x) = \sum_{k=0}^{n} kx^k.$$

Utilize that

$$\begin{aligned} S_{n+1}(x) = S_n(x) + (n+1)x^{n+1} &= \sum_{k=0}^{n}(k+1)x^{k+1} = \sum_{k=0}^{n} kx^{k+1} + \sum_{k=0}^{n} x^{k+1} \\ &= xS_n(x) + \sum_{k=0}^{n} x^{k+1} \end{aligned}$$

and a case study with $x = 1$ and $x \neq 1$.

Problem 21. The *Fibonacci numbers* are given by $x_{t+2} = x_{t+1} + x_t$ with $t = 0, 1, \ldots$ and $x_0 = x_1 = 1$. Consider

$$F = \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \cdots \equiv \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \cdots.$$

Show that the series converges. Is $F < 4$?

Problem 22. Show that

$$\frac{1}{4} = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots$$

Problem 23. Calculate the sum

$$S = \sum_{j=1}^{\infty} \frac{2n+1}{n^2(n+1)}.$$

Problem 24. Show that the *Cantor series*

$$c_1 + \frac{c_2}{2!} + \frac{c_3}{3!} + \frac{c_4}{4!} + \cdots$$

with $0 \leq c_n \leq n-1$ is convergent.

Problem 25. Let $n \geq 1$. Show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Problem 26. Let $k \geq 0$ and $s > 0$. Show that

$$\prod_{\ell=0}^{k} \frac{s+\ell+1}{s+\ell} = \frac{s+k+1}{s}.$$

Problem 27. Let $N \geq 1$ be an integer and $k = -4N, -4N+1, \ldots, 4N$. We define

$$f(k) = \frac{1}{N} \sum_{j=1}^{N} \cos\left(\left(j - \frac{1}{2}\right) \frac{k\pi}{N}\right).$$

Show that $f(0) = 1$, $f(\pm 2N) = -1$, $f(\pm 4N) = 1$ and 0 otherwise.

Problem 28. Show that

$$\sum_{k \in \mathbb{Z}} \frac{1}{(x-k)} = \pi \cot(\pi x).$$

Note that $\cot(0) = \infty$ and $\cot(\pi/2) = 0$.

Problem 29. Let $c > 0$. Apply the *Poisson summation formula*

$$\sum_{n=-\infty}^{+\infty} f(n+a) = \sum_{k=-\infty}^{+\infty} \exp(2\pi i k a) \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i k x) dx$$

for the function

$$f(x) := \begin{cases} x \exp(-cx^2) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Problem 30. Let $k, m = 1, 2, \ldots$. Write a C++ program that implements the sum

$$E(k,m) = \frac{1}{2^{k+m-1}} \sum_{i=0}^{m-1} \binom{k+m-1}{i}$$

for a given k and m. The sum plays a role for the arithmetic triangle.

Problem 31. Let $L \in \mathbb{N}_0$ and κ be a nonnegative real number.
(i) Show that

$$Z(\kappa) = \left(\sum_{n=0}^{L} \binom{L}{n} e^{-\kappa L n} \right)^2 = (1 + e^{-\kappa L})^{2L}.$$

(ii) Show that for $\kappa > 0$ one has for large L that $Z \approx 1 + 2Le^{-\kappa L}$. Show that at $\kappa = 0$ one has $Z(\kappa = 0) = 2^{2L}$.

Problem 32. Let $n > 2$. Show that

$$\sum_{j=2}^{n} \frac{1}{j} \approx \ln(n).$$

Problem 33. Let ω be the primitive nth root of 1. Show that

$$\sum_{j=0}^{n-1} \omega^j = 0.$$

Problem 34. Show that

$$e^x + e^{-x} + 2\cos(x) = 4 \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}.$$

Problem 35. Show that

$$\left(\sum_{j=0}^{\infty} \frac{1}{j!} \right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right) = 1$$

Problem 36. Let $n \in \mathbb{N}$. Applying the principle of mathematical induction show that

$$A(n) = \sum_{k=0}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

Note that $A(0) = 0$.

Problem 37. Let $N \geq 2$. Find the sum

$$\sum_{\substack{j_1,j_2,j_3=1 \\ j_3 \geq j_2 \geq j_1}}^{N} (j_1 j_2 j_3 + (j_1 + j_2 + j_3)).$$

Problem 38. Let $h, r > 0$ and fixed and $m, n \in \mathbb{N}$. Consider

$$\begin{aligned} S_{m,n} &= mnr \sin(\pi/n) \sqrt{\left(\frac{h}{m}\right)^2 + (r - r\cos(\pi/n))^2} \\ &= \pi r \frac{\sin(\pi/n)}{\pi/n} \sqrt{h^2 + \frac{1}{4}\pi^4 r^2 \frac{m^2}{n^4} \left(\frac{\sin(\pi/n)}{\pi/n}\right)^4}. \end{aligned}$$

(i) Consider $\lim_{m,n\to\infty} S_{m,n}$ with $m = n^2$.
(ii) Consider $\lim_{m,n\to\infty} S_{m,n}$ with $m = n^3$.
(iii) Consider $\lim_{m,n\to\infty} S_{m,n}$ with $\lim_{m\to\infty,n\to\infty}(m^2/n^4) = 0$ and show that $\lim_{m,n\to\infty} S_{m,n} = \pi r h$ which is a surface area of a half cylinder with radius r and length h.

Problem 39. (i) Let $a > 0$. Show that

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2 + a^2} = \frac{\pi}{2a} \frac{\cosh(a(\pi - x))}{\sinh(a\pi)} - \frac{1}{2a}.$$

(ii) Let $a > 0$. Show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2 + a^2} = \frac{\pi}{2a} \tanh\left(\frac{a\pi}{2}\right).$$

Problem 40. Let $k \geq 2$. Consider

$$f(k) = k!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^k}{k!}\right) \equiv k! \sum_{j=2}^{k} \frac{(-1)^j}{j!}.$$

Thus $f(2) = 1$. Find $f(3)$, $f(4)$, $f(5)$, $f(6)$, ... Which of these numbers is a prime number?

Problem 41. The q-exponential function is defined by

$$e_q^x := \sum_{j=0}^{\infty} \frac{x^n}{[j]!}$$

where

$$[j] := \frac{q^j - q^{-j}}{q - q^{-1}}.$$

Find e_q^x for $x = 1$ and $q = 1/2$.

Problem 42. Let $a, b, c \in \mathbb{R}$. Factorize

$$(b-c)^3 + (c-a)^3 + (a-b)^3.$$

The following Maxima may be helpful

```
T: (b-c)^3 + (c-a)^3 + (a-b)^3;
T: expand(T);
T: ratsimp(T);
```

Problem 43. Let $n \geq 0$. Starting from

$$\sum_{j=0}^{n} c_j x^j = (1+x)^n$$

show that

$$\begin{aligned} c_0 + c_1 + \cdots + c_n &= 2^n \\ c_0 - c_1 + c_2 - c_3 + \cdots + (-1)^n c_n &= 0 \\ c_1 + 2c_2 + 3c_3 + \cdots n c_n &= n 2^{n-1} \\ c_0^2 + c_1^2 + \cdots + c_n^2 &= \frac{(2n)!}{(n!)^2}. \end{aligned}$$

Problem 44. Assuming that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln(2).$$

Show that

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\ln(2).$$

Note that the convergence of the first series is not absolute.

Problem 45. Show that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} - \frac{1}{2}\ln(n)$$

tends to a finite value as $n \to \infty$. Find this value.

Problem 46. Let $\beta > 0$. Show that

$$\sum_{n=0}^{\infty} e^{-\beta n} = \frac{1}{1 - e^{-\beta}}.$$

Problem 47. Let n be a positive integer and $f(\theta_1, \ldots, \theta_N)$ be a periodic function, i.e. periodic 2π for each θ_j ($j = 1, \ldots, N$). Show for large n we have

$$\left(\prod_{k=1}^{N} \int_0^{2\pi} d\theta_k\right) \left(\sum_{j=1}^{N} \cos\theta_j\right)^n f(\theta_1, \ldots, \theta_N)$$
$$\approx N^n \left(\prod_{k=1}^{N} \int_{-\infty}^{\infty} d\theta_k e^{-n(\theta_k)^2/(2N)}\right) f(\theta_1, \ldots, \theta_N).$$

Problem 48. Let A, B be finite sets and $n(A)$, $n(B)$ the numbers of elements in A and B, respectively. Is

$$n(A) + n(B) = n(A \cup B) + n(A \cap B)\,?$$

Prove or disprove.

Problem 49. Let $n \in \mathbb{N}$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}_0$. Consider the function

$$f(m_1, m_2, \ldots, m_n) :=$$
$$\begin{cases} \delta_{m_1,0} & \text{for } n = 1 \\ \delta_{m_2,0}\delta_{m_1,1} & \text{for } n = 2 \\ \delta_{m_n,0}\delta_{(m_1+\cdots+m_n),n-1} \prod_{j=1}^{n-2} H(j - \sum_{k=1}^{j} m_{n-k}) & \text{for } n \geq 3 \end{cases}$$

where H is the step function

$$H(x) := \begin{cases} 1 \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases}$$

Give an implementation of this function in SymbolicC++ for a given n using the `Verylong` class.

Problem 50. Let $f : [0,1] \to [0,1]$ be a differentiable function, for example the logistic map $f(x) = 4x(1-x)$. Let $f^{(k)}$ be the k-th iterate of the function f. Let

$$b_k := \sup_{0 \leq x \leq 1} \left| \frac{d}{dx} f^{(k)}(x) \right|.$$

Show that
$$\lim_{k\to\infty}(b_k)^{1/k}$$
exist. Apply the chain rule.

Problem 51. Let $n \geq 2$. Show that
$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Problem 52. Show that
$$\prod_{j=1}^{\infty}\cos\left(\frac{x}{2^j}\right) = \frac{\sin(x)}{x}$$
for $x \in \mathbb{R}$.

Problem 53. Is the series
$$S = \sum_{k=1}^{\infty}\frac{k}{1+2k^3}$$
convergent?

Problem 54. Give a non-trival infinite sequence $(x_0, x_1, x_2, \ldots)$ such that
$$\sum_{j=0}^{\infty}\frac{|x_j|}{1+|x_j|}$$
is finite.

Problem 55. (i) Let $n \geq 2$. Show that
$$\sum_{j=0}^{n-1}\frac{1}{(j+1)(j+2)} = 1 - \frac{1}{n+1}$$
and thus show that
$$\sum_{j=0}^{\infty}\frac{1}{(j+1)(j+2)} = 1.$$

Problem 56. Let $a > 0$. Show that
$$\sum_{j=-\infty}^{\infty}(-1)^j\frac{1}{j^2+a^2} = \frac{\pi}{a\sinh(\pi a)}.$$

Problem 57. Let $n = 0, 1, 2, \ldots$. The *Fermat numbers* are given by

$$F_n = 2^{(2^n)} + 1.$$

Show that the Fermat numbers satisfy the recurrence relation

$$F_{n+1} = (F_n - 1)^2 + 1$$

with $F_0 = 3$

Problem 58. Let $N \geq 1$. Consider the Hilbert space $L_2([-1, 1])$. The Chebyhev-Gauss-Lobatto points are given by

$$x_j = \cos(j\pi/N), \quad j = 0, 1, \ldots, N.$$

Find the four points for $N = 3$.

Problem 59. Let $|z| < 1$. Show that

$$\sum_{j=1}^{\infty} jz^j = (1 - z)^{-1} \sum_{j=1}^{\infty} z^j.$$

Show that

$$\sum_{j=1}^{\infty} j^2 z^j = (1 + z)(1 - z)^{-2} \sum_{j=1}^{\infty} z^j$$

Problem 60. Show that

$$\frac{1}{3} + \frac{2}{9} + \cdots + \frac{2^n}{3^{n+1}} + \cdots = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \frac{1}{1 - 2/3} = 1.$$

Problem 61. (i) Let $a > 0$. Show that

$$\sum_{n=0}^{\infty} \frac{1}{(a + n)(a + n + 1)} = \frac{1}{a}.$$

(ii) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n + 1)(n + 2)} = \frac{1}{4}.$$

Problem 62. Let $a > 0$ and $a \notin \mathbb{N}$. Show that

$$\cos(ax) = \frac{2a \sin(ax)}{\pi} \left(\frac{1}{2a^2} - \frac{\cos(x)}{a^2 - 1^2} + \frac{\cos(2x)}{a^2 - 2^2} - \frac{\cos(3x)}{a^2 - 3^2} + \cdots \right)$$

for all $x \in [-\pi, \pi]$.

Problem 63. Let $n \in \mathbb{N}$. Show that

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1).$$

Problem 64. Let $N \geq 1$. Find the sums

$$v_k = \sum_{j=0}^{N-1} \left(\exp\left(\frac{2\pi}{N} jk \right) - 1 \right), \quad k = 0, 1, \ldots, N-1.$$

Chapter 2

Maps

2.1 Solved Problems

Problem 1. Newton's sequence takes the form of a difference equation

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}$$

where $t = 0, 1, 2, \ldots$ and x_0 is the initial value at $t = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2 - 1$$

and $x_0 \neq 0$. Find the fixed points of f. Find the fixed points of the difference equation. Let $x_0 = 1/2$. Find x_1, x_2, x_3. Find Newton's sequence for this function. Obtain the exact solution of the difference equation.

Problem 2. (i) Solve the nonlinear recurrence relation

$$x_{n+1} = x_n^2, \qquad n = 0, 1, \ldots$$

where $x_0 = 2$.
(ii) Solve the linear recurrence relation

$$x_{n+1} = x_n + x_{n-1} + x_{n-2}, \qquad n = 2, 3, \ldots$$

and the initial values $x_0 = x_1 = x_2 = 1$.

(iii) Solve the linear recurrence relation

$$x_{n+1} = 1 + \sum_{j=0}^{n-1} x_j, \qquad x_0 = 1.$$

Problem 3. Let $x > 0$ and $p > 0$. Consider the map

$$f(x) = xe^{p-x}.$$

(i) Find the fixed points. Study the stability of the fixed points.
(ii) Show that f has a least one periodic point x^* with $x^* \neq 0$ or p.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a positive, continously differentiable function, defined for all real numbers and whose derivative is always negative. Show that for any real number x_0 (initial value) the sequence (x_k) obtained by *Newton's method*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

has always limit ∞.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuosly differentiable map. Let $f^{(n)}$ be the n-th iterate of f. Calculate the derivative of $f^{(n)}$ at x_0.

Problem 6. (i) Solve the second order linear difference equation

$$x_{t+2} = x_{t+1} + x_t \qquad t = 0, 1, 2, \ldots$$

where $x_0 = 0$ and $x_1 = 1$.
(ii) Give the definition of the golden mean number and derive this number.
(iii) Calculate

$$\lim_{t\to\infty} \frac{x_{t+1}}{x_t}.$$

Problem 7. The recursion relation

$$F_{n+2} = F_n + F_{n+1}$$

with the initial values $F_0 = F_1 = 1$ provides the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. A generalization of the Fibonacci sequence is the q-analogue of the sequence defined by the recursion relation

$$F_{n+2}(q) = F_n(q) + qF_{n+1}(q)$$

and the initial condition $F_0(q) = 1$ and $F_1(q) = q$. Here, q is a real or complex number.
(i) Give the first five terms of the sequence.
(ii) Find a generating function of $F_n(q)$.
(iii) Find an explicit expression for $F_n(q)$.

Problem 8. (i) Find the linear map $f : \{0,1\} \to \{-1,1\}$ such that

$$f(0) = -1, \qquad f(1) = 1. \tag{1}$$

(ii) Find a linear map $g : \{-1,1\} \to \{0,1\}$ such that

$$g(-1) = 0, \qquad g(1) = 1. \tag{2}$$

This is obviously the inverse map of f.

Problem 9. Consider the differentiable function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1 - \cos(2\pi x)}{x}$$

where using L'Hospital $f(0) = 0$.
(i) Find the zeros of f.
(ii) Find the maxima and minima of f.

Problem 10. Given two manifolds M and N, a bijective map ϕ from M to N is called a *diffeomorphism* if both $\phi : M \to N$ and its inverse ϕ^{-1} are differentiable. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the analytic function

$$f(x) = 4x(1-x)$$

and the analytic function $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = 1 - 2x^2.$$

(i) Can one find a diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$g = \phi \circ f \circ \phi^{-1}\,?$$

(ii) Consider the diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$

$$\psi(x) = \sinh(x).$$

Calculate $\psi \circ g \circ \psi^{-1}$.

Problem 11. Consider the map $f : (-1,1) \to \mathbb{R}$

$$f(x) = \frac{x}{1 - |x|}.$$

Show that the inverse $f^{-1} : \mathbb{R} \to (-1,1)$ is

$$f^{-1}(x) = \frac{x}{1+|x|}.$$

Problem 12. Consider the polynomial $p(x) = x^3 - 3x + 3$. Show that for any positive integer N, there is an initial value x_0 such that the sequence x_0, x_1, x_2, ... obtained from *Newton's method*

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = \frac{2x_n^3 - 3}{3(x_n^2 - 1)}, \qquad n = 0, 1, 2, \ldots$$

has period N.

Problem 13. Consider the linear recursion equation for $F(n)$ with $n = 0, 1, 2, \ldots$

$$(n+1)F(n+1) - nF(n-1) + F(n) = 0$$

where $F(0) = 0$, $F(1) = 1$. We define

$$f(z) := \sum_{n=1}^{\infty} F(n) z^n$$

where z is an undeterminate. Find the differential equation for f with the initial condition. Solve the differential equation.

Problem 14. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be an analytic function. Consider the map

$$\mathbf{x}_j = \mathbf{f}(\mathbf{x}_{j-1}) = \cdots = \mathbf{f}(\mathbf{x}_0).$$

To study the evolution of phase-space distributions, we can introduce the evolution operator $U(\mathbf{x}', \mathbf{x}, j)$ such that any initial phase-space distribution $\rho(\mathbf{x}, 0)$ evolves into

$$\rho(\mathbf{x}'; j) = \int_{\Omega} U(\mathbf{x}', \mathbf{x}; j) \rho(\mathbf{x}, 0) d\mathbf{x}$$

where Ω is the phase space area. Find $U(\mathbf{x}', \mathbf{x}; j)$.

Problem 15. Find the solution of the recursion relation

$$a_{j+2} = \frac{2}{3} a_{j+1} + \frac{1}{3} a_j, \qquad j = 1, 2, \ldots$$

with $a_1 = 5$, $a_2 = 1$. Calculate $\lim_{j \to \infty} a_j$.

Problem 16. Solve the linear difference equation

$$x_{t+1} = 2x_t - t, \qquad t = 0, 1, 2, \dots$$

with the initial value $x_0 = 1$.

Problem 17. Let

$$\mathbb{N}_0 := \{ 0, 1, 2, \dots \}$$

Find an invertible map $f : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ with

$$f(0,0,0) = 0, \ f(0,0,1) = 1, \ f(0,1,0) = 2, \ f(0,1,1) = 3,$$

$$f(1,0,0) = 4, \ f(1,0,1) = 5, \ f(1,1,0) = 6, \ f(1,1,1) = 7$$

etc. Find the inverse function.

Problem 18. Consider the second order difference equation

$$x_{t+2} = x_{t+1} x_t$$

with the initial values $x_0 = a$, $x_1 = b$. Give the solution. What are the fixed points?

Problem 19. Let A be a set. Suppose that it is possible to define subsets A_1, A_2, ... of A which have the properties that
(i) the sets are pairwise disjoint; that is $A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \dots$ and $j \neq i$

(ii) $A_1 \cup A_2 \cup \cdots = A$.

Then the family of sets $\{ A_1, \dots \}$ is called a *partition* of A.

Let $S = \{1, 2, 3, \dots, 9\}$ and $A = \{1, 4, 7\}$. $B = \{2, 3, 5, 6\}$, $C = \{7, 8, 9\}$. Is $\{A, B, C\}$ a partition of S?

Problem 20. Let S be a finite set. Let $\mathcal{P}(S)$ be the power set of S. Then $\mathcal{P}(S)$ has $2^{|S|}$ elements.
(i) Show that the composition

$$A \circ B := (A \cup B) \cap (\complement A \cup \complement B)$$

defines a group, where $A, B \in \mathcal{P}(S)$. Here $\complement$ denotes the complement.
(ii) Show that the composition

$$A \circ B := (A \cap B) \cup (\complement A \cap \complement B)$$

defines a group, where $A, B \in \mathcal{P}(S)$. Here $\complement$ denotes the complement.

Problem 21. Find a function $f : [0,1] \to (0,1]$ that is one-to-one.

Problem 22. Let $n \in \mathbb{N}$. Consider the map

$$u_{t+1} = \frac{1}{2}\left(u_t + \frac{n}{u_t}\right), \qquad t = 0, 1, 2, \ldots$$

given the intial value u_0 with $u_0 > 0$. Show that

$$\frac{u_{t+1} - \sqrt{n}}{u_{t+1} + \sqrt{n}} = \left(\frac{u_t - \sqrt{n}}{u_t + \sqrt{n}}\right)^2 .$$

Show that $u_t \to \sqrt{n}$ as $t \to \infty$. Show that $\sqrt{n}$ is a fixed point.

Problem 23. Let $0 \leq \alpha < \pi/4$. Consider the transformation

$$X(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}}(x\cos(\alpha) + iy\sin(\alpha))$$
$$Y(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}}(-ix\sin(\alpha) + y\cos(\alpha)).$$

(i) Show that $X^2 + Y^2 = x^2 + y^2$.
(ii) Do the matrices

$$\frac{1}{\sqrt{\cos(2\alpha)}}\begin{pmatrix} \cos(\alpha) & i\sin(\alpha) \\ -i\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

form a group under matrix multiplication?

Problem 24. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree r if

$$f(\epsilon x_1, \ldots, \epsilon x_n) = \epsilon^r f(x_1, \ldots, x_n). \tag{1}$$

Show that (Euler's identity)

$$\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} x_j = rf.$$

Problem 25. Find the invariance group of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sin(x).$$

Problem 26. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \cos(\sin(x)) - \sin(\cos(x)).$$

Show that f admits at least one critical point. By calculating the second order derivative find out whether this critical point refers to a maxima or minima.

Problem 27. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$ be analytic functions. We define the *star product*

$$f(x_1, x_2) \star g(x_1, x_2) :=$$

$$\lim_{x_1' \to x_1, x_2' \to x_2} \exp\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2'} - \frac{\partial}{\partial x_1'}\frac{\partial}{\partial x_2}\right) f(x_1, x_2) g(x_1', x_2').$$

Let

$$f(x_1, x_2) = \sin(x_1 + x_2), \quad g(x_1, x_2) = \sin(x_1 - x_2).$$

Find the star product.

Problem 28. Let N_1, N_2 be given positive integers. Let $n_1 = 0, 1, \ldots, N_1 - 1$, $n_2 = 0, 1, \ldots, N_2 - 1$. There are $N_1 \cdot N_2$ points. The points (n_1, n_2) are a subset of $\mathbb{N}_0 \times \mathbb{N}_0$ and can be mapped one-to-one onto a subset of $\mathbb{N}_0$

$$j(n_1, n_2) = n_1 N_2 + n_2$$

where $j = 0, 1, \ldots, (N_1 - 1)(N_2 - 1)$. Find the inverse of this map.

Problem 29. Let $a \in \mathbb{R}$. Consider the transformation

$$\widetilde{t}(t, x) = \frac{1}{a} e^{ax} \sinh(at), \quad \widetilde{x}(t, x) = \frac{1}{a}(e^{ax} \cosh(at) - 1)$$

with $\lim_{a \to 0} \widetilde{t} = t$, $\lim_{a \to 0} \widetilde{x} = x$. Find the inverse of the transformation.

Problem 30. Show that the map $f : (0, 1) \to \mathbb{R}$

$$x \mapsto f(x) = \frac{x - 1/2}{x(x - 1)}$$

is bijective.

Problem 31. Consider the map

$$f : \mathbb{R} \to \{\, (x_1, x_2) \, : \, x_1^2 + (x_2 - 1)^2 = 1 \, \wedge \, x_2 \neq 2 \,\}$$

defined by

$$f(t) = \left(\frac{4t}{t^2+4}, \frac{2t^2}{t^2+4} \right).$$

Show that f is bijective. Show that f and f^{-1} are continuous.

Problem 32. Solve the initial value problem of the system of linear difference equations

$$x_{1,t+1} = -2x_{1,t}, \quad x_{2,t+1} = \frac{8}{9}x_{1,t} - x_{2,t}$$

where $t = 0, 1, \ldots$.

Problem 33. Let $N \geq 2$ and $\delta := 1/(N+1)$. Solve the linear one-dimensional linear difference equation

$$x_{j+1} - 2x_j + x_{j-1} = -12\delta^4(j+1)^2, \quad j = 0, 1, \ldots, N-1$$

with the boundary conditions $x_{-1} = x_N = 0$.

Problem 34. (i) Consider the analytic map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x(1-x).$$

Find the fixed points of f. Are the fixed points stable? Prove or disprove.
(ii) Let $x_0 = 1/2$ and iterate

$$f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \ldots$$

Does this sequence tend to a fixed point? Prove or disprove.
(iii) Let $x_0 = 2$ and iterate

$$f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \ldots$$

Does this sequence tend to a fixed point? Prove or disprove.
(iv) Find the critical points of f. Then find the extrema of f.
(v) Find the roots of f, i.e. solve $f(x) = 0$.
(vi) Find the minima of the function $g(x) := |f(x)|$.

Problem 35. Let $c > 0$ and $0 \leq v_1 < c$, $0 \leq v_2 < c$. We define the composition

$$v_1 \star v_2 := \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

Is the composition associative?

Problem 36. Solve the initial value problem of the system of first order difference equations

$$x_{1,t+1} = -2x_{1,t}$$
$$x_{2,t+1} = \frac{8}{9}x_{1,t} - x_{2,t}.$$

The first difference equation is independent of $x_{2,t}$ and we find the solution of the initial value problem

$$x_{1,t+1} = (-2)^t x_{1,0}.$$

Problem 37. Let m_A, m_B, $\mathbf{R}_A$, $\mathbf{R}_B$ be the masses and centre-of mass coordinates of mass A and B, respectively. We set $m = m_A + m_B$. Find the inverse of the transformation

$$\mathbf{r}(\mathbf{R}_A, \mathbf{R}_B) = \mathbf{R}_A - \mathbf{R}_B, \qquad \mathbf{R}(\mathbf{R}_A, \mathbf{R}_B) = \frac{1}{m}(m_A\mathbf{R}_A + m_B\mathbf{R}_B).$$

Problem 38. Let $x \in [0,1]$ and $\lfloor a \rfloor$ be the integer part of a. Show that the sequence x_t $(t = 0, 1, \ldots)$ given by

$$x_0 = 0, \text{ and } x_{t+1} = x_t + \frac{1}{2^{t+1}}\lfloor 2^{t+1}(x - x_t) \rfloor$$

converges to x.

Problem 39. Consider the map $\mathbf{f} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$

$$f_1(n_1, n_2) = |n_1 - n_2|, \qquad f_2(n_1, n_2) = n_1 + n_2.$$

Is the map invertible?

Problem 40. Let $n, m \geq 0$. Consider the differential operators

$$K_n := \sum_{j=0}^{n} u_j(x) \frac{d^j}{dx^j}, \quad L_m := \sum_{j=0}^{m} v_j(x) \frac{d^j}{dx^j}$$

where the $u_j(x)$'s and $v_j(x)$'s are smooth functions. If the two differential operators K_n and L_m commute, then there is a nonzero polynomial $R(z, w)$ such that $R(K_n, L_m) = 0$. The curve Γ defined by $R(z, w) = 0$ is called the spectral curve. If we consider the eigenvalue problem

$$K_n\psi = z\psi, \qquad L_m\psi = w\psi$$

then $(z,w) \in \Gamma$.
(i) Let $(\alpha \in \mathbb{R})$

$$K = \left(\frac{d^2}{dx^2} + x^3 + \alpha\right)^2 + 2x, \quad L = \left(\frac{d^2}{dx^2} + x^3 + \alpha\right)^3 + 3x\frac{d^2}{dx^2} + 3\frac{d}{dx} + 3x(x^2+\alpha).$$

Show that K and L commute with $w^2 = z^3 - \alpha$.
(ii) Show that

$$K = \left(\frac{d^3}{dx^3} + x^2 + \alpha\right)^2 + 2\frac{d}{dx}$$

and

$$L = \left(\frac{d^3}{dx^3} + x^2 + \alpha\right)^3 + 3\frac{d^4}{dx^4} + 3(x^2+\alpha)\frac{d}{dx} + 3x$$

commute.

Problem 41. Let $F_{GG} : (\mathbb{N} \to \mathbb{N}) \to ((\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}))$ be defined by

$$[F_{GG}(f)](g) := f \circ g \circ g.$$

Let $Y : (\mathbb{N} \to \mathbb{N}) \to ((\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}))$ be defined by

$$Y(f) := F_{GG}(f) \circ F_{GG}(f).$$

Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$. Show that

$$[Y(f)](g) = f \circ [Y(f)](g).$$

Problem 42. Let A, B, C be nonempty sets. Let $f : A \to B$ and $g : B \to C$ are invertible maps. Then the composition of the functions $gcircf : A \to C$ in invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Problem 43. Consider the map $f : (-1,1) \to \mathbb{R}$

$$f(x) = \frac{x}{1-|x|}.$$

Show that the map $f^{-1} : \mathbb{R} \to (-1,1)$ is given by

$$f^{-1}(x) = \frac{x}{1+|x|}.$$

Problem 44. Let $\tau \in \mathbb{R}$. Show that the curves

$$\mathbf{x}(\tau) = \begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} = \begin{pmatrix} 1 \\ \tau \end{pmatrix}, \quad \mathbf{y}(\tau) = \begin{pmatrix} y_1(\tau) \\ y_2(\tau) \end{pmatrix} = \begin{pmatrix} \tau \\ \tau^2 \end{pmatrix}$$

are linearly independent.

Problem 45. Let r, k be positive integers with $r \leq k$ and $M = \{x_1, \ldots, x_r\}$ and $N = \{y_1, \ldots, y_k\}$. Then there

$$\frac{k!}{(k-r)!}$$

surjective maps $f : M \to N$. Find all the maps for $r = k = 2$.

2.2 Supplementary Problems

Problem 1. The *Kustaanheimo-Stiefel transformation* is defined by the map from $\mathbb{R}^4$ (coordinates u_1, u_2, u_3, u_4) to $\mathbb{R}^3$ (coordinates x_1, x_2, x_3)

$$\begin{aligned} x_1(u_1, u_2, u_3, u_4) &= 2(u_1 u_3 - u_2 u_4) \\ x_2(u_1, u_2, u_3, u_4) &= 2(u_1 u_4 + u_2 u_3) \\ x_3(u_1, u_2, u_3, u_4) &= u_1^2 + u_2^2 - u_3^2 - u_4^2 \end{aligned}$$

together with the constraint

$$u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4 = 0.$$

(i) Show that

$$r^2 = x_1^2 + x_2^2 + x_3^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

(ii) Show that

$$\Delta_3 = \frac{1}{4r}\Delta_4 - \frac{1}{4r^2}V^2$$

where

$$\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_4 = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}$$

and V is the vector field

$$V = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} - u_4 \frac{\partial}{\partial u_3} + u_3 \frac{\partial}{\partial u_4}$$

(iii) Consider the differential one form

$$\alpha = u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4.$$

Find $d\alpha$. Find $L_V\alpha$, where $L_V(.)$ denotes the Lie derivative.
(iv) Let $g(x_1(u_1,u_2,u_3,u_4), x_2(u_1,u_2,u_3,u_4), x_3(u_1,u_2,u_3,u_4))$ be a smooth function. Show that $L_V g = 0$.

Problem 2. The Kustaanheimo-Stiefel map $\mathbb{R}^4 \to \mathbb{R}^3$ $((u_1,u_2,u_3,u_4) \to (x_1,x_2,x_3))$ is given by

$$x_1 = 2(u_1u_3 - u_2u_4), \quad x_2 = 2(u_1u_4 + u_2u_3), \quad x_3 = u_1^2 + u_2^2 - u_3^2 - u_4^2$$

together with the constraint

$$\alpha \equiv u_2du_1 - u_1du_2 - u_4du_3 + u_3du_4 = 0.$$

Show that applying the Kustaanheimo-Stiefel map the Laplacian operator Δ_3 in $\mathbb{R}^3$ can be written as

$$\Delta_3 = \frac{1}{4r}\Delta_4 - \frac{1}{4r^2}V^2$$

where

$$r = (x_1^2+x_2^2+x_3^2)^{1/2} = u_1^2+u_2^2+u_3^2+u_4^2, \quad V = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}.$$

Problem 3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $\times$ be the *vector product*

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Hence the vector product is a map from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3$. Show that the map is differentiable.

Problem 4. Study the map (Legendre map) $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1,x_2) = e^{x_1}\cos(x_2), \quad f_2(x_1,x_2) = -e^{x_1}\sin(x_2).$$

Find the functional determinant.

Problem 5. Consider the four symbols $\{A,B,C,D\}$. The *Rudin-Shapiro substitution* is

$$A \mapsto AC, \quad B \mapsto DC, \quad C \mapsto AB, \quad D \mapsto DB.$$

Find the sequence starting of with B.

Problem 6. Solve the difference equation

$$x_{t+1} = tx_t + t^2 \mod 2$$

with $t = 0, 1, 2, \ldots$ and $x_0 = 1$.

Problem 7. Study the map $\mathbf{f} : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

$$f_1(x_1, x_2, x_3) = x_1 - x_2 x_3, \quad f_2(x_1, x_2, x_3) = -x_2 + x_1 x_3,$$

$$f_3(x_1, x_2, x_3) = x_3 - x_1 x_2.$$

(i) Show that the fixed points are

$$(0,0,0), \ (1,0,0), \ (0,0,1), \ (-1,0,0), \ (0,0,-1).$$

(ii) Show that $(x_1 = 2, x_2 = 2, x_3 = 2)$ provides a periodic orbit.
(iii) Show that $(x_1 = 1, x_2 = 2, x_3 = 1)$ provides an eventually periodic orbit.
(iv) Is the map invertible?

Problem 8. Consider the map $\mathbf{f} : [0,1]^3 \to [0,1]^3$ given by

$$f_1(x_1, x_2, x_3) = 2x_1, \ \ f_2(x_1, x_2, x_3) = \frac{1}{2}x_2, \ \ f_3(x_1, x_2, x_3) = \frac{1}{2}x_3$$

for $0 \le x_1 \le 1/2$ and

$$f_1(x_1, x_2, x_3) = 2x_1 - 1, \ \ f_2(x_1, x_2, x_3) = \frac{1}{2}(x_2 + 1), \ \ f_3(x_1, x_2, x_3) = \frac{1}{2}(x_3 + 1)$$

for $1/2 < le \le 1$. Find the orbit for $x_1 = x_2 = x_3 = 1/3$.

Problem 9. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = \sinh(x_1)\sin(x_2), \qquad f_2(x_1, x_2) = \cosh(x_1)\cos(x_2).$$

Find the fixed points of the map and study their stability.

Problem 10. Study the system of difference equations

$$\begin{aligned}
x_{0,t+1} &= 2x_{0,t} - x_{1,t} - x_{2,t} - x_{3,t} - x_{4,t} \\
x_{1,t+1} &= -\frac{1}{2}x_{1,t} + x_{0,t} \\
x_{2,t+1} &= -\frac{1}{2}x_{2,t} + x_{0,t} \\
x_{3,t+1} &= -\frac{1}{2}x_{3,t} + x_{0,t} \\
x_{4,t+1} &= -\frac{1}{2}x_{4,t} + x_{0,t}
\end{aligned}$$

with the initial conditions

$$x_{0,0} = 1, \quad x_{1,0} = x_{2,0} = x_{3,0} = x_{4,0} = 0.$$

Problem 11. Consider the difference equation

$$x_{t+1} = 2x_t - 2, \quad t = 0, 1, 2, \ldots$$

where $x_0 = 1$. Find the general solution. Show that $x^* = 2$ is a particular solution (the fixed point).

Problem 12. The Fibonacci numbers can be defined by

$$\begin{pmatrix} x_{t+2} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix}$$

with $x_0 = x_1 = 1$. Find the eigenvalues of the 2×2 matrix. Study the sequence

$$\begin{pmatrix} x_{t+4} \\ x_{t+3} \\ x_{t+1} \\ x_t \end{pmatrix} = (A \otimes A) \begin{pmatrix} x_{t+3} \\ x_{t+2} \\ x_{t+1} \\ x_t \end{pmatrix}$$

with $x_0 = x_1 = x_2 = x_3 = 1$.

Problem 13. Let $x_0 \in \mathbb{R}^+$ and $r \in \mathbb{R}^+$. Show that

$$x_{j+1} = \frac{1}{2}\left(x_j + \frac{r}{x_j}\right), \quad j = 0, 1, 2, \ldots$$

tends to $\sqrt{r}$.

Problem 14. Let $A = (a_j k)$ be a 3×3 matrix with $\det(A) \neq 0$. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_1(x_1, x_2) = \frac{a_{11}x_1 + a_{12}x_2 + a_{13}}{a_{31}x_1 + a_{32}x_2 + a_{33}}, \quad f_2(x_1, x_2) = \frac{a_{21}x_1 + a_{22}x_2 + a_{23}}{a_{31}x_1 + a_{32}x_2 + a_{33}},$$

Find the inverse of the map.

Problem 15. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = x_1 + x_2, \quad f_2(x_1, x_2) = x_1 x_2$$

or written as difference equation

$$x_{1,\tau+1} = x_{1,\tau} + x_{2,\tau}, \quad x_{2,\tau+1} = x_{1,\tau} x_{2,\tau}.$$

Find the fixed points of the map. Then solve the difference equation for $x_{1,0} = 1/4$, $x_{2,0} = 1/5$.

Problem 16. Show that

$$\left(\frac{\partial F(x,y)}{\partial x}\right)^2 + \left(\frac{\partial F(x,y)}{\partial y}\right)^2$$

expressed in polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ takes the form

$$\left(\frac{F(x(r,\theta),y(r,\theta)}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial F(x(r,\theta),y(r,\theta))}{\partial \theta}\right)^2 .$$

Problem 17. Let $r > 1$ and $\tau = 0, 1, 2, \ldots$. Study the system of difference equation

$$x_{1,\tau+1} = \frac{1}{r}e^{x_{2,\tau}}, \quad x_{2,\tau+1} = re^{-x_{1,\tau}}$$

with the initial conditions $x_{1,0} = x_{2,0} = 0$. Are there fixed points?

Problem 18. Let $m \in \mathbb{N}$. Solve the recurrence relation

$$f(n) = 1 + f(n/2) \text{ for } n \geq 2, n = 2^m$$

with the initial condition $f(1) = 1$.

Problem 19. Study the initial value problem second order difference equation

$$x_{\tau+2} = x_{\tau+1} + x_\tau - x_{\tau+1}x_\tau$$

with $x_0 = 1$, $x_1 = 1/2$.

Problem 20. Study the initial value problem second order difference equation

$$x_{\tau+2} = x_{\tau+1} + x_\tau - x_{\tau+1}x_\tau$$

with $x_0 = 1$, $x_1 = 1/2$.

Chapter 3

Functions

3.1 Solved Problems

Problem 1. Let $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, n$ be real-valued functions with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leq i \leq n$. Prove that there is a function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f_i + \partial g/\partial x_i$ is linear for all $i = 1, 2, \ldots, n$. A linear function $p : \mathbb{R}^n \to \mathbb{R}$ is of the form

$$p(\mathbf{x}) = a_0 + a_1 x + a_2 x_2 + \cdots + a_n x_n.$$

Problem 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $\mathbf{x} \in \mathbb{R}^n$. Consider the invertible $n \times n$ matrix R and the transformation $\mathbf{x}' = R\mathbf{x}$. The transformation operator $\hat{P}_R$ associated with the invertible matrix R is defined by

$$\hat{P}_R f(\mathbf{x}) := f(R^{-1}\mathbf{x}).$$

Note that the operator acts upon the coordinates $\mathbf{x}$ and not on the argument of f. This means $\hat{P}_R \hat{P}_S f(\mathbf{x}) = f(S^{-1}R^{-1}\mathbf{x})$, where S is another invertible $n \times n$ matrix. Show that

$$\hat{P}_R \hat{P}_S f(\mathbf{x}) = \hat{P}_{RS} f(\mathbf{x}).$$

Problem 3. A ratio list is a finite list of positive numbers, $(r_1, r_2, \ldots, r_n)$. An iterated function system realizing a ratio list $(r_1, r_2, \ldots, r_n)$ in a metric space S is a list $(f_1, f_2, \ldots, f_n)$, where $f_j : S \to S$ is a similarity with ratio r_j. A nonempty compact set $K \subseteq S$ is an invariant set for the iterated function system $(f_1, f_2, \ldots, f_n)$ iff $K = f_1(K) \cup f_2(K) \cup \cdots \cup f_n(K)$. The triadic Cantor set is an invariant set for an iterated function system realizing the ratio list $(1/3, 1/3)$. The Sierpinski gasket is an invariant set for an iterated system realizing the ratio list $(1/2, 1/2, 1/2)$. The dimension associated with a ratio list $(r_1, r_2, \ldots, r_n)$ is the positive number s such that $r_1^s + r_2^s + \cdots + r_n^s = 1$. Let $(r_1, r_2, \ldots, r_n)$ be a ratio list. Suppose each $r_j < 1$. Show that there is a unique nonnegative number s satisfying

$$\sum_{j=1}^{n} r_j^s \equiv \sum_{j=1}^{n} e^{s \ln(r_j)} = 1.$$

The number s is 0 iff $n = 1$.

Problem 4. The arc length of the equilateral hyperbola

$$h(t) = \sqrt{t^2 - 1}, \qquad t \geq 1$$

starting at $t = 1$ is given by

$$L_h(x) = \int_1^x \sqrt{\frac{2t^2 - 1}{t^2 - 1}} dt$$

as a function of the terminal point $t = x$. The tangent line to the hyperbola at $t = x$ is

$$T_h(t) = \sqrt{x^2 - 1} + \frac{x}{\sqrt{x^2 - 1}}(t - x)$$

whose intersection with the t-axis is $t = 1/x$ ($t \in (0, 1)$). The line

$$N_h(t) = -\frac{\sqrt{x^2 - 1}}{x} t$$

is perpendicular to T_h passing through the origin.
(i) Find the point P_h where the lines T_h and N_h intersect.
(ii) Calculate the distance from $(x, h(x))$ to the common point P_h.

Problem 5. Given a differentiable function f. The *logarithmic derivative* of f is defined as

$$\frac{1}{f} \frac{df}{dx}.$$

When f is a real function of real variables and takes strictly positive values then the chain rule provides

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f}\frac{df}{dx}.$$

The logarithmic derivative has the following properties.
1. The logarithmic derivative of the product of functions is the sum of their logarithmic derivatives

$$\frac{1}{fg}\frac{d}{dx}(fg) = \frac{1}{f}\frac{df}{dx} + \frac{1}{g}\frac{dg}{dx}$$

2. The logarithmic derivative of the quotient of functions is the difference of their logarithmic derivatives

$$\frac{1}{f/g}\frac{d}{dx}(f/g) = \frac{1}{f}\frac{df}{dx} - \frac{1}{g}\frac{dg}{dx}$$

3. The logarithmic derivative of the α-th power of a function f is α times the logarithmic derivative of the function

$$\frac{1}{f^\alpha}\frac{d}{dx}(f^\alpha) = \alpha\frac{1}{f}\frac{df}{dx}$$

4. The logarithmic derivative of the exponential of a function equals the derivative of a function

$$\frac{1}{e^f}\frac{d}{dx}e^f = \frac{df}{dx}$$

(i) Find the logarithmic derivative of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cosh(x)$.
(ii) Let f be a meromorphic function in the open and conncted set $D \subseteq \mathbb{C}$. Let $G \subseteq D$ be a region such that its closure $\overline{G} \subseteq D$ and its boundary ∂G is a continuous curve not containing a zero or pole of f. Let N be the number of zeros of f lying inside G and P be the number of poles of f lying inside G. Then

$$N - P = \frac{1}{2\pi i}\int_{\partial G}\frac{1}{f(z)}\frac{df(z)}{dz}dz$$

where ∂G is an oriented boundary of G. Calculate the left and right-hand side for the function $f(z) = 1/z$ and $G = \{\,(x,y) \,:\, x^2 + y^2 \leq 1\,\}$.

Problem 6. Let f be an analytic function. Let $p \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Show that

$$\left(\frac{d}{dx} + \alpha x\right)^p f \equiv \exp\left(-\frac{\alpha x^2}{2}\right)\frac{d^p}{dx^p}\left(\exp\left(\frac{\alpha x^2}{2}\right)f\right).$$

Problem 7. Consider

$$K(t,s) = \begin{cases} 1 \text{ if } 0 \le t \le s \\ 0 \text{ if } s \le t \le 1 \end{cases}.$$

Show that this *kernel* satisfies the functional equation

$$K(cu + a, t) = K\left(u, \frac{t-a}{c}\right)$$

where $c > 0$.

Problem 8. Find a continuous differentiable function $f : \mathbb{R} \to \mathbb{R}$ which has no fixed points and no critical points.

Problem 9. Let $a, b, p, q, r \in \mathbb{R}$, $b > a$, $a < p$, $p < r$, $r < q$, $q < b$ and

$$p - a = r - p, \qquad q - r = b - q.$$

In fuzzy logic the following membership function plays an important role

$$f(x; a, b, r, p, q) = \begin{cases} 0 & x \le a \\ 2^{m-1}((x-a)/(r-a))^m & a < x \le p \\ 1 - 2^{m-1}((r-x)/(r-a))^m & p < x \le r \\ 1 - 2^{m-1}((x-r)/(b-r))^m & r < x \le q \\ 2^{m-1}((b-x)/(b-r))^m & q < x < b \\ 0 & x \ge b \end{cases}$$

where m is the fuzzifier. Im most cases $m = 2$. Where are the crossover points? What is the value at the centre?

Problem 10. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping of $(x, y) \to (u, v)$ given by

$$u(x,y) = e^x \cos(y), \qquad v(x,y) = e^x \sin(y).$$

What is the range of f? Show that the Jacobian determinant is not zero at any point of $\mathbb{R}^2$. Thus every pont of $\mathbb{R}^2$ has a neighbourhood in which f is one-to-one. Nevertheless f is not one-to-one on $\mathbb{R}^2$. What are the images under f of lines parallel to the coordinate axes?

Problem 11. Let $x, y \in \mathbb{R}$. Calculate the square root

$$\sqrt{x^2 + y^2 - 2xy}.$$

Problem 12. A criterion for linearly independence of functions $f_0, f_1, \ldots, f_n \in C^n[a,b]$ is that the *Wronski determinant* is nonzero

$$\det \begin{pmatrix} f_0 & f_1 & \dots & f_n \\ f_0' & f_1' & \dots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \dots & f_n^{(n)} \end{pmatrix} \neq 0.$$

Here $'$ denotes derivative. Apply this criterion to the functions

$$f_0(x) = \cos(x), \qquad f_1(x) = \sin(x).$$

Problem 13. For a sphere of radius r and mass density ρ the mass that must be concentrated at its centre is ($\lambda \geq 0$)

$$M(\lambda) = \frac{4\pi r \rho}{\lambda^2}(\cosh(\lambda r) - \sinh(\lambda r)/(\lambda r)).$$

Find $\lim_{\lambda \to 0} M(\lambda)$.

Problem 14. Let $-1 < a < 1$. Find the inverse of the transformation

$$\lambda(z) = \frac{z-a}{1-az}.$$

Problem 15. Consider the Jacobi elliptic functions

$$\mathrm{sn}(x,k), \quad \mathrm{cn}(x,k), \quad \mathrm{dn}(x,k)$$

where $x \in \mathbb{R}$ and $k^2 \in [0,1]$. We have

$$\mathrm{sn}(x,0) = \sin(x), \quad \mathrm{cn}(x,0) = \cos(x), \quad \mathrm{dn}(x,0) = 1$$

and

$$\mathrm{sn}(x,1) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x), \quad \mathrm{cn}(x,1) = \mathrm{dn}(x,1) = \frac{2}{e^x + e^{-x}} \equiv \mathrm{sech}(x).$$

(i) Find an expression using Jacobi elliptic functions that interpolates between $\sin(x)$ for $k=0$ and $\sinh(x)$ for $k=1$.
(ii) Find an expression using Jacobi elliptic functions that interpolates between $\cos(x)$ for $k=0$ and $\cosh(x)$ for $k=1$.
(iii) Use this result to interpolate between the matrices

$$\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}.$$

Problem 16. Let f_1, f_2 be differentiable functions and $f_1(x) > 0$, $f_2(x) > 0$ for all x. Let $f(x) = f_1(x)f_2(x)$. Find

$$\frac{d}{dx}(\ln(f(x)).$$

Problem 17. Let $x \in \mathbb{R}$. The sequence of functions $\{ f_k(x) \}$ is defined by $f_1(x) = \cos(x/2)$ and for $k > 1$ by

$$f_k(x) = f_{k-1}(x)\cos(x/2^k).$$

Thus

$$f_k(x) = \cos(x/2)\cos(x/2^2)\cdots\cos(x/2^k).$$

Obviously, we have $f_k(0) = 1$ for every k. Calculate $\lim_{k\to\infty} f_k(x)$ as a function of x for $x \neq 0$.

Problem 18. (i) Consider the transformation in $\mathbb{R}^3$

$$\begin{aligned}
x_0(a,\theta_1) &= \cosh(a)\\
x_1(a,\theta_1) &= \sinh(a)\sin(\theta_1)\\
x_2(a,\theta_1) &= \sinh(a)\cos(\theta_1)
\end{aligned}$$

where $a \geq 0$ and $0 \leq \theta_1 < 2\pi$. Find

$$x_0^2 - x_1^2 - x_2^2.$$

(ii) Consider the transformation in $\mathbb{R}^4$

$$\begin{aligned}
x_0(a,\theta_1,\theta_2) &= \cosh(a)\\
x_1(a,\theta_1,\theta_2) &= \sinh(a)\sin(\theta_2)\sin(\theta_1)\\
x_2(a,\theta_1,\theta_2) &= \sinh(a)\sin(\theta_2)\cos(\theta_1)\\
x_3(a,\theta_1,\theta_2) &= \sinh(a)\cos(\theta_2)
\end{aligned}$$

where $a \geq 0$, $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_2 \leq \pi$. Find

$$x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Extend the transformation to $\mathbb{R}^n$.

Problem 19. A fixed charge Q is located on the z-axis with coordinates $\mathbf{r}_a = (0, 0, d/2)$, where d is interfocal distance of the *prolate spheroidal* *coordinates*

$$
\begin{aligned}
x(\eta,\xi,\phi) &= \frac{1}{2}d((1-\eta^2)(\xi^2-1))^{1/2}\cos\phi \\
y(\eta,\xi,\phi) &= \frac{1}{2}d((1-\eta^2)(\xi^2-1))^{1/2}\sin(\phi) \\
z(\eta,\xi,\phi) &= \frac{1}{2}d\eta\xi
\end{aligned}
$$

where $-1 \le \eta \le +1$, $1 \le \xi \le \infty$, $0 \le \phi \le 2\pi$. Express the *Coulomb potential*

$$V = \frac{Q}{|\mathbf{r}-\mathbf{r}_a|}$$

in prolate spheroidal coordinates.

Problem 20. *Toroidal coordinates* are defined by

$$
\begin{aligned}
x_1(\alpha,\beta,\phi) &= \frac{c\sinh(\alpha)\cos(\phi)}{\cosh(\alpha)-\cos(\beta)} \\
x_2(\alpha,\beta,\phi) &= \frac{c\sinh(\alpha)\sin(\phi)}{\cosh(\alpha)-\cos(\beta)} \\
x_3(\alpha,\beta,\phi) &= \frac{c\sin(\beta)}{\cosh(\alpha)-\cos(\beta)}.
\end{aligned}
$$

Use the L'Hosptial rule to find x_1, x_2, x_3 for $\alpha \to 0$ and $\beta \to 0$.

Problem 21. Let $c, \theta \in \mathbb{R}$ and

$$f(\theta) = c(e^{i\theta} + e^{-i\theta}).$$

Calculate $\exp(f(\theta))$.

Problem 22. A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is called an *alternating function* if

$$f(x,y) = -f(y,x).$$

Give an example of an analytic alternating function. Find the minima and maxima of the function.

Problem 23. Consider the functions

$$
\begin{aligned}
j_1(x) &= \frac{1}{x^2}(\sin(x) - x\cos(x)) \\
j_2(x) &= \frac{1}{x^3}((3-x^2)\sin(x) - 3x\cos(x)).
\end{aligned}
$$

Use the L'Hospital rule to find $j_1(0)$ and $j_2(0)$.

Problem 24. Find the invariance group of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sin(x).$$

Problem 25. Let $a > 0$. Consider the transformation

$$u(x,y) = \frac{a\sin(2ax)}{2(\sin^2(ax) + \sinh^2(ay))}, \qquad v(x,y) = \frac{a\sinh(2ay)}{2(\sin^2(ax) + \sinh^2(ay))}.$$

Find the inverse transformation.

Problem 26. (i) Find an analytic function $f : \mathbb{R} \to \mathbb{R}$ that has no fixed point and no critical point. Draw the function.
(ii) Find an analytic function $f : \mathbb{R} \to \mathbb{R}$ that has no fixed point and exactly one critical point at $x = 0$. Draw the function.

Problem 27. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Calculate the commutator

$$\left[\cos(x)\frac{d}{dx}, \sin(x)\frac{d}{dx}\right] f.$$

Problem 28. (i) Consider the analytic function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1,x_2) = \sinh(x_2), \quad f_2(x_1,x_2) = \sinh(x_1).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(ii) Consider the analytic function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$

$$g_1(x_1,x_2) = \sinh(x_1), \quad g_2(x_1,x_2) = -\sinh(x_2).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(iii) Multiply the two matrices found in (i) and (ii).

(iv) Find the composite function $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{h}(\mathbf{x}) = (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x})).$$

Show that this function also admits the fixed point $(0, 0)$. Find the functional matrix at this fixed point

$$\left.\begin{pmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{pmatrix}\right|_{(0,0)} .$$

Compare this matrix with the matrix found in (iii).

Problem 29. An approximation of e^{-x} $(x \geq 0)$ as a rational polynomial using a 3rd order Padé approximation is given by

$$e^{-x} \approx \frac{1 - x/2 + x^2/10 - x^3/120}{1 + x/2 + x^2/10 + x^3/120}.$$

Note that $e^{-x} \geq 0$ for all x. For which value of $x > 0$ does the right-hand side takes negative values?

Problem 30. Let $x \in \mathbb{R}$. Consider the function

$$f(x) = \frac{x - i}{x + i}.$$

Find $f(0)$, $f(1)$, $f(-1)$ and $f(x \to \infty)$. Is there an inverse?

Problem 31. Consider the function

$$f(x) = x^3 - x^2.$$

Find an approximation of the derivative of f by using

$$f'(x) \approx \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} - \frac{1}{30}h^4 f^{(5)}(\xi)$$

$$f'(x) \approx \frac{-3f(x-h) - 10f(x) + 18f(x+h) - 6f(x+2h) + f(x+3h)}{12h} + \frac{1}{20}h^4 f^{(5)}(\xi)$$

$$f'(x) \approx \frac{-25f(x) + 48f(x+h) - 36f(x+2h) + 16f(x+3h) - 3f(x+4h)}{12h} - \frac{1}{5}h^4 f^{(5)}(\xi)$$

where $x \leq \xi \leq x + h$.

Problem 32. Let $z \in \mathbb{C}$. The *Airy functions* $Ai(z)$ and $Bi(z)$ are defined as sums of the power series

$$\begin{aligned} Ai(z) &= c_1 f(z) - c_2 g(z) \\ Bi(z) &= \sqrt{3}(c_1 f(z) + c_2 g(z)) \end{aligned}$$

where

$$f(z) = 1 + \frac{1}{3!}z^3 + \frac{1 \cdot 4}{6!}z^6 + \frac{1 \cdot 4 \cdot 7}{9!}z^9 + \cdots$$
$$g(z) = z + \frac{2}{4!}z^4 + \frac{2 \cdot 5}{7!}z^7 + \frac{2 \cdot 5 \cdot 8}{10!}z^{10} + \cdots$$

with

$$c_1 = \frac{1}{2\pi}\Gamma\left(\frac{1}{3}\right)3^{-1/6}, \qquad c_2 = \frac{1}{2\pi}\Gamma\left(\frac{2}{3}\right)3^{1/6}.$$

Show that the radius of convergence of these infinite series is infinite.

Problem 33. Find the inverse of the function $y = \tanh(x)$. Note that $y \in (-1, 1)$.

Problem 34. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{j=0}^{\infty} \frac{x^j}{(j+1)!} \equiv \frac{e^x - 1}{x}.$$

Find the fixed points and critical points of the function. Note that $f(0) = 1$.

Problem 35. Find non-negative analytic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 0.$$

Problem 36. The *Euler dilogarithm function* $\mathrm{Li}_2(x)$ is defined for $x \in (0, 1)$ as

$$\mathrm{Li}_2(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^2} = -\int_0^x \frac{\ln(1-t)}{t} dt.$$

It can be analytically continued to the complex plane with the branch cut from 1 to ∞ along the real axis using the integral. Calculate $\mathrm{Li}_2(1/2)$.

Problem 37. Let Ai be the Airy function and J_n, I_n are the Bessel and modified Bessel functions of order n. Show that

$$\frac{d^2 Ai}{dx^2} = xAi(x)$$
$$Ai(x) = \frac{1}{3}x^{1/2}(I_{-1/3}(z) - I_{1/3}(z))$$
$$Ai(-x) = \frac{1}{3}x^{1/2}(J_{1/3}(z) + J_{-1/3}(z))$$

where $z := 2x^{3/2}/3$.

Problem 38. Let $k \in \mathbb{R}$, $x, y \in \{+1, -1\}$ such that $xy = \pm 1$. Show that

$$e^{kxy} = \cosh(k) + xy \sinh(k).$$

Problem 39. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(-x), \quad f(x) = f(x + 2\pi), \quad f(0) = 0.$$

Problem 40. Let N be an integer and $N \geq 2$. The generalized hyperbolic function for a given N is defined as

$$f_j^{(N)}(x) := \sum_{k=0}^{\infty} \frac{x^{j+kN}}{(j+kN)!}, \qquad j = 0, 1, \ldots, N-1.$$

The functions $f_j^{(N)}$ are analytic.
(i) Show that

$$f_j^{(N)}(x) = f_{j+N}^{(N)}(x)$$

i.e. $f_j^{(N)}$ is periodic with respect to the index j.
(ii) Show that

$$\frac{d}{dx} f_j^{(N)}(x) = f_{j-1}^{(N)}(x).$$

(iii) Let $\omega^N = 1$, i.e. ω is the N-th primitive root of unity. Show that

$$f_j^{(N)}(x) = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-jk} \exp(\omega^k x).$$

Problem 41. Let $n = 2, 3, 4, \ldots$. Find

$$\frac{\Gamma(n/2)}{\Gamma(n)}$$

where Γ denotes the gamma function.

Problem 42. Consider the hyperspherical coordinates

$$\begin{aligned}
x_1(\theta, \phi, \psi) &= \cos(\theta) \\
x_2(\theta, \phi, \psi) &= \sin(\theta)\cos(\phi) \\
x_3(\theta, \phi, \psi) &= \sin(\theta)\sin(\phi)\cos(\psi) \\
x_4(\theta, \phi, \psi) &= \sin(\theta)\sin(\phi)\sin(\psi)
\end{aligned}$$

with $x_1^2+x_2^2+x_3^2+x_4^2=1$. Show that the *angular distance* can be calculated as

$$\cos(d_{jk})=\cos(\theta_j)\cos(\theta_k)+\sin(\theta_j)\sin(\theta_k)(\cos(\phi_j)\cos(\phi_k)+\sin(\phi_j)\sin(\phi_k)\cos(\psi_j-\psi_k)).$$

Problem 43. Consider the function

$$f(x)=1+x+2x^2+3x^3.$$

Find the second order derivative of f at $a=1$ applying

$$f''(a)=\lim_{\epsilon\to 0}\frac{f(a+\epsilon)-2f(a)+f(a-\epsilon)}{\epsilon^2}.$$

Problem 44. Let $-1<x<1$ and

$$\sum_{j=0}^{\infty}a_jx^j=\frac{1}{\sqrt{1-x}}.$$

Find the expansion coefficients a_j.

Problem 45. Show that the function

$$f(x)=\cos(x)\cosh(x)+1$$

has infinitely many roots, i.e. solutions of $f(x)=0$. What happens for x large?

Problem 46. The Jacobi elliptic functions $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ with $k\in[0,1]$ and $k^2+k'^2=1$ have the properties

$$\mathrm{sn}(x,0)=\sin(x),\quad \mathrm{cn}(x,0)=\cos(x),\quad \mathrm{dn}(x,0)=1$$

and

$$\mathrm{sn}(x,1)=\frac{e^x-e^{-x}}{e^x+e^{-x}},\quad \mathrm{cn}(x,1)=\frac{2}{e^x+e^{-x}}=\mathrm{dn}(x,1).$$

We define

$$\begin{aligned}u_1(x,y,k,k')&=\mathrm{sn}(x,k)\mathrm{dn}(y,k')\\ u_2(x,y,k,k')&=\mathrm{cn}(x,k)\mathrm{cn}(y,k')\\ u_3(x,y,k,k')&=\mathrm{dn}(x,k)\mathrm{sn}(y,k').\end{aligned}$$

(i) Find $u_1(x,y,0,1)$, $u_2(x,y,0,1)$, $u_3(x,y,0,1)$ and calculate $u_1^2(x,y,0,1)+u_2^2(x,y,0,1)+u_3^2(x,y,0,1)$.

(ii) Find $u_1(x,y,1,0)$, $u_2(x,y,1,0)$, $u_3(x,y,1,0)$ and calculate $u_1^2(x,y,1,0)+u_2^2(x,y,1,0)+u_3^2(x,y,1,0)$.

Problem 47. Let m be an integer with $m \geq 1$. Then the *gamma function* Γ is given by

$$\Gamma(m) = (m-1)!.$$

Thus $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 6$, $\Gamma(5) = 24$. Furthermore

$$\Gamma(m + \frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2^m}\sqrt{\pi}.$$

Thus $\Gamma(3/2) = \sqrt{\pi}/2$, $\Gamma(5/2) = 3\sqrt{\pi}/4$, $\Gamma(7/2) = 15\sqrt{\pi}/8$, $\Gamma(9/2) = 105\sqrt{\pi}/16$. Calculate

$$\frac{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

for $m = 5$, $m = 10$, $m = 20$.

Problem 48. Let $c_1 > 0$, $c_2 > 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2 + \frac{c_1 x^2}{1 + c_2 x^2}.$$

Find the minima and maxima. Find the fixed points.

Problem 49. Let $c > 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f_c(x) = \frac{4}{x^2}\sin^2\left(\frac{cx}{2}\right).$$

Find $f_c(0)$. Show that f_c has a maximum at $x = 0$.

Problem 50. (i) Find the minima and maxima of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \cosh(x)\cos(x).$$

Find the fixed points.
(ii) Find the minima and maxima of the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sinh(x)\sin(x).$$

Find the fixed points.

Problem 51. Consider the coordinate transformation

$$x_1(r,\alpha,\beta,\gamma) = r\sin(\beta/2)\sin((\gamma-\alpha)/2)$$

$$
\begin{aligned}
x_2(r,\alpha,\beta,\gamma) &= r\sin(\beta/2)\cos((\gamma-\alpha)/2) \\
x_3(r,\alpha,\beta,\gamma) &= r\cos(\beta/2)\sin((\gamma+\alpha)/2) \\
x_4(r,\alpha,\beta,\gamma) &= r\cos(\beta/2)\sin((\gamma+\alpha)/2).
\end{aligned}
$$

Show that $x_1^2+x_2^2+x_3^2+x_4^2=r^2$.

Problem 52. Let S_1 and S_2 be two finite sets with n_1 and n_2 elements, respectively. The number of functions $f: S_1 \to S_2$ is given by

$$n_2^{n_1}.$$

(i) Let $n_1 = n_2 = 2$. Find all possible functions.
(ii) Let $n_1 = n_2 = n$. Then there are $n!$ one-to-one functions $f: S_1 \to S_2$. Let $n = 3$. Find all one-to-one functions.

Problem 53. Let $\mu > 0$ and

$$\mathbf{R} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{R}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}.$$

(i) Show that

$$\frac{\exp(-|\mathbf{R}-\mathbf{R}'|/\mu)}{|\mathbf{R}-\mathbf{R}'|} =$$

$$\sum_{n=0}^{\infty}\epsilon_n\cos(n(\phi-\phi'))\int_0^{\infty}\frac{1}{k^2+1/\mu}J_n(kr)J(kr')\exp(-\sqrt{k^2+1/\mu^2}|x_3-x_3'|)kdk$$

where $\varepsilon_n = 1$ for $n = 0$ and $\varepsilon_n = 2$ for $n > 0$ and $J_m(kr)$ is ordinary Bessel function of order m.
(ii) Consider the functions

$$f_{s,k,n}(\mathbf{R}) = \frac{\sqrt{k}}{2\pi}J_n(kr)e^{in\phi+isx_3}$$

where $0k\infty$, $-\infty s\infty$, and $n = 0, \pm 1, \pm 2, \ldots$. Show that

$$\int_0^{2\pi}\int_{-\infty}^{+\infty}dx_3\int_0^{\infty}f_{s,k,n}(\mathbf{R})\bar{f}_{s',k',n'}(\mathbf{R})rdr = \delta_{nn'}\delta(s-s')\delta(k-k')$$

Problem 54. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be analytic functions and

$$
\begin{aligned}
g_1(x) &= \frac{1}{2}(f_1(x)+f_2(x)-|f_1(x)-f_2(x)| \\
g_2(x) &= \frac{1}{2}(f_1(x)+f_2(x)+|f_1(x)-f_2(x)|.
\end{aligned}
$$

Let $f_1(x) = \sin(x)$ and $f_2(x) = \cos(x)$. Find the maxima and minima of g_1. Find the maxima and minima of g_2.

Problem 55. Let the function f be continuous on a closed interval $[a, b]$ and is continuous differentiable in the open interval (a, b). Then there exists a point c in (a, b) such that (mean value theorem)

$$f(b) - f(a) = f'(c)(b - a).$$

Apply the theorem to the function

$$f(x) = 1 + 2x + 3x^2$$

and the interval $[-1, 1]$.

Problem 56. The plane $\{(x_1, x_2, x_3) : x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1\}$ and the cylinder $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 4\}$ intersect. Find the shortest distance from $(0, 0, 0)$ to this curve.

Problem 57. Consider the plane $\{(x_1, x_2, x_3) : 2x_1 + 3x_2 - x_3 = 5\}$ in the three dimensional Euclidean space $\mathbb{E}^3$. Find the shortest distance from $(0, 0, 0)$ to this plane.

Problem 58. We denotes by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the rational, real and complex fields, respectively. Let B_1 and B_2 be Banach spaces. A map $f : B_1 \to B_2$ is said to be additive if and only if

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in B_1$. Show that f is additive implies that $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$. Show that f is additive and continuous implies that $f(rx) = rf(x)$ for all $r \in \mathbb{R}$.

Problem 59. Let $a_1d_1 - b_1c_1 \neq 0$, $a_2d_2 - b_2c_2 \neq 0$ and the functions $f_1 : \mathbb{C} \to \mathbb{C}$

$$f_1(z) = \frac{a_1z + b_1}{c_1z + d_1}, \qquad f_2(z) = \frac{a_2z + b_2}{c_2z + d_2}.$$

Find $f_2 \circ f_1$, i.e. the function compositon.

Problem 60. Let $x_1, x_2 \geq 0$. Consider

$$f(x_1, x_2) = \frac{\sqrt{1 + x_1}}{\sqrt{1 + x_1}\sqrt{1 + x_2}} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} F(j_1, j_2) x_1^{j_1} x_2^{j_2}.$$

Find the expansion coefficients.

Problem 61. Consider the function

$$f(x) = \frac{3}{x^3}(\sin(x) - x\cos(x)).$$

Find $f(0)$ using the L'Hospital rule.

Problem 62. Let

$$f(x) = \frac{\sin(x)}{\sinh(x)}.$$

Find $f(0)$.

Problem 63. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^3.$$

Show that f is not convex. Show that f is convex if we restrict to the domain $x \geq 0$.

Problem 64. Consider the convex functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that $f + g$ is convex. Show that $\max\{\, f, g \,\}$ is convex.

Problem 65. **Definition (Convex Set)**. A subset C of $\mathbb{R}^n$ is said to be *convex* if for any $\mathbf{a}$ and $\mathbf{b}$ in C and any θ in $\mathbb{R}$, $0 \leq \theta \leq 1$, the n-tuple $\theta\mathbf{a} + (1-\theta)\mathbf{b}$ also belongs to C. In other words, if $\mathbf{a}$ and $\mathbf{b}$ are in C then

$$\{\, \theta\mathbf{a} + (1-\theta)\mathbf{b} \,:\, 0 \leq \theta \leq 1 \,\} \subset C.$$

Definition (Convex Polyhedron). Let $\mathbf{a}_1, \ldots, a_p$ be p points in $\mathbb{R}^n$. The n-tuple

$$\sum_{j=1}^{p} \theta_j a_j, \quad \theta_j \geq 0, \quad j = 1, \ldots, p, \qquad \sum_{j=1}^{p} \theta_j = 1$$

is called a convex combination (or a convex sum) of $\mathbf{a}_1, \ldots, \mathbf{a}_p$. If $X \subset \mathbb{R}^n$ then the set of all (finite) convex combination of points of X is called the convex hull of X and is denoted by $H(X)$. If X is finite, $X = \{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$, then $H(X)$ is called the convex polyhedron spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_p$ and is also denoted by $H(\mathbf{a}_1, \ldots, \mathbf{a}_p)$.

Defintion. Let S be a nonempty convex set in $\mathbb{R}^n$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space. The function $f : S \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda)f(\mathbf{x}_2)$$

for each $\mathbf{x}_1, \mathbf{x}_2 \in S$ and for each $\lambda \in [0,1]$. The function is said to be strictly convex if the above inequality holds as a strict inequality for each distinct $\mathbf{x}_1, \mathbf{x}_2 \in S$ and for each $\lambda \in (0,1)$.

Let

$$\mathbf{a}_1 = (1,-1), \qquad \mathbf{a}_2 = (2,2), \qquad \mathbf{a}_3 = (3,1)$$

be points of $\mathbb{R}^2$. Find the convex polyhedron.

Problem 66. Show that the function $\phi : [0,\infty) \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \ln x & \text{if } x \neq 0 \end{cases} \tag{1}$$

is strictly convex, i.e.,

$$\phi(\alpha x + \beta y) \leq \alpha\phi(x) + \beta\phi(y) \tag{2}$$

if $x, y \in [0,\infty), \alpha\beta \geq 0,\ \alpha + \beta = 1$ with equality only when $x = y$ or $\alpha = 0$ or $\beta = 0$. By induction we get

$$\phi\left(\sum_{i=1}^{k} \alpha_i x_i\right) \leq \sum_{i=1}^{k} \alpha_i \phi(x_i) \tag{3}$$

if

$$x_i \in [0,\infty), \quad \alpha_i \geq 0, \quad \sum_{i=1}^{k} \alpha_i = 1. \tag{4}$$

Equality holds only when all the x_i, corresponding to non-zero α_i, are equal.

Problem 67. Let $x \geq 0$ and

$$f(x) = x\ln(x) - x + 1. \tag{1}$$

(i) Show that

$$f(x) \geq 0. \tag{2}$$

(ii) Show that from L'Hospital rule we find that $f(0) = 1$.
(iii) Show that the function is convex.

Problem 68. Show that $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |x| \tag{1}$$

is convex.

Problem 69. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = ax^2 + by^2 + 2cxy + d$$

where $a, b, c, d \in \mathbb{R}$. For what values of a, b, c, d is f concave?

Problem 70. Let $r_1, r_2 \in \mathbb{R}$. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = r_1 x_1 (1 - x_1 - x_2), \quad f_2(x_1, x_2) = r_2 x_1 x_2.$$

Find df_1, df_2, $df_1 \wedge df_2$.

Problem 71. Find $x = \arctan(-1/\sqrt{3})$.

Problem 72. Let $f, g : \mathbb{R} \to \mathbb{R}$ be analytic functions. Find

$$e^{f(x)} \frac{d^2}{dx^2} (e^{-f(x)} g(x)).$$

Problem 73. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = \sin(x_1^2 + x_2^2), \quad f_2(x_1, x_2) = \cos(x_1^2 + x_2^2).$$

Find the Jacobian matrix and the Jacobian determinant.

Problem 74. Consider the map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = -\frac{1}{2} x^2 - x + \frac{1}{2}.$$

Then $f(1) = -1$, $f(-1) = 1$. Thus we have a periodic orbit with period 2. Is the orbit attracting or repelling? We set $x_1 = 1$, $x_2 = -1$. We have to test

$$|(df(x = x_1)/dx)(df(x = x_2)/dx)| < 1$$

for attracting and

$$|(df(x = x_1)/dx)(df(x = x_2)/dx)| > 1$$

for repelling.

Problem 75. Let $\alpha \in \mathbb{R}$ and $f(x) = e^{\alpha x}$. Find

$$\frac{1}{h^2}(f(x + h) - 2f(x) + f(x - h)).$$

Then consider $h \to 0$.

Problem 76. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$. The equation $\cos(x) = x$ has one solution, i.e. we have one fixed point for f. Consider $f(f(x)) = \cos(\cos(x))$. Does the equation $\cos(\cos(x)) = x$ admits other solutions besides the one of $\cos(x) = x$.

Problem 77. Let $a, b \in \mathbb{N}$. Consider the 2×2 matrix

$$C = \begin{pmatrix} 1 & a \\ b & 1 + ab \end{pmatrix}$$

with $\det(C) = 1$. Consider the map (generalized Arnold cat map)

$$\begin{pmatrix} x_{1,\tau+1} \\ x_{2,\tau+1} \end{pmatrix} = C \begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \end{pmatrix} \mod 1$$

where $\tau = 0, 1, 2, \ldots$. Find the eigenvalues of C which relates to the one-dimensional Liapunov exponents.

Problem 78. Consider the functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$

$$f(x) = 2x, \qquad g(x) = 3x^2 - 1.$$

Find $(f \circ g)(x) = f(g(x))$, $(g \circ f)(x) = g(f(x))$ and the extrema of these functions.

Problem 79. Let $a \in \mathbb{R}$. Given the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}(x_1, x_2) = (x_1 + x_2, 2x_1 + ax_2).$$

(i) Find the fixed points of the map.
(ii) Find the matrix $D\mathbf{f}(x_1, x_2)$ and show that the matrix $D\mathbf{f}(x_1, x_2)$ is invertible if and only if $a \neq 2$.

Problem 80. Let A, B be non-empty sets. If $f : A \to B$ is (1,1) and onto (i.e. bijective) one can define the inverse function $f^{-1} : B \to A$ as the unique function from B to A such that

$$(f \circ f^{-1})(y) = y \text{ for all } y \in B$$

$$(f^{-1} \circ f)(x) = \text{ for all } x \in A.$$

The function $f : (0, 1) \to \mathbb{R}$

$$x \mapsto f(x) = \frac{x - 1/2}{x(x - 1)}$$

is bijective. Find f^{-1}. Note that $f(1/2) = 0$, $f(1/4) = 4/3$, $f(3/4) = -4/3$.

Programming Problems

Problem 81. Let $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable function. Find the determinant of the 3×3 matrix $A = (a_{jk})$

$$a_{jk} := \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j}.$$

Apply computer algebra.

Problem 82. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2 + x + 1.$$

Find the minima and maxima of

$$g(x) = f(f(x)) - f(x)f(x).$$

Apply computer algebra.

Problem 83. Consider the polynomials

$$f(x) = x^2 + 2x + 1, \qquad g(x) = x + 4;$$

Find $(f \circ g)(x) = f(g(x))$, $(g \circ f)(x) = g(f(x))$ and $f(g(x)) - g(f(x))$.

Problem 84. Let $f : \mathbb{R} \to \mathbb{R}$ be an analatic function. The *Schwarzian derivative* is defined by

$$S(f(x)) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

where $'$ denotes the derivative with respect to x. Let $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$ and

$$g(x) = \frac{af(x) + b}{cf(x) + d}.$$

Show that $S(g(x)) = S(f(x))$. Apply computer algebra.

3.2 Supplementary Problems

Problem 1. Find

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

Problem 2. Consider the analytic function $f:\mathbb{R}^2\to\mathbb{R}$

$$f(x,y)=\frac{xe^x(e^y-1)-ye^y(e^x-1)}{xy(e^x-e^y)}.$$

(i) Show that $f(0,0)=\frac{1}{2}$ applying the L'Hospital rule.
(ii) Show that

$$f(x,x)=\frac{e^x-x-1}{x^2}.$$

(iii) Show that

$$f(x,-x)=\frac{\tanh(x/2)}{x}.$$

Problem 3. Consider the map from $\mathbb{R}^4$ to the vector space of 2×2 matrices over $\mathbb{R}$

$$f(x_1,x_2,x_3,x_4)=\begin{pmatrix} x_1+x_4 & x_2+x_3 \\ -x_2+x_3 & x_1-x_4 \end{pmatrix}.$$

Find the inverse of the map. Note that the determinant of the matrix is given by $x_1^2+x_2^2-x_3^2-x_4^2$.

Problem 4. Let

$$D=\{\,(x_1,x_2,x_3)\,:\,x_1^2+x_2^2+x_3^2=1,\;x_1\geq 0,x_2\geq 0,x_3\geq 0\,\}.$$

Let $\mathbf{v}$ be a normalized vector in $\mathbb{R}^3$ with nonnegative entries and A be a 3×3 matrix over $\mathbb{R}$ with strictly positive entries. Show that the map $f:D\to D$

$$f(\mathbf{v})=\frac{A\mathbf{v}}{\|A\mathbf{v}\|}$$

has a fixed point, i.e. there is a normalized vector $\mathbf{v}_0$ such that

$$\frac{A\mathbf{v}_0}{\|A\mathbf{v}_0\|}=\mathbf{v}_0.$$

Problem 5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be analytic functions. Find

$$(e^{f(x)} \frac{d^2}{dx^2} e^{-f(x)})x.$$

Problem 6. Let $\mathbb{Z}$ be the set of integers and $\mathbb{N}_0$ be the set of natural numbers including 0. Consider the one-to-one map $f : \mathbb{Z} \to \mathbb{N}_0$

$$f(j) = \begin{cases} 0 & \text{if} \quad j = 0 \\ 2j - 1 & \text{if } j \text{ is positive} \\ -2j & \text{if } j \text{ is negative} \end{cases} .$$

Give a C++ implementation with the class `Verylong` of the function f and its inverse.

Problem 7. Let $\{ f_1, f_2, \ldots, f_n \}$ be a set of convex functions from $\mathbb{R}^n \to \mathbb{R}$. Show that the nonnegative linear combination

$$f(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x}) + \cdots + \alpha_n f_n(\mathbf{x}), \qquad \alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$$

is convex.

Problem 8. Study the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = \sqrt{1 - \sin(x_2)}, \quad f_2(x_1, x_2) = \sqrt{1 + \sin(x_1)}.$$

First find the fixed points if there are any.

Problem 9. Let $k \in \mathbb{N}$ and $x \geq 0$. Find the fixed points of the map

$$f_k(x) = \sqrt{k + x}$$

and study their stability.

Problem 10. Consider the continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$f(x) = \begin{cases} 2x & x < 1 \\ \max\{4 - 2x, \frac{1}{4}\} & x \geq 1 \end{cases}$$

Find the fixed points and study their stability.

Problem 11. Let

$$D = \{(x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 + x_3^2 = 1, \;\; x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

Let $\mathbf{v}$ be a normalized vector in $\mathbb{R}^3$ with nonnegative entries and A be a 3×3 matrix over $\mathbb{R}$ with strictly positive entries. Show that the map $f : D \to D$

$$f(\mathbf{v}) = \frac{A\mathbf{v}}{\|A\mathbf{v}\|}$$

has fixed point, i.e. there is a $\mathbf{v}_0$ such that

$$\frac{A\mathbf{v}_0}{\|A\mathbf{v}_0\|} = \mathbf{v}_0.$$

Problem 12. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two analytic functions and S be the Schwarzian derivative. Show that if $Sf < 0$ and $Sg < 0$, then

$$S(f \circ g) < 0.$$

Problem 13. Let $x, y \in \mathbb{R}^+$. Consider the function $f : (\mathbb{R}^+)^2 \to \mathbb{R}$

$$f(x,y) = \frac{\sqrt{1+x}}{\sqrt{1+x} + \sqrt{1+x}}.$$

Then $f(0,0) = 1/2$ and $f(x,y) + f(y,x) = 1$. Set

$$\frac{\sqrt{1+x}}{\sqrt{1+x} + \sqrt{1+x}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F(m.n) x^m y^n.$$

Show that $F(m,n) = -F(n,m)$ unless $m = n = 0$ and $F(m,m) = 0$ for $m \geq 1$. Show that

$$F(0,0) = 1/2, \quad F(0,1) = -F(1,0) = -1/8, \quad F(0,2) = -F(2,0) = 1/16,$$

$$F(0,3) = -F(3,0) = -5/128, \quad F(1,2) = -F(2,1) = -1/128.$$

Problem 14. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = e^{x_1} \cos(x_2), \qquad f_2(x_1, x_2) = e^{x_1} \sin(x_2).$$

Find the fixed points of the map.

Problem 15. Consider the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{pmatrix} = \begin{pmatrix} x_1 \cos(x_2) \sin(x_3) \\ x_1 \sin(x_2) \sin(x_3) \\ x_1 \cos(x_3) \end{pmatrix}.$$

Find the fixed points and study their stability.

Problem 16. Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1 x_2 \end{pmatrix}.$$

The Jacobian matrix is

$$\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}$$

with determinant equal to $4(x_1^2 + x_2^2)$. Find the fixed points and study their stability.

Problem 17. Consider the differential operators

$$\begin{aligned} D_1 &= \frac{d^2}{dx^2} + \frac{c}{x^2} + \frac{x^2}{16} \\ D_2 &= \sinh(\alpha)\left(\frac{d^2}{dx^2} + \frac{c}{x^2} - \frac{x^2}{16}\right) + \cosh(\alpha)\left(-\frac{i}{2}x\frac{d}{dx} - \frac{i}{4}\right) \\ D_3 &= \cosh(\alpha)\left(\frac{d^2}{dx^2} + \frac{c}{x^2} - \frac{x^2}{16}\right) + \sinh(\alpha)\left(-\frac{i}{2}x\frac{d}{dx} - \frac{i}{4}\right). \end{aligned}$$

(i) Show the differential operator satisfy (Lie algebra $su(1,1)$)

$$[D_1, D_2] = -iD_3, \quad [D_2, D_3] = iD_1, \quad [D_3, D_1] = iD_2.$$

For the actual calculations let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and one shows that $[D_1, D_2]f(x) = -iD_3 f(x)$ etc.
(ii) Show that

$$D_3^2 - D_1^2 - D_2^2 = -\frac{3}{16} - \frac{c}{4}.$$

Problem 18. Let

$$\mathbf{v}(\tau) = \begin{pmatrix} v_1(\tau) \\ v_2(\tau) \\ v_3(\tau) \end{pmatrix}, \quad \mathbf{w}(\tau) = \begin{pmatrix} w_1(\tau) \\ w_2(\tau) \\ w_3(\tau) \end{pmatrix}$$

where $v_j(\tau)$, $w_j(\tau)$ ($j = 1, 2, 3$) are smooth functions of τ. Calculate

$$\frac{d}{d\tau}(\mathbf{v} \times (\mathbf{v} \times \mathbf{w})).$$

Problem 19. Show that the $(2,2)$ *Padé approximant* of the cosine function is given by

$$\cos(x) \approx \frac{12 - 5x^2}{12 + x^2}.$$

Problem 20. Let $f(x) = x^2$ and $g(x) = x\ln(x)$. Show that

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

Problem 21. Let $c > 0$. Find

$$\lim_{x\to 0} \frac{\sinh(cx)}{x}, \qquad \lim_{x\to 0} \frac{x}{\sinh(cx)}.$$

Problem 22. Show that the function $f : (0,1) \to \mathbb{R}$

$$f(x) = \frac{x - 1/2}{x(x-1)}$$

is bijective.

Problem 23. Show that the function

$$f(x) = \frac{\pi}{\sin(\pi x)}$$

has poles at $0, \pm 1, \pm 2, \ldots$.

Problem 24. For any real number x one defines $\lfloor x \rfloor$ (floor of x) as the largest integer less than or equal to x. One defines $\lceil x \rceil$ (ceiling of x) as the smallest integer greater than or equal to x. Find the minima and the maxima of the function

$$f(x) = 2x - \lfloor x \rfloor.$$

Problem 25. Let $0 \le t < 2$. Show that

$$\ln\left(\frac{2+t}{2-t}\right) = \ln\left(\frac{1+t/2}{1-t/2}\right) = 2\mathrm{arctanh}(t/2).$$

Problem 26. Consider the continuous function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1}{2}\left(\frac{x}{|x|+1} + 1\right).$$

Find the fixed points and study their stability.

Problem 27. Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2^2), \quad f_2(x_1, x_2) = x_1 x_2.$$

Find the fixed points and study their stability. Is their an inverse $\mathbf{f}^{-1}$.

Problem 28. Explain

$$\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}, \quad \cos(\alpha/2) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}.$$

Problem 29. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that the function is not analytic. Note that $f(x) = f(-x)$ and

$$\frac{df}{dx} = \frac{2}{x^3} f(x), \quad \frac{d^2 f}{dx^2} = \left(\frac{4}{x^6} - \frac{6}{x^4}\right) f(x)$$

Furthermore $\lim_{x \to \pm\infty} f(x) = 1$.

Problem 30. Let $N > 0$. Show that

$$\lim_{\epsilon \to 0} \epsilon^{-N} e^{-(1/\epsilon)} = 0.$$

Problem 31. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 \cos(x_2) \\ x_1 \sin(x_2) \end{pmatrix}.$$

Find the fixed points and study their stability. Show that the functional matrix is given by

$$\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_1 \end{pmatrix} = \begin{pmatrix} \cos(x_2) & -x_1 \sin(x_2) \\ \sin(x_2) & x_1 \sin(x_2) \end{pmatrix}.$$

Is the map $\mathbf{f}$ invertible?

Problem 32. Let $a > 0$. Show that

$$\lim_{x\to 0}\frac{\sinh(2ax)}{\sinh(x)} = 2a.$$

Problem 33. (i) Write down the first four terms of the Taylor expansion of $\cos(x)$. Then find $(2,2)[x]$ of the Padé approximant. Discuss the case $x\to\infty$ for both functions.
(ii) Write down the first four terms of the Taylor expansion of $\cosh(x)$. Then find $(2,2)[2]$ of the Padé approximant. Discuss the case $x\to\infty$ for both functions.

Problem 34. Consider the Hénon map $(a > 0, b > 0)$

$$x_{1,t+1} = a - bx_{2,t} + x_{1,t}^2, \quad x_{2,t+1} = x_{1,t}, \quad t = 0,1,2,\ldots\,.$$

Let $|x_{1,0}| < R$, $|x_{2,0}| < R$, where R is the larger root of $\lambda^2-(|b|+1)\lambda-a = 0$. Show that all points $(x_{1,0}, x_{2,0})$ outside of this domain tend to ∞ or $-\infty$ for $t\to\infty$ or $t\to-\infty$.

Problem 35. Consider the one-dimensional map $(t = 0,1,2,\ldots)$

$$x_{t+1} = \frac{x_t+2}{x_t+1}, \quad x_0 \ge 0.$$

Find the fixed points of the map. Let $x_0 = 0$. Does $\lim_{t\to\infty} x_t$ tend to a fixed point?

Problem 36. Consider the analytic functions $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$

$$f(x) = \sinh(x), \qquad g(x) = 2x^3.$$

Find

$$f^{-1}\circ g\circ f, \qquad g^{-1}\circ f\circ g.$$

Problem 37. Let $R > r > 0$. Consider the function $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^2$ (battle of the sexes)

$$f_1(x_1,x_2) = \begin{pmatrix} x_1 & 1-x_1\end{pmatrix}\begin{pmatrix} R & 0\\ 0 & r\end{pmatrix}\begin{pmatrix} x_2\\ 1-x_2\end{pmatrix} = ((R+r)x_2-r)x_1+r(1-x_2)$$

$$f_2(x_1,x_2) = \begin{pmatrix} x_1 & 1-x_1\end{pmatrix}\begin{pmatrix} r & 0\\ 0 & R\end{pmatrix}\begin{pmatrix} x_2\\ 1-x_2\end{pmatrix} = ((R+r)x_2-R)x_2+R(1-x_2).$$

Find the fixed points and study their stability.

Problem 38. Let $c_1, c_2 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Show that

$$\exp\left(c_1 \frac{d}{dx}\right) \exp(c_2 x) f(x) \equiv \exp(c_2 x) \exp\left(c_1 \frac{d}{dx}\right) \exp(c_1 c_2) f(x).$$

Problem 39. Let n be a non-negative integer. Let $k = 0, 1, \ldots, n$. Find the derivative

$$\frac{\partial^n}{\partial x^k \partial x^{n-k}} (x+y)^n.$$

Show that

$$\frac{\partial^n}{\partial x^k \partial x^{n-k}} (x+y)^n = n!.$$

Problem 40. The content (n-dimensional volume) bounded by a hypersphere of radius r is known to be

$$V_n = \frac{2r^n \pi^{n/2}}{n\Gamma(n/2)}$$

where Γ is the *gamma function.* Let $r = 1$. Show that

$$\lim_{n \to \infty} V_n = 0.$$

Problem 41. Let $n \geq 1$. Show that the vector space spanned by

$$x^n, yx^{n-1}, \ldots, y^{n-1}x, y^n$$

is $n+1$ dimensional.

Problem 42. Let $\epsilon, x \in \mathbb{R}$. Show that

$$\lim_{\epsilon \to 0} \frac{\sinh(\epsilon x)}{\sinh(\epsilon)} = x, \qquad \lim_{\epsilon \to 0} \frac{\sin(\epsilon x)}{\sin(\epsilon)} = x.$$

Problem 43. The *Hurwitz zeta-function* $\zeta_H(s,a)$ is defined by

$$\zeta_H(s,a) := \sum_{k=0}^{\infty} (n+a)^{-s}, \quad 0 < a \leq 1, \quad \Re(s) > 1.$$

(i) Show that the Hurwitz zeta-function can be written in the form

$$\zeta_H(s,a) = a^{-s} + \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt$$

where $\Gamma(s)$ is the Gamma function.
(ii) Show that

$$\frac{\partial}{\partial a}\zeta_H(s,a) = -s\zeta_H(s+1,a)$$

$$\frac{\partial}{\partial s}\zeta_H(s,a)\bigg|_{s=0} = \ln(\Gamma(a)) - \frac{1}{2}\ln(2\pi).$$

Problem 44. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Show that (differential identity)

$$\frac{d^n}{dx^n} f(x) = x^{-n} \frac{d^n}{d\epsilon^n} f(\epsilon x)\bigg|_{\epsilon=1}.$$

Problem 45. Find an analytic function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\infty) = 1, \quad f(-\infty) = 0$$

and

$$f(x_1) \le f(x_2) \ \text{whenever} \ x_1 \le x_2.$$

Find the derivative of the function f and determine the maxima of the function df/dx.

Problem 46. Let

$$\Theta(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that

$$\Theta(x_1 - x_3) - \Theta(x_2 - x_3) = \begin{cases} 1 & \text{for } x_1 > x_3 > x_2 \\ -1 & \text{for } x_2 > x_3 > x_1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 47. Let

$$f(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}.$$

Find $f(\theta)$ for $\theta = n\pi$ $(n \in \mathbb{Z})$ using the L'Hospital rule. Plot $f(\theta)$ as a function of θ.

Problem 48. Applying the L'Hospital rule to show that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Problem 49. Show that

$$\lim_{x \to 0} \frac{\sinh(x)}{x} = 1, \qquad \lim_{x \to 0} \frac{x}{\sinh(x)} = 1.$$

Problem 50. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{x\cosh(x) - \sinh(x)}{2x\sinh^2(x)}.$$

Find

$$\lim_{x \to 0} f(x), \qquad \lim_{x \to \infty} f(x).$$

Problem 51. Consider the functions

$$f(x) = \frac{x}{\sinh(x)}, \qquad g(x) = \frac{\sinh(x)}{x}.$$

Find

$$\lim_{x \to 0} f(x), \qquad \lim_{x \to 0} g(x).$$

Problem 52. Let $x \in \mathbb{R}$. Show that

$$\lim_{x \to \infty} \frac{1 - e^{-x}}{x} = 1.$$

Problem 53. (i) Show that

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0.$$

(ii) Show that

$$\lim_{x \to 1} \frac{x - 1}{\ln(x)} = 1.$$

Show that

$$\lim_{x\to 0}\frac{\cosh(x)-\cosh(2x)}{\sinh(x)}=0.$$

Problem 54. Let $\alpha\in\mathbb{R}$. Find

$$\lim_{\alpha\to 0}\frac{\sin(\alpha)\cos(\alpha)}{\alpha}.$$

Problem 55. Let $k>0$ and dimension 1/length. Find the extrema of the functions

$$\begin{aligned}
f_1(x) &= \ln(\cosh(kx))\\
f_2(x) &= \cosh(kx)+\alpha\sinh(kx), \quad -1\le\alpha\le+1\\
f_3(x) &= A\mathrm{sech}(kx)\equiv A\frac{1}{\cosh(kx)}\\
f_4(x) &= a\mathrm{sech}^2(kx)\equiv a\frac{\sinh^2(kx)}{\cosh^2(kx)}.
\end{aligned}$$

Problem 56. Consider the analytic function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\cos(x)+\sin(x)$.
(i) Find the critical points of f and the minima and maxima of the function.
(ii) Find the roots of f, i.e. solve $f(x)=0$.
(iii) Find the fixed points of f, i.e. solve $f(x)=x$.
(iv) Find the differential equation together with the initial conditions the function f satisfies.

Problem 57. The function $f:(0,1)\to\mathbb{R}$

$$f(x)=\frac{x-1/2}{x(x-1)}$$

is bijective. Note that

$$f(1/4)=4/3.\quad f(1/2)=0,\quad f(3/4)=-4/3.$$

Describe in detail the construction of the inverse function $f^{-1}:\mathbb{R}\to(0,1)$.

Problem 58. Consider the polynomial $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=x^3-\frac{3}{2}x^2+x-\frac{3}{2}.$$

Show that there is a root, i.e. a solution of $f(x) = 0$ in the interval $[1, 2]$. Apply the Newton method (say 3 steps) to find an approximate solution for the root. Apply the two initial condition $x_0 = 1$ and $x_0 = 2$. After finding this root how would you proceed to find the other two roots?

Problem 59. A real valued function f defined on an open subset G of a Banach space E is said to be Fréchet differentiable at a point $x \in G$ if there is $f'(x) \in E^*$ such that

$$\lim_{v \to 0} \frac{|f(x+v) - f(v) - \langle f'(x), v \rangle|}{\|v\|} = 0.$$

Then $f'(x)$ is called the Fréchet derivative of f at x. Let $E = \mathbb{R}$. Let $f(x) = x^3 + 2x^2 + 3x + 4$. Find the Fréchet derivative.

Problem 60. Consider the function

$$f(x) = \frac{\exp(-x^2/2) \int_0^x \exp(\tau^2/2) d\tau}{x}.$$

Show that

$$\lim_{x \to 0} f(x) = 1.$$

Problem 61. Consider the functions $f : \mathbb{R} \to [-1, 1]$, $g : \mathbb{R} \to [-1, 1]$

$$f(x) = \sin(x), \qquad g(x) = \cos(x).$$

(i) Find the minima and maxima of the function

$$h_1(x) = \max(f(x), g(x)).$$

Is the function h_1 differentiable?
(ii) Find the minima and maxima of the function

$$h_2(x) = \min(f(x), g(x)).$$

Is the function h_2 differentiable?

Chapter 4

Polynomial

4.1 Solved Problems

Problem 1. The *Chebyshev polynomials* are defined by

$$T_k(x) = \cos(k \arccos(x)), \quad k = 0, 1, \ldots \qquad x \in [-1, 1]$$

Thus the first six polynomials are

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x.
\end{aligned}$$

Find $1, x, x^2, x^3, x^4, x^5$ as functions of T_0, T_1, T_2, T_3, T_4, T_5.

Problem 2. Let $L_n(x)$, $H_n(x)$ be the *Laguerre polynomials* and *Hermite polynomials*, where $n = 0, 1, \ldots$. Let

$$L_n^{(\alpha)}(x) := \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha})$$

be the associated Laguerre polynomials with $\alpha > -1$ and $n = 0, 1, \ldots$. The Laguerre polynomials are recovered by setting $\alpha = 0$. We have

$$H_{2n}(x) = (-4)^n n! L_n^{(-1/2)}(x^2) \tag{1}$$

and the following addition formula for the associated Laguerre polynomials $L_n^\alpha(x)$

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_{n-k}^{(\alpha)}(x) L_k^{(\beta)}(y)\,. \tag{2}$$

(i) Find a new sum rule by inserting (2) into (1).
(ii) Consider the sum rule

$$\frac{1}{n!2^n} H_n(\sqrt{2}x) H_n(\sqrt{2}y) = \sum_{k=1}^{n} (-1)^k L_{n-k}^{(-1/2)}((x+y)^2) L_k^{(-1/2)}((x-y)^2). \tag{3}$$

Insert (1) into (3) to find a sum rule for Hermite polynomials.

Problem 3. The *Hermite polynomial* of degree n can be written as

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}.$$

Express x^n using the Hermite polynomials.

Problem 4. Given the differentiable function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = 4x(1-x).$$

(i) Find the *fixed points* of $f(x)$ and $f(f(x))$.
(ii) Find the *critical points* of $f(x)$ and $f(f(x))$.

Problem 5. Let $P(z)$ be a polynomial of degree $n \geq 2$ with distinct zeros $\zeta_1, \ldots, \zeta_n$. Show that

$$\sum_{j=1}^{n} \frac{1}{P'(\zeta_j)} = 0$$

where $'$ denotes the derivative, i.e. $P'(\zeta) \equiv dP(z = \zeta)/dz$.

Problem 6. Given a set of N real numbers $x_1, x_2, \ldots, x_N$. It is often useful to express the sum of the j powers

$$s_j = x_1^j + x_2^j + \cdots + x_N^j, \qquad j = 0, 1, 2, \ldots$$

in terms of the *elementary symmetric functions*

$$\sigma_1 = \sum_{i=1}^{N} x_i$$
$$\sigma_2 = \sum_{i<j}^{N} x_i x_j$$
$$\sigma_3 = \sum_{i<j<k}^{N} x_i x_j x_k$$
$$\vdots \quad \vdots$$
$$\sigma_N = x_1 x_2 \cdots x_N .$$

Consider the special case with three numbers x_1, x_2, x_3. Then the elementary symmetric functions are given by

$$\sigma_1 = x_1 + x_2 + x_3, \quad \sigma_2 = x_1 x_2 + x_1 x_3 + x_2 x_3, \quad \sigma_3 = x_1 x_2 x_3.$$

We know that the elementary symmetric functions are the coefficients (up to sign) of the polynomial with the roots x_1, x_2, x_3. In other words the values of x_1, x_2, x_3 each satisfy the polynomial equation

$$x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3 = 0. \tag{1}$$

Find a *recursion relation* for

$$s_j := x_1^j + x_2^j + x_3^j, \qquad j = 0, 1, 2, \ldots$$

and give the initial values s_0, s_1, s_2. Calculate s_3 and s_4.

Problem 7. The *dominant tidal potential* at position (r, ϕ, λ) due to the moon or sun is given by

$$U(\mathbf{r}) = \frac{GM^* r^2}{r^{*3}} P_2^0(\cos(\psi))$$

where M^* is the mass of the moon or sun located at (r^*, ϕ^*, λ^*). Moreover, ψ is the angle between mass M^* and the observation point at (r, ϕ, λ), where ϕ is the latitude and λ is the longitude. By the *spherical cosine theorem* we have

$$\cos(\psi) = \sin(\phi)\sin(\phi^*) + \cos(\phi)\cos(\phi^*)\cos(\lambda - \lambda^*).$$

The *Legendre polynomials* are defined as

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

with $n = 0, 1, 2, \ldots$ and the *associated Legendre polynomials* are defined as

$$P_n^m(x) := (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) = \frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n$$

with $P_n^0(x) = P_n(x)$ and $P_n^m = 0$ if $m > n$.
(i) Show that $U(\mathbf{r})$ can be written as

$$U(\mathbf{r}) = \frac{GM^* r^2}{r^{*3}} \Bigg(P_2^0(\sin\phi) P_2^0(\sin\phi^*) + \frac{1}{3} P_2^1(\sin\phi) P_2^1(\sin\phi^*) \cos(\lambda - \lambda^*)$$
$$+ \frac{1}{12} P_2^2(\sin\phi) P_2^2(\sin\phi^*) \cos(2(\lambda - \lambda^*)) \Bigg).$$

(ii) Give an interpretation (maxima and nodes) of the terms in the parenthesis.

Problem 8. Consider the cubic equation

$$y^3 + py + q = 0, \qquad p, q \in \mathbb{R}, \quad pq \neq 0. \tag{1}$$

Show that applying the nonlinear transformation

$$y(z) := z - \frac{p}{3z}$$

equation (1) can be reduced to

$$z^6 + qz^3 - \frac{p^3}{27} = 0$$

and with $u = z^3$ to a quadratic equation.

Problem 9. Find all integers c for which the cubic equation

$$x^3 - x + c = 0$$

has three integer roots.

Problem 10. Let Φ be an endomorphism of the space $\mathbb{C}^n[X]$ of polynomials of degree n with complex coefficients, which maps a polynomial $p(X)$ to the polynomial $p(X+1)$. Let Ψ be an endomorphism of the space $\mathbb{C}^n[X]$ which maps a polynomial $p(X)$ to $(1-X)^n p\left(\frac{X}{1-X}\right)$, which of course is also a polynomial. Show that we have *braid-like relation*

$$\Phi \circ \Psi \circ \Phi = \Psi \circ \Phi \circ \Psi.$$

Problem 11. Let $a \in \mathbb{R}$. Let r and s be the roots of the quadratic equation

$$x^2 + ax + \frac{a^2 - 1}{2} = 0.$$

Find $r^3 + s^3$ in terms of a, and express it as a polynomial in a with rational coefficients.

Problem 12. Consider the polynomial $p(x) = x^3 - x^2 + x - 2$. Does there exist a nontrivial polynomial $q(x)$ with real coefficients such that the degree of every term of the product $p(x)q(x)$ is a multiple of 3? If so, find one. If not, show there is none.

Problem 13. (i) Let n be a positive integer. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The *Bernstein polynomials* of degree n associated with the continuous function f are given by

$$B_n(f(x), x) := \frac{1}{(b-a)^n} \sum_{j=0}^{n} \binom{n}{j} (x-a)^j (b-x)^{n-j} f(x_j)$$

where

$$x_j = a + j\frac{b-a}{n}, \qquad j = 0, 1, \ldots, n.$$

Consider the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \sin(4x)$. Show that $B_2(f, x)$ is not a "good approximation" for f. Consider $x = \pi/8$.
(ii) The *Bernstein basis polynomials* are defined as

$$B_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}, \quad j = 0, 1, 2, \ldots, n, \quad x \in [0, 1].$$

Show that

$$\sum_{j=0}^{n} B_{n,j}(x) = 1.$$

Show that $B_{n,j}$ satisfies the recursion relations

$$\begin{aligned} B_{n,j}(x) &= (1-x)B_{n-1,j}(x) + xB_{n-1,j-1}(x), \qquad j = 1, 2, \ldots, n-1 \\ B_{n,0}(x) &= (1-x)B_{n-1,0}(x) \\ B_{n,n}(x) &= xB_{n-1,n-1}(x). \end{aligned}$$

Problem 14. A one-dimensional map f is called an *invariant* of a two-dimensional map g if

$$g(x, f(x)) = f(f(x)).$$

Let
$$f(x) = 2x^2 - 1.$$
Show that f is an invariant for
$$g(x, y) = y - 2x^2 + 2y^2 + d(1 + y - 2x^2).$$

Problem 15. Consider the functions $f(z) = z^3$ and $h(z) = z + 1/z$. Find a function p such that
$$h(f(z)) = p(h(z)). \tag{1}$$

Problem 16. Consider the map $f_c(z) = z^2 + c$, where $c \in \mathbb{C}$. Find all complex c-values where the map f_c has a fixed point z^* with $f_c'(z^*) = -1$.

Problem 17. Let $p(x, y)$ be a real polynomial. Show that if $p(x, y) = 0$ for infinitely many (x, y) on the unit circle $x^2 + y^2 = 1$, then $p(x, y) = 0$ on the unit circle.

Problem 18. Show that the equation
$$z^n = a \tag{1}$$
where n is a positive integer and a is any nonzero complex number, has exactly n roots. Hint. Set
$$a = \rho(\cos(\phi) + i\sin(\phi)). \tag{2}$$

Problem 19. Show that every polynomial $\alpha(x) \in C[x]$ of degree $m \geq 1$ has precisely m zeros over $\mathbb{C}$, where any zero of multiplicity is to be counted as n of the m zeros.

Problem 20. Show that if $r \in \mathbb{C}$ is a zero of any polynomial $\alpha(x)$ with real coefficients, then $\bar{r}$ is also a zero of $\alpha(x)$, where $\bar{r}$ denotes the complex conjugate of r.

Problem 21. Let A be a 2×2 matrix over the real numbers $\mathbb{R}$. The *trace* is defined as
$$\mathrm{tr}(A) = a_{11} + a_{22}.$$
It can be proved that the trace is the sum of the eigenvalues of A, i.e.
$$\mathrm{tr}(A) = \lambda_1 + \lambda_2.$$

Thus we have

$$\mathrm{tr}(A^2) = \lambda_1^2 + \lambda_2^2.$$

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find the eigenvalues using the two equations.

Problem 22. Find the zeros of the cubic polynomial

$$\alpha(x) := a_0 + a_1 x + a_2 x^2 + x^3 \tag{1}$$

over $\mathbb{C}$, where $a_0, a_1, a_2 \in \mathbb{R}$ and $a_0 \neq 0$.

Problem 23. Show that the zeros of

$$z^3 - 6z^2 + 11z - 6 = 0 \tag{1}$$

are given by 1, 2, and 3.

Problem 24. Find the zeros of the quartic polynomial

$$\alpha(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + x^4 \tag{1}$$

over $\mathbb{C}$ when $a_0 \neq 0$.

Problem 25. Find the zeros of

$$\alpha(x) = 35 - 16x - 4x^3 + x^4. \tag{1}$$

Problem 26. The variational equation of the *Lorenz model*

$$\frac{dX}{dt} = -\sigma X + \sigma Y \tag{1a}$$

$$\frac{dY}{dt} = -XZ + \tau X - Y \tag{1b}$$

$$\frac{dX}{dt} = XY - bZ \tag{1c}$$

is given by

$$\begin{pmatrix} dx_0/dt \\ dy_0/dt \\ dz_0/dt \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (\tau - Z) & -1 & -X \\ Y & X & -b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}. \tag{2}$$

(i) Show that Lorenz model possess the steady-state solution $X = Y = Z = 0$, representing the state of no convection.
(ii) Show that with this basic solution, the characteristic equation of the variational matrix is

$$(\lambda + b)(\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - \tau)) = 0. \tag{3}$$

(iii) Show that this equation has three real roots when $\tau > 0$; all are negative when $\tau < 1$, but one is positive when $\tau > 1$. The criterion for the onset of convection is therefore $\tau = 1$.
(iv) Show that when $\tau > 1$, system (1) possess two additional steady state solutions

$$X = Y = \pm\sqrt{b(\tau - 1)}, \qquad Z = \tau - 1. \tag{4}$$

(v) Show that for either of these solutions, the characteristic equation of the matrix in (2) is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (\tau + \sigma)b\lambda + 2\sigma b(\tau - 1) = 0. \tag{5}$$

(vi) Show that this equation possesses one real negative root and two complex conjugate roots when $\tau > 1$. Show that the complex conjugate roots are pure imaginary if the product of the coefficients of λ^2 and λ equals the constant term, or

$$\tau = \sigma(\sigma + b + 3)(\sigma - b - 1)^{-1}. \tag{6}$$

Problem 27. The variational equation of the *Lotka Volterra model*

$$\frac{du_1}{dt} = u_1 - u_1u_2, \qquad \frac{du_2}{dt} = -u_2 + u_1u_2 \tag{1}$$

is given by

$$\begin{pmatrix} dv_1/dt \\ dv_2/dt \end{pmatrix} = \begin{pmatrix} 1 - u_2 & -u_1 \\ u_2 & -1 + u_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2}$$

where $u_1 > 0$ and $u_2 > 0$.
(i) Show that (1) possess the steady-state solution $u_1 = u_2 = 1$.
(ii) Show that with this basic solution, the characteristic equation of the matrix in (2) is

$$\lambda^2 + 1 = 0. \tag{3}$$

(iii) Find the solution of the characteristic equation and discuss.

Problem 28. Show the following. Let $P(z)$ be a polynomial. Then either

1. $P(z)$ has a fixed point q with $P'(q) = 1$,

2. $P(z)$ has a fixed point q with $|P'(q)| > 1$.

Problem 29. Let

$$P = \{f_i(x), i = 1, \ldots, k\}, \quad \mathbf{x} = (x_1 \ldots x_n) \tag{1}$$

be a set of multivariate polynomials in the ring $Q[x_1 \ldots x_n]$, with a solution set

$$S = S(f_i \ldots f_k) = \{x \mid f_i(x) = 0, \forall i = 1, \ldots, k\} \tag{2}$$

All polynomials

$$u(x) = \sum_j g_j(x) f_j(x) \tag{3}$$

for arbitrary polynomials g_j will vanish in all points of S. The set of all u establishes a polynomial ideal

$$I = I(f_1 \cdots f_k) = \left\{ \sum_j g_j(x) f_j(x) \right\} \tag{4}$$

and classical algebra tells that S is invariant if we replace the set $\{f_1, \cdots, f_k\}$ by any other basis for the ideal $I(f_1, \cdots f_k)$.

The *Buchberger algorithm* allows one to transform the set of polynomials into a canonical basis of the same ideal, the Gröbner basis $GB = GB(I)$. For the purpose of the equation solving especially Gröbner bases computed under lexicographical term ordering are important. They allow one to determine the set S directly. If I has dimension zero (S is a finite set of isolated points, GB has in most cases the form

$$\begin{aligned}
g_1(x_1, x_k) &= x_1 + c_{1,m-1} x_k^{m-1} + c_{1,m-2} x_k^{m-2} + \cdots + c_{1,0} \\
g_2(x_2, x_k) &= x_2 + c_{2,m-1} x_k^{m-1} + c_{2,m-2} x_k^{m-2} + \cdots + c_{2,0} \\
&\cdots \\
g_{k-1}(x_{k-1}, x_k) &= x_{k-1} + c_{k-1,m-1} x_k^{m-1} + c_{k-1,m-2} x_k^{m-2} + \cdots + c_{k-1,0} \\
g_k(x_k) &= x_k^m + c_{k,m-1} x_k^{m-1} + c_{k,m-2} x_k^{m-2} + \cdots + c_{k,0}.
\end{aligned}$$

A basis in this form has the elimination property: the variable dependency has been reduced to a triangular form, just as with a Gaussian elimination in the linear case. The last polynomial is univariate in x_k. It can be solved with usual algebraic or numeric techniques; its zero $\bar{x}_k$ then are propagated into the remaining polynomials, which then immediately allow one to determine the corresponding coordinates $(\bar{x}_1, \ldots, \bar{x}_{k-1})$.

Consider the system

$$\{y^2 - 6y, xy, \quad 2x^2 - 3y - 6x + 18, \quad 6z - y + 2x\}. \tag{4}$$

(i) Show that that this system has for $\{x, y, z\}$ the lexicographical Gröbner basis

$$\{g_1(x,z) = x - z^2 + 2z - 1, \quad g_2(y,z) = y - 2z^2 - 2z - 2, \quad g_3(z) = z^3 - 1\}. \tag{5}$$

(iii) Show that the roots of the third polynomial are given by

$$\left\{ z = 1, \quad z = \frac{1 - \sqrt{3}i}{2}, \quad z = \frac{\sqrt{3}i - 1}{2} \right\}. \tag{6}$$

If we propagate one of them into the basis, we generate univariate polynomials of degree one which can be solved immediately; e.g. Selecting $\frac{1-\sqrt{3}i}{2}$ for z the basis reduces to

$$\left\{ x = \frac{3\sqrt{3}i + 3}{2}, y, 0 \right\} \tag{7}$$

such that the final solution for this branch is

$$\left\{ x = \frac{3\sqrt{3}i + 3}{2}, \quad y = 0, \quad z = \frac{1 - \sqrt{3}i}{2} \right\}. \tag{8}$$

For zero dimensional problems the last polynomial will always be univariate. However, in degenerate cases the other polynomials can contain their leading variable in a higher degree, then containing more mixed terms with the following variables. And there can be additional polynomials with mixed leading terms (of lower degree) imposing some restrictions. But the varible dependency pattern will remain triangular (evenutally with more than k rows). There is a special algorithm for decomposing such ideals using ideal quotients.

Problem 30. Let S_n be the *symmetric group*. The symmetric group S_n acts naturally on polynomials in n variables. For a polynomial p in n variables, define the symmetrized polynomial associated to p, $\mathrm{Sym}_n(p)$, by

$$\mathrm{Sym}_n(p) := \sum_{\sigma \in S_n} \sigma(p).$$

Let $n = 2$ and

$$p(x_1, x_2) = x_1^2 x_2 + 2x_2.$$

Find $\mathrm{Sym}_2(p)$.

Problem 31. Consider the system of equations

$$\begin{aligned} z_1 + z_2 + \cdots + z_{n-1} + z_n &= 0 \\ z_1 z_2 + z_2 z_3 + \cdots + z_{n-1} z_n + z_n z_1 &= 0 \\ &\vdots \\ z_1 z_2 \cdots z_{n-1} + z_2 z_3 \cdots z_n + \cdots + z_{n-1} z_n \cdots z_{n-3} + z_n z_1 \cdots z_{n-2} &= 0 \\ z_1 z_2 \cdots z_n &= 1. \end{aligned}$$

Find the solutions for the case $n = 2$ and $n = 3$. This system of equations arise as follows. Let p be a prime number. A vector $\mathbf{x} = (x_0, x_1, \ldots, x_{p-1}) \in \mathbb{C}^p$ viewed as a function $\mathbb{Z}/p \to \mathbb{C}$ has discrete Fourier transform $\hat{\mathbf{x}} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{p-1})$, where

$$\hat{x}_j = \sum_{k=0}^{p-1} \omega^{jk} x_k, \qquad \omega := e^{2\pi i/p}.$$

The vector $\mathbf{x}$ is called equimodular if all its coordinates have the same absolute value, and $\mathbf{x}$ is called bi-equimodular if both $\mathbf{x}$ and $\hat{\mathbf{x}}$ are equimodular. The question is: Which vectors are bi-equimodular?

Problem 32. Let $\alpha \in \mathbb{R}$. Find the roots of the characteristic equation

$$\lambda^6 - 2\cos(3\alpha)\lambda^3 + 1 = 0.$$

Problem 33. Show that the n-th order polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

which goes exactly through $n+1$ data points is unique.

Problem 34. Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial that satisfies

$$p(1-x) + 2p(x) = 3x$$

for all $x \in \mathbb{R}$. Find the values of $p(0)$ and $p(1)$. Give an example of a polynomial that satisfies this condition.

Problem 35. (i) Let $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$. Find the solutions of the system of equations

$$x_1 - \frac{1}{(x_1 - x_2)^2} = 0, \qquad x_2 + \frac{1}{(x_1 - x_2)^2} = 0.$$

(ii) Let $x_1, x_2, x_3 \in \mathbb{R}$ and $x_1 \neq x_2$, $x_1 \neq x_3$, $x_2 \neq x_3$. Find the solutions of the system of equations

$$\begin{aligned} x_1 - \frac{1}{(x_1-x_2)^2} - \frac{1}{(x_1-x_3)^2} &= 0 \\ x_2 + \frac{1}{(x_1-x_2)^2} - \frac{1}{(x_2-x_3)^2} &= 0 \\ x_3 + \frac{1}{(x_1-x_3)^2} + \frac{1}{(x_2-x_3)^2} &= 0. \end{aligned}$$

Problem 36. Let $z, w \in \mathbb{C}$. Find the solution of the system of equations

$$|z|^2 + |w|^2 = 1, \qquad z^2 + w^3 = 0.$$

Problem 37. Consider the two polynomials

$$p_1(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad p_2(x) = b_0 + b_1 x + \cdots + b_m x^m$$

where $n = \deg(p_1)$ and $m = \deg(p_2)$. Assume that $n > m$. Let $r(x) = p_2(x)/p_1(x)$. We expand $r(x)$ in powers of $1/x$, i.e.

$$r(x) = \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots$$

From the coefficients c_1, c_2, $\dots$, c_{2n-1} we can form an $n \times n$ *Hankel matrix*

$$H_n = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}.$$

The determinant of this matrix is proportional to the *resultant* of the two polynomials. If the resultant vanishes, then the two polynomials have a non-trivial greates common divisor. Apply this theorem to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \qquad p_2(x) = x^2 + 4x + 3.$$

Problem 38. Let p_1 and p_2 two polynomials with $n = \text{degree}(p_1)$, $m = \text{degree}(p_2)$ and $n > m$. We expand the rational function

$$r(x) = \frac{p_2(x)}{p_1(x)}$$

with respect to powers of $1/x$, i.e.,

$$r(x) = d_1 x^{-1} + d_2 x^{-2} + \cdots$$

The coefficients $d_1, d_2, \ldots, d_{2n-1}$ are inserted into the $n \times n$ Hankel matrix

$$H_n = \begin{pmatrix} d_1 & d_2 & \ldots & d_n \\ d_2 & d_3 & \ldots & d_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ d_n & d_{n+1} & \ldots & d_{2n-1} \end{pmatrix}.$$

A *Hankel matrix* is a diagonal matrix in which all the elements are the same along any diagonal that slopes from northeast to southwest. If the determinant of this matrix is zero, then the two polynomials have a non-trival common divisor. Apply this algorithm to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \qquad p_2(x) = x^2 + 6x + 8$$

to test whether they have a common non-trivial divisor.

Problem 39. Let p be a polynomial with coefficients in $\mathbb{R}$. If the equation $p(x) = 0$ has repeated roots, then $p(x)$ and $dp(x)/dx$ have a highest common factor. Apply this to the polynomial

$$p(x) = 32x^4 - 64x^3 + 24x^2 + 8x - 3.$$

Problem 40. Let p be a polynomial with real coefficients. The equation $p(x) = 0$ cannot have more positive roots than there are changes of sign from $+$ to $-$ and from $-$ to $+$ in the coefficients of the polynomial $p(x)$, or more negative roots than there are changes of sign in $p(-x)$ (Descartes rule of signs). Apply the rule to the polynomiol

$$p(x) = x^6 + 7x^3 + x - 2 = 0.$$

Problem 41. The Liouville-Riemann definition for the fractional integral operator D_x^{-q} is given by

$$D_x^{-q} f(x) := \frac{1}{\Gamma(q)} \int_0^x (x-y)^{q-1} f(y) dy, \quad q > 0.$$

The fractional differential operator D_x^ν for $\nu > 0$ is given by the definition

$$D_x^\nu f(x) := \frac{d^n}{dx^n} (D_x^{\nu - n} f(x)), \qquad \nu - n < 0.$$

Let $f(x) = x^2$. Find $D_x^{-q} f(x)$.

Problem 42. Let $n \in \mathbb{N}$. Show by induction that $x^n - y^n$ is divisible without remainder by $x - y$ for all values of n. We have

$$x^{n+1} - y^{n+1} \equiv x(x^n - y^n) + y^n(x - y).$$

Problem 43. Consider the polynomial

$$P_n(x) = \sum_{j=0}^{n} (-1)^j a_j x^{n-j}, \qquad a_0 = 1.$$

The homogeneous product sums symmetric functions $h_k(x_1, \ldots, x_n)$ of the zeros of this polynomial are defined as follows

$$\prod_{j=1}^{n} (1 - x_j x) = (1 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots)^{-1}.$$

Show that the first few sums are given explicitly by

$$\begin{aligned} h_1(x_1, \ldots, x_n) &= \sum_{j=1}^{n} x_j \\ h_2(x_1, \ldots, x_n) &= \sum_{j=1}^{n} x_j^2 + \sum_{j<k}^{n} x_j x_k \\ h_3(x_1, \ldots, x_n) &= \sum_{j=1}^{n} x_j^3 + \sum_{j \neq k}^{n} + \sum_{j<k<\ell}^{n} x_j x_k x_\ell. \end{aligned}$$

Problem 44. Consider the quintic equation

$$p(x) = x^5 - 5x^3 + 5x - 5 = 0.$$

Is p irreducible over the rational numbers? What is the Galois group of p? Look for solutions of the form $r + 1/r$.

Problem 45. The *Bernoulli polynomials* $B_n(x)$ ($n = 0, 1, \ldots$) can be defined recursively by

$$\frac{dB_n(x)}{dx} = nB_{n-1}, \quad n = 1, 2, \ldots$$

with $B_0(x) = 1$ and the condition

$$\int_0^1 B_n(x)dx = 0, \quad n \geq 1.$$

The *Bernoulli numbers* B_n are defined by $B_n := B_n(x = 0)$.
(i) Find the first four Bernoulli polynomials.
(ii) Show that

$$B_m(x) = \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^m.$$

(iii) Show that

$$B_n(1-x) = (-1)^n B_n(x).$$

Problem 46. Consider the operators

$$D_1 = x, \quad D_2 = \frac{d}{dx}, \quad D_3 = x\frac{d}{dx}$$

which apply to the function of the *Bargmann space*

$$B := \{\, f(n) = \frac{x^n}{\sqrt{n!}},\ n \in \mathbb{N}_0,\ x \in \mathbb{R}\,\}.$$

Show that

$$D_1 f(n) = \sqrt{n+1} f(n+1), \quad D_2 f(n) = \sqrt{n} f(n-1), \quad D_3 f(n) = n f(n).$$

Find the commutators $[D_1, D_2]$, $[D_1, D_3]$, $[D_2, D_3]$.

Problem 47. Express the polynomial

$$p(x) = 7x_1^2 + 6x_2^2 + 5x_3^2 - 4x_1x_2 - 4x_2x_3 + 14x_1 - 8x_2 + 10x_3 + 6$$

in matrix form

$$p(x) = (\, x_1 \quad x_2 \quad x_3 \,)\, A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathbf{v}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + c$$

Find the eigenvalues and normalized eigenvectors of the 3×3 matrix.

Problem 48. Find the polynomials generated by

$$\frac{dp_{j+1}(x)}{dx} = (j+1)p_j(x), \quad j = 0, 1, 2, \ldots$$

with $p_0(x) = 1$. The constants of integration we set to 0.

Problem 49. The complete *Bell polynomials* $B_j(x_1, x_2, \ldots, x_j)$ are given by the exponential generating function

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \ldots, x_n) \frac{t^n}{n!}. \tag{1}$$

Taking the n-th derivative with respect to t we obtain

$$\frac{d^n}{dt^n} \exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right)\Bigg|_{t=0} = B_n(x_1, \ldots, x_n) \tag{2}$$

with $B_0 = 1$. Find the first four Bell polynomials.

Programming Problem

Problem 50. Consider the polynomials $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \mathbb{R} \to \mathbb{R}$

$$f_1(x) = x^2 + 2x + 1, \quad f_2(x) = 2x.$$

Apply Maxima to find

$$h_1(x) = f_1(f_2(x)), \quad h_2(x) = f_2(f_1(x))$$

and $C = f_1 - f_2$.

4.2 Supplementary Problems

Problem 1. Construct a polynomial

$$p(x) = x^2 + ax + b$$

that admits the roots

$$\frac{1}{6}(\sqrt{3} + 3), \qquad -\frac{1}{6}(\sqrt{3} - 3)$$

and the fixed points

$$\frac{1}{6}(\sqrt{30} + 6), \qquad -\frac{1}{6}(\sqrt{30} - 6).$$

Problem 2. Let $a, b \in \mathbb{R}$. Consider the *quartic equation*

$$(\lambda - 1)^2(\lambda - b^2)^2 - \lambda^2 a^2 (1 - b^2)^2 = 0$$

Show that the roots are given by

$$\lambda_1(a,b) = \frac{1}{2}(1 - a + (1+a)b^2 + (T_+)^{1/2})$$
$$\lambda_2(a,b) = \frac{1}{2}(1 - a + (1+a)b^2 - (T_+)^{1/2})$$
$$\lambda_3(a,b) = \frac{1}{2}(1 + a + (1-a)b^2 + (T_-)^{1/2})$$
$$\lambda_4(a,b) = \frac{1}{2}(1 + a + (1-a)b^2 - (T_-)^{1/2})$$

where $T_\pm = (1+a^2)(1-b^2)^2 \pm 2a(b^4 - 1)$.

Problem 3. Let n be a positive integer. Show that $x^n - y^2$ has $x - y$ as a factor for all n.

Problem 4. Let $s = 1/2, 1, 3/2, 2, \ldots$ be the spin values. The *Brillouin function* is defined as

$$B_s(x) := \frac{2s+1}{2s} \coth\left(\frac{2s+1}{2s}x\right) - \frac{1}{2s} \coth\left(\frac{1}{2s}x\right)$$

Find $B_s(x=0)$ applying L'Hospital.

Problem 5. Let $n \in \mathbb{N}_0$ and $x, m \in \mathbb{R}$. The *Sonine polynomials* are defined by

$$S_m^n(x) = \sum_{j=0}^{n} (-1)^j \frac{\Gamma(m+n+1)x^j}{\Gamma(m+j+1)(n-j)!j!}$$

where Γ denotes the gamma function. Show that

$$S_m^0(x) = 1, \qquad S_m^1(x) = m + 1 - x.$$

Show that the Sonine polynomials satisfy

$$\int_0^\infty e^{-x} x^m S_m^j(x) S_m^k(x) dx = \frac{\Gamma(m+j+1)}{j!} = \delta_{j,k}.$$

Problem 6. (i) Show that the cubic roots of unity $z^3 = 1$ are

$$1, \qquad w = -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \qquad w^2 = -\frac{1}{2} - \frac{\sqrt{3}i}{2}. \tag{2}$$

(ii) Show that the zeros of $z^3 + 1 = 0$ are given by

$$-1, \qquad \frac{1}{2} + \frac{i}{2}\sqrt{3}, \qquad \frac{1}{2} - \frac{i}{2}\sqrt{3}. \tag{1}$$

(iii) Show that the set

$$S = \{\, \omega_1 = -\frac{1}{2} + \frac{i}{2}\sqrt{3}, \quad \omega_2 = -\frac{1}{2} - \frac{i}{2}\sqrt{3}, \quad \omega_3 = 1 \,\}$$

of the cubic roots of 1, forms an abelian group with respect to multiplication on the set of complex numbers $\mathbb{C}$, i.e. show that

$$\omega_1\omega_2 = \omega_3, \quad \omega_2\omega_1 = \omega_3, \quad \omega_1\omega_3 = \omega_1,$$

$$\omega_3\omega_1 = \omega_1, \quad \omega_2\omega_3 = \omega_2, \quad \omega_3\omega_2 = \omega_2. \tag{1}$$

Obviously, the neutral element is ω_3. From (1) we see that each element has an inverse (which is unique), i.e.

$$(\omega_1)^{-1} = \omega_2, \quad (\omega_2)^{-1} = \omega_1, \quad (\omega_3)^{-1} = \omega_3.$$

The associative law is true for all complex numbers.

Problem 7. Let $a, b \in \mathbb{R}$ and z be denote a root of the quadratic equation

$$z^2 + az + b = 0. \tag{1}$$

Show that the sequence of powers of z, $\{z^n\}$, $n \geq 2$, satisfies the linear difference equation

$$w_n + aw_{n-1} + bw_{n-2} = 0, \qquad n \geq 2. \tag{2}$$

Set $w_n = x^n$. Then $w_{n-1} = x^{n-1}$.

Problem 8. A real number is said to be *algebraic* if it is a zero of a polynomial

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$$

where the coefficients c_j are integers. The height of such polynomials is defined as the positive integer number

$$h = n + |c_n| + |c_{n-1}| + \cdots + |c_0|.$$

Show that there are only finitely many polynomials of height h. Show that the set of all algebraic real numbers is enumerable.

Chapter 5

Equations

5.1 Solved Problems

Problem 1. Solve the equation

$$\frac{1}{3} = \frac{1}{x} + \frac{1}{4}.$$

Problem 2. Let x_1, x_2 be positive real numbers. Consider the equation

$$x_2 \sin(\theta) = x_1 \cos(\theta).$$

Find $\sin(\theta)$.

Problem 3. Let L be a given positive real number. Solve the system of two coupled nonlinear equations

$$1 = x^2 L + 2x^2 y + Lx^2 y^2$$
$$0 = -x^2 + 2x^2 y + Lx^2 y^2.$$

Problem 4. Let $c, y \in \mathbb{R}$. Solve the quadratic equation

$$x = cy + yx - x^2$$

with respect to x.

Problem 5. Consider a triangle in the plane. Consider a point within the triangle. We draw lines from this point to the three vertices, thereby dividing the triangle into three triangles of area A_1, A_2, A_3. The sides of the triangle are designated by the same number as the opposite vertex. The areas are identified by the number of the adjacent side. The quantities L_j $(j = 1, 2, 3)$

$$L_1 = \frac{A_1}{A}, \qquad L_2 = \frac{A_2}{A}, \qquad L_3 = \frac{A_3}{A}$$

where A is the area of the original triangle, are defined to be the triangular coordinates. Show that

$$L_1 + L_2 + L_3 = 1.$$

The relationship between the Cartesian coordinates x, y which are the coordinates of the points in the elements, and the triangular coordinates L_1, L_2, L_3 are

$$x = L_1x_1 + L_2x_2 + L_3x_3, \qquad y = L_1y_1 + L_2y_2 + L_3y_3$$

where x_j, y_j $(j = 1, 2, 3)$ are the coordinates of the nodes. The triangular coordinates can be expressed in terms of the known locations of the vertices, i.e.

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}.$$

Find L_1, L_2, L_3 as function of x_j, y_j, x, y.

Problem 6. Let $x, y \neq 0$. Find all solutions of the equation

$$\frac{4x^2y^2}{(x^2+y^2)^2} = 1.$$

Problem 7. Solve the equation

$$71x^2 = 133 \pmod{11}.$$

Problem 8. Let $x, y \in \mathbb{N}$. Find all solutions of

$$16x + 7y = 601.$$

Problem 9. Let $x, y \in \mathbb{Z}$. Find all solutions of

$$18x + 12y = 4.$$

Problem 10. Consider the two hyperplane ($n \geq 1$)

$$x_1 + x_2 + \cdots + x_n = 2, \qquad x_1 + x_2 + \cdots + x_n = -2.$$

The hyperplanes do no intersect. Find the shortest distance between the hyperplanes. First consider the cases $n = 1$ and $n = 2$ and then the general case. What happens if $n \to \infty$?

Problem 11. Let d be a positive distance, v, V be velocities $v \neq V$ and T_1, T_2 time-intervals. Assume that

$$\frac{d}{V+v} = T_1, \qquad \frac{d}{V-v} = T_2.$$

Find d/V.

Problem 12. Solve the quadratic equation

$$x^2 - ix - (1+i) = 0.$$

Problem 13. Let $z \in \mathbb{C}$. Solve

$$\left(\frac{z+1}{z-1}\right) = i.$$

Problem 14. Solve the system of nonlinear equations

$$\begin{aligned} x^2 - (y-z)^2 &= a^2 \\ y^2 - (z-x)^2 &= b^2 \\ z^2 - (x-y)^2 &= c^2. \end{aligned}$$

Problem 15. Show that the quartic equation

$$x^4 + qx^2 + rx + s = 0$$

can be written as

$$(x^2 - ex + f)(x^2 + ex + g) = 0$$

where e^2 is the root of the cubic equation

$$z^3 + 2qz^2 + (q^2 - 4s)z - r^2 = 0.$$

Problem 16. Let

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be elements of $\mathbb{C}^2$. Solve the equation $\mathbf{z}^*\mathbf{w} = \mathbf{w}^*\mathbf{z}$.

Problem 17. A special set of coordinates on S^n called spheroconical (or elliptic spherical) coordinates are defined as follows: For a given set of real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}$ and nonzero $x_1, \ldots, x_{n+1}$ the coordinates λ_j $(j = 1, \ldots, n)$ are the solutions of the equation

$$\sum_{j=1}^{n+1} \frac{x_j^2}{\lambda - \alpha_j}.$$

Find the solutions for $n = 2$.

Problem 18. Find all solutions of the system of equations

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha)\cos(\beta) \\ \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\sin(\beta) \end{pmatrix}.$$

Problem 19. Let $\phi \in [0, 2\pi)$. Solve the *cubic equation*

$$4x^3 - 3x - \cos(\phi) = 0$$

over the real numbers.

Problem 20. Let $\epsilon > 0$. Find the solution of the coupled two non-linear equations

$$\epsilon x^2 + x - y - 1 = 0, \qquad \epsilon y^2 + x - y - 1 = 0.$$

Study $\epsilon \to 0$ for these solutions.

Problem 21. Consider the six 3×3 matrices

$$X_{12} = \begin{pmatrix} a_{12} & b_{12} & 0 \\ c_{12} & d_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X'_{12} = \begin{pmatrix} a'_{12} & b'_{12} & 0 \\ c'_{12} & d'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{13} = \begin{pmatrix} a_{13} & 0 & b_{13} \\ 0 & 1 & 0 \\ c_{13} & 0 & d_{13} \end{pmatrix}, \quad X'_{13} = \begin{pmatrix} a'_{13} & 0 & b'_{13} \\ 0 & 1 & 0 \\ c'_{13} & 0 & d'_{13} \end{pmatrix},$$

$$X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{23} & b_{23} \\ 0 & c_{23} & d_{23} \end{pmatrix}, \quad X'_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a'_{23} & b'_{23} \\ 0 & c'_{23} & d'_{23} \end{pmatrix}.$$

Find the 9 conditions on the entries such that (local Yang-Baxter equation)

$$X_{12}X_{13}X_{23} = X'_{23}X'_{13}X'_{12}.$$

Find solutions of these 9 equations for the 24 unkowns $a_{12}, \ldots, d'_{23}$.

Problem 22. Solve the equations

$$2\arcsin(x) + \arcsin(2x) - \pi/2 = 0$$

$$2\arcsin(x) - \arccos(3x) = 0$$

$$\arccos(x) - \arctan(x) = 0$$

$$\arccos(2x^2 - 4x - 2) - 2\arcsin(x) = 0$$

$$2\arctan(x) - \arctan\left(\frac{2x}{1-x^2}\right) = 0.$$

Problem 23. Let $x \in \mathbb{R}$. Find all values of x that satisfy

$$|5x + 7| = 3.$$

Then find the smallest and largest values.

Problem 24. Let $\theta \in [0, 2\pi)$. Can one find $x \in \mathbb{R}$ such that

$$\sin(\theta) = \frac{e^{-x/2}}{\sqrt{1+e^{-x}}} \; ?$$

Problem 25. Let $x < 1$. Find a solution of the equation

$$f^2(x) - 2f(x) + x = 0.$$

Problem 26. What are the conditions on $c_1 > 0$ and $c_2 > 0$ such that the system of equations

$$x_1 + x_2 = c_1, \qquad x_1x_2 = c_2$$

has real solutions?

Problem 27. Let m, n be positive integers. Find all solution of the system of equations

$$2mn + (m^2 + n^2) = (m^2 - n^2)^2, \quad m - n = 1.$$

Problem 28. Let $x \in \mathbb{R}$.
(i) Find all solutions of $2x = |x| + 1$.
(ii) Find all solutions of $2x = -|x| + 1$.

Problem 29. Let $x \in \mathbb{Z}$. Solve the equation

$$x^2 - 2x + 2 = 0 \bmod 5.$$

Problem 30. Let $a > b > 0$. Find the points of intersections of the two ellipses

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \qquad \frac{x_1^2}{b^2} + \frac{x_2^2}{a^2} = 1.$$

Problem 31. Let $r_1 \geq 0$, $r_2 \geq 0$, $r_3 \geq 0$. Find the solutions of the system of equations

$$r_1 + r_2 + r_3 = 1, \qquad r_1^2 + r_2^2 + r_3^2 = 1.$$

Problem 32. Consider the cubic equation

$$p(x) = x^3 - 6x^2 + 11x - 6 = 0.$$

Let x_1, x_2, x_3 be the roots. Given the equations

$$x_1 + x_2 + x_3 = -a_1, \quad x_1x_2 + x_2x_3 + x_3x_1 = a_2, \quad x_1x_2x_3 = -a_3$$

Find a_1, a_2, a_3.

Problem 33. Let $z \in \mathbb{C}$. Find all solutions of

$$z + z^* + zz^* = 0.$$

Set $z = re^{i\phi}$ with $r \geq 0$ and $\phi \in [0, 2\pi)$. An alternative would be setting $z = x + iy$ with $x, y \in \mathbb{R}$.

Problem 34. We want to find the positive root of $f(x) = 0$ with

$$f(x) = x^3 - x^2 - x - 1.$$

We write $x^3 - x^2 - x - 1 = 0$ as $x = 1 + 1/x + 1/x^2$ and set

$$x_{t+1} = 1 + 1/x_t + 1/x_t^2, \qquad t = 0, 1, \ldots$$

with $x_0 = 1$. Do we find the positive root?

Problem 35. Find all 2×2 invertible matrices S over $\mathbb{R}$ with $\det(S) = 1$ such that

$$S \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} S \;\; S \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus we have to solve the three equations

$$s_{21} = 0, \quad s_{11} + s_{12} = s_{22}, \quad s_{11}s_{22} = 1.$$

Problem 36. Let $z \in \mathbb{C}$. Find all solutions of

$$z + z^* + zz^* = 0.$$

Set $z = re^{i\phi}$ with $r \geq 0$.

Problem 37. Let a be a real constant. Solve the two equations

$$-a^3 + x_3 - 3ax_2 + 3a^2x_1 = 0, \qquad x_2 - 2ax_1 + a^2 = 0$$

with respect to x_1, x_2, x_3.

Programming Problems

Problem 38. Let $\epsilon \in \mathbb{R}$. Consider the quadratic equation

$$x^2 - \epsilon x + \epsilon - 1 = 0.$$

Find the roots $x_1(\epsilon)$, $x_2(\epsilon)$. Then find the mimimum of $x_1^2 + x_2^2$ with respect to ϵ.

A SymbolicC++ program to solve this problem is:

```
// quadratic.cpp

#include <iostream>
#include "symbolicc++.h"
using namespace std;
```

```
int main(void)
{
 Symbolic x("x"), eps("eps"), f = 0;
 Equations soln = solve((x^2)-eps*x+eps-1==0,x);
 cout << "Solutions: " << endl << soln << endl;
 Equations::iterator i;
 for(i=soln.begin();i!=soln.end();i++) f += (i->rhs^2);
 cout << "f(eps) = " << f << endl;
 Equations min = solve(df(f,eps)==0,eps);
 for(i=min.begin();i!=min.end();i++)
  if(double(df(f,eps,2)[*i]) > 0)
   cout << "Minimum at " <<  *i << endl;
 return 0;
}
/*
Solutions:
[ x == eps-1,
  x == 1 ]
f(eps) = eps^(2)-2*eps+2
Minimum at eps == 1
*/
```

5.2 Supplementary Problems

Problem 1. Let $x, y \in \mathbb{R}$. Find all solutions of

$$e^{-x-y} = e^{-x} + e^{-y}.$$

Problem 2. Solve the system of nonlinear equations

$$x_1 x_2 = 1, \quad x_2 x_3 = 2, \quad x_3 x_1 = 3.$$

Extend to the general case

$$x_1 x_2 = c_1, \quad x_2 x_3 = c_2, \ldots, x_{n-1} x_n = c_{n-1}, \quad x_n x_1 = c_n$$

where c_j $(j = 1, \ldots, n)$ are positive constants.

Problem 3. Find the solution of the equation

$$e^{-x} - 1 = 2(\sqrt{2} + 1).$$

Problem 4. Let n be a positive integer. Solve the nonlinear equation

$$\frac{1-ne^{-x}}{1+ne^{-x}} = \frac{x}{2}$$

for $n = 1, 2, 3, 4, 5$.

Problem 5. Let $c_1 > 0$ and $c_2 > 0$. What is the condition on c_1, c_2 such that

$$x_1 + x_2 = c_1, \qquad x_1 x_2 = c_2$$

has a real solution?

Problem 6. Let n be a positive integer. Solve

$$\frac{1-ne^{-x}}{1+ne^{-x}} = \frac{x}{2}$$

for $n = 1, 2, 3, 4, 5$.

Problem 7. Let $x > 0$. Show that solution of the equation

$$x(1 + x^{1/2}) = (1 - x^{1/2})$$

is given by

$$x = \frac{1}{9}((17 + 3\sqrt{33})^{1/3} + (17 - 3\sqrt{33})^{1/3} - 1)^2 \approx 0.2955977\ldots$$

Problem 8. Let $x_1, x_2, x_3 \in \mathbb{R}$. Find solutions of the equation

$$x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 = x_1 x_2 x_3$$

with $x_1 \neq 0$, $x_2 \neq 0$, $x_3 \neq 0$. Note that

$$x_1 = x_2 = x_3 = 1/3, \quad x_1 = 1/3, x_2 = x_3 = -1/3, \quad x_1 = x_2 = -1/3, x_3 = 1/3, \quad x_1 = x_3 = -1/3, x_2 = 1/3$$

are solutions.

Problem 9. (i) Consider the function

$$f(z, \bar{z}) = z + \bar{z} + z\bar{z}.$$

Find all solutions of $f(z, \bar{z}) = 0$.
(ii) Let A be an $n \times n$ matrix over $\mathbb{C}$. Find all solutions of

$$A + A^* + AA^* = 0_n.$$

Find all 2×2 matrices A such that

$$A + A^* + AA^* = I_2.$$

Problem 10. (i) Let $\phi, \theta \in [0, 2\pi)$. Consider the equation

$$e^{i\phi} \sin(\theta) = \frac{2\eta}{|\eta|^2 + 1}.$$

Let $\eta = 0, 1, -1, i, -i$. Find ϕ and θ.
(ii) Let $\theta \in [0, 2\pi)$. Consider the equation

$$\cos(\theta) = \frac{|\eta|^2 - 1}{|\eta|^2 + 1}.$$

Let $\eta = 0, 1, -1, i, -i$. Find ϕ and θ.

Problem 11. Show that roots of the polynomial $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^4 - x^3 - 10x^2 - x + 1$$

are given by

$$x_1 = \frac{1}{2}(\sqrt{5} - 3), \quad x_2 = \frac{1}{2}(-\sqrt{5} - 3), \quad x_3 = 2 + \sqrt{3}, \quad x_4 = 2 - \sqrt{3}.$$

Problem 12. Find the real root of the polynomial $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^3 + x - 1.$$

The real root lie in the interval $[0, 1]$. Note that $f(0) = -1$ and $f(1) = 1$.

Problem 13. Find solution solutions of the equation (Scherk's surface)

$$\cos(x_2)e^{x_3} = \cos(x_1).$$

For example $(x_1, x_2, x_3) = (0, 0, 0)$ is a solution and $(x_1, x_2, x_3) = (\pi/3, \pi/3, 0)$ is a solution.

Problem 14. Let $x > 0$. Find all solutions of

$$x = \sqrt{1 + \sqrt{2 + x}}.$$

First show that $x^4 - 2x^2 - x - 1 = 0$.

Problem 15. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z - 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Problem 16. Are there solutions of the equation ($z \in \mathbb{C}$)

$$\sin(z) = e^z \quad \text{and} \quad \cos(z) = e^z\,?$$

Problem 17. (i) Show that $\cos(\pi/7)$ is a real root of

$$8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0.$$

(ii) Show that

$$\cos(\pi/7)\cos(2\pi/7)\cos(4\pi/7) = -1/8.$$

Problem 18. Draw the curve in the plane given by

$$\sinh(x) = \exp(-2y).$$

Note that if $y = 0$, then $\sinh(2x) = 1$ and therefore $2x = \text{arcsinh}(1)$.

Problem 19. Solve the system of integral equations

$$\int_0^{2\pi} (f(\phi))^3 \cos(\phi) d\phi = 0, \quad \int_0^{2\pi} (f(\phi))^3 \sin(\phi) d\phi = 0.$$

Chapter 6

Normed Spaces

6.1 Solved Problems

Problem 1. Consider the Hilbert space $\mathbb{C}^2$ and the vectors

$$\mathbf{v}_1 = \begin{pmatrix} \cos(i) \\ \sin(i) \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -\sin(i) \\ \cos(i) \end{pmatrix}.$$

Find the distance $\|\mathbf{v}_1 - \mathbf{v}_2\|$.

Problem 2. (i) Let $a, b \in \mathbb{R}$. Show that

$$d(a,b) := |\arctan(a) - \arctan(b)|$$

defines a distance in $\mathbb{R}$.
(ii) Show that $x_n = \arctan(n)$, $(n \in \mathbb{N})$ is a *Cauchy sequence*. Is the metric space $\{\mathbb{R}\,,\, d\}$ complete?

Problem 3. Let $n \geq 1$, $0 \leq a < b$ and $p \in \mathbb{R}^n$. Show that there exists a map $k : C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $k(\mathbf{x}) = 0$ for $\|\mathbf{x} - \mathbf{p}\| \geq b$, $k(\mathbf{x}) = 1$ for $\|\mathbf{x} - \mathbf{p}\| \leq a$, and $0 < k(\mathbf{x}) \leq 1$ for $\|\mathbf{x} - \mathbf{p}\| \leq b$.

Problem 4. Let $\mathbf{x} \in \mathbb{R}^2$. Is

$$\|\mathbf{x}\| := |x_1 x_2|^{1/2}$$

a norm on $\mathbb{R}^2$?

Problem 5. Show that if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq M(|x - y|)^a$$

for some fixed $M > 0$ and $a > 1$, then f is a constant function, i.e., f is identically equal to some real number b for all $x \in \mathbb{R}$.

Problem 6. Let f, g be continuously differentiable functions on the interval $[0, 1]$. One defines

$$\langle f, g \rangle = \int_0^1 \left(f(x)\overline{g(x)} + \frac{df}{dx}\overline{\frac{dg}{dx}} \right) dx.$$

Show that this satisfies the properties of an inner product. Calculate $\langle f, g \rangle$ for $f(x) = \sin(x)$, $g(x) = \cos(x)$. Extend it to

$$\langle f, g \rangle = \int_0^1 \left(\sum_{j=0}^{n} \frac{d^j f(x)}{dx^j} \overline{\frac{d^j g(x)}{dx^j}} \right) dx.$$

Problem 7. The p-norm of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}$$

where $p \in \mathbb{R}^+$. Find the norm for $p \to \infty$.

Problem 8. Let $\mathbf{x} \in \mathbb{R}^n$ and $\|\mathbf{x}\|$ be the Euclidean norm of $\mathbf{x}$. If $B \subset \mathbb{R}^m$ is a nonempty compect set and $\mathbf{x} \in \mathbb{R}^n$, then we define the distance

$$\text{dist}(\mathbf{x}, B) := \min\{\, \|\mathbf{x} - \mathbf{b}\| \,:\, \mathbf{b} \in B \,\}.$$

If $A, B \subset \mathbb{R}^n$ are nonempty compact sets then we define the distance

$$\text{dist}(A, B) := \max\{\, \text{dist}(\mathbf{a}, B) \,:\, \mathbf{a} \in A \,\}.$$

Show that

$$\text{dist}(A, B) = 0 \;\Leftrightarrow\; A \subset B$$

and $\text{dist}(A, B) < \epsilon$ means that $A \subset N_\epsilon(B)$ where

$$N_\epsilon(B) := \{\, \mathbf{x} \,:\, \text{dist}(\mathbf{x}, B) < \epsilon \,\}$$

is the ϵ-neighborhood of B.

Problem 9. Let $\mathbf{v}$ and $\mathbf{w}$ be two normalized column vectors in $\mathbb{C}^n$. Does

$$D(\mathbf{v},\mathbf{w}) := 2\arccos\left(\sqrt{\frac{(\mathbf{v}^*\mathbf{w})(\mathbf{w}^*\mathbf{v})}{(\mathbf{v}^*\mathbf{v})(\mathbf{w}^*\mathbf{w})}}\right)$$

provide a distance measure between $\mathbf{v}$ and $\mathbf{w}$.

Problem 10. Let $z_1, z_2 \in \mathbb{C}$. One can define a distance measure via

$$\rho(z_1,z_2) = \frac{|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}$$

Let $z_1 = e^{i\phi_1}$ and $z_2 = e^{i\phi_2}$. Find $\rho(z_1, z_2)$.

Problem 11. The *chordal distance* between to complex numbers z_1, z_2 is defined as

$$d(z_1,z_2) := \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

Let $z_1 = e^{i\pi/2}$ and $z_2 = e^{-i\pi/2}$. Find $d(z_1, z_2)$.

Problem 12. Find $n \times n$ matrices A, B such that

$$\|[A,B] - I_n\| \to \min$$

where $\|.\|$ denotes the norm and $[,]$ denotes the commutator.

6.2 Supplementary Problems

Problem 1. Let $x, y \in \mathbb{R}$. Is

$$d(x,y) = \frac{|x-y|}{2+|x-y|}$$

a metric on $\mathbb{R}$? Note that $d(x,x) = 0$ and $d(x,y) \geq 0$.

Problem 2. Let $\mathbb{R}$ be the set of real numbers and $x, y \in \mathbb{R}$. Show that

$$d(x,y) = |x-y|$$

provides a distance function. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ Show that

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\left(\sum_{j=1}^{n} (x_j - y_j)^2\right)}$$

provides a distance function. Of course one always assume the positive square root.

Problem 3. Consider the functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$

$$f(x) = \exp(-x^2), \qquad g(x) = \exp(-|x|).$$

Find the distance $d(f,g)$ between f and g given by

$$d(f,g) = \int_{\mathbb{R}} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

Chapter 7

Complex Numbers and Complex Functions

7.1 Solved Problems

Problem 1. Let $z \in \mathbb{C}$. Solve the nonlinear equation

$$z^3 = z|z|^2.$$

Problem 2. (i) Find the complex numbers z satisfying $z^2 = \bar{z}$.
(ii) Find the complex numbers z satisfying $z^3 = \bar{z}$.

Problem 3. Solve the cubic equation $z^3 = -1$. Do the solutions form a group under multiplication? If not, what numbers have to be added to form a group. Find them by multiplication of the solutions of the cubic equation.

Problem 4. Find the square root of $z = 4\exp(i\pi/4)$.

Problem 5. Let $\phi \in [0, 2\pi)$. Consider the complex number $z = 1 - e^{i\phi}$. Find the condition on ϕ such that $|z| = 1$.

Problem 6. Let $x, y \in \mathbb{R}$ and $z = x + iy$. Find the real and imaginary part of $i/(\pi z)$.

Problem 7. Let $\phi \in [0, 2\pi)$ and $\theta \in (-\pi, \pi)$. Can any complex number be represented by

$$z = e^{i\phi} \tan\left(\frac{1}{2}\theta\right) ?$$

Problem 8. Find

$$zd\bar{z} - \bar{z}dz, \quad (zd\bar{z} - \bar{z}dz) \otimes (zd\bar{z} - \bar{z}dz).$$

Problem 9. Let $x, c \in \mathbb{R}$ and $c \neq 0$. Find the real and imaginary part of the function

$$f_c(x) = \frac{x - ic}{x + ic}.$$

Problem 10. Let $x, y \in \mathbb{R}$. Solve

$$\frac{x - i}{x + i}\frac{y - i}{y + i} = \frac{xy - i}{xy + i}.$$

Problem 11. Find $\sqrt{i}$.

Problem 12. (i) Calculate

$$p(\alpha, \beta, \theta, \phi) = |\cos(\alpha)\cos(\beta)\sin(\theta)e^{i\phi} + \sin(\alpha)\sin(\beta)\cos(\theta)|^2.$$

(ii) Show that

$$p(\alpha, \beta, \theta, \phi) \leq 1.$$

(iii) Simplify the result from (i) for $\theta = \pi/4$ and $\phi = 0$.

Problem 13. Let $z_1, z_2 \in \mathbb{C}$. Consider the distance measures

$$d_1(z_1, z_2) := |z_1 - z_2|, \quad d_2(z_1, z_2) := \frac{|z_1 - z_2|}{1 + |z_1 - z_2|},$$

$$d_3(z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}.$$

Here d_3 is the chordal distance. Let $z_1 = e^{i\phi}$ and $z_2 = e^{i\phi_2}$. Find d_1, d_2, d_3.

Problem 14. Let $f(z_1)$ and $g(z_2)$ be a pair of analytic functions of z_1 and z_2, respectively. We define

$$f(z_1) \circ g(z_2) := \frac{1}{2\pi} \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta}) d\theta.$$

Let

$$f(z_1) = \sum_{k=0}^{\infty} a_k z_1^k, \qquad g(z_2) = \sum_{k=0}^{\infty} b_k z_2^k$$

for $|z_j| < R$, $j = 1, 2$. Find $f(z_1) \circ g(z_2)$.

Problem 15. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Find

$$\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}.$$

Problem 16. Study the behaviour (stability) of the fixed points of the complex map $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z^2.$$

Problem 17. Let $z = x + iy$ ($x, y \in \mathbb{R}$) be a nonzero complex number. We define the *principal argument* by $z = |z| \exp(i\arg(z))$, where $\arg(z) \in (-\pi, \pi]$ and we define the imaginary remainder $\text{Imr}(z)$ and the imaginary quotient $\text{Imq}(z)$ by

$$\Im(z) = \text{Imr}(z) + 2\pi \text{Imq}(z)$$

where $\text{Imr}(z) \in (-\pi, \pi]$ and $\text{Imq}(z) \in \mathbb{Z}$. Show that

$$\ln(e^z) = \Re(z) + i\text{Imr}(z)$$

and in particular

$$\ln(e^z) = z \quad \text{iff} \quad \Im(z) \in (-\pi, \pi].$$

Problem 18. Let $z \in \mathbb{C}$ and consider the analytic map

$$f(z) = \exp(z).$$

Find the solutions (fixed points) of the equation

$$z = f(z).$$

We set $z = x + iy$ $(x, y \in \mathbb{R}$. Then

$$x + iy = \exp(x + iy) \equiv e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

Thus we have to solve

$$e^x \cos(y) - x = 0, \qquad e^x \sin(y) - y = 0.$$

Problem 19. Let A be an $n \times n$ matrix. Suppose f is an analytic function inside on a closed contour Γ which encircles $\lambda(A)$, where $\lambda(A)$ denotes the eigenvalues of A. We define $f(A)$ to be the $n \times n$ matrix

$$f(A) = \frac{1}{2\pi i} \oint_\Gamma f(z)(zI_n - A)^{-1} dz.$$

This is a matrix version of the *Cauchy integral theorem*. The integral is defined on an element-by-element basis $f(A) = (f_{jk})$, where

$$f_{jk} = \frac{1}{2\pi i} \oint_\Gamma f(z)\mathbf{e}_j^T (zI_n - A)^{-1} \mathbf{e}_d z.$$

Let $f(z) = z^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate $f(A)$.

Problem 20. Let a, b, ϕ be fixed real numbers. Consider the function

$$w(z) = z + az^2 + be^{i\phi} z^3.$$

Write it as real and imginary part, with $w = u + iv$ and $z = x + iy$.

Problem 21. Consider the z-transform

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \qquad x(n) = \frac{1}{2\pi i} \oint x(z) z^{n-1} dz.$$

Let

$$S(N) := \sum_{n=1}^{N} x(n).$$

Then

$$S(N) = \sum_{n=1}^{N} x(n) = \frac{1}{2\pi i} \oint x(z) \sum_{n=1}^{N} z^{n-1} dz.$$

It follows that (geometric series)

$$S(N) = \frac{1}{2\pi i} \oint \frac{x(z)(z^N - 1)}{z - 1} dz.$$

Apply this expression and the *residue theorem* to calculate

$$S(N) = \sum_{n=1}^{N} n^3.$$

Problem 22. (i) Consider the complex number $z = e^{i\phi}$. Let $n \in \mathbb{N}$. Find $\sqrt[n]{z}$.
(ii) Let $z = re^{i\phi}$ and $w = x + iy$ $(x, y \in \mathbb{R})$. Find z^w.

Problem 23. Consider the complex numbers

$$z_1 = 0.4 + 0.3i, \qquad z_2 = 5 + 2i.$$

Calculate $z_1^{z_2}$. Hint: Set $z_1 = r_1 e^{i\phi_1}$.

Problem 24. Consider the complex numbers

$$z_1 = x_1 + iy_1 = 1 + 4i, \qquad z_2 = x_2 + iy_2 = 3 - 2i.$$

Calculate $\log_{z_2} z_1$.

Problem 25. Let $A > 0$ and $B \geq 0$. Consider the quadratic conformal map in the complex w-plane of the unit disc in the complex z-plane

$$w(z) = Az + Bz^2.$$

Let θ be the polar angle in the complex z-plane.
(i) Show that with the notation $w := u + iv$, $z := x + iy$ (with u, v, x, y real) one has the parametric equation of the boundary

$$u(\theta) = A\cos(\theta) + B\cos(2\theta), \qquad v(\theta) = A\sin(\theta) + B\sin(2\theta).$$

(ii) Let $C := B/A$. Show that for $C = 0$ (i.e. $B = 0$) one obtains a circular disc. Show that for $C = 1/4$ the curvature vanishes at $\theta = \pi$. Show that for $C = 1/2$ the derivative dw/dz vanishes at the boundary, i.e. at $\theta = \pi$.

Problem 26. (i) Solve the equation

$$\left(\frac{z + \frac{i}{2}}{z - \frac{i}{2}} \right)^4 = 1.$$

(ii) Solve the system of equations

$$\left(\frac{z_1+\frac{i}{2}}{z_1-\frac{i}{2}}\right)^4 = \frac{z_1-z_2+i}{z_1-z_2-i}, \qquad \left(\frac{z_2+\frac{i}{2}}{z_2-\frac{i}{2}}\right)^4 = \frac{z_2-z_1+i}{z_2-z_1-i}.$$

These equations play a role for the Bethe ansatz for spin systems.

Problem 27. Let z be a complex number such that $|z| < \infty$ and $\Re(z) > 0$. Consider

$$Z(z) = \int_0^\infty \exp(-zx - x^2)dx.$$

Show that

$$Z(z) = \frac{1}{2}\sum_{k=0}^\infty (-1)^k \frac{\Gamma(\frac{1}{2}+\frac{1}{2}k)}{k!} z^k.$$

Problem 28. Let $z \neq 0$. Show that the function

$$f(z) = \frac{\ln(z)}{z-1}$$

is analytic near $z = 1$ and admits the Taylor expansion

$$f(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n+1}(z-1)^n \quad \text{for} \quad |z-1| < 1.$$

Problem 29. Is the function $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z + |z|$$

continuous? Find the fixed points of f.

Problem 30. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ with $z_1\bar{z}_1 \neq 0$, $z_2\bar{z}_2 \neq 0$. Find the conditions such that

$$(\,\bar{z}_1 \quad \bar{z}_2\,)\begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} = 0.$$

Problem 31. Let $z = \rho e^{i\theta}$ and $\zeta = Re^{i\omega}$ with with $R > 0$ and $\rho > 0$. Show that

$$\frac{\rho^2 - R^2}{\rho^2 - 2R\rho\cos(\omega-\theta) + R^2} = \Re\left(\frac{z+\zeta}{z-\zeta}\right).$$

7.2 Supplementary Problems

Problem 1. Show that

$$\frac{1}{5}(-1+2i) = \frac{1}{\sqrt{5}}e^{i(\pi-\arctan(2))}.$$

Problem 2. Let $z_1 = r_1 e^{i\phi_1}$, $z_2 = r_2 e^{i\phi_2}$ with $r_1, r_2 \geq 0$. Show that

$$\Re\left(\frac{z_1+z_2}{z_1-z_2}\right) = \frac{r_1^2+r_2^2}{|z_1-z_2|}.$$

Problem 3. Let $z = x+iy$ with $x, y \in \mathbb{R}$. Show that

$$\Re(z^2) = x^2 - y^2.$$

Problem 4. Let $0 < r < 1$ and $\phi \in [0, 2\pi)$. Consider the map

$$f(z) = re^{i\phi} + (1-re^{i\phi})z.$$

Find the fixed points. Find $f(0)$, $f(f(0))$, $f(f(f(0)))$.

Problem 5. Let $x, y \in \mathbb{R}$. Find real and imaginary part of

$$z = \sqrt{\frac{x-iy}{x+iy}}e^{i\phi}.$$

Problem 6. Let $z \in \mathbb{C}$. Show that

$$\sin(z) = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right).$$

Problem 7. Show that the *Bessel function*

$$J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{2n+\nu}$$

is an entire function of z for $\nu = 0, 1, \ldots$. Show that

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu\frac{1}{z}J_\nu(z).$$

Problem 8. (i) Show that the radius of convergence of the function

$$f(z_1, z_2) = \sum_{k=0}^{\infty} z_1^k z_2$$

is given by

$$\{ (z_1, z_2) \, : \, |z_1| < 1 \vee z_2 = 0 \}.$$

(ii) Show that the radius of convergence of the function

$$f(z_1, z_2) = \sum_{k=0}^{\infty} (z_1 z_2)^k$$

is given by

$$\{ (z_1, z_2) \, : \, |z_1| \cdot |z_2| < 1 \}$$

(iii) Show that the radius of convergence of the function

$$f(z_1, z_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} z_1^{k_2} z_2^{k_1-k_2}$$

is given by

$$\{ (z_1, z_2) \, : \, |z_1| + |z_2| < 1 \}.$$

Problem 9. (i) Show that the sum

$$\sum_{k=0}^{\infty} k! z^k$$

only converges for $z = 0$.
(ii) Show that the sum

$$\sum_{k=0}^{\infty} \frac{z^k}{k^2}$$

converges for $|z| \leq 1$ and diverges for $|z| > 1$.
(iii) Show that the sum

$$\sum_{k=0}^{\infty} (z_1 z_2)^k$$

converges for $\{ (z_1, z_2) \, : \, |z_1| \cdot |z_2| < 1 \}$.
(iv) Show that the sum

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} z_1^{k_2} z_2^{k_1-k_2}$$

converges for $\{ (z_1, z_2) : |z_1|+|z_2| < 1 \}$. The sum appears at the expansion of $1/(1-(z_1+z_2))$.

Problem 10. Let $x \in \mathbb{R}^+$ and $f(x) = \sqrt{x}$. Show that real function can be extended with a power series expansion around 1 into the complex domain

$$f(z) = \sqrt{1+(z-1)} = \sum_{k=0}^{\infty} \binom{1/2}{k} (z-1)^k$$

and for $z = re^{i\phi}$ with $-\pi/2 < \phi < \pi/2$ we have $f(z) = \sqrt{r}e^{i\phi/2}$.

Problem 11. (i) Is the complex function

$$f(z) = \overline{z} = x_1 - ix_2$$

holomorph?
(ii) Is the complex function

$$f(z) = z^2 - \overline{z}^2 = 4ix_1x_2$$

holomorph?

Problem 12. (i) Show that if n is a positive integer then

$$(r(\sin(\phi) + i\sin(\phi)))^n \equiv r^n(\cos(n\phi) + i\sin(n\phi)). \tag{1}$$

Hint. Apply $\exp(i\phi) \equiv \cos(\phi) + i\sin(\phi)$.
(ii) Show that the n n-th roots of unity are

$$\rho = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right), \quad \rho^2, \quad \rho^3, \dots, \rho^{n-1}, \qquad \rho^n \equiv 1. \tag{1}$$

Problem 13. Let $\epsilon \in \mathbb{R}$ and $z \in \mathbb{C}$. Consider the product

$$f(z) = \prod_{k=1}^{\infty} \frac{1-e^{-\epsilon k}}{1-e^{-\epsilon(k+z)}}.$$

Find $f(0)$. Show that

$$f(z) = (1-e^{-\epsilon z})f(z-1).$$

Show that

$$f(m) = \prod_{k=1}^{m} (1-e^{-\epsilon k}), \qquad m = 1, 2, \dots$$

Show that f is periodic with period $2\pi i/\epsilon$. Show that f has simple poles at $z = -m$ $(m = 1, 2, \ldots)$. Show that the residues are given by

$$\lim_{z \to -m} (z+m) f(z) = \frac{(-1)^{m+1} \exp(-\frac{1}{2}\epsilon m^2 + \frac{1}{2}\epsilon m)}{\epsilon f(m-1)}.$$

Problem 14. Show that

$$(1+i)^{1/2} = \pm 2^{1/4} \left(\cos(\pi/8) + i \sin(\pi/8)\right).$$

Problem 15. Let $n = 0, 1, \ldots$ and $x \in \mathbb{R}$. Find the real and imaginary part of the functions

$$f_n(x) = \frac{1}{\sqrt{\pi}} \frac{(ix-1)^n}{(ix+1)^{n+1}}.$$

Problem 16. Show that

$$\frac{1}{(n-k)!} = \frac{1}{2\pi i} \oint dt \frac{e^t}{t^{n-k+1}}$$

where the integration contour is a small circle around the origin in the complex plane.

Problem 17. We know that $0 \le |\sin(x)| \le 1$ and $0 \le |\cos(x)| \le 1$ for $x \in \mathbb{R}$. This is no longer true for $\sin(z)$ and $\cos(z)$ with $z \in \mathbb{C}$.
(i) Let $a > 0$. Show that

$$|\sin(az)| = \sqrt{\sinh^2(ay) + \sin^2(ax)}, \quad |\cos(az)| = \sqrt{\sinh^2(ay) + \cos^2(ax)}.$$

(ii) Find all solutions of $|\sin(z)| = 2$ and $|\cos(z)| = 2$.
(iii) Find all solutions of $\cos(z) = i$.
(iv) Let $n \in \mathbb{N}$. Show that $(x, y \in \mathbb{R})$

$$\begin{aligned} |\sin(n(x+iy))| &= \sqrt{\sin^2(nx)\cosh(^2(ny) + \cos^2(nx)\sinh^2(ny)} \\ &> |\sinh^2(ny)| \to \infty \end{aligned}$$

as $n \to \infty$ for any $y \neq 0$.

Chapter 8

Integration

8.1 Solved Problems

Problem 1. The time-average of a continuous function f is

$$\langle f \rangle := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt.$$

Find the time-average of the functions

$$f_1(t) = \cos(\omega t)\sin(\omega t), \quad f_2(t) = \cos^2(\omega t), \quad f_3(t) = \sin^2(\omega t).$$

Problem 2. Let $j, k = 1, 2, \ldots$. Consider the function

$$f(j,k) = \int_0^1 x^{j-1}(1-x)^{k-1} dx, \qquad j,k = 1,2,\ldots.$$

Thus $f(1,1) = 1$. Is

$$f(j-1,k+1) = \frac{k}{j-1} f(j,k), \qquad j \geq 2$$

and

$$f(j+1,k) = f(j,k) - f(j,k+1)\,?$$

Prove or disprove.

Problem 3. Draw the functions

$$\begin{aligned} f_1(x) &= \cos(2\pi x) \\ f_2(x) &= \cos(2\pi(\cos(2\pi x))) \\ f_3(x) &= \cos(2\pi(\cos(2\pi(\cos(2\pi x))))). \end{aligned}$$

Extend it to $f_n(x)$. Find the integral

$$\int_0^{2\pi} f_n(x)dx.$$

Problem 4. Let

$$x(\tau) = \begin{cases} 1 & \text{for} \quad 0 \le \tau \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(\tau) = \begin{cases} 1 & \text{for} \quad 0 \le \tau \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Problem 5. Let $T > 0$ and $\omega = 2\pi/T$. Let $m, n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. Calculate

$$I(\alpha, \beta) = \frac{1}{T}\int_0^T c_m \sin(m\omega t + \phi_m - \alpha)c_n \sin(n\omega t + \phi_n - \beta)dt.$$

Problem 6. A *cubic B-spline* with uniform knot spacing, centered at the orign, is given by

$$B(x) = \begin{cases} \frac{1}{6}(2-|x|)^3 & \text{if} \quad 1 \le |x| < 2 \\ \frac{1}{6}((2-|x|)^3 - 4(1-|x|)^3) & \text{if} \quad 0 \le |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the integral

$$\int_{-\infty}^{\infty} B(x)dx.$$

Problem 7. Find the normalization constant K from the condition

$$1 = 4K \int_0^{2\pi} \sin^2(\omega/2)d\omega \int_0^{\pi} \sin(\theta)d\theta \int_{-\pi}^{\pi} d\phi.$$

Problem 8. Let $c \in \mathbb{R}$. Calculate the integral

$$f(c) = \int_0^{2\pi} \exp(ce^{i\phi})d\phi$$

by finding an ordinary differential equation for f together with the initial conditions. Obviously $f(0) = 2\pi$.

Problem 9. Let $b > a$. Find the mean and variance of random variable x with uniform probabilty density function p

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Problem 10. Consider a one-dimensional lattice (chain) with lattice constant a. Let k be the sum over the first Brioullin zone we have

$$\frac{1}{N} \sum_{k \in 1.BZ} F(\epsilon(k)) \to \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} F(\epsilon(k))dk = G$$

where

$$\epsilon(k) = \epsilon_0 - 2\epsilon_1 \cos(ka).$$

Using the identity

$$\int_{-\infty}^{\infty} \delta(E - \epsilon(k))F(E)dE \equiv F(\epsilon(k))$$

we can write

$$G = \frac{a}{2\pi} \int_{-\infty}^{\infty} F(E) \left(\int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k))dk \right) dE.$$

Calculate

$$g(E) = \int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k))dk$$

where $g(E)$ is called the density of states.

Problem 11. Let

$$i_1(t) = I^2 \sin^2(\omega t), \qquad i_2(t) = I^2 \sin^2(\omega t + \phi).$$

Calculate

$$\langle i_1(t) i_2(t) \rangle := \lim_{T \to \infty} \frac{1}{T} \int_0^T i_1(t) i_2(t) dt$$

and

$$\langle i_1(t) i_2(t) \rangle - \langle i_1(t) \rangle \langle i_2(t) \rangle.$$

Problem 12. Let $f(t)$ be a continuous function. Show that

$$\int_0^x d\zeta \int_0^\zeta f(t) dt = \int_0^x (x-t) f(t) dt.$$

Problem 13. (i) Calculate

$$\int_{-\infty}^{\infty} \mathrm{sech}(t) \mathrm{tanh}(t) \cos(t + t_0) dt.$$

(ii) Calculate

$$\int_{-\infty}^{\infty} \mathrm{sech}^2(t) \mathrm{tanh}^2(t) dt.$$

Problem 14. Let $\lambda > 0$. Calculate

$$\int_{-\infty}^{\infty} \frac{\sin(\lambda t)}{\lambda \sinh(t)} dt.$$

Problem 15. Calculate

$$\int_{-\infty}^{\infty} \exp(-x^2 + 2ixy) dx.$$

Problem 16. Calculate the integral

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{e^t}{(1+e^t)^2} \cos(t + \lambda) dt$$

using the residue technique.

Problem 17. Let m be an non-negative integer. Find

$$\int_0^{\pi} d\theta |\cos^{2m+1}(\theta)|.$$

Problem 18. Let $\epsilon > 0$. Calculate

$$\int_0^{\infty} \frac{k^2 dk}{e^{\epsilon k^2} - 1}.$$

Problem 19. The mother *Haar wavelet* is given by

$$f(t) = \begin{cases} -1 & \text{for} \quad 0 \leq t < 1/2 \\ +1 & \text{for} \quad 1/2 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt.$$

Problem 20. The *Poisson wavelet* is given by

$$f(t) = \left(t\frac{d}{dt} + 1 \right) P(t)$$

where

$$P(t) = \frac{1}{\pi} \frac{1}{1+t^2}.$$

Find the Fourier transform of f.

Problem 21. Let $m \in \mathbb{Z}$. Calculate

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} e^{im\phi} 2\cos(\phi) d\phi.$$

Problem 22. Show that (Fresnel's integral)

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Problem 23. Let $0 < \alpha < 1$. Find the integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx.$$

Problem 24. The linear one-dimensional *diffusion equation* is given by

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0, \quad -\infty < x < \infty$$

where $u(x,t)$ denotes the concentration at time t and position $x \in \mathbb{R}$. D is the diffusion constant which is assumed to be independent of x and t. Given the initial condition $c(x,0) = f(x)$, $x \in \mathbb{R}$ the solution of the one-dimensional diffusion equation is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t|x',0) f(x') dx'$$

where

$$G(x,t|x',t') = \frac{1}{\sqrt{4\pi D(t-t')}} \exp\left(-\frac{(x-x')^2}{4D(t-t')}\right).$$

Here $G(x,t|x',t')$ is called the fundamental solution of the diffusion equation obtained for the initial data $\delta(x-x')$ at $t=t'$, where δ denotes the Dirac delta function.
(i) Let $u(x,0) = f(x) = \exp(-x^2/(2\sigma))$. Find $u(x,t)$.
(ii) Let $u(x,0) = f(x) = \exp(-|x|/\sigma)$. Find $u(x,t)$.

Problem 25. Let $\omega_0 > 0$ be a fixed frequency and t the time. Calculate

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|\omega t|} e^{-i\omega t} dt.$$

Problem 26. Let $\omega_0 > 0$ be a fixed frequency and t the time. Calculate

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\omega_0 t|} e^{-i\omega t} dt.$$

Problem 27. The sum

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \exp\left(2\cos\left(\frac{k\pi}{n+1}\right)\right)$$

can be cast into the integral

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n \exp(2\cos(\pi x))dx. \qquad (1)$$

Calculate this integral.

Problem 28. Let $f:\mathbb{R}\to\mathbb{R}$ be an analytic function. The *Dirichlet integral identity* is given by

$$\int_0^u\int_0^{u-w_2} f(u-w_1-w_2)w_1^{\mu_1-1}w_2^{\mu_2-1}dw_1dw_2 =$$
$$\frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2)}\int_0^u f(u-w)w^{\mu_1+\mu_2-1}dw.$$

Let $f(x)=e^{-x}$. Calculate the left and right-hand side of this identity.

Problem 29. Let $r_1>0, r_2>0$. Find the integral

$$I(r_1,r_2)=\int_{-1}^1\frac{dx}{\sqrt{r_1^2+r_2^2-2r_1r_2x}}.$$

Problem 30. Show that

$$e^{a^2}=\int_{\mathbb{R}}dx\exp\left(-\frac{x^2}{2}+\sqrt{2}ax\right).$$

Problem 31. Calculate the integral

$$\int_0^1|\cos(2\pi x)|dx$$

using the random numnber generator described in problem 7, chapter 10, page 250, Problems and Solutions in Scientific Computing. Compare to the exact result by solving the integral.

Problem 32. Let $u\geq 0$ and

$$\rho(u)=\frac{1}{2}\exp(-\sqrt{u}).$$

Find

$$\rho_n=\int_0^{-\infty}u^n\rho(u)du.$$

Problem 33. Calculate the integral

$$I = \int\int_D \sqrt{x^2+4y^2}\,dydx$$

where D is the domain bounded by the positive x-axis, the positive y-axis and the parabola $y^2 = 1-x$.

Problem 34. Show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{sech}(\alpha-\beta)e^{i\omega\beta}d\beta = \mathrm{sech}(\frac{1}{2}\pi\omega)e^{i\omega\alpha}.$$

Problem 35. Calculate the definite integral

$$\int_0^1 \sin(x^2)dx.$$

Problem 36. (i) Consider the wavelet ($\omega_0 > 0$)

$$\psi_0(t) = (e^{i\omega_0 t} - e^{-\omega_0^2/2})e^{-t^2/2}.$$

Show that

$$\int_{-\infty}^{\infty} \psi_0(t)dt = 0.$$

Hint: Use ($a > 0$)

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/4a}.$$

(ii) We define

$$\psi_n(t) := \frac{d}{dt}\psi_{n-1}(t), \quad n = 1,2,\ldots$$

Show that

$$\int_{-\infty}^{\infty} t^k\psi_n(t)dt = 0, \qquad 0 \le k \le n.$$

Problem 37. Let $b > a$. Consider the integral

$$\int_{y=a}^{y=b} f(y)dy$$

where f is a continuous function in $[a, b]$. Apply the transformation

$$y(x) = \frac{1}{2}((b-a)x + a + b)$$

so that the integration range is between -1 and $+1$. Then the Gauss quadrature can be applied which extends over the interval $[-1, +1]$.

Problem 38. Let $\epsilon > 0$. Find f of the equation

$$\exp(-\epsilon t) = 1 - \int_0^t f(s)ds.$$

Problem 39. (i) Find the area of the set

$$S_2 := \{\, (x_1, x_2) \,:\, 1 \geq x_1 \geq x_2 \geq 0 \,\}.$$

(ii) Find the volume of the set

$$S_3 := \{\, (x_1, x_2, x_3) \,:\, 1 \geq x_1 \geq x_2 \geq x_3 \geq 0 \,\}.$$

Extend the n-dimensions.

Problem 40. Let ω_1, ω_2 be real and positive. Find

$$J(\omega_1, \omega_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2 + i\omega_1 x + i\omega_2 x^2)dx.$$

Problem 41. Let e be the eccentricity of an ellipse, i.e. $\sqrt{1-e^2} = b/a$. Let $n \geq 1$. Calculate the integral

$$\int_0^{2\pi} \frac{dx}{\left(\frac{1}{e} - \cos(x)\right)^n}$$

by making the substitution $z = \exp(ix)$. Apply the residue theorem.
Hint. We have

$$\int_0^{2\pi} \frac{dx}{\left(\frac{1}{e} - \cos(x)\right)^n} = \int_{|z|=1} \frac{dz}{\left(\frac{1}{e} - \frac{1}{2}\left(z + \frac{1}{z}\right)\right)^n iz} = \int_{|z|=1} \frac{i(-1)^{n+1}2^n z^{n-1} dz}{\left(z^2 - \frac{2}{e}z + 1\right)^n}.$$

Problem 42. Let $q^2 > 0$ and $(p-q)^2 > 0$. Find

$$\int_0^{\infty} \exp(-xq^2)dx, \qquad \int_0^{\infty} \exp(-x(p-q)^2)dx.$$

Problem 43. Find $\alpha > 0$ such that

$$\int_{-\infty}^{\infty} \exp(-\alpha|x|)dx = 1.$$

Afterwards calculate

$$\int_{-\infty}^{\infty} x\exp(-\alpha|x|)dx, \qquad \int_{-\infty}^{\infty} x^2\exp(-\alpha|x|)dx.$$

Problem 44. Let $c > 0$. Show that

$$\exp\left(\frac{1}{2}cy^2\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}x^2 + \sqrt{c}yx\right).$$

Problem 45. The Gauss invariant for two given closed loops C_α and C_β in $\mathbb{R}^3$ parametrized by $\mathbf{r}_\alpha(s)$, $\mathbf{r}_\beta$ is defined by

$$G(C_\alpha, C_\beta) := \frac{1}{4\pi}\oint_{C_\alpha} ds \oint_{C_\beta} ds' \frac{d\mathbf{r}_\alpha(s)}{ds} \times \frac{d\mathbf{r}_\beta(s')}{ds'} \frac{\mathbf{r}_\alpha(s) - \mathbf{r}_\beta(s')}{|\mathbf{r}_\alpha(s) - \mathbf{r}_\beta(s')|^3}$$

where $\times$ denotes the vector product. Find $G(C_\alpha, C_\beta)$ for the two curves

$$C_\alpha \,:\, x_1^2 + x_3^2 = 1, \qquad C_\beta \,:\, (x_1 - 1)^2 + x_2^2 = 1.$$

Problem 46. Let $x \in \mathbb{R}$. We define $[x]$ as the integer part of x and $\{x\} := x - [x]$. Calculate the integrals

$$\int_0^{7/2} [x]dx, \qquad \int_0^{7/2} \{x\}dx.$$

Problem 47. Let $x \in \mathbb{R}$. Consider the integral

$$f(x) = \int_x^{\infty} \frac{e^{iy}}{y} dy.$$

(i) Show that $f(-|x|) = f(|x|) - i\pi$.
(ii) Show that for large x

$$f(x) = e^{ix}\left(\frac{i}{x} + \frac{1}{x^2} + \cdots\right).$$

Problem 48. Let $\sigma > 0$. Show that

$$\frac{1}{4\sigma^2}\int_0^\infty x^{5/2}\exp\left(-\frac{x^2}{8\sigma^2}\right)dx = \left(\frac{32}{10}\right)^{1/4}\Gamma(3/4)\sigma^{3/2}.$$

Problem 49. Let $n = 0, 1, 2, \ldots$. We define

$$a_n := \int_0^{\pi/2}\cos^{2n}(x)dx, \qquad b_n := \int_0^{\pi/2}x^2\cos^{2n}(x)dx.$$

Then $a_0 = \pi/2$ and $b_0 = \pi^3/24$. Show that using integration by parts

$$a_n = (2n-1)(a_{n-1} - a_n).$$

Show that for $n \geq 1$ we have

$$a_n = (2n-1)nb_{n-1} - 2n^2 b_n.$$

Show that $(n \geq 1)$

$$0 \leq \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2} = 2\frac{b_n}{a_n} \leq \frac{\pi^2}{4(n+1)}.$$

Problem 50. Calculate

$$I = \int_0^{10}(x - \mathrm{int}(x))dx$$

where $\mathrm{int}(x)$ defines the largest integer less than x.

Problem 51. Let $0 < a < b < 1$. Find

$$\int_a^b \frac{\ln(1-x)}{x}dx.$$

Problem 52. Show that

$$\int_0^{\pi/2}\cos^{10}(x)\cos^8(2x)\cos^6(4x)\cos^4(6x)\cos^2(8x)dx =$$

$$\frac{5166673\pi}{536870912} + \frac{2966549762512816}{98120709987525225}.$$

Problem 53. Let $z \in \mathbb{C}$. Let $\ln(1+z)$ be the branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, -1]$. Calculate

$$I_n(r) = Pv \int_{|z|=r} z^{n-1} \ln(1+z) dz, \quad r > 0, \ n \in \mathbb{Z}$$

where Pv is the principal value.

Problem 54. Let $b > a$ and a, b be finite. Consider the integral

$$\int_a^b f(x) dx.$$

(i) Apply the transformation

$$x = \frac{1}{2}(a + b + (b-a)\tanh(y) \Leftrightarrow y = \tanh^{-1}\left(\frac{2x - a - b}{b - a}\right)$$

to the integral.
(ii) Apply the transformation

$$x = \frac{1}{2}\left(a + b + (b-a)\tanh\left(\frac{\pi}{2}\sinh(y)\right)\right)$$

with

$$\frac{dx}{dy} = \frac{(b-a)\pi\cosh(y)/4}{\cosh^2(\pi\sinh(y)/2)}$$

to the integral.

Problem 55. Calculate the integral

$$\int_0^\pi \frac{1+\sin(x)}{1+\cos(x)} dx$$

utilizing the transformation $t = \tan(x/2)$ with $-\pi < x < \pi$. From this transformation it follows that

$$x = 2\arctan(t), \qquad dx = \frac{2}{1+t^2} dt$$

and $x = 0 \to t = 0$, $x = \pi/2 \to t = 1$.

Problem 56. Calculate

$$P \int_0^1 \frac{y^m}{(y-x)} dy$$

for $m = 0, 1, 2, 3$. Here P denotes the Cauchy principal value.

Problem 57. Simplify the integral

$$I = \int_0^1 \frac{\cos(x)}{\sqrt{x}} dx$$

for numerical calculation.

Problem 58. Let $a > 0$. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be continuous functions with $f(-x) = f(x)$ and $g(-x) = -g(x)$. Show that

$$\int_{-a}^{a} f(x)g(x)dx = 0.$$

Problem 59. Show that

$$\int_0^\infty e^{-x}x^2 dx = 2!, \qquad \int_0^\infty e^{-x}x^3 dx = 3!.$$

Let $n = 4, 5, \ldots$. Show that

$$\int_0^\infty e^{-x}x^n dx = n!.$$

Problem 60. Let $k_1 > 0$, $k_2 > 0$, $k_3 > 0$. Find the integral

$$\int_0^\infty \sin(k_1 r)\sin(k_2 r)\sin(k_3 r)\frac{dr}{r}.$$

Problem 61. Let $\alpha > 0$. Show that

$$\int_0^\infty x e^{-\alpha x^2} J_0(\beta x)dx = \frac{1}{2\alpha} e^{-\beta^2/(4\alpha)}.$$

Problem 62. Calculate

$$F(s,t) = \int_{\mathbb{R}} e^{-ixt-|x-s|}dx.$$

Problem 63. Let $x \in (0,1)$. Calculate

$$P\int_0^1 \frac{y^m}{(y-x)}dy$$

for $m = 0$, $m = 1$, $m = 2$, $m = 3$.

Problem 64. Let $a, b > 0$ and $b > |a|$. Show that

$$\int_{\mathbb{R}} \frac{dx}{x^2 + 2ax + b^2} = \frac{\pi}{\sqrt{b^2 - a^2}}.$$

Problem 65. Let $a, b > 0$. Show that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln\left(\frac{b}{a}\right).$$

Problem 66. Let $x > 0$. Show that

$$\int_0^\infty e^{-tx} dt = \frac{1}{x}.$$

Problem 67. $a, b \in \mathbb{R}$. Let

$$f(x) = a \exp(bx).$$

Find the conditions on a and b such that

$$\int_0^1 f(x) dx = 1, \qquad \int_0^1 x f(x) dx = \frac{1}{2}.$$

Problem 68. *Dawson's integral* is given by

$$f(x) = \int_0^x e^{t^2 - x^2} dt, \quad x \geq 0.$$

(i) Show that for all complex values z the function f satisfies the linear differential equation

$$\frac{df(z)}{dz} + z f(z) = 1.$$

(ii) Let $j = 1, 2, \ldots$. Show that

$$f^{(j+1)}(z) + 2z f^{(j)} + 2j f^{(j-1)}(z) = 0, \quad j = 1, 2, \ldots$$

where $f^{(j)}$ indicates the jth derivative.

Problem 69. Let $a, b, c \in \mathbb{R}$ and $a + b\cos(\theta) + c\sin(\theta) \neq 0$. Show that

$$\int_{-\pi}^{+\pi} \frac{b\sin(\theta) - c\cos(\theta)}{a + b\cos(\theta) + c\sin(\theta)} d\theta = 0.$$

Problem 70. Let $a > 0$. Show that

$$\int_0^\infty \frac{\cos(ax)}{1 + x^2} dx = \frac{\pi}{2} e^{-a}.$$

Problem 71. Show that

$$\int_0^\infty \frac{dx}{\cosh^3(x)} = \frac{\pi}{4}.$$

Problem 72. Show that

$$\int_0^\infty \frac{\ln(\cosh(x))}{\cosh^3(x)} dx = \frac{\pi}{4}(\ln(2) - 1/2).$$

Problem 73. Let $m, n = 0, 1, 2, \ldots$. Find the integral

$$f_{mn}(t) = \int_0^t (t - \tau)^m \tau^n d\tau.$$

Problem 74. Let $a, b > 0$. Find the integral

$$\int_0^\infty \frac{\cos(at) - \cos(bt)}{t} dt.$$

Problem 75. Let $q^2 > 0$. Calculate the integral

$$\int_0^\infty \exp(-q^2 x) dx.$$

Problem 76. Let $x > 0$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is integrable over $[0, x]$ for all $x > 0$ and $\lim_{x \to \infty} f(x) = a$. Show that

$$\lim_{x \to \infty} \frac{1}{x} \int_0^x f(s) ds = a.$$

Utilize

$$\left|\frac{1}{x}\int_0^x f(s)ds - a\right| = \left|\frac{1}{x}\int_0^x (f(s)-a)ds\right| \le \left|\frac{1}{x}\right|\int_0^x |f(s)-a|ds.$$

Problem 77. Consider the function $f:[0,1]\to[0,1]$

$$f(x) = \begin{cases} 1/x - \mathrm{int}(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where $\mathrm{int}(y)$ denotes the integer part of y. The function is integrable.
(i) Let $k \ge 1$ be a positive integer. Find

$$\int_{1/(k+1)}^{1/k} f(x)dx.$$

(ii) Find

$$\int_{1/k}^{1} f(x)dx.$$

(iii) Find

$$\int_0^1 f(x)dx.$$

Problem 78. Show that

$$\int_0^{\pi/2} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx = \frac{\pi}{4}.$$

Problem 79. Calculate

$$\int_0^3 (x - \lfloor x \rfloor + \frac{1}{2})dx$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Hilbert Transform

Problem 80. The *Hilbert transform* H and its inverse is given by

$$g(y) = H(f(x)) = \frac{1}{\pi}P\int_{\mathbb{R}} \frac{f(x)}{x-y}dx$$

$$f(x) = H^{-1}(g(y)) = \frac{1}{\pi}P\int_{\mathbb{R}} \frac{g(y)}{y-x}dy$$

where the *Cauchy principal value* is defined by

$$P\int_{\mathbb{R}} f(x)dx := \lim_{R\to\infty}\int_{-R}^{+R} f(x)dx.$$

The Hilbert transform relates parts of the function in the same domain. Let $k > 0$. Find the Hilbert transform of $f(x) = \cos(kx)$.

Problem 81. Consider the Hilbert transform

$$H[f] = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{f(x')}{x'-x}dx'.$$

Find H^2.

Problem 82. The *Hilbert transform* acts on a one-dimensional function $g(s)$ by a convolution with the kernel $1/(\pi s)$. The singularity at $s = 0$ is handled in the Cauchy principal value sense. The Fourier transform of the Hilbert kernel is $-i\,\mathrm{sgn}\sigma$. Thus the Hilbert transform of g is

$$Hg(s) = \int_{-\infty}^{\infty} \frac{g(s-s')}{\pi s'}ds' = \int_{-\infty}^{\infty}(-i\,\mathrm{sgn}\sigma)G(\sigma)e^{i2\pi s\sigma}d\sigma$$

where $G(\sigma)$ is the Fourier transform of g, i.e.

$$G(\sigma) = \int_{-\infty}^{\infty} g(s)e^{-i2\pi\sigma s}ds.$$

Suppose g is a function whose support is strictly less than radius R, i.e. $g(s) = 0$ for all $|s| > R - \epsilon$ for some small positive ϵ. Find an inversion formula.

Problem 83. The *Hilbert transform* of a function $f \in L_2(\mathbb{R})$ is defined as

$$H(f)(y) = \frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{f(x)}{x-y}dx$$

where PV stands for *principal value*. Calculate the Hilbert transform of $f(x) = 1/(1+x^4)$.

Problem 84. The *Radon transform* for a function $f(x, y)$ is given by the integral transform

$$P(r,\theta) = Rf(x,y) = \int_{-\infty}^{+\infty} f(r\cos(\theta) - s\sin(\theta), r\sin(\theta) + s\cos(\theta))ds.$$

The function $P(r,\theta)$ desribes the values of points on projections. Show that the inverse Radon transform can be given by

$$f(x,y) = R^{-1}P(r,\theta) = -\frac{1}{2\pi} BHDP(r,\theta)$$

where D is the partial differential operator $Dg(r,\theta) = \partial g/\partial r$ with respect to r, H is the Hilbert transform operator

$$Hg(r,\theta) = -\frac{1}{\pi}\int \frac{g(u,\theta)}{r-u}du$$

and B is the backprojection operator

$$Bg(r,\theta) = \int_0^\pi g(x\cos(\theta) + y\sin(\theta),\theta)d\theta.$$

Problem 85. The *Hilbert transform* of a function $f \in L_2(\mathbb{R})$ is defined as

$$H(f)(y) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(x)}{x-y}dx$$

where PV stands for *principal value.* Calculate the Hilbert transform of

$$f(x) = \exp(-x^2/2).$$

Problem 86. Let $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{x}{\sinh(x)}.$$

(i) Find $f(0)$.
(ii) Let $c > 0$. Find

$$\int_0^c f(x)dx.$$

Problem 87. Let $p_1, p_2, p_3 > 0$. Calculate

$$f(p_1,p_2,p_3) = \frac{4}{\pi}\int_0^\infty \sin(p_1 r)\sin(p_2 r)\sin(p_3 r)\frac{dr}{r}.$$

Problem 88. Find the constant K (normalization) from the condition

$$1 = 4K\int_0^{2\pi} \sin^2(\omega/2)d\omega \int_0^\pi \sin(\theta)d\theta \int_{-\pi}^{\pi} d\phi.$$

Problem 89. Consider

$$I = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d\alpha \int_0^{\pi} \sin(\beta) d\beta \int_{-\pi}^{\pi} F(\alpha, \beta, \gamma) d\gamma.$$

(i) Find I for $F(\alpha, \beta, \gamma) = 1$.
(ii) Find I for $F(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$.
(iii) Find I for $F(\alpha, \beta, \gamma) = \alpha\beta\gamma$.

Problem 90. Consider the circle around $(0, 0, 0)$ in the $x_1 - x_2$ plane

$$\mathbf{r}_1(t) = \begin{pmatrix} x_{1,1}(t) \\ x_{1,2}(t) \\ x_{1,3}(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}, \quad t \in [0, 2\pi]$$

and the circle around $(0, 1, 0)$ in the $x_2 - x_3$ plane

$$\mathbf{r}_2(s) = \begin{pmatrix} x_{2,1}(s) \\ x_{2,2}(s) \\ x_{2,3}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + \cos(s) \\ \sin(s) \end{pmatrix}.$$

Then the derivatives are

$$\frac{d\mathbf{r}_1(t)}{dt} = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix}, \qquad \frac{d\mathbf{r}_2(t)}{dt} = \begin{pmatrix} 0 \\ -\sin(s) \\ \cos(s) \end{pmatrix}.$$

Calculate (Gauss formula)

$$\frac{1}{4\pi} \oint \oint dt ds \left(\frac{d\mathbf{r}_1(t)}{dt} \times \frac{d\mathbf{r}_2(s)}{ds} \right) \cdot \frac{\mathbf{r}_1(t) - \mathbf{r}_2(s)}{|\mathbf{r}_1(t) - \mathbf{r}_2(s)|^3}$$

where $\times$ denotes the vector product, $\cdot$ denotes the scalar product and contour integrations run from 0 to 2π.

Problem 91. Show that

$$\int_0^{\infty} e^{-x} x^2 dx = 2!, \qquad \int_0^{\infty} e^{-x} x^3 dx = 3!.$$

Problem 92. (i) Find the area of the set

$$S_2 := \{ (x_1, x_2) \, : \, 1 \geq x_1 \geq x_2 \geq 0 \}.$$

(ii) Find the volume of the set

$$S_3 := \{ (x_1, x_2, x_3) \, : \, 1 \geq x_1 \geq x_2 \geq x_3 \geq 0 \}.$$

Extend the n-dimensions.

Problem 93. Let n be a positive integer. Find

$$\int_0^\pi \cos^2(nx)dx.$$

Problem 94. Let $n \geq 0$. The Legendre polynomial of degree n is defined as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)$$

with $P_0(x) = 1$. Let $n \geq 1$. Show that

$$\int_0^1 x^k P_n(x)dx = 0$$

for $k = 0, 1, \ldots, n-1$

Problem 95. Let $k = 0, 1, \ldots$. Calculate

$$c_k = \frac{2}{\pi}\int_0^\pi \arcsin(\cos(\theta))\cos(k\theta)d\theta.$$

Problem 96. Find $\alpha > 0$ such that

$$\int_0^\infty 2\alpha x e^{-2\alpha x^3/3}dx = 1.$$

Problem 97. The complete elliptic integral of first kind $K(m)$ can be defined by

$$K(m) = \int_0^1 ((1-t^2)(1-mt^2))^{-1/2}dt, \quad |m| < 1.$$

The *beta function* can be defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad \Re(a) > 0, \;\; \Re(b) > 0.$$

Show that $2\sqrt{2}K(1/2) = B(1/4, 1/2)$.

Problem 98. Consider the function $f : [0,1] \to [0,1]$, $f(x) = \sqrt{x}$ with

$$I = \int_0^1 \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}.$$

Find a approximation (upper bound) of the integral in the sense of Lebesgue by partioning the ordinate interval into the intervals $[0, 1/4]$, $(1/4, 1/2]$, $(1/2, 3/4]$, $(3/4, 1]$ with $a_0 = 0$, $a_1 = 1/4$, $a_2 = 1/2$, $a_3 = 3/4$, $a_4 = 1$.

Problem 99. Let $a > 0$. Calculate

$$\int \frac{dx}{(a^2 - x^2)^{3/2}}.$$

Set $x = a\sin(\theta)$. Then $dx = a\cos(\theta)d\theta$.

Double Integrals

Problem 100. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $f(x_1, x_2) = f(x_2, x_1)$ for all $x_1, x_2 \in \mathbb{R}$. Let $b, a \in \mathbb{R}$ and $b > a$. Calculate

$$\int_a^b \int_a^b f(x_1, x_2)\sin(x_1 - x_2)dx_1 dx_2.$$

Problem 101. Let f be a continuous function.
(i) Show that the double integral

$$\int_0^x d\xi \int_0^\xi f(t)dt$$

can be expressed by a single integral.
(ii) Show that $(n \geq 2)$

$$\int_0^x d\xi_1 \int_0^{\xi_1} d\xi_2 \cdots \int_0^{\xi_{n-1}} f(\xi_n)d\xi_n$$

can be expressed by a single integral.

Problem 102. Calculate the integral

$$I = \int_0^{\pi/2} \sin^{2n}(\theta)\cos^{2n+1}(\theta)d\theta$$

by considering the double integral

$$\int\int_D (r\sin(\theta))^{2n}(r\cos(\theta)^{2n+1}e^{-r^2}rdrd\theta$$

where D is the first quadrant.

Problem 103. Find the integral

$$I(\ell) = \int_0^{2\pi} \left(\int_0^{|\cos(\theta)|(\ell/2)} dp \right) d\theta.$$

Problem 104. Evaluate the quadruple integral

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 dx' \int_0^1 dy' \left(\frac{1}{((x-x')^2+(y-y')^2)^{1/2}} - \frac{1}{((x-x')^2+(y-y')^2+1)^{1/2}} \right).$$

Problem 105. Let $z \in (0,1]$ and $x \in (0,1]$. Find the double integral

$$f(z) = 1 - \int_z^1 dx \int_{z/x}^1 dy.$$

Problem 106. (i) Let $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that the integral

$$\int \int_{r \geq 1} \cdots \int \frac{dx_1 \cdots dx_n}{r^\alpha}$$

is finite for $\alpha > n$ and infinite for $\alpha \leq n$.
(ii) Let $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that the integral

$$\int \int_{r \leq 1} \cdots \int \frac{dx_1 \cdots dx_n}{r^\alpha}$$

is finite for $\alpha < n$ and infinite for $\alpha \geq n$.

Problem 107. Consider a three-dimensional probability distribution $f(x_1, x_2, x_3)$ such that for all j

$$f_x(x_j) = \frac{1}{2\sqrt{\pi}} \left(1 + 2x_j^2 - \frac{-1+2x_j^2}{\sqrt{2}} \right) e^{-x_j^2}$$

where $f_x(x)$ is the probability density associated with an individual variable. This means

$$f_x(x_1) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2, x_3) dx_2 dx_3$$

etc. Is it possible that the probabilty density associated with the sum of these variables $s = x_1 + x_2 + x_3$ is given by

$$f_s(s) = \frac{1}{2\sqrt{\pi}}\left(1 + 2s^2 + \frac{-1+2s^2}{\sqrt{2}}\right)e^{-s^2}$$

provided that $f(x_1, x_2, x_3)$ is non-negative?

Problem 108. Consider a one-dimensional chain of length N with open end boundary conditions. The counting is from left to right starting at 0. The canonical partition function $Z(\beta)$ ($\beta > 0$) is given by the multiple integral

$$Z_N(\beta) = \int_{-1}^{1} ds_0 \int_{-1}^{1} ds_1 \cdots \int_{-1}^{1} ds_{N-1} e^{\beta|s_0-s_1|} e^{\beta|s_1-s_2|} \cdots e^{\beta|s_{N-2}-s_{N-1}|}.$$

Show that there is a coordinate transformation which decouples the sites. Find $Z_2(\beta)$ and $Z_3(\beta)$.

Problem 109. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be an analytic function. Consider the map

$$\mathbf{x}_j = \mathbf{f}(\mathbf{x}_{j-1}) = \cdots = \mathbf{f}(\mathbf{x}_0).$$

To study the evolution of phase-space distributions, we can introduce the evolution operator $U(\mathbf{x}', \mathbf{x}, j)$ such that any initial phase-space distribution $\rho(\mathbf{x}, 0)$ evolves into

$$\rho(\mathbf{x}'; j) = \int_{\Omega} U(\mathbf{x}', \mathbf{x}; j)\rho(\mathbf{x}, 0)d\mathbf{x}$$

where Ω is the phase space area. Find $U(\mathbf{x}', \mathbf{x}; j)$.

Problem 110. Let

$$S := \{\,(x, y) \in \mathbb{R}^2 \,:\, x, y \geq 0,\ \ 0 \leq x_1^2 + x_2^2 \leq 1\,\}.$$

Let m, n be nonnegative integers. Find the integral

$$\int_S x^{2m+1} y^{2n+1} dxdy.$$

Problem 111. Calculate

$$\int\int_A dxdy$$

where

$$A = \{(x,y) \, : \, x, y \geq 0, \; x + y \leq 1, \; x \geq 1/3 \}.$$

Problem 112. Calculate the integral

$$I = \int_1^\infty dx \int_1^\infty dy \frac{\sqrt{x^2-1}\sqrt{y^2-1}}{(x+y)^6}$$

using the substitution $x = \cosh(\alpha)$, $y = \cosh(\beta)$.

Problem 113. Find the integral

$$I = \int_1^\infty dx \int_1^\infty dy \frac{\sqrt{x^2-1}\sqrt{y^2-1}}{(x+y)^6}.$$

Hint. Use the substitutions $x = \cos(\alpha)$, $y = \cosh(\beta)$ and show that the integral can be written as

$$I = \int_0^\infty d\alpha \int_0^\infty d\beta \frac{\sinh^2(\alpha)\sinh^2(\beta)}{(\cosh(\alpha)+\cosh(\beta))^6}.$$

Problem 114. Calculate the double integral

$$\frac{1}{2\pi^2}\int_0^\pi \int_0^\pi d\alpha d\alpha' \ln(2 - \cos(\alpha) - \cos(\alpha'))$$

utilizing the identity

$$\cos(\alpha) + \cos(\alpha') \equiv 2\cos\left(\frac{\alpha+\alpha'}{2}\right)\cos\left(\frac{\alpha-\alpha'}{2}\right)$$

the transformation $x = (\alpha+\alpha')/2$, $y = (\alpha-\alpha')/2$ and the integral

$$\int_0^\pi \ln(1+\sin(x))dx = -\pi\ln(2) + G$$

where G is the Catalan constant.

Problem 115. Let $a > 0$ and $R \geq 0$. Find

$$I_a(R) = \int_0^R \int_0^R \frac{dxdy}{(x-y)^2+a^2} \equiv \int_0^R \int_0^R \frac{dxdy}{x^2+y^2-2xy+a^2}.$$

Problem 116. Calculate the integral

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{dx_1 dx_2 dx_3 dx_4 dx_5 dx_6}{1 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6}.$$

Problem 117. Let $n_1, n_2 \in \mathbb{Z}$. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ with a period 2π for both x_1 and x_2

$$f(x_1, x_2) = \sum_{n_1, n_2 = -\infty}^{\infty} c_{n_1,n_2} e^{i(n_1 x_1 + n_2 x_2)}$$

and

$$\int_0^{2\pi} \int_0^{2\pi} f(x_1, x_2) dx_1 dx_2 = 0.$$

Then

$$c_{n_1,n_2} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x_1, x_2) e^{-i(n_1 x_1 + n_2 x_2)} dx_1 dx_2$$

(i) Let

$$f(x_1, x_2) = \sin(x_1) + \cos(x_2).$$

Find c_{n_1,n_2}.
(ii) Let

$$f(x_1, x_2) = \sin(x_1)\cos(x_2).$$

Find c_{n_1,n_2}.

Problem 118. Find a non-negative analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) dx dy = 1$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} x^2 dx dy = \frac{1}{2}, \qquad \int_{\mathbb{R}} \int_{\mathbb{R}} y^2 dx dy = \frac{1}{2}.$$

Problem 119. Consider the compact set

$$S := \{ (x, y) \, : \, y \geq x^2, \, y \leq \sqrt{x}, \, x, y \geq 0 \}.$$

Thus $S \subset [0, 1] \times [0, 1]$. Find

$$I(S) = \int_S d\mu = \int_S dx dy.$$

Problem 120. Let $a > 0$. Consider the compact set (*lemniscate*)

$$S := \{ (x,y) \, : \, (x^2+y^2)^2 \leq 2a^2xy \}.$$

Find

$$I(S) = \int_S d\mu = \int_S dxdy.$$

Introduce polar coordinates $x(r,\phi) = r\cos(\phi)$, $y(r,\phi) = r\sin(\phi)$. Thus we have

$$x^2+y^2 = r^2, \qquad xy = r^2\cos(\phi)\sin(\phi) \equiv \frac{1}{2}\sin(2\phi).$$

Problem 121. Calculate the integrals

$$\int_0^1 dx_2 \left(\int_0^{1-x_2} dx_1 \right) dx_2, \qquad \int_0^1 dx_3 \left(\int_0^{1-x_3} dx_2 \left(\int_0^{1-x_3-x_2} dx_1 \right) \right).$$

Extend to n-dimensions. Give an interpretation of the result.

Problem 122. (i) Is the subset of $\mathbb{R}^2$

$$S_2 = \{ (x_1,x_2) \, : \, x_1 \geq 0, \; x_2 \geq 0, \; x_1+x_2 \leq 1 \}$$

convex?
(ii) Calculate

$$A = \int_{S_2} dx_1 dx_2.$$

(iii) The area of a triangle in the plane $\mathbb{R}^2$ with vertices at (x_1,y_1), (x_2,y_2), (x_3,y_3) is given by

$$A = \pm\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

where the sign is chosen so that the area is nonnegative. Let $(x_1,y_1) = (0,0)$, $(x_2,y_2) = (1,0)$, $(x_3,y_3) = (0,1)$. Find A. Compare to (ii).
(iv) Is the subset of $\mathbb{R}^3$

$$S_3 = \{ (x_1,x_2,x_3) \, : \, x_1 \geq 0, \, x_2 \geq 0, \, x_3 \geq 0 \}$$

convex?
(v) Calculate

$$V = \int_{S_3} dx_1 dx_2 dx_3.$$

Problem 123. Consider

$$I = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d\alpha \int_0^{\pi} \sin(\beta) d\beta \int_{-\pi}^{\pi} F(\alpha, \beta, \gamma) d\gamma.$$

Find I for $F(\alpha, \beta, \gamma) = 1$, $F(\alpha, \gamma, \beta) = \alpha + \beta + \gamma$, $F(\alpha, \beta, \gamma) = \alpha\beta\gamma$.

Problem 124. Let $\gamma > 0$. A random variable X is said to be Lorentzian with parameters (α, γ) if its probability density is given by

$$f_X(x) = \frac{1}{\pi} \frac{\gamma}{(x-\alpha)^2 + \gamma^2}, \quad x \in \mathbb{R}.$$

Let X, Y be two independent Lorentzian random variables with parameters (α, γ) and (β, δ), respectively. Let $\epsilon \geq 0$. Show that
(a) ϵX is distributed Lorentzian $(\epsilon\alpha, \epsilon\gamma)$
(b) $\epsilon + X$ is distributed Lorentzian $(\epsilon + \alpha, \gamma)$
(c) $-X$ is distributed Lorentzian $(-\alpha, \gamma)$
(d) $X + Y$ is distributed Lorentzian $(\alpha + \beta, \gamma + \delta)$
(e) X^{-1} is distributed Lorentzian $\left(\frac{\alpha}{\alpha^2+\gamma^2}, \frac{\gamma}{\alpha^2+\gamma^2}\right)$

Problem 125. A random variable X is said to be Lorentzian if its probability density p_X is a the form

$$p_X(x) = \frac{1}{\pi} \frac{\gamma}{(x-\alpha)^2 + \gamma^2}$$

where $\gamma > 0$. We say that X is (α, γ) to indicate that the random variable is Lorentzian with the probability density given above. Let X, Y be two independent random variables with (α, γ) and (β, δ). Let ϵ be a nonnegative real number. Show that

$$\begin{aligned} \epsilon X \quad &\text{is} \quad (\epsilon\alpha, \epsilon\gamma) \\ \epsilon + X \quad &\text{is} \quad (\epsilon + \alpha, \gamma) \\ -X \quad &\text{is} \quad (-\alpha, \gamma) \\ X + Y \quad &\text{is} \quad (\alpha + \beta, \gamma + \delta) \\ X^{-1} \quad &\text{is} \quad \left(\frac{\alpha}{\alpha^2 + \gamma * 2}, \frac{\gamma}{\alpha^2 + \gamma^2}\right). \end{aligned}$$

Problem 126. Consider the set

$$M = \{ (x_1, x_2, x_3) : x_1^2 + x_2^2 \leq 1 \text{ and } x_1^2 + x_3^2 \leq 1 \}$$

which is subset of $\mathbb{R}^3$. Obviously M is compact and measureable

$$I(M) = \int_M d\mu = \int_{\mathbb{R}^3} \chi_M d\mu = \int_{\mathbb{R}^3} \chi_M dx_1 dx_2 dx_3$$

where χ_M is the indicator function. Find $I(M)$.

Problem 127. Given a smooth Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{n} \frac{p_j^2}{2} + U(\mathbf{q})$$

with n degrees of freedom ($\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{q} = (q_1, \ldots, q_n)$. Let $V(E)$ be the classical phase space volume at energy E of a smooth Hamilton function is given by

$$V(E) = \int_{\mathbb{R}^{2n}} \Theta(E - H(\mathbf{p}, \mathbf{q})) d^n\mathbf{p} d^n\mathbf{q}$$

where Θ is the step function. Assume that $U(\epsilon\mathbf{q}) = \epsilon^m U(\mathbf{q})$.
(i) Consider the transformation

$$\mathbf{p} = E^{1/2}\mathbf{p}', \qquad \mathbf{q} = E^{1/n}\mathbf{q}'$$

with the inverse transformation

$$\mathbf{p}' = E^{-1/2}\mathbf{p}, \qquad \mathbf{q}' = E^{-1/n}\mathbf{q}.$$

Find $d^n\mathbf{p}' d^n\mathbf{q}'$ and $H(\mathbf{p}', \mathbf{q}')$.
(ii) Calculate $V(E)$ with the assumption that $E > 0$. Find the asymptotic behaviour.

8.2 Supplementary Problems

Problem 1. Let $\alpha > 0$. Show that

$$\int_{-\infty}^{\infty} \operatorname{sech}(\alpha t) dt = \frac{2}{\alpha}.$$

Problem 2. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$

$$\phi(x) := \begin{cases} 1 & \text{for} \quad x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find

$$\psi(x) := \phi(2x) - \phi(2x-1).$$

Draw the function. Calculate

$$\int_{-\infty}^{+\infty} \psi(x)dx.$$

Problem 3. Show that

$$e^{-x^2/2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2 + ixy\right) dy.$$

Problem 4. Let $\ell > 0$ and $r_0 > 0$. Find the integral

$$\int_0^{r_0} \frac{r^3}{\sqrt{1+r^2/\ell^2}} dr.$$

Problem 5. Let $0 \le r < 1$. Consider the Hilbert space $L_2[0,2\pi]$ and $f(\theta) \in L_2[0,2\pi]$. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta + \frac{1}{\pi}\int_0^{2\pi}\sum_{j=1}^{\infty} r^j f(\theta)\cos(j(\phi-\theta))d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}d\theta.$$

Problem 6. Show that

$$\int_0^{\infty} e^{-kx}\sin(ky)\sin(k\gamma)\frac{dk}{k} = \frac{1}{4}\ln\left(\frac{x^2+(y-\gamma)^2}{x^2+(y+\gamma)^2}\right).$$

Note that

$$f(x) = \int_0^{\infty} e^{-kx}\frac{dk}{k} \quad \Rightarrow \quad \frac{df}{dx} = -\frac{1}{x}.$$

Problem 7. Show that

$$\int_0^{\infty} \frac{x^2}{(x^2+a^2)^3} = \frac{x(x^2-a^2)}{8a^2(x^2+a^2)^3} + \frac{1}{8a^2}\arctan(x/a).$$

Problem 8. Show that

$$\frac{3}{\pi}\int_0^{\pi/6} \ln(2\cos(x))dx = 0.338314\ldots .$$

Problem 9. The content (n-dimensional volume) bounded by a hypersphere of radius r is known to be

$$V_n = \frac{2r^n \pi^{n/2}}{n\Gamma(n/2)}$$

where Γ is the gamma function. Let $r = 1$. Show that

$$\lim_{n\to\infty} V_n = 0.$$

Problem 10. Let $k = 0, 1, 2, \ldots$ and

$$y_k := \int_0^1 \frac{x^k}{1+x+x^2}dx.$$

Show that

$$y_{k+2} + y_{k+1} + y_k = \frac{1}{k+1}.$$

Note that

$$y_0 = \int_0^1 \frac{1}{1+x+x^2}dx = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\Bigg|_{x=0}^{x=1}$$

$$y_1 = \int_0^1 \frac{x}{1+x+x^2}dx = \frac{1}{2}\ln(1+x+x^2)\Bigg|_0^1 - \frac{1}{2}\int_0^1 \frac{1}{1+x+x^2}dx = \frac{1}{2}\ln(1+x+x^2)\Bigg|_0^1 - \frac{1}{2}y_0.$$

Problem 11. Let $c \in \mathbb{C}$ with $|c| < \infty$ and $\Re(c) > 0$. Consider the integral

$$I(c) = \int_0^\infty \exp(-cx - x^2)dx.$$

Show that

$$I(c) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(1/2 + k/2)}{k!}c^k.$$

Problem 12. Find $\alpha > 0$ such that

$$\int_0^\infty 2\alpha x^2 e^{-2\alpha x^3/3}dx = 1.$$

Problem 13. Let $c_1 > 0$ and $c_2 > 0$. Find c_1, c_2 such that

$$\int_{\mathbb{R}} c_1 e^{-c_2|x|} dx = 1.$$

Problem 14. Let $\mathbf{x} = (x_1 \ x_2 \ \cdots \ x_n)$, $\mathbf{y} = (y_1 \ y_2 \ \cdots \ y_n)$ and $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$. Show that

$$\exp(a\mathbf{y}^2) = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \exp(-\mathbf{x}^2 + 2a^{1/2}\mathbf{x} \cdot \mathbf{y}) dx_1 \dots dx_n.$$

Problem 15. Let $b > a$. Show that

$$\int_a^b \sqrt{\frac{y-a}{b-y}} \left(1 - \frac{\gamma}{y}\right) \frac{dy}{y-x} = \pi \left(1 - \frac{\gamma}{x}\sqrt{\frac{a}{b}}\right).$$

Problem 16. Let $c > 0$ and $k \in \mathbb{R}$. Show that

$$\int_0^\infty \exp(-cs^2)\cos(2ks) ds = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{c}} \exp(-k^2/c).$$

This integral plays a role in optics.

Problem 17. Let $\alpha > 0$. Show that

$$int_0^\infty \exp(-\alpha x) dx = \frac{1}{\alpha}.$$

Problem 18. Let $x > 0$. Show that

$$2\int_0^\infty e^{-s^2} ds = \int_{x^2}^\infty \frac{e^u}{\sqrt{u}}$$

Problem 19. Let $x \geq 0$. Consider the function f defined by

$$f(x) = \int_0^x \frac{\ln(1+y)}{y} dy.$$

Show that for small X we can write

$$f(x) = x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \cdots$$

Show that
$$f(x)=\frac{\pi^2}{6}+\frac{1}{2}\ln^2(x)-f\left(\frac{1}{x}\right).$$
Give a value of $f(1)$.

Problem 20. Calculating
$$\int \cos^3(x)\sin^3(x)dx$$
student Alice tells you the result is
$$\frac{1}{4}\sin^4(x)-\frac{1}{6}\sin^6(x)+C_1$$
and student Bob tells that
$$\frac{1}{6}\cos^6(x)-\frac{1}{4}\cos^4(x)+C_2$$
is the result. C_1, C_2 are the constants of integration. Discuss.

Problem 21. Let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ be an analytic function, $g_j:\mathbb{R}\to\mathbb{R}$ $(j=1,\dots,n)$ are analytic functions and $c_1,\dots,c_n$ are constants. Consider
$$I(\epsilon)=\int_{c_1}^{g_1(\epsilon)}dx_1\int_{c_2}^{g_2(\epsilon)}dx_2\cdots\int_{c_n}^{g_n(\epsilon)}f(x_1,x_2,\dots,x_n,\epsilon).$$
Show that
$$\begin{aligned}\frac{dI(\epsilon)}{d\epsilon}&=\int_{c_1}^{g_1(\epsilon)}dx_1\int_{c_2}^{g_2(\epsilon)}dx_2\cdots\int_{c_n}^{g_n(\epsilon)}\frac{\partial f(x_1,x_2,\dots,x_n,\epsilon)}{\partial\epsilon}\\ &+\frac{dg_1}{d\epsilon}\int_{c_2}^{g_2(\epsilon)}dx_2\int_{c_3}g_3(\epsilon)\cdots\int_{c_n}^{g_n(\epsilon)}f(x_1,x_2,\dots,x_n,\epsilon)\\ &+\frac{dg_2}{d\epsilon}\int_{c_1}^{g_1(\epsilon)}dx_1\int_{c_3}^{g_3(\epsilon)}dx_3\cdots\int_{c_n}^{g_n(\epsilon)}f(x_1,x_2,\dots,x_n,\epsilon)\\ &+\cdots\\ &+\frac{dg_n}{d\epsilon}\int_{c_1}^{g_1(\epsilon)}dx_1\int_{c_2}^{g_2(\epsilon)}dx_2\cdots\int_{c_{n-1}}^{g_{n-1}(\epsilon)}dx_{n-1}f(x_1,x_2,\dots,x_n,\epsilon).\end{aligned}$$

Problem 22. Show that
$$\int_0^\infty\frac{tdt}{e^{2\pi t}-1}=\frac{1}{4\pi^2}\int_0^\infty\frac{\tau d\tau}{e^\tau-1}=\frac{1}{4\pi^2}\frac{\pi^2}{6}=\frac{1}{24}.$$

Problem 23. Let $k \in \mathbb{N}$. Show that

$$\int_0^{2\pi} \cos^k(\theta)d\theta = 2(1+(-1)^k)\pi \frac{(k-1)!!}{k!!}.$$

Problem 24. Let $k \in \mathbb{N}$. Show that

$$\int_0^{2\pi} \cos^k(\theta)d\theta = 2(1+(-1)^k)\pi \frac{(k-1)!!}{k!!}.$$

Problem 25. Show that

$$\int_0^1 \sqrt{1-\sqrt{x}}dx = \frac{8}{15}.$$

Hint. Set $x = \sin^4(y)$. Then $dx = 4\sin^3(y)\cos(y)dy$.

Problem 26. Let $a > 0$. Show that

$$\int \frac{x^2 dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin(x/a) + C$$

Problem 27. Show that

$$2\int_0^{\infty} \frac{\sin(2px)\sin(qx)}{x}dx = \ln \frac{|2p+q|}{|2p-q|}.$$

Problem 28. For a $\lambda/2$ antenna we obtain the expression

$$E(r,\theta) = -\frac{\omega I_0 \sin\theta}{4\pi\epsilon_0 c^2 r}\int_{-\lambda/4}^{\lambda/4} \cos(k\ell)\sin(\omega(t-c^{-1}(r-\ell\cos\theta)))d\ell.$$

Calculate $E(r,\theta)$.

Problem 29. Let $\Re(a) > 0$ and $n = 0, 1, \ldots$. Show that

$$\begin{aligned}\int_{-\infty}^{\infty} x^n e^{-ax^2+px}dx &= \frac{\partial^n}{\partial p^n}\int_{-\infty}^{\infty} e^{-ax^2+px}dx \\ &= \frac{\partial^n}{\partial p^n}\sqrt{\frac{\pi}{a}}e^{p^2/(4a)}.\end{aligned}$$

Problem 30. Let $a \neq 0$. Show that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x} dx = \pi \mathrm{sgn}(a).$$

Problem 31. Let $c > 1$. Show that

$$\int_1^c \sin(x^2) dx = \frac{1}{2} \int_1^{c^2} \frac{\sin(\tau)}{\sqrt{\tau}} d\tau.$$

Problem 32. Show that

$$\int_0^1 x(1+x^2)^{1/2} dx = \frac{1}{3}(2^{3/2} - 1)$$

setting $u = 1 + x^2$ and hence $du = 2xdx$.

Problem 33. Let $(B_\tau,\, \tau \geq 0)$ be the linear Brownian motion starting from 0. Show that

$$\Gamma_+ := \int_0^1 ds\, 1_{(B_s > 0)}$$

follows the arcsine distribution, i.e.

$$P(\Gamma_+ \in d\tau) = \frac{1}{\pi} \frac{d\tau}{\sqrt{\tau(1-\tau)}}.$$

Problem 34. Let $a > 0$. Show that

$$\int_{\mathbb{R}} \frac{\sin^2(ax/2)}{x^2} dx = \frac{1}{2} \pi a.$$

Chapter 9

Functional Equations

9.1 Solved Problems

Problem 1. To solve a number of nonlinear functional equation it is helpful to have the solution of the linear functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

where we assume that the function f is continouos.

Problem 2. Find the solution of the functional equations

$$f(x+y) = f(x)f(y) \tag{1}$$

where we assume that f is continuous.

Problem 3. Find the solution of the functional equation

$$f(xy) = f(x) + f(y) \tag{1}$$

where we assume that the function f is continuous.

Problem 4. Find the solution of the functional equation

$$f(xy) = f(x)f(y). \tag{1}$$

We assume that the function f is continuous.

Problem 5. Find the solution of the *Jensen equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \tag{1}$$

where we assume that f is continuous.

Problem 6. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$$

admits the solutions $f(x) = \tan(cx)$, where c is a constant.

Problem 7. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)}{1+\frac{f(x)f(y)}{C^2}} \tag{1}$$

admits the solution

$$f(x) = C\tanh(cx). \tag{2}$$

Problem 8. Show that the functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$$

admits the solution

$$f(x) = \frac{c}{x}.$$

Problem 9. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)-1}{2f(x)+2f(y)-2f(x)f(y)-1}$$

admits the solution

$$f(x) = \frac{1}{1+\tan(cx)}.$$

Problem 10. Show that the functional equation

$$f(x+y) = f(x)f(y) + \sqrt{f(x)^2}\sqrt{f(y)^2-1}$$

admits the solution

$$f(x) = \cosh(cx).$$

Problem 11. Show that the functional equation

$$f(x + y + axy) = f(x)f(y)$$

admits the solution

$$f(x) = (1 + ax)^c.$$

Problem 12. Show that the functional equation

$$f(x + y) - f(x - y) = 4\sqrt{f(x)f(y)}$$

admits the solution

$$f(x) = cx^2.$$

Problem 13. Solve the functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

assuming that f is a continuous function.

Problem 14. Show that the trigonometric functions $f(x) = \cos(x)$ and $g(x) = \sin(x)$ satisfy the system of functional equations

$$\begin{aligned}
g(x + y) &= g(x)f(y) + f(x)g(y) \\
f(x + y) &= f(x)f(y) - g(x)g(y) \\
g(x - y) &= g(x)f(y) - g(y)f(x) \\
f(x - y) &= f(x)f(y) + g(x)g(y).
\end{aligned}$$

Problem 15. Show that the Jacobi elliptic functions satisfy the system of functional equations

$$f(x \pm y) = \frac{f(x)g(y)h(y) \pm f(y)g(x)h(x)}{1 - k^2 f(x)^2 f(y)^2}$$

$$g(x \pm y) = \frac{g(x)g(y) \mp f(x)f(y)h(x)h(y)}{1 - k^2 f(x)^2 f(y)^2}$$

$$h(x \pm y) = \frac{h(x)h(y) \mp k^2 f(x)f(y)g(x)g(y)}{1 - k^2 f(x)^2 f(y)^2}.$$

Problem 16. Let a be a positive integer with $a \geq 2$. Let $1 \leq x \leq a$. Consider the equation

$$g(x-1) - 2g(x) + g(x+1) = -\lambda g(x).$$

Show that

$$g_j(x) = \sin(j\pi x/(a+1)), \qquad \lambda_j(x) = 2(1 - \cos(j\pi/(a+1)))$$

satisfy this equation.

Problem 17. Let a, c, ϵ be positive constants. Solve the functional equation

$$g((\theta + c)(\mathrm{mod}\, 1)) = a g(\theta) + \epsilon \sin(2\pi\theta).$$

9.2 Supplementary Problems

Problem 1. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)+2f(x)f(y)}{1-f(x)f(y)}$$

admits the solution

$$f(x) = \frac{cx}{1-cx}.$$

Problem 2. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \arctan(x).$$

Show that

$$f(x)+f(y) = f\left(\frac{x+y}{1-xy}\right).$$

What happens at $xy = 1$?

Problem 3. Consider the functional equation

$$g(x) = \alpha g(g(x/\alpha))$$

with $g(0) = 0$ and $g'(x=0) = 1$. Show that

$$g(x) = \frac{x}{1-cx}$$

is a solution with c an arbitarary constant.

Problem 4. We know that

$$\tan(\alpha+\beta) \equiv \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

Show that

$$\tan(\alpha+\beta+\gamma) =$$
$$\frac{\tan(\alpha)+\tan(\beta)+\tan(\gamma)-\tan(\alpha)\tan(\beta)\tan(\gamma)}{1-\tan(\alpha)\tan(\beta)-\tan(\alpha)\tan(\gamma)-\tan(\beta)\tan(\gamma)}.$$

Chapter 10

Inequalities

10.1 Solved Problems

Problem 1. Let $a, b \in \mathbb{R}$. Show that

$$2ab \leq a^2 + b^2.$$

Problem 2. Let $a, b \in \mathbb{R}$. Show that

$$|a + b| \leq |a| + |b|.$$

Problem 3. Let $a, b \in \mathbb{R}^+$. Show that

$$\frac{1}{2}(a + b) \geq \sqrt{ab}.$$

Problem 4. Let $x \geq 0$ and $0 < p < 1$. Show that

$$\frac{1}{p}(1 - x^p) \geq 1 - x.$$

Problem 5. Let $x \in (0, 1)$. Show that

$$x(1 - x) < x.$$

Problem 6. Let $n \in \mathbb{N}$.
(i) Show that $n < 2^n$.
(ii) Show that $n^2 < 4^n$.
(iii) Show that if $n \geq 4$ then $2^n < n!$.

Problem 7. Let x, y be two nonnegative real numbers. Show that

$$xy \leq \left(\frac{x+y}{2}\right)^2 \equiv \frac{1}{4}(x^2 + y^2 + 2xy).$$

Problem 8. Let a, b, c, d be nonnegative real numbers. Show that

$$(abcd)^{1/4} \leq \frac{1}{4}(a+b+c+d).$$

Problem 9. Let x_j $(j = 1, \ldots, n)$ be nonegative real numbers. Show that

$$x_1 x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n.$$

Problem 10. Let a, b, c, d be nonnegative real numbers. Show that

$$a^4 + b^4 + c^4 + d^4 \geq 4abcd.$$

Problem 11. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that (*Bernoulli inequality*)

$$(1+x)^n \geq 1 + nx.$$

Problem 12. Let $x \geq 0$. Show that

$$1 - e^{-x} \geq \frac{x}{1+x}.$$

Problem 13. Let

$$e_n := 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

where $n = 1, 2, \ldots$. Let $m > n$. Show that

$$|e_m - e_n| = e_m - e_n \leq \frac{2}{(n+1)!}.$$

Problem 14. Does the inequality

$$1 + 2\cos(\theta) - \cos(2\theta) \leq 2$$

hold for all $\theta \in [0, 2\pi)$?

Problem 15. Let $x > 0$. Show that

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}.$$

Problem 16. Let $x > 0$. Show that

$$1 - \frac{1}{x} \leq \ln(x) \leq x - 1$$

with equality iff $x = 1$.

Problem 17. Show that if a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a second order derivative which is non-negative (positive) everywhere, then the function is convex (strictly convex).

Problem 18. Let $a_1, a_2, \ldots, a_n$ be positive numbers and $b_1, b_2, \ldots, b_n$ be nonnegative numbers such that

$$\sum_{j=1}^{n} b_j > 0.$$

Show that (*log-sum inequality*)

$$\sum_{j=1}^{n} \left(a_j \log \frac{a_j}{b_j} \right) \geq \left(\sum_{j=1}^{n} a_j \right) \log \frac{\left(\sum_{j=1}^{n} a_j \right)}{\left(\sum_{j=1}^{n} b_j \right)}$$

with the conventions based on continuity arguments

$$0 \cdot \log 0 = 0, \qquad 0 \cdot \log \frac{p}{0} = \infty, \quad p > 0.$$

Show that equality holds if and only if $a_j/b_j = \text{constant}$ for all $j = 1, 2, \ldots, n$.

Problem 19. Let x_j $(j = 1, 2, \ldots, n)$ be positive real numbers. Show that

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{\sum_{k=1}^{n} x_k}{n}.$$

This is the arithmetic-geometric mean inequality.

Problem 20. Let n be a positive integer and $a, b \geq c/2 > 0$. Show that

$$|a^{-n} - b^{-n}| \leq 4nc^{-n-1}|a - b|.$$

Problem 21. Let $X \subset \mathbb{R}$ be an interval. A function $\psi : X \to \mathbb{R}$ is *convex* if for all $x_1, x_2 \in X$ and numbers $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$,

$$\psi(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 \psi(x_1) + \alpha_2 \psi(x_2). \tag{1}$$

This means that every chord of the graph of ψ lies above the graph. Let $\psi : X \to \mathbb{R}$ be convex, let $x_1, x_2, \ldots, x_n \in X$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ satisfy $\sum_{j=1}^{n} \alpha_j = 1$. Show that (Jensen's inequality)

$$\psi\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \psi(x_i). \tag{2}$$

Problem 22. Consider the differentiable function $f : [0, \infty) \to \mathbb{R}$

$$f(x) = \frac{x}{1+x}.$$

Let $a, b \in \mathbb{R}^+$. Show that

$$f(|a+b|) \leq f(|a| + |b|). \tag{1}$$

Problem 23. Let $a, b \in \mathbb{R}$. Show that

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

Problem 24. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is convex.

Problem 25. Let $n \in \mathbb{N}$ and $n \geq 2$. Show that

$$e^n \ln(n+1) < e^{n+1} \ln(n).$$

Problem 26. Show that the function $f : (0, \infty) \to \mathbb{R}$

$$f(x) = x \ln(x)$$

is convex.

Problem 27. Let $x, y \in \mathbb{R}$. Show that

$$\frac{x+y}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Problem 28. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\|AB\| \leq \|A\| \cdot \|B\|.$$

Problem 29. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\|A+B\| \leq \|A\| + \|B\|.$$

Problem 30. Let $\mathcal{H}$ be a Hilbert space. Let $\psi, \phi \in \mathcal{H}$. Assume that $\psi \neq 0, \phi \neq 0$. Show that

$$|\langle \phi, \psi \rangle| \leq \sqrt{\langle \phi, \phi \rangle} \sqrt{\langle \psi, \psi \rangle} \tag{1}$$

where $\langle \, , \, \rangle$ denotes the scalar product in the Hilbert space.

Problem 31. Let $n \geq 2$. Let $x_1, x_2, \ldots, x_n$ be given positive real number with

$$x_1 < x_2 < \cdots < x_n.$$

Let $\lambda_1, \ldots, \lambda_n \geq 0$ and $\sum_{j=1}^n \lambda_j = 1$. Show that

$$\left(\sum_{j=1}^n \lambda_j x_j \right) \left(\sum_{j=1}^n \lambda_j x_j^{-1} \right) \leq A^2 G^{-2}$$

where

$$A = \frac{1}{2}(x_1 + x_n), \qquad G = (x_1 x_n)^{1/2}.$$

Problem 32. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (column vectors) and $\epsilon > 0$. Show that

$$2\mathbf{x}^T \mathbf{y} \leq \epsilon \mathbf{x}^T \mathbf{x} + \frac{1}{\epsilon} \mathbf{y}^T \mathbf{y}.$$

Problem 33. Let $a, b \in \mathbb{R}$ and $\epsilon > 0$. Show that

$$2ab \leq \epsilon a^2 + \epsilon^{-1} b^2.$$

Problem 34. Show that the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(\alpha, \beta) = \sin(\alpha)\sin(\beta)\cos(\alpha - \beta)$$

is bounded between $-1/8$ and 1.

Problem 35. Let n be an integer and $n \geq 2$. Show that

$$e^n \ln(n+1) < e^{n+1} \ln(n).$$

Problem 36. Let a, b, c be positive numbers. Assume that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Show that

$$x + y + z \leq \sqrt{a^2 + b^2 + c^2}.$$

Use the Lagrange multiplier method.

Problem 37. Let a, b, c, d be positive numbers and $s := a + b + c + d$. Show that

$$\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} + \frac{s}{s-d} \geq \frac{16}{3}.$$

Problem 38. Let $x \geq 0$ and $0 < p < 1$. Show that

$$\frac{1}{p}(1 - x^p) \geq 1 - x.$$

Problem 39. Let b be a real number such that the elements of the infinite sequence $(a_k)_{k=1}^{\infty}$ satisfy

$$a_{k+m} \leq a_k + a_m + b$$

for all $k, m = 1, 2, \ldots$. Show that

$$a := \lim_{k \to \infty} \frac{a_k}{k}$$

exists and $a_k \geq ka - b$ for all k.

Problem 40. Let $n \in \mathbb{N}$ and $n \geq 2$. Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Problem 41. Let $a, b \geq 0$. Show that

$$\frac{a+b}{1+a+b} \leq \frac{a+b+ab}{1+a+b+ab} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Problem 42. Show that

$$\frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin^2(x-y)}{(x-y)^2} dxdy < \frac{1}{4}.$$

Problem 43. Let $x_1, x_2, x_3 \in \mathbb{R}$. Show that

$$x_1^2 + x_2^2 + x_3^2 \geq x_1 x_2 + x_2 x_3 + x_3 x_1.$$

Why could it be useful to consider the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 - x_3 x_1 \, ?$$

Problem 44. Let x_1, x_2, $\ldots$, x_n be positive numbers. Assume that the numbers satisfy

$$\sum_{j=1}^{n} x_j = 1, \qquad \sum_{j=1}^{n} x_j^2 = b^2.$$

Show that

$$\max\{\, x_j \;:\; 1 \leq j \leq n \} \leq \frac{1}{n}(1 + \sqrt{n-1}\sqrt{nb^2 - 1}).$$

Problem 45. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Show that

$$|f(x_2) - f(x_1)| \leq 4|x_2 - x_1|$$

for all x_1 and x_2 in $[-2, 2]$. Hint. Apply

$$x_1^2 - x_2^2 \equiv (x_1 + x_2)(x_1 - x_2).$$

Problem 46. Let $n \in \mathbb{N}$. Show that

$$\frac{n}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1} \leq n.$$

Problem 47. Let

$$0 \le a_1 \le a_2 \le \cdots \le a_n, \qquad 0 \le b_1 \le b_2 \le \cdots \le b_n$$

and $r \ge 1$. Show that (*Chebyshev inequality*)

$$\left(\frac{1}{n}\sum_{j=1}^{n} a_j^r\right)^{1/r} \left(\frac{1}{n}\sum_{j=1}^{n} b_j^r\right)^{1/r} \le \left(\frac{1}{n}\sum_{j=1}^{n} (a_j b_j)^r\right)^{1/r}.$$

Problem 48. Let $a, x, y \in \mathbb{R}$. Show that

$$|\sqrt{a^2+x^2} - \sqrt{a^2+y^2}| \le |x-y|.$$

Problem 49. Let $b > a > 0$. Show that

$$\left(1 - \frac{a}{b}\right) < \ln\left(\frac{b}{a}\right) < \frac{b}{a} - 1.$$

Problem 50. Show that

$$\left|\int_0^1 \frac{\cos(nx)}{x+1}\right| \le \ln(2)$$

for $n \in \mathbb{N}$.

Problem 51. Let $n \ge 2$. Show that

$$2^n < \binom{2n}{n} < 2^{2n}.$$

Problem 52. Consider the two manifolds

$$x_1^2 + x_2^2 = 1, \qquad y_1^2 + y_2^2 = 1.$$

Show that

$$|x_1 y_1 + x_2 y_2| \le 1.$$

Hint. Set

$$x_1(t) = \cos(t), \quad x_2(t) = \sin(t), \quad y_1(t) = \cos(\tau), \quad y_2(t) = \sin(\tau).$$

Problem 53. Let $n \in \mathbb{N}$ and $x > 0$. Show that

$$x^{n+1} + \frac{1}{x^{n+1}} > x^n + \frac{1}{x^n}.$$

Problem 54. Find all $x \in \mathbb{R}$ and $x > 0$ such that

$$|\sqrt{x} - \sqrt{2}| < 1.$$

Problem 55. Let j, k be positive intgers. Find all pairs (j,k) such that the following four inequalities are satisfied

$$j + k < 10, \quad j + k \geq 6, \quad \frac{j}{k} > 1, \quad \frac{j}{k} < 2.$$

Problem 56. Let n be a positive integer. Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

Problem 57. Let $z = x + iy$ with $x, y \in \mathbb{R}$. Show that

$$\frac{1}{\sqrt{2}}(|x| + |y|) \leq |z| \leq |x| + |y|.$$

Matrices

Problem 58. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$|\mathrm{tr}(AB^*)|^2 \leq \mathrm{tr}(AA^*)\mathrm{tr}(BB^*).$$

Problem 59. Let A and B be two $n \times n$ matrices over $\mathbb{R}$. It can be shown that

$$\mathrm{tr}e^{A+B} \leq \mathrm{tr}(e^A e^B) \leq \frac{1}{2}\mathrm{tr}(e^{2A} + e^{2B})$$
$$\mathrm{tr}e^{A+B} \leq \mathrm{tr}(e^A e^B) \leq (\mathrm{tr}e^{pA})^{1/p}(\mathrm{tr}e^{qB})^{1/q}$$

where $p > 1$, $q > 1$ with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Is

$$(\mathrm{tr}e^{pA})^{1/p}(\mathrm{tr}e^{qB})^{1/q} \leq \frac{1}{2}\mathrm{tr}(e^{2A} + e^{2B})\,?$$

Prove or disprove.

Problem 60. Let A, B be hermitian matrices. Then

$$\mathrm{tr}(e^{A+B}) \leq \mathrm{tr}(e^A e^B).$$

Assume that

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate the left and right-hand side of the inequality. Does equality hold?

Problem 61. Let A, B be positive definite matrices. Show that

$$\mathrm{tr}(AB)^{2^{p+1}} \leq \mathrm{tr}(A^2B^2)^{2^p}, \qquad p \text{ a non-negative integer}$$

Problem 62. Let A be an $n \times n$ matrix with $\|A\| < 1$.
(i) Show that $(I_n + A)^{-1}$ exists.
(ii) Show that

$$\|(I_n + A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Problem 63. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $a_{jj} \geq 1$ for all j and

$$\sum_{j \neq k}^{n} a_{jk}^2 < 1.$$

Show that A is inververtible.

Problem 64. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let I_n be the $n \times n$ identity matrix. Assume that

$$\sum_{k=1}^{n} |a_{jk}| < 1$$

for each j. Show that $I_n - A$ is invertible.

Problem 65. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$. Let B be an $n \times n$ positive semidefinite matrix over $\mathbb{R}$. Show that

$$\det(A + B) \geq \det(A).$$

Problem 66. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that there exists nonnull vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^n$ such that

$$\frac{\mathbf{x}_1^T A \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \leq \frac{\mathbf{x}_2^T A \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2}$$

for every nonnull vector $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 67. Let A be an $n \times n$ skew-symmetric matrix over $\mathbb{R}$. Show that

$$\det(I_n + A) \geq 1$$

with equality holding if and only if $A = 0$.

Problem 68. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $a_{jj} \geq 1$ for $j = 1, 2, \ldots, n$ and

$$\sum_{j \neq k}^{n} a_{jk}^2 < 1.$$

Show that A is invertible.

Problem 69. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that

$$|a_{jj}| > \sum_{k \neq j}^{n} |a_{jk}|$$

for all $j = 1, 2, \ldots, n$. Show that A is invertible.

Problem 70. Let A, B be $n \times n$ positive definite matries. Show that

$$\mathrm{tr}(A \ln(A)) - \mathrm{tr}(A \ln(B)) \geq \mathrm{tr}(A - B).$$

Problem 71. Let $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$ and let A be an $n \times n$ hermitian matrix. Is

$$\mathbf{v}^* e^A \mathbf{v} \geq e^{\mathbf{v}^* A \mathbf{v}}$$

for all normalized $\mathbf{v}$? Prove or disprove.

10.2 Supplementary Problems

Problem 1. Let $n \in \mathbb{N}$ and $n \geq 2$. Show that

$$\frac{1}{\sqrt{n-1}} > 2\sqrt{n} - 2\sqrt{n-1} > \frac{1}{\sqrt{n}}.$$

Use this reult to show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

diverges.

Problem 2. Let $b > a$. Show that

$$\frac{b-a}{1+b^2} < \arctan(b) - \arctan(a) < \frac{b-a}{1+a^2}.$$

Note that $\arctan(-x) = -\arctan(x)$.

Problem 3. Let n be is positive integer. Is

$$\sqrt[n+1]{(n+1)!} > \sqrt[n]{n!}$$

for all n?

Problem 4. Let $a, b, c \in \mathbb{C}$ with $|a| = |b| = |c| = 1$. Suppose that $\Im(a) \geq 0$, $\Im(b) \geq 0$, $\Im(c) \leq 0$. Show that

$$\left| \frac{-1 + ab + bc + ca}{4} \right| \leq \frac{1}{\sqrt{2}}.$$

Problem 5. Let n be a positive integer with $n \geq 2$. Show that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} < 1.$$

Problem 6. Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ be positive numbers. Assume that the numbers satisfy

$$\sum_{j=1}^{n} x_j = 1, \qquad \sum_{j=1}^{n} x_j^2 = b^2.$$

Show that

$$\max\{\, x_j \, : \, 1 \leq j \leq n \,\} \leq \frac{1}{n} \left(1 + \sqrt{n-1}\sqrt{nb^2 - 1} \right).$$

Problem 7. Show that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Problem 8. Let $x \geq 0$. Show that $1 + x \leq 3^x$.

Problem 9. Let A be an $n \times n$ matrix. Is

$$|\det(I_n + A)| \leq \exp(\|A\|)?$$

Does it depend on the chosen norm?

Problem 10. Let z_j $(j = 1, \ldots, p)$ be fixed complex numbers. Is

$$\prod_{j=1}^{p} |z - z_j| > \prod_{j=1}^{p} (|z_j| - |z|)\,?$$

Problem 11. Let $x \in (0, 1)$. Show that

$$x > \ln(1 + x) > x - \frac{x^2}{2}.$$

Problem 12. Let $x, y \in \mathbb{R}$. Show that

$$\frac{|x + y|}{1 + |x + y|} \leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}.$$

Problem 13. Let $x_1 > 0$, $x_2 > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$. Show that

$$x_1^{\alpha_1} x_2^{\alpha_2} \leq \alpha_1 x_1 + \alpha_2 x_2.$$

Apply $\ln(x)$.

Bibliography

Abramowitz M. and Stegun (editors)
Pocketbook of Mathematical Functions
Harry Deutsch (1984)

Aldous J. M. and Wilson R. J.
Graphs and Applications: An Introductory Approach
Springer (2000)

Armstrong M. A.
Groups and Symmetry
Springer (1988)

Barbeau E. J.
Polynomials, Springer-Verlag (1989)

Bender C. M. and Orszag S. A. *Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New Yrok 1978

Bredon G. E.
Introduction to Compact Transformation Groups
Academic Press (1972)

Bronson R.
Matrix Operations
Schaum's Outlines, McGraw-Hill (1989)

Chern S. S., Chen W. H. and Lam K. S.
Lectures on Differential Geometry
World Scientific (1999)

DasGupta Ananda
American J. Phys. **64** 1422–1427 (1996)

de Souza P. N. and Silva J.-N.
Berkeley Problems in Mathematics
Springer (1998)

Englefield M. J.
Group Theory and the Coulomb Problem
Wiley-Interscience, New York (1972)

Erdmann K. and Wildon M.
Introduction to Lie Algebras,
Springer (2006)

Fuhrmann, P. A.
A Polynomial Approach to Linear Algebra
Springer (1996)

Golub G. H. and Van Loan C. F.
Matrix Computations, Third Edition,
Johns Hopkins University Press (1996)

Gradshteyn I. S. and Ryzhik I. M.
Tables of Integrals, Series and Products
Academic Press, New York (1980)

Hall B. C.
Lie Groups, Lie Algebras, and Representations: an elementary introduction
Springer (2003)

Hardy G. H., Littlewood J. E. and Pólya G.
Inequalities, 2nd edition, Cambridge University Press, Cambridge, 1952

Helgason S.
Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators and Spherical Functions
Academic Press (1984)

Horn R. A. and Johnson C. R.
Topics in Matrix Analysis
Cambridge University Press (1999)

Kedlaya K. S., Poonen B. and Vakil R.
The William Lowell Putnam Mathematical Competition 1985–2000,

The Mathematical Association of America (2002)

Keener J. P.
Principles of Applied Mathematics: Transformation and Approximation, Perseus Books, second revised edition (2000)

Lang S.
Linear Algebra
Addison-Wesley, Reading (1968)

Lay S. R.
Analysis: with an introduction to proof
Pearson Prentice Hall (2005)

Letac G.
Integration and Probability by Paul Malliavin: Exercises and Solution
Springer (1995)

Malik S. C. and Arora S.
Mathematical Analysis, 2nd edition
New Age International (1992)

Miller W.
Symmetry Groups and Their Applications
Academic Press, New York (1972)

Posamentier A. S. and Salkind C. T.
Challenging Problems in Algebra, Dover Publications, New York 1996

Riley K. F. and Hobson M. P.
Essential Mathematical Methods for the Physical Sciences: Student Solution Manual
Cambridge Unversity Press, 2011

Rosenlicht M.
Introduction to Analysis
Dover (1968)

Sneddon I.
Elements of Partial Differential Equations, McGraw-Hill (1957)

Spiegel M. R.
Advanced Calculus, Schaum's Outline Series, McGraw-Hill (1962)

Spiegel M. R.
Complex Variables, Schaum's Outline Series, McGraw-Hill (1964)

Steeb W.-H.
Matrix Calculus and Kronecker Product with Applications and C++ Programs
World Scientific Publishing, Singapore (1997)

Steeb W.-H.
Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra
World Scientific Publishing, Singapore (1996)

Steeb W.-H.
Hilbert Spaces, Wavelets, Generalized Functions and Quantum Mechanics
Kluwer Academic Publishers, Dordrecht (1998)

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics,
Second Edition, Volume I: Introductory Level
World Scientific Publishing, Singapore (2003)

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics,
Second Edition, Volume II: Advanced Level
World Scientific Publishing, Singapore (2003)

Steeb W.-H., Hardy Y., Hardy A. and Stoop R.
Problems and Solutions in Scientific Computing with C++ and Java Simulations
World Scientific Publishing, Singapore (2004)

Steele J. M.
The Cauchy-Schwarz Master Class, Cambridge University Press, 2000

Stewart J.
Calculus
Brooks/Cole (2012)

Watkins D. S.
Fundamentals of matrix computations
Wiley

Whittaker E. T.
A treatise on the analytical dynamics of particles and rigid bodies

Whittaker E. T. and Watson G. N. *A Course of Modern Analysis* 4th edition, Cambridge University Press (1962)

Wybourne B. G.
Classical Groups for Physicists
John Wiley, New York (1974)

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