Problems and Solutions in Nonlinear Dynamics, Chaos and Fractals

by
Willi-Hans Steeb
International School for Scientific Computing
at
University of Johannesburg, South Africa

Charles Villet
Department of Applied Mathematics
at
University of Johannesburg, South Africa

Yorick Hardy
Department of Mathematical Sciences
at
University of South Africa, South Africa

Ruedi Stoop
Institute of Neuroinformatik
University / ETH Zürich

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Chapter 1

One-Dimensional Maps

1.1 Notations and Definitions

We consider exercises for nonlinear one-dimensional maps. In particular we consider one-dimensional maps with chaotic behaviour. We first summarize the relevant definitions such as fixed points, stability, periodic orbit, Ljapunov exponent, invariant density, topologically conjugacy, etc.. Ergodic maps are also considered.

We use the notation $f : D \to C$ to indicate that a function f with domain D and codomain C. The notation $f : D \to D$ indicates that the domain and codomain of the function are the same set.

We also use the following two definitions: A mapping $g : A \mapsto B$ is called *surjective* if $g(A) = B$. A mapping g is called *injective* (one-to-one) when $\forall a, a' \in A, g(a) = g(a') \Rightarrow a = a'$. If a the mapping f is surjective and injective the mapping f is called bijective.

Definition. If $B \subset C$, then $f^{-1}(B)$ is called the *inverse image* or *preimage* of B and consists of all elements of D whose image is contained in B. That is

$$f^{-1}(B) := \{\, x \in D : f(x) \in B \,\}.$$

Thus the use of the notation f^{-1} does not necessarily imply that f is an invertible function.

Definition. Consider a map $f : S \to S$. A point $x^* \in S$ is called a *fixed point* of f if

$$f(x^*) = x^*$$

Definition. Let $f : A \to A$ and $g : B \to B$ be two maps. The maps f and g are said to be *topologically conjugate* if there exists a homeomorphism $h : A \to B$ such that, $h \circ f = g \circ h$.

Definition. Consider a map $f : S \to S$. A point $x \in S$ is an eventually fixed point of the function, if there exists $N \in \mathbb{N}$ such that

$$f^{(n+1)}(x) = f^{(n)}(x)$$

whenever $n \geq N$. The point x is eventually periodic with period k, if there exists N such that $f^{(n+k)}(x) = f^{(n)}(x)$ whenever $n \geq N$.

Definition. Let f be a function and p be a periodic point of f with prime period k. Then the point x is *forward asymptotic* to p if the sequence

$$x, \quad f^{(k)}(x), \quad f^{(2k)}(x), \quad f^{(3k)}(x), \ldots$$

converges to p. In other words,

$$\lim_{n\to\infty} f^{(nk)}(x) = p.$$

Definition. The stable set of p, denoted by $W^s(p)$, consists of all points which are forward asymptotic to p. If the sequence

$$|x|, \quad |f(x)|, \quad |f^{(2)}(x)|, \quad |f^{(3)}(x)|, \ldots$$

grows without bound, then x is forward asymptotic to ∞. The stable set of ∞, denoted by $W^s(\infty)$, consists of all points which are forward asymptotic to ∞.

Definition. Let p be a periodic point of the differentiable function f with prime period k. Then p is a *hyperbolic periodic point* if

$$\left| \frac{df^{(k)}}{dx}(x = p) \right| \neq 1.$$

If

$$\left| \frac{df^{(k)}}{dx}(x = p) \right| = 1$$

then p is a nonhyperbolic periodic point.

Definition. Let f be a map of an interval into itself. Consider the one-dimensional difference equation

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots$$

with chaotic behaviour. Assume that in its chaotic regime the map f has a unique invariant measure which is absolutely continuous with respect to the Lebesgue measure. By virtue of ergodicity, the invariant density, denoted by ρ, is determined as a unique solution to the equation

$$\rho(x) = \int_I dy \delta(x - f(y))\rho(x).$$

This equation is called the *Frobenius-Perron integral equation.*

Definition. Consider one-dimensional maps $f : I \to I$. We assume that f is differentiable. One defines the average rate of growth as

$$\lambda(x_0, \delta x_0) = \lim_{n\to\infty} \frac{1}{n} \ln |D_{x(0)} f^{(n)} \delta x(0)|$$

where δx satisfies the variational equation. By a theorem of Oseledec, this limit exists for almost all $x(0)$ with respect to the invariant measure. The average expansion value depends on the direction of the initial perturbation $\delta x(0)$, as well on $x(0)$. If the invariant measure is ergodic, the largest λ with respect to changes of $\delta x(0)$ is independent of $x(0)$, μ-almost everywhere. The number λ_1 is called the largest Liapunov exponent of the map f with respect to the measure μ.

Definition. The *topological entropy*, $H(f)$ gives a measure of the number of distinct trajectories generated by a map f. Unlike the metric entropy, $h(\mu, f)$, the topological entropy is a property of f alone and is not associated with any metric properties of the dynamics. It provides a measure of the number of trajectories, or orbits, $\{x, f(x), f^{(2)}(x) \ldots\}$ the map f has. This appears to be infinite, like the number of choices for x. However, orbits $\{x, f(x), f^{(2)}(x) \ldots\}$ and $\{y, f(y), f^{(2)}(y) \ldots\}$ are only considered distinct if

$$|f^{(k)}(x) - f^{(k)}(y)| > \epsilon \text{ for some } k > 0.$$

If one observes up to the nth iterate there will now only exist an enumerable number of orbits. If $m(\epsilon, n)$ is the maximum number of different orbits (that is trajectories separated by greater than ϵ) of length n, the topological entropy is defined as a measure of the exponential growth of M with n in the limit of arbitrarily fine discrimination between trajectories

$$H \equiv \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \ln M(n, \epsilon).$$

This indicates $M \sim \exp(Hn)$ in the limit. The topological entropy gives the rate of growth of orbits with finite length as their allowed length goes to infinity ($n \to \infty$) and resolution fidelity becomes arbitrarily fine ($\epsilon \to 0$).

The topological entropy is also determined by the number of fixed points of $f^{(n)}$, and the following is equivalent

$$H(f) = \lim_{n\to\infty} \frac{1}{n} \ln(\text{number of fixed points under the map} f^{(n)}).$$

There is a close relationship between Kolmogorov's metric entropy, $h(\mu, f)$, and the topological entropy. In particular, if f preserves several finite invariant measures μ_i, then a metric entropy $h_i(\mu_i, f)$ can be associated with each. The topological entropy can be shown to be equal to the largest metric entropy and the corresponding maximal measure is referred to as the Gibbs measure,

$$H(f) = \sup_{\mu}\{h(\mu, f)\}.$$

If f preserves a unique invariant measure, then h and H will be equal. An important feature of both the metric and topological entropies is their stable character. If one slightly alters some parameter determining the evolution of a chaotic dynamical system, a large change in its behaviour will generally result because of its exponental trajectory instability.

Definition. Suppose μ is invariant with respect to the map f, then the metric entropy $h(\mu, f)$ is defined as follows:
let $\alpha = \{A_i\}$ and $\beta = \{B_i\}$ ($i = 1, 2, \ldots n(\alpha)$ or $n(\beta)$) be partitions of the phase space I and let $\alpha^{(n)}$ be defined as

$$\alpha^{(n)} = \bigvee_{i=0}^{n-1} f^{-i}(\alpha)$$

where $f^{(-i)}(\alpha)$ is a partition of I into $f^{(-i)}(A_1)$, $f^{(-i)}(A_2)$, ..., $f^{-i}(A_{n(\alpha)})$ and $\alpha \vee \beta$ is the partition of I into the sets $A_i \cap B_j$ with independent i and j. The entropy of the partition α is defined as $H_\mu(\alpha)$ where

$$H_\mu(\alpha) := -\sum_{i=1}^{n(\alpha)} \mu(A_i) \ln \mu(A_i).$$

The entropy per unit step-length of the partition α is defined as

$$h_\alpha(\mu, f) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha(n))$$

and the *metric entropy* is defined as

$$h(\mu, f) := \sup h_\alpha(f, \mu)$$

where the supremum is taken over all finite (or countable), measurable partitons of the phase space. We call a partition a generator if the diameters of the members of $\alpha^{(n)}$ tend to zero as $n \to \infty$. If α is a generator then

$$h(f, \mu) = h_\alpha(f, \mu).$$

1.2 One-Dimensional Maps

1.2.1 Solved Problems

Problem 1. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 4x(1-x).$$

(i) The fixed points of the function f are the solutions of the equation $f(x^*) = x^*$. Find the fixed points.
(ii) The critical points of f are the solutions of the equation $df(x)/dx = 0$. Find the critical points of f. If there are critical points determine whether they relate to minima or maxima.
(iii) The roots of the function f are the solutions of $f(x) = 0$. Find the roots of f.
(iv) Find the fixed points of the analytic function $g(x) = f(f(x))$.
(v) Find the critical points of the analytic function $g(x) = f(f(x))$. If there are critical points of g determine whether they relate to minima and maxima.
(vi) Find the roots of the analytic function $g(x) = f(f(x))$.

If $x \in [0,1]$, then $f(x) \in [0,1]$. So the function f could be restricted to $f : [0,1] \to [0,1]$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuosly differentiable map. Let $f^{(n)}$ be the n-th iterate of f.
(i) Calculate the derivative of $f^{(n)}$ at x_0.
(ii) Apply it to $f(x) = 2x(1-x)$.

Problem 3. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1.$$

(i) Find the fixed points.
(ii) Show that $\{\, 0,\, 1,\, 2\,\}$ form an orbit of period three.

Problem 4. Let $f : \mathbb{Z} \to \mathbb{Z}$ with $f(n) = |n-1|$.
(i) Show that 0 and 1 form a periodic cycle.
(ii) Show that $n = 2$ is eventually periodic.
(iii) Show that every integer is eventually periodic.
(iv) Does the map f admit a fixed point?

Problem 5. Let $g : \mathbb{Z} \to \mathbb{Z}$

$$g(n) = |n^2 - 1|.$$

(i) Find $g(0)$, $g(g(0))$. Discuss.
(ii) Find the fixed points of g. Find the fixed points of $g \circ g$.

Problem 6. (i) Consider the *negation map* $f : \mathbb{R} \to \mathbb{R}$, i.e. $f(x) = -x$. Find all fixed points. Find all its periodic points.
(ii) Let $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = -x^3$. Show that g has a fixed point at 0 and a periodic cycle consisting of 1 and -1. Find the stable set.

Problem 7. Give an analytic map $f : \mathbb{R} \to \mathbb{R}$ which is eventually periodic.

Problem 8. (i) Consider the cubic map $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$. Find all its periodic points and the stable set of each.
(ii) Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. Calculate the iterate $f^{(n)}(x)$. Assume that $x \in (-1, 1)$. Find $\lim_{n\to\infty} f^{(n)}(x)$.

Problem 9. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x|x|.$$

(i) Is the function differentiable? If so find the derivative.
(ii) Find the fixed points of the function and study their stability.

Problem 10. (i) Show that the analytic function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \sin(x)$$

admits the fixed point $x^* = 0$. Study the stability of this fixed point.
(ii) Show that the analytic function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \sinh(x)$$

admits the fixed point $x^* = 0$. Study the stability of this fixed point.
(iii) Show that the analytic function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \tanh(x)$$

admits the fixed point $x^* = 0$. Study the stability of this fixed point.

Hint. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function and x^* be a fixed point, i.e. $x^* = f(x^*)$. The fixed point x^* is called non-hyperbolic if

$$|f'(x = x^*)| = 1$$

where $'$ denotes derivative. If $f'(x = x^*) = 1$, then three cases have to be studied: (i) If $f''(x = x^*) \neq 0$, then the fixed point x^* is semi-asymptotically from the left if $f''(x = x^*) > 0$ and from the right if

$f''(x = x^*) < 0$. (ii) If $f''(x = x^*) = 0$ and $f'''(x = x^*) < 0$, then the fixed point x^* is asymptotically stable. (iii) If $f''(x = x^*) = 0$ and $f'''(x = x^*)$, then x^* is unstable.
If $f'(x = x^*) = -1$, then two cases have to be studied: (i) If $Sf(x = x^*) < 0$, then the fixed point is asymptotically stable. (ii) If $Sf(x = x^*) > 0$, then the fixed point x^* is unstable. Here $Sf(x)$ denotes the *Schwarzian derivative* defined by

$$Sf(x) := \frac{f'''(x)}{f''(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Problem 11. Give examples of maps $f : [0,1] \to [0,1]$ where f is 1 to 1 and monotone on the interval $[0, 0.5]$ and satisfy the conditions

$$f(0) = 0, \quad f(0.5) = 1, \quad f(1) = 0, \quad f(x) = f(1-x).$$

Problem 12. Can one find polynomials $p : \mathbb{R} \to \mathbb{R}$ such that one critical point of p and one fixed point of p coincide? Start of with

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \qquad (1)$$

where $n \geq 2$.

Problem 13. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \mathbb{R} \to \mathbb{R}$ be continuous function. Assume that $x^* = 0$ is a fixed point of both f_1 and f_2. Let $g_1(x) = f_1(f_2(x))$ and $g_2(x) = f_2(f_1(x))$. Show that the functions $h_1(x) = g_1(x) - g_2(x)$ and $h_2(x) = g_1(x) + g_2(x)$ also admit this fixed point.

Problem 14. Let $a \neq 0$. Consider the polynomials

$$f(x) = ax^3 + bx^2 + cx + d, \qquad g(x) = x^3 + Ax + B$$

with $A = 9ac - 3b^2$, $B = 27a^2d + 2b^3 - 9abc$.
(i) Find (Newton map)

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \qquad N_g(x) = x - \frac{g(x)}{g'(x)}.$$

(ii) Let $h(x) = 3ax + b$. Show that $h \circ N_f = N_g \circ h$.

Problem 15. Let $n \in \mathbb{N}$. Consider the map (*Newton's method* to find the square root of n)

$$x_{t+1} = \frac{1}{2}\left(x_t + \frac{n}{x_t}\right), \qquad t = 0, 1, 2, \ldots$$

given the initial value x_0 with $x_0 > 0$.
(i) Find the fixed points.
(ii) Show that

$$\frac{x_{t+1}-\sqrt{n}}{x_{t+1}+\sqrt{n}} = \left(\frac{x_t-\sqrt{n}}{x_t+\sqrt{n}}\right)^2 .$$

(iii) Find $\lim_{t\to\infty} x_t$.

Problem 16. Let $r > 0$. Find the first iterate of the Newton map

$$f_r(x) = \frac{1}{2}\left(x + \frac{r}{x}\right).$$

Problem 17. *Newton's sequence* takes the form of a difference equation

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}$$

where $t = 0, 1, 2, \ldots$ and x_0 is the initial value at $t = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2 - 1$$

and $x_0 \neq 0$.
(i) Find the fixed points of f.
(ii) Find the fixed points of the difference equation.
(iii) Find the exact solution of

$$x_{t+1} = \frac{1}{2}\left(x_t + \frac{1}{x_t}\right).$$

(iv) Let $x_0 = 1/2$. Find x_1, x_2 and x_3.

Problem 18. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a positive, continously differentiable function, defined for all real numbers and whose derivative is always negative. Show that for any real number x_0 (initial value) the sequence (x_k) obtained by *Newton's method*

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)} \qquad t = 0, 1, 2, \ldots$$

has always limit ∞.

Problem 19. Consider the function $f : [0,1] \to [0,1]$

$$f(x) = \frac{1}{2} - \frac{1}{2}\sin(2\pi x).$$

Find the fixed points and study their stability.

Problem 20. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be analytic functions with the same fixed point x^*, i.e. $f(x^*) = x^*$, $g(x^*) = x^*$. Show that $f \circ g$ and $g \circ f$ admit this fixed point.

Problem 21. Consider the polynomial $f(x) = x^3 - 3x + 3$. Show that for any positive integer N, there is an initial value x_0 such that the sequence $x_0, x_1, x_2, \ldots$ obtained from *Newton's method*

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)} = \frac{2x_t^3 - 3}{3(x_t^2 - 1)}, \qquad t = 0, 1, 2, \ldots$$

has period N.

Problem 22. Let $x > 0$ and $p > 0$. Consider the map

$$f(x) = xe^{p-x}.$$

(i) Find the fixed points. Study the stability of the fixed points.
(ii) Show that f has a least one periodic point x^* with $x^* \neq 0$ or p.

Problem 23. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 2x(1 - x).$$

(i) Find the fixed points. Are the fixed points stable?
(ii) Calculate

$$\lim_{n\to\infty} f^{(n)}(1/3).$$

Discuss.
(iii) Let n be a positive integer n with $n \geq 2$. Find the distances

$$|1/n - 1/(n+1)| \quad \text{and} \quad |f(1/n) - f(1/(n+1))|.$$

Discuss.

Problem 24. Consider the logistic map $f : [0, 1] \to [0, 1]$

$$f(x) = 4x(1 - x).$$

(i) Find the fixed points.
(ii) Let $\widetilde{x} = 3/4$. Find the preimage.
(iii) Let $\widetilde{x} = 0$. Find the preimage.
(iv) Let $\widetilde{x} = 1/2$. Find the preimage.

(v) Let n be an integer with $n \geq 3$. Show that

$$\left| \frac{1}{n} - \frac{1}{n+1} \right| < |f(1/n) - f(1/(n+1))| \, .$$

Problem 25. Consider the logistic maps $f_r : \mathbb{R} \to \mathbb{R}$ given by

$$f_r(x) = rx(1-x), \quad r > 0.$$

(i) Find the fixed points of f_r.
(ii) Establish for which values of the bifurcation parameter r the fixed points of f_r are attrative
(iii) Find the the periodic points of prime period 2 for f_r. In both cases, establish for which values of r the points will occur.
(iv) Establish for which values the periodic points of prime period 2 are attractive.

Problem 26. Consider the logistic family $f_r : \mathbb{R} \to \mathbb{R}$ given by

$$f_r(x) = rx(1-x).$$

Show that there exists an infinite number of eventually fixed points for $r > 4$.

Problem 27. The logistic family $f_r : \mathbb{R} \to \mathbb{R}$ is defined by $f_r(x) = rx(1-x)$.
(i) Show that f_r undergoes a period-doubling bifurcation.
(ii) Find the values of x and μ the period doubling occurs.

For f_r to undergo a period-doubling bifurcation for $x = x_0$, $r = r_0$, it must satisfy the following four conditions.

(i) $f_{r_0}(x_0) = x_0$
(ii) $f'_{r_0}(x_0) = -1$
(iii) $\left. \dfrac{\partial (f_r^{(2)})'(x_0)}{\partial r} \right|_{r=r_0} \neq 0$
(iv) $f'''_{r_0}(x_0) \neq -\frac{3}{2}[f''_r(x_0)]^2$.

Problem 28. The family of quadratic maps $f_r : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_r(x) = x^2 + r.$$

Find out whether f_r undergoes a tangent bifurcation, and if so, for which values of x and r it occurs.

For f_r to undergo a tangent bifurcation for $x = x_0$, $r = r_0$, it must satisfy the following four conditions.

(i) $f_{r_0}(x_0) = x_0$
(ii) $f'_{r_0}(x_0) = 1$
(iii) $f''_{r_0}(x_0) \neq 0$
(iv) $\left.\dfrac{\partial f_r(x_0)}{\partial r}\right|_{r=r_0} \neq 0$

Problem 29. The family of quadratic maps $g_r : \mathbb{R} \to \mathbb{R}$ is defined by

$$g_r(x) = x^2 + r.$$

(i) Establish for which values of r the quadratic map is topologically conjugate to the logistic map $f_\mu(x) = \mu x(1-x),\ \mu > 0$.
(ii) Make use of the topological conjugacy in order to establish for which values of r, g_r will have a single attractive fixed point.
(iii) What happens if $r = -\frac{3}{4}$?
(iv) Can any conclusions be drawn about the dynamics of g_r if $r > 1/4$?

Problem 30. The *tent map* $T : [0,1] \to [0,1]$ is defined by

$$T(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

(i) Sketch the graph of the tent map T. Draw the line $f(x) = x$ for $x \in [0,1]$ to locate the fixed points. Find the fixed points.
(ii) Sketch the graph for the second iterate $T^{(2)}$ of the tent map T. Draw the line the locate the fixed points.
(iii) Let $T^{(n)}$ be the n-th iterate. Show that $T^{(n)}$ has 2^n repelling periodic points of period n.
(iv) Show that these periodic points are dense on $[0,1]$.

Problem 31. The *tent map* $T : [0,1] \to [0,1]$ is defined by

$$T(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

Show that the map

$$h(x) = \frac{1}{2}\cos[\pi(1-x)] + \frac{1}{2} \equiv \cos^2[\frac{\pi}{2}(1-x)]$$

is a *topological conjugacy* between $f(x) = 4x(1-x)$ and the tent map.

Problem 32. Let $(s_0, s_1, ..., s_{n-1})^T \in \mathbb{R}^n$, where $n = 2^k$. This vector in $\mathbb{R}^n$ can be associated with a piecewise constant function f defined on $[0, 1)$

$$f(x) = \sum_{j=0}^{2^k-1} s_j \Theta_{[j2^{-k},(j+1)2^{-k})}(x)$$

where $\Theta_{[j2^{-k},(j+1)2^{-k})}(x)$ is the step function

$$\Theta_{[j2^{-k},(j+1)2^{-k})}(x) := \begin{cases} 1 \; x \in [j2^{-k}, (j+1)2^{-k}) \\ 0 \; x \notin [j2^{-k}, (j+1)2^{-k}) \end{cases}$$

with the support $[j2^{-k}, (j+1)2^{-k})$. Consider the logistic map $x_{j+1} = 4x_j(1 - x_j)$ with $j = 0, 1, 2, \ldots$ and $x_0 = 1/3$. Then

$$x_0 = \frac{1}{3}, \quad x_1 = \frac{8}{9}, \quad x_2 = \frac{32}{81}, \quad x_3 = \frac{6272}{6561}.$$

Find $f(x)$ for this data set and then calculate

$$\int_0^1 f(x)dx.$$

Problem 33. Consider the function $f : [0, \infty) \to [0, \infty)$

$$f(x) = x^4 e^{-x}$$

or written as difference equation

$$x_{t+1} = x_t^4 e^{-x_t}, \quad t = 0, 1, 2, \ldots$$

with $x_0 \geq 0$.
(i) Find the fixed points of f and study their stability.
(ii) Find the maxima and minima of the function f.

Problem 34. Consider the map $f : [0, \pi] \to [0, 2]$ defined by

$$f(x) = 2\sin(x).$$

Show that the map has two fixed points. Are these fixed point hyperbolic fixed points?

Problem 35. Consider the symmetric tent map f on the unit interval $[0, 1]$

$$f(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2 - 2x & \text{if } x \geq 1/2 \end{cases}$$

The map is chaotic and completely mixing. It als has a unique absolutely continuous invariant measure and cycles of all orders. Let N be a positive integer. As a discrete model g of f consider the restriction of f on the N-digital binary lattice

$$L_N = \left\{ 0, \frac{1}{2^N}, \frac{2}{2^N}, \frac{3}{3^N}, \dots, \frac{2^N - 1}{2^N} \right\}.$$

Show that g is asymptotically trivial.

Problem 36. Let $f : [0,1) \to [0,1)$. The *Bernoulli map* is defined by

$$f(x) := 2x \bmod 1.$$

The map can be written as a difference equation

$$x_{t+1} = \begin{cases} 2x_t & \text{for } 0 \leq x_t < 1/2 \\ (2x_t - 1) & \text{for } 1/2 \leq x_t < 1 \end{cases} \tag{2}$$

where $t = 0, 1, 2, \dots$ and $x_0 \in [0,1)$.
(i) Find the fixed points.
(ii) Study the stability of the fixed points.
(iii) Find a periodic orbit.
(iv) Find the exact solution.
(v) Evaluate the Ljapunov exponent.
(vi) Find the invariant density.
(vii) Evaluate the autocorrelation function.

Problem 37. Consider the first-order discrete time dynamical system

$$x_{t+1} = 2x_t \mod 1 \quad t = 0, 1, 2, \dots$$

and

$$s_t = \begin{cases} 1 \text{ if } x_t \geq 0.5 \\ 0 \text{ if } x_t < 0.5 \end{cases}$$

where $x_0 \in [0,1]$. We call $\mathbf{s} = s_0 s_1 s_2 \dots$ the output symbol. Show that if $x_0 \in [0.78125, 0.8125]$ then the output coincide for the first three bits.

Problem 38. The logistic map is given by

$$x_{t+1} = 4x_t(1 - x_t), \qquad t = 0, 1, \dots \tag{1}$$

where $x_0 \in [0,1]$. It can also be considered as a map $f : [0,1] \to [0,1]$, $f(x) = 4x(1-x)$.
(i) Show that $x_t \in [0,1]$ for $t = 1, 2, \dots.$

(ii) Find the fixed points of the equation.
(iii) Give the variational equation.
(iv) Show that the fixed points are unstable. Hint. Show that

$$\left|\frac{df(x)}{dx}\right|_{x=x^*} > 1. \tag{2}$$

(v) Find the periodic orbits.
(vi) Show that the exact solution of (1) is given by

$$x_t = \frac{1}{2} - \frac{1}{2}\cos(2^t \arccos(1-2x_0)). \tag{3}$$

where $x_0 \in [0,1]$ is the initial value.
(vii) Show that for the initial values

$$x_0 = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{r\pi}{2^s}\right) \tag{4}$$

where r and s are integers we find that the orbits are periodic or tend to a fixed point.
(viii) Show that for allmost all initial values we find that the autocorrelation function is given by

$$C_{xx}(\tau) = \begin{cases} \frac{1}{8} & \text{for} \quad \tau = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

(ix) Show that the invariant density is given by

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

(x) Show that for almost all initial values the Ljapunov exponent λ for the logistic map (1) is given by

$$\lambda = \ln(2).$$

Problem 39. (i) Let $f : [-1,1] \mapsto [-1,1]$ be defined by

$$f(x) := 1 - 2x^2. \tag{1}$$

Let $-1 \leq a \leq b \leq 1$ and

$$\mu([a,b]) := \frac{1}{\pi}\int_a^b \frac{dx}{\sqrt{1-x^2}}. \tag{2}$$

Calculate $\mu([-1,1])$.

(ii) Show that

$$\mu(f^{-1}([a,b])) = \mu([a,b]) \tag{3}$$

where $f^{-1}([a,b])$ denotes the set S which is mapped under f to $[a,b]$, i. e. $f(S) = [a,b]$. The quantity μ is called the *invariant measure* of the map f.
(iii) Let $g : [0,1] \mapsto [0,1]$ be defined by

$$g(x) := \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 < x \le 1 \end{cases} \tag{4}$$

This map is called the *tent map*. Let $0 \le a \le b \le 1$ and

$$\nu([a,b]) := \int_a^b dx.$$

Show that

$$\nu(g^{-1}([a,b])) = \nu([a,b])$$

where $g^{-1}([a,b])$ is the set S which is mapped under g to $[a,b]$, i. e. $g(S) = [a,b]$.
(iv) Find the Lyapunov exponent of the tent map (4).

Problem 40. The *Chebyshev polynomials* of first kind can be defined as $(n = 0, 1, 2, \ldots)$

$$\begin{aligned} T_n(x) &= \frac{1}{2}\left(\left(x - \sqrt{x^2-1}\right)^n + \left(x + \sqrt{x^2-1}\right)^n\right) \\ &= x^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (1 - x^{-2})^k \end{aligned}$$

or as the recurrence relation $(n = 1, 2, \ldots)$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

with $T_0(x) = 1$, $T_1(x) = x$.
(i) Find T_2 and T_3.
(ii) Find $T_n(T_m(x))$.
(iii) Find the fixed points of T_3 and study whether they are stable.

Problem 41. Consider the *cubic map* $f_r : [-1,1] \to [-1,1]$

$$f_r(x) = rx^3 + (1-r)x, \quad r \in [3.2, 4.0]$$

(i) Find the fixed points and study their stability.
(ii) Find the critical points of the map f_r in the interval $[-1,1]$ and test whether we have a minimum or maximum.

(iii) Find the linearized map (variational equation)
(iv) Find the exact solution for the case $r = 4$.

Problem 42. Consider the piecewise linear map $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 3x & \text{for } \ 0 \le x \le 1/3 \\ 2 - 3x & \text{for } 1/3 \le x \le 2/3 \\ 3x - 2 & \text{for } \ 2/3 \le x \le 1. \end{cases}$$

(i) Find the fixed points and study their stability.
(ii) Find $f(1/2)$, $f(f(1/2))$, $f(f(f(1/2)))$.
(iii) Find $f(1/3)$, $f(f(1/3))$, $f(f(f(1/3)))$.

Problem 43. Consider the piecewise linear map $f : [0,1] \to [0,1]$.

$$f(x) = \begin{cases} 4x & \text{for } \ 0 \le x < 1/4 \\ 4x - 1 & \text{for } 1/4 \le x < 1/2 \\ -4x + 3 & \text{for } 1/2 \le x < 3/4 \\ -4x + 4 & \text{for } \ 3/4 \le x \le 1. \end{cases}$$

(i) Find the fixed points and study their stability.
(ii) Find $f(1/2)$, $f(f(1/2))$, $f(f(f(1/2)))$.
(iii) Find $f(1/3)$, $f(f(1/3))$, $f(f(f(1/3)))$.

Problem 44. Consider the map $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} 7x/5 & \text{for } x \in [0, 1/2] \\ 14x(1-x)/5 & \text{for } x \in [1/2, 1]. \end{cases}$$

Find a lower as well as an upper limit for the value of the Liapunov exponent.

Problem 45. Consider the logistic maps $f : [0,1] \to [0,1]$

$$f(x) = 4x(1-x)$$

and $\phi : [0,1] \to [0,1]$

$$\phi(x) = \frac{2}{\pi} \arcsin \sqrt{x}.$$

(i) Show that f and ϕ are continuous.
(ii) Show that ϕ is a homeomorphism and

$$\phi^{-1}(x) = \sin^2\left(\frac{\pi x}{2}\right) \equiv \frac{1 - \cos(\pi x)}{2}.$$

(iii) Let $g : [0,1] \to [0,1]$ be defined by

$$g(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2(1-x) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

Show that

$$g(x) = (\phi \circ f \circ \phi^{-1})(x).$$

The maps f and g are called *topologically conjugacy.*

Problem 46. Consider the *tent map* given by

$$f(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2(1-x) & \text{for } 1/2 \le x \le 1. \end{cases} \tag{1}$$

The *Frobenius-Perron integral equation* is given by

$$\rho(x) = \int_0^1 \rho(y)\delta(x - f(y))dy. \tag{2}$$

Find ρ for the tent map.

Problem 47. Consider the map $f : [-1,1] \to [-1,1]$ with

$$f(x) = 1 - 2|x|^r. \tag{1}$$

This map is a fully developed chaotic map for $r \ge \frac{1}{2}$. The *Frobenius-Perron integral equation*

$$\rho(x) = \int_0^1 \rho(y)\delta(x - f(y))dy \tag{2}$$

becomes

$$\rho(x) = \frac{1}{2r}\left(\frac{1-x}{2}\right)^{(1-r)/r}\left[\rho\left(\left(\frac{1-x}{2}\right)^{1/r}\right) + \rho\left(-\left(\frac{1-x}{2}\right)^{1/r}\right)\right]. \tag{3}$$

Find the invariant density for $r = 1$, $r = 1/2$ and $r = 2$.

Problem 48. Let Σ be the set of all infinite sequences of 0's and 1's. This set is called the sequence space of 0 and 1 or the symbol space of 0 and 1. More precisly

$$\Sigma := \{\, (s_0 s_1 s_2 \ldots) \,:\, s_i = 0 \,\text{or}\, 1 \} \tag{1}$$

Let $s = s_0 s_1 s_2 \ldots$ and $t = t_0 t_1 t_2 \ldots$ be elements in Σ. We denote the distance between s and t as $d[s,t]$ and define it by

$$d[s,t] := \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}. \tag{2}$$

(i) Show that

$$0 \le d[s,t] \le \sum_{i=0}^{\infty} \frac{1}{2^i} = 2. \tag{3}$$

Note that $|s_i - t_i|$ is either 0 or 1.
(ii) Show that the distance bewteen $s = 0000\ldots$ and $t = 01010101010\ldots$ is $2/3$.
(iii) Let s and t be elements of Σ. Show that if the first $n+1$ digits in s and t are identical, then $d[s,t] \le 1/2^n$. Show that if $d[s,t] \le 1/2^n$, then the first n digits in s and t are identical.

Problem 49. Let $0 < r < 1$. Consider the map $f_r : [0,1] \to [0,1]$

$$f_r(x) = \begin{cases} x/r & \text{for } 0 \le x \le r \\ (x-r)/(1-r) & \text{for } r < x \le 1 \end{cases}$$

Find the Liapunov exponent.

Problem 50. Let Σ be the set of all bi-infinite sequences of the binary symbols $\{0,1\}$, i.e.

$$\Sigma := \{\, \sigma \; : \; \sigma : \mathbb{Z} \to \{0,1\} \,\}. \tag{1}$$

The elements, σ of Σ are called *symbol sequences* and they are defined by specifying $\sigma(n) = \sigma_n \in \{0,1\}$ for each $n \in \mathbb{Z}$. We write

$$\sigma := \{\sigma_n\}_{n=-\infty}^{\infty} = \{\ldots \sigma_{-2}\sigma_{-1} \cdot \sigma_0\sigma_1\sigma_2 \ldots\} \tag{2}$$

We consider the dynamics of the map $f : \Sigma \to \Sigma$ defined by

$$f(\sigma)_n := \sigma_{n-1} \tag{3}$$

$n \in \mathbb{Z}$. This is known as a *left-shift* on Σ because it corresponds to moving the binary point one symbol to the left. Show that the left shift $f : \Sigma \to \Sigma$ has periodic orbits of all period as well as aperiodic orbits.

Problem 51. Show that the map

$$f(x) = x + r \mod 1 \tag{1}$$

is not ergodic when r is rational. This means $r = k/m$, where $k \in \mathbb{Z}$, $m \in \mathbb{Z} \setminus \{0\}$.

Problem 52. Consider the map (so-called *Mixmaster return map*)

$$x_{n+1} = f(x_n) \equiv x_n^{-1} - \lfloor x_n^{-1} \rfloor \tag{1}$$

where $f(0) \equiv 0$ and $f : [0,1] \to [0,1]$. In analytic form this return map is the single-valued function,

$$f(x) = x^{-1} - k, \quad (k+1)^{-1} < x < k^{-1}; \quad k \in \mathbb{Z}^+. \tag{2}$$

The function possesses an infinite number of discontinuties and is not injective since each x_0 has a countable infinity of inverse images, one on each interval $[(k+1)^{-1}, k^{-1}]$ for integral k.
(i) Show that the return mapping is *expansive*,

$$|f'(x)| > 1 \tag{3}$$

on $x \in (0,1)$, everywhere.
(ii) Show that all the fixed points ($f(x^*) = x^*$) are unstable.
(iii) Find the invariant measure.
(iv) Find the metric entropy.

Problem 53. We consider one-dimensional smooth maps. Show that the Lyapunov exponents are invariant under conjugation.

Problem 54. Show that the topological entropy is invariant under conjugation.

Problem 55. Consider the unimodal map $f : [0,1] \to [0,1]$. We assume that f is continuous and reaches its maximal value at an interior point c of I. The point c is called the critical point of f. In both subintervals divided by $c, [0,c)$ and $(c,1]$, the map f is strictly monotonic. We assume that $f(0) = f(1) = 0$. A discrete dynamical system

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots$$

is defined from f by iteration. Given a starting point $x_0 \in I$, we use the notation

$$(x_0, x_1, \cdots, x_t, \cdots)$$

to denote the orbit from x_0. Using the *coarse-grained description*

$$A(x) := \begin{cases} 0 & \text{for} \;\; x < c \\ c & \text{for} \;\; x = c \\ 1 & \text{for} \;\; x > c \end{cases}$$

we transform the orbit $(x_0, x_1, x_2, \ldots)$ into an itinerary, that is,

$$I(x) = (A(x_0), A(x_1), \ldots, A(x_t), \ldots)$$

which is an infinite string over the alphabet $\{0, c, 1\}$. The kneading sequence of the unimodal map f is the itinerary $I(f(c))$, which decides nearly all other itineraries a given map f can have.

(i) Show that the logistic map $f(x) = 4x(1-x)$ is a unimodular map.
(ii) Obviously, $c = 1/2$ is a critical point of the map. Find a preimage of the map.
(iii) Find the orbit in the coarse-grained description for the initial value $x_0 = 1/3$.
(iv) Find the kneading sequence.

Problem 56. To construct the *symbolic dynamics* of a dynamical system, the determination of the partition and the ordering rules for the underlying symbolic sequences is of crucial importance. For one-dimensional mappings, the partition is composed of all the critical points. Consider the antisymmetric cubic map $f : [-1,1] \to [-1,1]$ $(t = 0,1,2,\ldots)$

$$x_{t+1} = f(x_t) \equiv r x_t^3 + (1-r)x_t, \qquad r \in [1,4]. \tag{1}$$

(i) Find the critical points of f. We denote the critical points by C and $\bar{C}$.
(ii) Show that the ternary partition marked by $\bar{C}$ and C divides the interval $[-1,1]$ into three monotonic branches.
(iii) The right branch to C is assigned 0, the left branch to $\bar{C}$ is assigned 2, whereas the part between $\bar{C}$ and C is 1. Show that nearly all trajectories are unambiguously encoded by infinite strings of bits $S(x) = (s_1 s_2 \cdots)$, where s_i is either 0, 1 or 2.
(iv) Referring to the natural ordering of the real numbers in the interval $[-1,1]$, show that the ordering rules for these symbolic strings can be defined, that is, considering two symbolic strings $S(x_1)$ and $S(x_2)$ from the initial points x_1 and x_2, $S(x_1) \geq S(x_2)$ if and only if $x_1 > x_2$.
(v) Find the kneading sequence K_g and K_s (i.e. the forward symbolic sequences from the maximal and minimal values $\bar{C}$ and C).

Problem 57. Consider the the family of maps $f_r : [0,1] \to [0,1]$ $(r \in (0,1))$

$$f_r(x) = \begin{cases} \frac{x}{r}, & x \in (0,r), \\ \frac{1-x}{1-r}, & x \in (r,1]. \end{cases}$$

(i) Show that for each r we have constant invariant density $\rho(x) = 1$.
(ii) Show that this can be used to calculate the Liapunov exponent λ_r

$$\lambda_r = (r-1)\ln(1-r) - r\ln r.$$

Problem 58. The *Bernoulli map* $f : [0,1) \to [0,1)$ is given by

$$f(x) = 2x \bmod 1. \tag{1}$$

We consider the time evolution of a probability density $\rho_n(x)$ (a "state") describing an ensemble of trajectories. The *Frobenius-Perron integral equation* is given by

$$\rho(x) = \int_0^1 \rho(y)\delta(x - f(y))dy.$$

Thus the time evolution of a state $\rho(x)$ under f is given by the *Frobenius-Perron operator* U, defined by

$$\rho_{n+1}(x) = U\rho_n(x) := \frac{1}{2}\left(\rho_n\left(\frac{1}{2}x\right) + \rho_n\left(\frac{1}{2}(x+1)\right)\right).$$

Since ρ is a probability density, it should be integrable, we require it to be in the Banach space $L_1(0,1)$ of Lebesgue integrable functions on $[0,1]$. However, it is common to restrict ρ to the Hilbert space $L_2(0,1)$ of square-integrable functions (Lebesgue measure). Consider the states $e_{n,s}$ defined by

$$e_{n,s}(x) := \exp(2\pi i 2^n(2s+1)x)$$

where n is a non-negative integer and s is an integer. Any integer ℓ can be written uniquely as $\ell = 2^n(2s+1)$ for integers $n \geq 0$ and $s \in \mathbb{Z}$.
(i) Show that U is a one-sided shift operator, i.e.,

$$Ue_{n,s}(x) := \begin{cases} e_{n-1,s}(x), & n > 0 \\ 0, & n = 0. \end{cases}$$

(ii) Show that the spectrum of U fills the unit disk in the complex plane.

Problem 59. Calculate the invariant density of the logistic map $g : [0,1] \to [0,1]$

$$g(x) = 4x(1-x)$$

without using the *Frobenius-Perron approach* and by making use of the fact that for the symmetric tent map $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x < 0.5 \\ 2 - 2x & \text{for } 0.5 \leq x \leq 1 \end{cases}$$

the invariant density is constant.

Problem 60. Let us consider the *Frobenius-Perron equation* given by

$$\rho_{n+1}(x) = \int_0^1 \delta(x - f(y))\rho_n(y)dy, \qquad n = 0, 1, 2, \cdots \tag{1}$$

with the map $f : [0,1] \to [0,1]$. Equation (1) implies that the probability density evolves toward the stationary invariant density ρ, provided that all eigenvalues except unity are located within the unit circle. Therefore, if the Frobenius-Perron is asymptotically stable, then we have

$$\rho(x) = \int_0^1 \delta(x - f(y))\rho(y)dy. \tag{2}$$

Find f if the invariant density ρ is given, where we assume that $f(0) = 0$. Furthermore we assume that

$$f(1-x) = f(x), \qquad \text{(type 1)} \tag{3}$$

$$f(x+1/2) = f(x), \qquad \text{(type 2)}. \tag{4}$$

In type 1, the map is symmetric about the value $x = 1/2$. Type 2 corresponds to the translationally symmetric map. These restrictions are mainly due to the difficulty for the arbitrary form of the map f. It is difficult to obtain the inverse image of f.

Problem 61. Let X be a random variable, absolutely continuously distributed on the interval $[0, 1]$ with probability density $\rho(x)$. The random variable is expanded into a simple continued fraction, i.e. a sequence of positive integer valued random variables a_k and a sequence of random variables $X_k \in [0, 1]$ are defined as follows

$$X_1 = X, \qquad a_k = [X_k^{-1}], \qquad X_{k+1} = X_k^{-1} - a_k. \tag{1}$$

For each k, X_k and a_k are random variables, X_k being absolutely continuously distributed in the interval $[0, 1]$ with probability density $\rho_k(x)$, and a_k being a discrete variable. Show that

$$\lim_{k\to\infty} \rho_k(x) = \frac{1}{(1+x)\ln 2}. \tag{2}$$

Show that the density

$$\rho_0 = \frac{1}{(1+x)\ln 2}$$

is a fixed point of A.

Problem 62. Let S^1 be the unit circle and identify each point on the circle by the radian measure of the angle between the positive x-axis and the ray beginning at the orign and passing through the point. We measure angles in a counterclockwise direction. We identify the point α with the point $\alpha + 2n\pi$, where n is an integer. We define a metric on S^1 be letting $d[\alpha, \beta]$ be the length of the shortest arc on the circle from α to β. More precisly, if α and β are in the interval $[0, 2\pi)$, then

$$d[\alpha, \beta] := \begin{cases} |\alpha - \beta| & \text{for } |\alpha - \beta| \leq \pi \\ |\alpha - \beta| & \text{for } |\alpha - \beta| > \pi \end{cases} \tag{1}$$

Define the doubling function $D : S^1 \to S^1$ by

$$D(\theta) = 2\theta. \tag{2}$$

(i) Show that d is a metric for S^1.
(ii) Show that if

$$\alpha = \frac{\pi 2k}{2^n - 1} \tag{3}$$

where k and n are natural numbers, then α is periodic with period n under the doubling function.
(iii) Show that the map is chaotic on S^1.

Problem 63. Consider the logistic map $f_r : \mathbb{R} \to \mathbb{R}$ given by

$$f_r(x) = rx(1-x) \tag{1}$$

where $r > 2 + \sqrt{5}$. Let us restrict the map to $f_r : \Lambda \to \Lambda$ (Cantor set). Consider the shift map $\sigma : \Sigma \to \Sigma$ given by

$$\sigma(s_0 s_1 s_2 \ldots) := s_1 s_2 s_3 \ldots \tag{2}$$

In other words, the shift map forgets the first digit of the sequence. Construct a topological conjugacy from f_r to σ.

Problem 64. The *discrete Fourier transform* is used when a set of sample function values, $x(i)$, are available at equally spaced time intervals numbered $i = 0, 1, \ldots, N-1$. The discrete Fourier transform maps the given function values into the sum of a discrete number of sine and cosine waves whose frequencies are numbered $k = 0, 1, \ldots, N-1$, and whose amplitudes are given by

$$\hat{x}(k) = \frac{1}{N} \sum_{l=0}^{N-1} x(l) \exp\left(-i2\pi k \frac{l}{N}\right). \tag{1}$$

Equation (2) can be written as

$$\hat{x}(k) = \frac{1}{N} \sum_{l=0}^{N-1} x(l) \cos\left(2\pi k \frac{l}{N}\right) - \frac{i}{N} \sum_{l=0}^{N-1} x(l) \sin\left(2\pi k \frac{l}{N}\right). \tag{3}$$

The inverse transformation is given by

$$x(l) = \sum_{k=0}^{N-1} \hat{x}(k) \exp\left(i2\pi k \frac{l}{N}\right). \tag{4}$$

To find (4), we use the fact that

$$\sum_{k=0}^{N-1} \exp\left(i2\pi k \frac{n-m}{N}\right) = N\delta_{nm} \tag{5}$$

where n, m are integers. Consider the tent map

$$f(x) := \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2 - 2x & \text{for } x \in (1/2, 1] \end{cases} \tag{6}$$

(i) Show that the initial value $x = 6/17$ leads to the periodic orbit.
(ii) Find the Fourier transform of this orbit.

Problem 65. Consider a one-dimensional ergodic and chaotic map $f : [0, 1] \to [0, 1]$. The Frobenius-Perron integral equation is given by

$$\rho(x) = \int_0^1 \delta(x - f(y))\rho(y)dy \tag{1}$$

where ρ is the invariant density, i.e.

$$\int_0^1 \rho(x)dx = 1, \qquad \rho(x) > 0 \quad \text{for} \quad x \in [0, 1]. \tag{2}$$

Assume that

$$f(0) = 0, \qquad f(1 - x) = f(x), \quad x \in [0, 1] \tag{3a}$$

and

$$\frac{df}{dx} > 0 \quad x \in [0, 1/2]. \tag{3b}$$

Thus f is symmetric about the value $1/2$. Find f if ρ is given.

Problem 66. (i) Consider the map $f : [-1, 1] \to [-1, 1]$

$$f(x) = 1 - 2(|x|)^{1/2}$$

or written as difference equation

$$x_{t+1} = 1 - 2(|x_t|)^{1/2}, \qquad t = 0, 1, \ldots$$

with $x_0 \in [-1, 1]$. Find the fixed points and show that they are unstable.
(ii) Find the invariant density and under the assumption that the system is ergodic calculate the Liapunov exponent.

Problem 67. Consider the nonlinear difference equation

$$x_t x_{t+1} + c_1 x_t + c_2 = 0, \qquad t = 0, 1, 2, \ldots \tag{1}$$

where c_1 and c_2 are constant and $y_0 \neq 0$. Show that the substitution

$$x_t = \frac{1}{y_t} + \alpha \tag{2}$$

reduces (1) to the linear difference equation

$$(\alpha + c_1)y_{t+1} + \alpha y_t + 1 = 0 \tag{3}$$

if α satisfies the quadratic equation

$$\alpha^2 + c_1\alpha + c_2 = 0. \tag{4}$$

Problem 68. Apply the substitution

$$y_t = \frac{x_t}{x_{t+1}} - b_t, \qquad t = 0, 1, 2, \ldots$$

to the *Riccati equation*

$$y_{t+1}y_t + a_{t+1}y_{t+1} + b_{t+1}y_t = c_{t+1}.$$

Problem 69. (i) Consider the nonlinear difference equation

$$y_{t+1} = 2y_t^2 - 1 \tag{1}$$

where $t = 0, 1, 2, \ldots$. Show that the substitution $y_t = \cos(x_t)$ reduces (1) to the equation

$$\cos(x_{t+1}) = \cos(2x_t).$$

Thus deduce that either

$$x_{t+1} = 2x_t + 2m\pi$$

or

$$x_{t+1} = -2x_t + 2n\pi$$

where m and n are arbitrary integers.
(ii) Show that from the second alternative we obtain the solution

$$y_t = \cos\left(2^t\theta + (-1)^t\frac{2n\pi}{3}\right)$$

where θ is an arbitrary constant and n an arbitrary integer.
(iii) Show that the solution corresponding to the first alternative is contained in this one.

Problem 70. Let $x_0 > 0$. Discuss the solution of

$$x_{t+1} = \frac{2\sqrt{x_t}}{1 + x_t}, \quad t = 0, 1, \ldots.$$

First find the fixed points.

Problem 71. The *W-map* $f : [0,1] \to [1/4, 3/4]$ is given by

$$f(x) = \begin{cases} -2x + 3/4 & \text{for} \quad 0 \le x < 1/4 \\ 2x - 1/4 & \text{for} \ \ 1/4 \le x < 1/2 \\ -2x + 7/4 & \text{for} \ \ 1/2 \le x < 3/4 \\ 2x - 5/4 & \text{for} \quad 3/4 \le x \le 1 \end{cases}$$

The graph of the map looks like a W. Find the invariant measure ρ.

Problem 72. Consider the one-dimensional map $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 1/2 - 2x & \text{for} \quad 0 \le x \le 1/4 \\ -1/2 + 2x & \text{for} \ \ 1/4 \le x \le 3/4 \\ 5/2 - 2x & \text{for} \quad 3/4 \le x \le 1 \end{cases}$$

Find the invariant measure.

Problem 73. Consider the *tent map* $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 2x & \text{for} \ \ 0 \le x \le 1/2 \\ -2x + 2 & \text{for} \ \ 1/2 < x \le 1 \end{cases}$$

Show that this map can be modelled by the one-sided 2-shift.

Problem 74. A map $f : [-1,1] \to [-1,1]$ is called S-unimodal if it satisfies the following conditions:

(a) $f(0) = 1$.
(b) $f([f(1), 1]) = [f(1), 1]$.
(c) f is monotonically increasing in $[-1, 0]$ and monotonically decreasing in $[0, 1]$.
(d) f is at least three times continously differentiable.
(e) $f''(0) < 0$.
(f) The *Schwarzian derivative* of f is negative, i.e.

$$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0 \ \text{ for all } \ x \ne 0.$$

Does the function

$$g_r(x) = 1 - rx^2, \quad r \in (0, 2]$$

satisfies these conditions?

Problem 75. The *Schwarzian derivative* of a C^3 function f of one complex variable is defined by

$$(Sf)(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian derivative can also be written as

$$(Sf)(y) := 6 \lim_{x \to y} \left(\frac{f'(x)f'(y)}{(f(x)-f(y))^2} - \frac{1}{(x-y)^2} \right).$$

(i) Find the Schwarzian derivative of

$$g(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$

(ii) Show that if f and g have negative Schwarzian derivative, then $f \circ g$ has negative Schwarzian derivative.

Problem 76. Consider the map ($t = 0, 1, 2, \ldots$)

$$x_{t+1} = \frac{x_t + 2}{x_t + 1}, \qquad x_0 \geq 0.$$

(i) Find the fixed points.
(ii) Find

$$\lim_{t \to \infty} x_t.$$

Problem 77. (i) Show that the difference equation

$$x_{t+1} = x_t(3 - 4x_t^2), \qquad t = 0, 1, 2, \ldots \tag{1}$$

with $x_0 \in [-1, 1]$ admits the three fixed points

$$x^* = 0, \qquad x^* = \pm\frac{1}{\sqrt{2}}. \tag{2}$$

(ii) Show that the solution of the initial value problem is given by

$$x_t = \sin(3^t \arcsin(u_0)). \tag{3}$$

Problem 78. (i) Consider the difference equation

$$x_{t+1} = 16x_t(1 - x_t)(1 - 2x_t^2)^2, \qquad t = 0, 1, 2, \ldots$$

with $x_0 \in [0, 1]$. Find the fixed points.

(ii) Find the exact solution.

Problem 79. (i) Consider the difference equation

$$x_{t+1} = \frac{4x_t(1-x_t)(1-k^2x_t)}{(1-k^2x_t^2)^2}, \qquad t = 0,1,2,\ldots, \qquad 0 \le k^2 \le 1 \qquad (1)$$

with $x_0 \in [0,1]$. Find the fixed points.

$$x^* = 0, \qquad k^4x^{*4} - 3x^{*2} + 6x^* - 3 = 0. \qquad (2)$$

(ii) Find the solution of the initial value problem using Jacobi elliptic functions. Apply the addition theorems for Jacobi elliptic functions

$$\mathrm{sn}(u \pm v) = \frac{\mathrm{sn}(u)\mathrm{cn}(v)\mathrm{dn}(v) \pm \mathrm{cn}(u)\mathrm{sn}(v)\mathrm{dn}(u)}{1 - k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)}$$

Problem 80. (i) Find the fixed points of the difference equation

$$x_{t+1} = 16x_t(1 - 2\sqrt{x_t} + x_t), \qquad t = 0,1,2,\ldots \qquad (1)$$

with $x_0 \in [0,1]$.
(ii) Find the solution of (1) of the initial value problem.

Problem 81. (i) Consider the nonlinear difference equation

$$x_{t+1} = \sqrt{2}x_t(1 - x_t^4)^{1/2}, \qquad t = 0,1,2,\ldots \qquad (1)$$

with $x_0 \in [0,1]$. Find the fixed points.
(ii) Find the solution of the initial value problem.

Problem 82. (i) Consider the nonlinear difference equation

$$x_{t+1} = (x_t^{2/3} - 1)^3, \qquad t = 0,1,2,\ldots$$

with $x_0 \in [-1,1]$. Find the fixed points $x^* = -\frac{1}{8}$.
(ii) Find the solution of the initial value problem.

Problem 83. Consider the skew-tent map $f_r : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} x/r & \text{for } 0 \le x \le r \\ (1-x)/(1-r) & \text{for } r < x \le 1 \end{cases}$$

where $0.5 \le r < 1$.

(i) Find the Ljapunov exponent λ_r.
(ii) Find the probability density ρ. Use the probability density to calculate the Ljapunov exponent.

Problem 84. Consider the *decimal map* $T : [0,1) \to [0,1)$ defined by

$$T(x) = 10x \mod 1$$

or

$$T(x) = 10x - j \text{ for } \frac{j}{10} \le x < \frac{j+1}{10}, \quad j = 0, 1, \ldots, 9.$$

(i) Let

$$x = \sum_{k=1}^{\infty} \frac{a_k}{10^k} = .a_1 a_2 \cdots .$$

Find $T(x)$. Find $T^{(n)}(x)$.
(ii) Find the fixed points of T.
(iii) Show that under iteration of T no infinite string of 9's can occur in the expansion of any point.

Problem 85. Consider the logistic map

$$x_{t+1} = 4x_t(1 - x_t), \qquad t = 0, 1, 2, \ldots$$

Apply the transformation $x_t = \sin^2(\pi\theta_t)$, i.e. find the map for θ_t, where $\theta_t \in [0,1)$.

Problem 86. Consider the map

$$\theta_{t+1} = 2\theta_t \mod 1$$

with initial value $\theta_0 \in [0,1)$.
(i) Find the solution.
(ii) We can express the initial value $\theta_0 \in [0,1)$ as binary number

$$\theta_0 = \frac{b_0}{2} + \frac{b_1}{2^2} + \frac{b_2}{2^3} + \cdots = \sum_{j=0}^{\infty} \frac{b_j}{2^{j+1}}, \qquad b_j \in \{0, 1\}.$$

Let θ_0 given in binary as

$$\theta_0 = 0.b_0 b_1 \ldots b_7 = 0.10110101.$$

Find $\theta_1, \ldots, \theta_8$.

Problem 87. Consider the map $f_n : [-1,1] \times [-1,1]$

$$f_n(x) = \cos(nx \arccos(x)), \qquad n = 1, 2, \ldots .$$

Find the fixed points and study their stability.

Problem 88. Let $(s_0, s_1, ..., s_{n-1})^T \in \mathbb{R}^n$, where $n = 2^k$. This vector in $\mathbb{R}^n$ can be associated with a piecewise constant function f defined on $[0,1)$

$$f(x) = \sum_{j=0}^{2^k-1} s_j \Theta_{[j2^{-k},(j+1)2^{-k})}(x)$$

where $\Theta_{[j2^{-k},(j+1)2^{-k})}(x)$ is the step function with the support $[j2^{-k},(j+1)2^{-k})$

$$\Theta_{[j2^{-k},(j+1)2^{-k})}(x) := \begin{cases} 1 \; x \in [j2^{-k},(j+1)2^{-k}) \\ 0 \; x \notin [j2^{-k},(j+1)2^{-k}) \end{cases}$$

Let $x_{j+1} = 4x_j(1-x_j)$ with $j = 0,1,2,\ldots$ and $x_0 = 1/3$. Then

$$x_0 = \frac{1}{3}, \quad x_1 = \frac{8}{9}, \quad x_2 = \frac{32}{81}, \quad x_3 = \frac{6272}{6561}.$$

Find $f(x)$ and calculate

$$\int_0^1 f(x)dx.$$

Problem 89. Consider the map $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 4x & \text{for} \quad 0 \le x \le 1/4 \\ -x/4+2 & \text{for} \quad 1/4 \le x \le 1/2 \\ 2x-1 & \text{for} \quad 1/2 \le x \le 3/4 \\ -2x+2 & \text{for} \quad 3/4 \le x \le 1 \end{cases}$$

Does the map show chaotic behaviour.

Problem 90. Let f be a continuous map from the unit interval $[0,1]$ onto itself, i.e. $f([0,1]) = [0,1]$.
(i) Show that the map f must have at least one fixed point.
(ii) Show that $f^{(2)}$ mus have at least two fixed points.

Problem 91. Let $f_r(x) = rx(1-x)$ with $r > 2 + \sqrt{5}$. Show that the Ljapunov exponent of any orbit that remains in $[0,1]$ is greater than 0 if it exists.

Problem 92. Consider the map $f : [0,\infty) \to [0,\infty)$. Assume that f is analytic and $f(0) = 0$. Let $p > 0$ be a fixed point such that $f'(p) \ge 0$. Furthermore assume that $f'(x)$ is decreasing. Show that all positive x_0 converges to the fixed point p under the iteration of the map f.

Problem 93. Consider the one-dimensional map $f:[0,1]\to[0,1]$

$$f(x)=\begin{cases} 1/2-2x & \text{if } 0\le x\le 1/4 \\ -1/2+2x & \text{if } 1/4\le x\le 3/4 \\ 5/2-2x & \text{if } 3/4\le x\le 1 \end{cases}$$

Find the invariant measure.

Problem 94. Find an approximation for *Feigenbaum's universal constant* α in his equation

$$g(x)=-\alpha g(g(-x/\alpha)) \qquad g(0)=1$$

by approximating g by a fourth degree polynomial. Also estimate the maximum error in $g(x)$ which results from this approximation. By construction,

$$g:[-\frac{1}{2},\frac{1}{2}]\to[-\frac{1}{2},\frac{1}{2}].$$

Problem 95. Consider the logistic equation

$$x_{t+1}=rx_t(1-x_t), \qquad x_0\in[0,1], \qquad t=0,1,2,\ldots \tag{1}$$

Using the *renormalization technique* show that how the accumulation point (r_∞) and the structural universalities (δ and α) can be determined approximatively for (1).

Problem 96. Let $x\in[0,1]$. Then $4x(1-x)\in[0,1]$. Apply the transformation $x\mapsto 4x(1-x)$ to the differential one form

$$\alpha=\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}dx.$$

Discuss.

Problem 97. Consider a one-dimensional chaotic map

$$x_{t+1}=f(x_t), \qquad t=0,1,2,\ldots$$

Then the *invariant measure* $\rho(x)$ with the *multiplication factor* m satisfies

$$\rho(x_{t+1})|dx_{t+1}|=m\cdot\rho(x_t)|dx_t|.$$

Thus

$$\left|\frac{df(x)}{dx}\right|=m\cdot\frac{\rho(x)}{\rho(f(x))}.$$

(i) Find the multiplication factor for the logistic map $f : [0,1] \to [0,1]$, $f(x) = 4x(1-x)$ with the invariant measure

$$\rho(x) = \frac{1}{\sqrt{x(1-x)}}.$$

(ii) Find the multiplication factor m for Baker map

$$f_3(x) = \begin{cases} 3x & 0 \le x \le 1/3 \\ 2-3x & 1/3 \le x \le 2/3 \\ 3x-2 & 2/3 \le x \le 1 \end{cases}$$

with the invariant measure $\rho(x) = 1$. The map f_3 is piecewise linear and continous.

Problem 98. A one-parameter family of analytic functions $f_r : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_r(x) = -x(x+r), \quad r > 0$$

where r is the bifurcation parameter.
(i) Find the fixed points of f_r.
(ii) Show that f_r has an infinite number of eventually fixed points in the interval $[-(1+r), 1]$ for $r > 2$.
Hint. It is sufficient to show that any $x \in [-(1+r), 1]$ has two preimages, both of which lie in $[-(1+r), 1]$.

Problem 99. Let $a, b, c, d \in \mathbb{R}$ and $a - bc \neq 0$, $c \neq 0$. Consider the map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{ax+b}{cx+d}$$

or written as a difference equation

$$x_{t+1} = \frac{ax_t + b}{cx_t + d}, \quad t = 0, 1, \ldots$$

(i) Find a $v(x)$ such that

$$v(f(x)) = v(x)\frac{df(x)}{dx}.$$

(ii) Find the fixed points of f and study their stability.

Problem 100. Consider the tent map

$$f_r(x) = \begin{cases} rx & \text{for } 0 \le x \le 0.5 \\ r(1-x) & \text{for } 0.5 < x \le 1 \end{cases}$$

where $r = \frac{1}{2}(1+\sqrt{5})$ (golden mean number). For symbolic dynamics we partition the unit interval $[0,1]$ into $L = [0, 0.5)$ and $R = (0.5, 1]$. Show that this tent map generates a first order Markov string with stationary probabilities

$$p(L) = \frac{1}{1+r^2}, \qquad p(R) = \frac{r^2}{1+r^2}.$$

Problem 101. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 1 - (1-x^2)^2.$$

Find the fixed points and study their stability.

Problem 102. Let $x_0 = 1$. Study the nonlinear difference equation

$$x_{t+1} = 1 + 1/x_t, \quad t = 0, 1, 2, \ldots$$

First find the fixed points.

Problem 103. Consider the nonlinear difference equation

$$x_{t+1} = \frac{x_t}{1+x_t}, \qquad t = 0, 1, 2, \ldots \tag{1}$$

with $x_0 > 0$ as a prescribed positive number (initial value).
(i) Find the fixed points of the difference equation (1).
(ii) Show that if $x_0 \neq 0$, then $x_t \neq 0$ for every t-value.
(iii) Let

$$v_t = \frac{1}{x_t}, \qquad t = 0, 1, 2, \ldots. \tag{2}$$

Show that (1) is transformed into the linear difference equation

$$v_{t+1} = v_t + 1, \qquad t = 0, 1, 2, \ldots. \tag{3}$$

(iv) Show that

$$v_t = v_0 + t$$

is the solution of the initial value problem of (3).
(v) Then use (2) to find the solution of the initial value problem of the original difference equation.

Problem 104. The *thermodynamic formalism* is as follows. Consider a chaotic map f. We define the partition function

$$Z_n(f, \beta) := \sum_{\mathrm{Fix}(f^{(n)}) \in Z} \exp\left[-\beta \ln |(f^{(n)})'(z)|\right] \tag{1}$$

where Fix denotes the number of fixed points of the iterated map $f^{(n)}$. Next we introduce the free energy $F(f,\beta)$ per unit time (or unit "site")

$$F(f,\beta) \equiv -\beta^{-1} \lim_{n\to\infty} \frac{1}{n} Z_n(f,\beta). \tag{2}$$

We have introduced the (inverse) temperature β, so we can formally develop a thermodynamics of chaos following the equilibrium thermodynamics. The internal energy U is

$$U(\beta) \equiv \frac{(\partial \beta F(f,\beta))}{\partial \beta} = \langle \ln |f'| \rangle_\beta. \tag{3}$$

The partition function $Z_n(f,\beta)$ can be characterized by the variational principle

$$-\ln Z_n(f,\beta) = \min(\beta(LCN) - h(\mu)) \tag{4}$$

where LCN is the Liapounov characteristic number, and $h(\mu)$ is the Kolmogorov-Sinai entropy of the invariant measure μ. Therefore U is the LCN for the map f. The entropy S is

$$S(f,\beta) := \beta^2 \frac{\partial F(f,\beta)}{\partial \beta} = \beta(U(f,\beta) - F(f,\beta)). \tag{5}$$

If f exhibits chaotic behaviour supported by an absolutely continuous measure, $F(f,1) = 0$. Hence

$$S(1) = \langle \ln |f'| \rangle_1. \tag{6}$$

The Kolmogorov entropy in this case is equal to the LCN. Thus we may identify $S(1)$ with the Komogorov entropy, if g allows an absolutely continuous invariant measure. In the $\beta \to 0$ limit, we have

$$S(0) = \lim_{n\to\infty} \frac{1}{n} \ln(\#\text{Fix}(f^{(n)}))$$

which is the formula for the topological entropy for g. *Dinaburg's theorem* asserting the inequality, topological entropy $\geq$ Kolmogorov entropy, has the following expression

$$S(0) \geq S(1).$$

This is obvious since entropy increases as β decreases due to the positivity of the "heat capacity". We identify $S(\beta_c)$ with the Kolmogorov entropy, where β_c is the β such that $F(f,\beta_c) = 0$.
(i) Let $f_p : [0,1] \to [0,1]$ be a map defined by

$$f_p(x) = \begin{cases} x/p & x \in [0,p] \\ (1-x)/(1-p) & x \in [p,1]. \end{cases}$$

Find $Z_n(f_p,\beta)$.

(ii) Find $S(1)$.

Problem 105. Let $b > a$. Consider a differentiable map $f : [a, b] \to [a, b]$. A fixed point x^* of the map f is called *superstable* if $df(x = x^*)/dx = 0$. Let $r \in [1, 4]$ and $f_r : [0, 1] \to [0, 1]$

$$f_r(x) = rx(1 - x).$$

Does the function f_r for some $r \in [1, 4]$ admits a superstable fixed point?

Problem 106. Let $g : \mathbb{R} \to \mathbb{R}$ be a nonconstant analytic function. Let $\widetilde{x} \in \mathbb{R}$ such that $g(\widetilde{x}) = 0$. Show that if $dg(x = \widetilde{x})/dx \neq 0$, then $\widetilde{x}$ is a superstable fixed point for Newton's method with

$$f(x) := x - g(x)/g'(x)$$

where $'$ denotes differentiation. Give an example.

Problem 107. Consider an integer sequence x_t $(t = 1, 2, \ldots)$ given by

$$\begin{aligned} x_{2t} &= 2x_t - 1 \quad \text{for } t \geq 1 \\ x_{2t+1} &= 2x_t + 1 \quad \text{for } t \geq 1 \end{aligned}$$

with the initial condition $x_1 = 1$. Find x_2, x_3, $\ldots$, x_{16} and the solution.

1.2.2 Supplementary Problems

Problem 108. Obtain the general solution of the equation

$$u_{t+1} = a u_t^n, \qquad n \neq 1$$

where $t = 0, 1, 2, \ldots$ in the form

$$u_t = c^{n^t} a^{1/(1-n)}$$

where c is an arbitrary constant. Take the logarithm of both sides. Show that this result can also be written in the form

$$u_t = u_0^{n^t} a^{(1-n^t)/(1-n)}.$$

Problem 109. Let $\omega \in [0,1]$ and $r > 0$ be the bifurcation parameters. The *sine circle map* is given by

$$\theta_{t+1} = \theta_t + \omega - \frac{r}{2\pi}\sin(2\pi\theta_t), \quad \bmod 1$$

where $t = 0, 1, \ldots$. The *rotation number* ρ is defined by

$$\rho := \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} |\theta_t - \theta_{t-1}|$$

whenever the limit exists. For $r < 1$ the limit always exists. It can be either a rational or an irrational number. The regions in the $r - \omega$ space where a unique rational number of ρ exists are called *Arnold tongues*. For $r < 1$ there is no overlap of the Arnold tongues. Show that for $r > 1$ the Arnold tongues begin to overlap.

Problem 110. Study the difference equation

$$e^{4x_{t+1}} = \frac{e^{8x_t} - e^{4x_t} + 4}{e^{4x_t} + 3}, \quad t = 0, 1, 2, \ldots$$

and $x_0 \geq 0$. The only fixed points are 0 and ∞. Let $x_0 = 1$. Find x_1, x_2, $\ldots$.

Problem 111. Given the two sequences of length n

$$\mathbf{S} = (S_0, S_1, \ldots, S_{n-1}), \quad \mathbf{T} = (T_0, T_1, \ldots, T_{n-1})$$

where $S_j, T_j \in \{-1, +1\}$. Implement the "delta function"

$$\delta(\mathbf{S}, \mathbf{T}) = \frac{1}{2^n}\prod_{j=0}^{n-1}(1 + S_j T_j).$$

Let $n > 2$. Calculate the autocorrelation function

$$C_k(\mathbf{S}) = \sum_{j=0}^{n-1-k} S_j S_{j+k}$$

where $k = 1, \ldots, n-1$.

Problem 112. The logistic map $x_{t+1} = 4x_t(1-x_t)$ is well studied. Study the map

$$\sinh(x_{t+1}) = 4x_t(1-x_t), \quad t = 0, 1, \ldots.$$

Is $x^* = 0$ a fixed point? Note that

$$\sinh(x) := \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!}$$

and $f(x) = \sinh(x)$ is a diffeomorphism.

Problem 113. Study the one-dimensional map

$$x_{t+1} = \frac{1}{2}\ln(\cosh(4x_t))$$

with $t = 0, 1, \ldots$ and the initial values $x_0 = 1/2, 1, 2$.

Problem 114. Consider the logistic map $f : [0,1] \to [0,1]$, $f(x) = 4x(1-x)$. Calculate

$$g(n) = \int_0^1 x^{n-1} f(x) dx, \quad n = 1, 2, \ldots$$

Reconstruct the function f from $g(n)$.

Problem 115. Proof the following theorem: If a continous function of the real numbers has a periodic point with prime period three, then it has periodic points of all prime periods.

Problem 116. Consider the Hilbert space of square integrable functions $L_2([-1,1])$. Then the chaotic map $f : [-1,1] \to [-1,1]$

$$f(x) = 1 - 2x^2$$

is an element of $L_2([-1,1])$. Using the *Legendre polynomials* we can form an orthonormal basis

$$B = \left\{ \phi_\ell(x) = \frac{\sqrt{2\ell+1}}{\sqrt{2}} \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell \, : \, \ell = 0, 1, 2, \ldots \right\}$$

in $L_2([-1,1])$. Consider the infinite dimensional matrix $F = (F_{jk})$ $(j,k = 0,1,2,\ldots)$

$$F_{jk} = \langle \phi_j | f(x) | \phi_k \rangle = \int_{-1}^{+1} \phi_j(x) f(x) \phi_k(x), \quad j,k = 0,1,2,\ldots$$

which acts as a linear bounded operator in the Hilbert space $\ell_2(\mathbb{N}_0)$. How can one reconstruct the function f from F and the orthonormal basis? Calculate the matrix F and find the spectrum of F.

Problem 117. Show that the only real solution of

$$\sin(x) = x$$

is $x = 0$, i.e. $f(x) = \sin(x)$ only admits the fixed point $x = 0$.

Problem 118. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = e^{-x} + 1.$$

(i) Show that $f(x^*) = x^*$ (fixed point equation) has only one real solution.
(ii) Show that this fixed point is given by

$$x^* = 1 + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} j^{j-1} e^{-j}}{j!}.$$

Problem 119. Consider the polynomial $T_5 : \mathbb{R} \to \mathbb{R}$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

which is one of the *Chebyshev polynomials* of first kind.
(i) Show that if $x \in [-1,1]$, then $T_5(x) \in [-1,1]$.
(ii) Find the fixed points in $[-1,1]$ and show that they are unstable.
(iii) Find the critical points of T_5 in $[-1,1]$ and study symbolic dynamics.
(iv) Find the exact solution of

$$x_{t+1} 16x_t^5 - 20x_t^3 + 5x_t.$$

First derive the variational equation.
(v) Find the Ljapunov exponent.

Problem 120. Show that the map $f : [-1,1] \to [-1,1]$

$$f(x) = 4x^3 - 3x$$

and the map $g : [-1,1] \to [-1,1]$

$$g(x) = \begin{cases} 4x+3 \text{ for } -1 \le x \le -1/2 \\ -2x \quad \text{for } -1/2 \le x \le 1/2 \\ 4x-3 \text{ for } \quad 1/2 \le x \le 1 \end{cases}$$

are topologically conjugate.

Problem 121. Let $r \ge 0$ be the bifurcation parameter and $x_0 \ge 0$. Study the map

$$x_{t+1} = x_t^5 e^{-x_t} + r, \quad t = 0,1,2,\ldots$$

Since $x_0 \ge 0$ we have $x_t \ge 0$ for all t. First find the fixed points and study their stability.

Problem 122. Let x be a positive integer. Now $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Let a and b be real numbers with $a > 1$. We define the sequence p_t as

$$p_t = \lfloor (t+1)/a + b \rfloor - \lfloor t/a + b \rfloor, \quad t = 1,2,\ldots$$

Consequently we have a sequence of 0's and 1's.
(i) Show that the sequence p_t takes its 1's on the set

$$S_1 \,:\, \{\, t \,:\, t = \lfloor (k-b)a \,\rfloor \,,\, k \in \mathbb{R} \,\}$$

and its 0's on the set

$$S_0 \,:\, \{\, t \,:\, t = \lfloor (\ell + b)c \rfloor \,,\, \ell \in \mathbb{N} \,\}$$

where c is defined as

$$\frac{1}{a} + \frac{1}{c} = 1.$$

(ii) Show that the two sets satisfy $S_0 \cap S_1 =$ and $S_0 \cup S_1 = \mathbb{N}$.
(iii) Let $a = 2$ and $b = 1/2$. Write a C++ program using the class `Verylong` that finds the sequence.

Problem 123. Let $\phi = \frac{1}{2}(\sqrt{5} - 1)$. Calculate the sequence

$$x_t = \lfloor (t+1)/\phi \rfloor - \lfloor t/\phi \rfloor, \quad t = 0,1,2,\ldots.$$

Problem 124. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 2x^3.$$

Then the inverse function f^{-1} is given by

$$f^{-1}(x) = \sqrt[3]{\frac{x}{2}}.$$

Find the fixed points of f and f^{-1}. Discuss.

Problem 125. Let $r \in [1,4]$. Consider the logistic map $f_r : [0,1] \to [0,1]$

$$f_r(x) = rx(1-x).$$

Let $r = r_\infty \approx 3.570$. Show that the corresponding invariant set $A \subset [0,1]$ has both Hausdorff and box-dimensions equal to ≈ 0.538.

Problem 126. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x + \sin(x).$$

(i) Show that f admits infinite many fixed points given by $x^* = n\pi$, where $n \in \mathbb{Z}$.
(ii) Study the stability of these fixed points.
(iii) Show that the function is a diffeomorphism. Find f^{-1}.
(iii) Find the inverse of f applying the *Lagrange inversion theorem*. Let $y = f(x)$, where f is analytic at a point p and $df(x=p)/dx \neq 0$. Then one can invert on a neighbourhood of $f(p)$, i.e. $x = g(y)$, where the function g is analytic at the point $f(p)$. The series expansion is

$$g(y) = p + \sum_{j=1}^{\infty} \left(\lim_{x \to p} \left(\frac{(y-f(p))^j}{j!} \frac{d^{j-1}}{dx^{j-1}} \left(\frac{x-p}{f(x)-f(p)} \right)^j \right) \right).$$

In the present case we have $p = 0$. Then $df(x=p)/dx = 2$ and $f(p=0) = 0$ and the series expansion simplifies to

$$g(y) = \sum_{j=1}^{\infty} \left(\lim_{x \to p} \left(\frac{y^j}{j!} \frac{d^{j-1}}{dx^{j-1}} \left(\frac{x}{f(x)} \right)^j \right) \right).$$

Problem 127. Show that the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = -2x - \sin(x)$$

is a diffeomorphism. Show that $x^* = 0$ is a fixed point. Is the fixed point stable?

Problem 128. Let $r > 0$. Show that the analytic function $f_r : \mathbb{R} \to \mathbb{R}$

$$f_r(x) = rx + \arctan(x)$$

is a diffeomorphism. Show that $df_r/dx > 0$ for all $x \in \mathbb{R}$. Note that

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Problem 129. (i) Show that the logistic map $x_{t+1} = 4x_t(1-x_t)$ can be written as

$$x_{t+1} = \begin{pmatrix} x_t & 1-x_t \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ 1-x_t \end{pmatrix}.$$

(ii) Show that the logistic map $x_{t+1} = 4x_t(1-x_t)$ can be written as

$$\begin{pmatrix} x_{t+1} \\ 1-x_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix} \left(\begin{pmatrix} x_t \\ 1-x_t \end{pmatrix} \otimes \begin{pmatrix} x_t \\ 1-x_t \end{pmatrix} \right)$$

where $\otimes$ denotes the Kronecker product.

Problem 130. (i) Construct a polynomial $p(x) = x^2 + ax + b$ that admits the roots

$$\frac{\sqrt{3}+3}{6}, \qquad -\frac{\sqrt{3}-3}{6}.$$

(ii) Construct a polynomial $p(x) = x^2 + ax + b$ that admits the fixed points

$$\frac{\sqrt{30}+6}{6}, \qquad -\frac{\sqrt{30}-6}{6}.$$

Problem 131. Let $x > 0$. Find the fixed points of the functions $f_1(x) = x + 1/x$, $f_2(x) = x - 1/x$ and study their stability.

Problem 132. Study the map $f : [0,1) \to [0,1)$

$$f(x) = \begin{cases} 2x \pmod 1 & 0 \le x < 1/2 \\ 4x \pmod 1 & 1/2 \le x < 1 \end{cases}$$

(i) Find the fixed points.
(ii) Show that f preserves Lebesgue measure.
(iii) Does the map show chaotic behaviour?

Problem 133. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{\pi}{2} + x - \arctan(x).$$

(i) Show that the function has no fixed points.

(ii) Show that

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x \neq y.$$

Problem 134. Consider the map $f : [-1, 1] \to [-1, 1]$

$$f(x) = 1 - 2\sqrt{|x|}$$

with $f(1) = -1$, $f(-1) = -1$ and $f(0) = 1$. Does the map show chaotic behaviour?

Problem 135. Consider the map $f_r : (1, \infty) \to \mathbb{R}$ given by

$$f_r(x) = \frac{1}{2}\left(x + \frac{r}{x}\right).$$

(i) Show that if $r \in (1, 3)$, then f_r maps $(1, \infty)$ into itself, i.e. $f(x) \in (1, \infty)$ for all $x \in (1, \infty)$.
(ii) Show that the map f_r is a contraction if $r \in (1, 3)$. Find the fixed points as a function of the parameter r.

Problem 136. Demonstrate the existence of an orbit of $f : [0, 1] \to [0, 1]$, $f(x) = 4x(1 - x)$ with prime period three.

Problem 137. (i) Consider the *Bernoulli map* $f : [0, 1) \to [0, 1)$ given by

$$f(x) = 2x \bmod 1. \tag{1}$$

Find a point $x_0 \in [0, 1)$ whose orbit is dense in $[0, 1)$.
(ii) A generalization of the Bernoulli map is the map

$$f(x) = Dx \quad \bmod 1 \tag{1}$$

with $D \geq 2$. Show that the Ljapunov exponent for almost all initial values is given by $\ln(D)$.

Problem 138. Let f_α be defined by

$$f_\alpha(x) := \begin{cases} x/\alpha & x \in [0, \alpha] \\ 1 & x \in [\alpha, 1-\alpha] \\ (1-x)/\alpha & x \in [1-\alpha, \alpha] \end{cases}$$

Show that this allows a Cantor set C as a maximal invariant set. Show that the Hausdorff dimension D_H of C is $-\ln(2)/\ln\alpha$. Show that there is an invariant measure with the Kolmogorov entropy $\ln(2)$ supported on C. Show that the partition function is

$$Z_n(f_\alpha, \beta) = (2\alpha^\beta)^n.$$

Show that from $Z_n(f_\alpha, \beta)$ we obtain

$$F(f_\alpha, \beta) = -\ln(\alpha) - \beta \ln 2$$

which implies $F(f_\alpha, D_H) = 0$. Show that $S(f_\alpha, \beta)$ is $\ln 2$. Show that $S(\infty) = \ln 2$.

Problem 139. Let f be a continuous map from the unit interval $[0,1]$ onto itself, i.e. $f([0,1]) = [0,1]$.
(i) Show that the map f must have at least one fixed point.
(ii) Show that $f^{(2)}$ mus have at least two fixed points.

Problem 140. Consider a continuous map $f : [0,1] \to [0,1]$. Show that there are points in $[0,1]$ that are not fixed points, perdiodic points, or eventually periodic points of the map f.

Problem 141. Let a sequence of functions $r_k(x)$ be defined as follows. The zeroth function if defined to be $r_0(x) = (x)$, the first function to be $r_1(x) = a_1/(b_1 + x)$, and the kth function is obtained from the preceeding function $r_{k-1}(x)$ by replacing x by $a_k/(b_k + x)$, where $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ are constants.
(i) Show that

$$r_0(x) = x, \qquad r_1(x) = \frac{a_1}{b_1 + x}, \qquad r_2(x) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + x}}, \cdots \tag{1}$$

and, in general,

$$r_k(x) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{\cdot}{\cdot + \cfrac{a_k}{b_k + x}}}}}.$$

The expression $r_k \equiv r_k(0)$, obtained by setting $x = 0$ in $r_k(x)$, is called a continued fraction of k stages.
(ii) Show that the result of clearing fractions in the expression for $r_k(x)$ is the ratio of two linear functions of x, of the form

$$r_k(x) = \frac{A_k + C_k x}{B_k + D_k x}, \qquad k = 0, 1, 2, \cdots,$$

(iii) Deduce that

$$\frac{A_k + C_k x}{B_k + D_k x} \equiv \frac{(b_k A_{k-1} + a_k C_{k-1}) + A_{k-1} x}{(b_k B_{k-1} + a_k D_{k-1}) + B_{k-1} x}, \qquad k = 1, 2, \ldots,$$

for all values of x for which $r_k(x)$ is defined, and that also

$$A_1 = Ca_1, \qquad C_1 = 0, \qquad B_1 = Cb_1, \qquad D_1 = C$$

where C is an arbitrary nonzero constant of proportionality. The right-hand member of the identity is the result of replacing k by $k-1$ and x by $a_k/(b_k + x)$ in the left-hand member.

Problem 142. Consider a system which prints out a sequence of symbols using a basic alphabet of m different characters. Show that the number of admissible character strings of length n is m^n, ϵ is the length of a sequence examined and $H = \log m$.

Problem 143. Consider the tent map ($t = 0, 1, \ldots$)

$$x_{t+1} = \begin{cases} rx_t & x_t \in [0, 1/2] \\ r(1 - x_t) & x_t \in [1/2, 1] \end{cases}$$

where $r = 2$. Derive the master equation

$$p_n(x) = \frac{1}{r}\left(p_{n-1}\left(\frac{x}{r}\right) + p_{n-1}\left(1 - \frac{x}{r}\right)\right), \quad n = 1, 2, \ldots$$

given some initial non-equilibrium density $p_0(x)$.

Problem 144. Let $\alpha = \sqrt{2}$. Consider the map

$$x_j = (j+1)^2 \alpha \mod 1$$

where $j = 0, 1, \ldots$. The sequence x_0, x_1, x_2, $\ldots$, is uniformly distributed on the unit interval (equidistribution theorem). Study numerically the Liapounov exponent and Hurst exponent for this map. Study also the case where $\alpha = (\sqrt{5} - 1)/2$.

Problem 145. Let $n = 0, 1, 2, \ldots$. The *Fermat numbers* are given by

$$F_n = 2^{(2^n)} + 1.$$

Show that the Fermat numbers satisfy the recurrence relation

$$F_{n+1} = (F_n - 1)^2 + 1$$

with $F_0 = 3$.

Problem 146. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = 1 + x + \cos(x)\cosh(x).$$

Show that the fixed point equation $f(x^*) = x^*$ has infinitely many solutions.

Problem 147. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |1 - |2 - |3 - x|||.$$

(i) Find the fixed points. Are the fixed points stable?
(ii) Is the function continuous?
(iii) Find minima and maxima of f.
(iv) Find $f(5)$, $f(f(5))$, $f(f(f(5)))$. Discuss.

Problem 148. Study the difference equation

$$x_{t+1} = e^{i\pi t}(1 - 2x_t^2), \quad t = 0, 1, 2, \ldots$$

where $x_0 \in [-1, 1]$.

Problem 149. Consider the function $f : [0, 1] \to [0, 1]$

$$f(x) = \frac{1}{2} - \frac{1}{2}\sin(2\pi x).$$

Find the fixed points and study their stability. The fixed $x^* = 1/2$ is obvious. The others two must be found numerically.

Problem 150. Consider the function $f : [0, 1] \to [0, 1]$

$$f(x) = 1 - \frac{1}{2}x^2 - \frac{1}{2}x^4.$$

Note that $f(0) = 1$ and $f(1) = 0$. If $x_1 < x_2$, then $f(x_1) > f(x_2)$. The function admits one fixed point. Find the fixed point and study its stability.

Problem 151. (i) Consider (*Catalan numbers*)

$$C_n = \frac{4n - 2}{n + 1}C_{n-1}, \quad n \geq 1$$

with $C_0 = 1$. Find C_1, C_2, C_3, C_4, C_5. Write a SymbolicC++ program utilizing the class `Verylong` to find these numbers. Write a Java program utilizing the class `BigInteger` to find these numbers.

(ii) Consider

$$C_n = \sum_{j=0}^{n-1} C_j C_{n-j-1}, \quad n \geq 1$$

with $C_0 = 1$. Find C_1, C_2, C_3, C_4, C_5. Write a SymbolicC++ program utilizing the class `Verylong` to find these numbers. Write a Java program utilizing the class `BigInteger` to find these numbers.
(iii) Let $n \geq 3$. Consider

$$T_n = \sum_{j=2}^{n-1} T_j T_{n-j+1}, \quad n \geq 3$$

with $T_2 = 1$. Find T_3, T_4, T_5, T_6. Write a SymbolicC++ program utilizing the class `Verylong` to find these numbers. Write a Java program utilizing the class `BigInteger` to find these numbers.

Problem 152. Construct a polynomial

$$p(x) = x^2 + ax + b$$

that admits the roots

$$\frac{1}{6}(\sqrt{3}+3), \qquad -\frac{1}{6}(\sqrt{3}-3)$$

and the fixed points

$$\frac{1}{6}(\sqrt{30}+6), \qquad -\frac{1}{6}(\sqrt{30}-6).$$

Problem 153. Consider the chaotic map $f : [-1,1] \to [-1,1]$

$$f(x) = 1 - 2x^2$$

and the Hilbert space $L_2([-1,1])$ of square integrable functions. Then $f \in L_2([-1,1])$. An orthonormal basis in this Hilbert space is given by the normalized Legrende polynomials

$$\mathcal{B} = \left\{ \frac{\sqrt{2\ell+1}}{\sqrt{2}} \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell \; \ell = 0,1,2 \right\}.$$

Find the infinite dimensional matrix (matrix representation of f)

$$F_{jk} = \int_{-1}^{+1} \phi_j(x) f(x) \bar{\phi}_k(x) dx$$

where in this case $\bar{\phi}_k = \phi_k$. Reconstruct f from the infinite dimensional matrix $F = (F_{jk})$.

Chapter 2

Higher-Dimensional Maps and Complex Maps

2.1 Introduction

Inverse function theorem. Let $\mathbf{f}$ be a continuously differentiable, vector-valued function mapping an open set $E \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Let $S = \mathbf{f}(E)$. If, for some point $\mathbf{p} \in \mathbf{E}$, the Jacobian determinant $\det(J_{\mathbf{f}}(\mathbf{p}))$ is nonzero, then tere exists a uniquely determined functional $\mathbf{g}$ and two open sets $X \subset E$ and $Y \subset S$ such that (i) $\mathbf{p} \in X$, $\mathbf{f}(\mathbf{p}) \in Y$, (ii) $Y = \mathbf{f}(X)$, (iii) $\mathbf{f} : X \to Y$ is one-one, (iv) $\mathbf{g}$ is continuously differentiable on Y and

$$\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$$

for all $\mathbf{x} \in X$.

2.2 Two-Dimensional Maps

2.2.1 Solved Problems

Problem 1. Study the coupled logistic maps $\mathbf{f} : [0,1] \times [0,1] \to [0,1] \times [0,1]$

$$f_1(x_1, x_2) = 4x_2(1 - x_2), \quad f_2(x_1, x_2) = 4x_1(1 - x_1)$$

or written as difference equations

$$x_{1,t+1} = 4x_{2,t}(1 - x_{2,t}), \qquad x_{2,t+1} = 4x_{1,t}(1 - x_{1,t})$$

where $x_{1,0} \in [0,1]$ and $x_{2,0} \in [0,1]$. First find the fixed points and study their stability. Does the system show chaotic and hyperchaotic behaviour?

Problem 2. The *Henon map* is given by The Henòn map $\mathbf{f}_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f_1(x_1, x_2) = 1 + x_2 - ax_1^2 \qquad f_2(x_1, x_2) = bx_1.$$

or written as system of difference equations

$$x_{1,t+1} = 1.0 + x_{2,t} - ax_{1,t}^2, \qquad x_{2,t+1} = bx_{1,t}$$

where a and b are bifurcation parameters and $t = 0, 1, 2, \ldots$. We assume that $b > 0$.
(i) Show that the determinant of the functional matrix is given by $-b$. What is the conclusion we can draw from this result?
(ii) Show that the fixed points of the Henon map are given by

$$x_1^* = \frac{1}{2}[-(1-b) \pm \sqrt{(1-b)^2 + 4a}], \qquad x_2^* = bx_1^*.$$

Show that the fixed points are real for $a > (1-b)^2/4$.
(iii) Show that for $a = 1.4$ and $b = 0.3$ the fixed points are unstable, i.e. find the values of a and b for which the fixed points are attractive, repellent or saddle points.
(iv) Find the periodic points of period 2 of $\mathbf{f}_{a,b}$ and the values of a and b for which they exist.

Problem 3. Let $a > 0$ and $b > 0$. Consider the Hénon map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = 1 + x_2 - ax_1^2, \quad f_2(x_1, x_2) = bx_1.$$

(i) Let $\omega = dx_1 \wedge dx_2$. Find $\mathbf{f}^*(\omega) = \mathbf{f}^*(dx_1 \wedge dx_2)$.
(ii) Let $\alpha_1 = x_1 dx_2 + x_2 dx_1$. Find $\mathbf{f}^*(\alpha)$.
(iii) Let $\alpha_2 = x_1 dx_2 - x_2 dx_1$. Find $\mathbf{f}^*(\alpha_2)$.

(iv) Let $\alpha_3 = x_1 dx_1 + x_2 dx_2$. Find $\mathbf{f}^*(\alpha_3)$.
(v) Let $\alpha_4 = x_1 dx_1 - x_2 dx_2$. Find $\mathbf{f}^*(\alpha_4)$.

Problem 4. (i) Consider the analytic function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = \sinh(x_2), \quad f_2(x_1, x_2) = \sinh(x_1).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(ii) Consider the analytic function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$

$$g_1(x_1, x_2) = \sinh(x_1), \quad g_2(x_1, x_2) = -\sinh(x_2).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(iii) Multiply the two matrices found in (i) and (ii).
(iv) Find the composite function $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{h}(\mathbf{x}) = (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x})).$$

Show that this function also admits the fixed point $(0,0)$. Find the functional matrix at this fixed point

$$\left.\begin{pmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

Compare this matrix with the matrix found in (iii).

Problem 5. Study the coupled map

$$x_{t+1} = 2y_t \mod 1, \qquad y_{t+1} = 2x_t \mod 1$$

where $t = 0, 1, 2, \ldots$ and $x_0, y_0 \in [0,1)$. Consider the cases that x_0, y_0 rational and irrational numbers in the interval $[0,1)$.

Problem 6. Study the coupled map

$$x_{t+1} = 2x_t \mod 1, \qquad y_{t+1} = 2x_t y_t \mod 1$$

where $t = 0, 1, 2, \ldots$ and $x_0, y_0 \in [0, 1)$. Consider the cases that x_0, y_0 rational and irrational numbers in the interval $[0, 1)$.

Problem 7. Study the coupled map

$$x_{t+1} = 2x_t y_t \mod 1, \qquad y_{t+1} = 2x_t y_t \mod 1$$

where $t = 0, 1, 2, \ldots$ and $x_0, y_0 \in [0, 1)$. Consider the cases that x_0, y_0 rational and irrational numbers in the interval $[0, 1)$.

Problem 8. The *Baker's transformation* $\mathbf{f} : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ is given by

$$\mathbf{f}(x_1, x_2) = \begin{cases} (2x_1, x_2/2) & 0 \leq x_1 < 1/2 \\ (2x_1 - 1, x_2/2 + 1/2) & 1/2 < x_1 \leq 1 \end{cases}$$

The horizontal direction is stretched by a factor 2, the vertical direction is contracted by a factor 1/2. The Baker's transformation is a completely chaotic (Bernoulli) area preserving map of the unit square onto itself.
(i) Find $\mathbf{f}(1/3, 2/3)$ and $\mathbf{f}(\mathbf{f}(1/3, 2/3))$.
(ii) Find the fixed points of $\mathbf{f}$.
(iii) Find the inverse of $\mathbf{f}$.
(iv) Find $\mathbf{f}(1/2, 1/2)$, $\mathbf{f}(\mathbf{f}(1/2, 1/2))$ etc. Does this sequence converges?
(v) Find the Frobenius-Perron operator for $\mathbf{f}$.
(vi) Find an explicit expression for the n-th iterate $\mathbf{f}^{(n)}$ of $\mathbf{f}$ in terms of certain permutations σ_n on sets of integers.

Problem 9. Let $t = 0, 1, 2, \ldots$. Solve the second order difference equation

$$x_{t+2} = \frac{1}{x_{t+1} + 1/x_t}$$

for (i) $x_0 = x_1 = 1$ and (ii) $x_0 = 1$, $x_1 = 1/2$.

Problem 10. The coupled logistic map $\mathbf{f}_{r,\epsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f_{1,r,\epsilon}(x, y) = rx(1 - x) + \epsilon(y - x) \qquad f_{2,r,\epsilon}(x, y) = ry(1 - y) + \epsilon(x - y)$$

with $r > 0$ and ϵ are bifurcation parameters.
(i) Find all its fixed points and the values of the bifurcation parameters r and ϵ for which they exist.
(ii) Analyse the stability of all its fixed points.

Problem 11. The coupled logistic map $\mathbf{f}_{r,\epsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$x_{k+1} = f_1(x_k, y_k), \qquad y_{k+1} = f_2(x_k, y_k)$$

where

$$f_1(x,y) = rx(1-x) + \epsilon(y-x), \qquad f_2(x,y) = ry(1-y) + \epsilon(x-y)$$

and $r > 0$ and ϵ are constants.
(i) Find all the fixed points of F and establish for which values of r and ϵ they exist. (Hint: First find the fixed points for which $x = y$. These can then be factored out of the fixed point equations.)
(ii) For $\epsilon = 0.1$ find one value of r for which the map will exhibit regular dynamics and also one value of r for which it exhibits chaotic dynamics. The maximal Lyapunov exponent for a two-dimensional set of difference equations,

$$x_{t+1} = f_1(x_t, y_t), \qquad y_{t+1} = f_2(x_t, y_t)$$

(where we assume that f_1 and f_2 are smooth functions), is calculated in the following way. The variational equations are defined as by

$$u_{t+1} = \frac{\partial f_1}{\partial x}(x = x_t, y = y_t)u_t + \frac{\partial f_1}{\partial y}(x = x_t, y = y_t)v_t$$

$$v_{t+1} = \frac{\partial f_2}{\partial x}(x = x_t, y = y_t)u_t + \frac{\partial f_2}{\partial y}(x = x_t, y = y_t)v_t.$$

The maximal one-dimensional Lyapunov exponent is then given by

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \ln\left(|u_T| + |v_T|\right).$$

In practice $T = 1000$ can be used together with arbitrary initial values of u and v.]

Problem 12. Let z be a complex number. The *Ikeda laser map* is given by

$$z \to \rho + c_2 z \exp\left(i(c_1 - \frac{c_3}{1+|z|^2}\right) \qquad (1)$$

where $t = 0, 1, 2, \ldots$ and z_0 is the initial value. The bifurcation parameters are ρ, c_1, c_2 and c_3. With $z = x + iy$ and $x, y \in \mathbb{R}$ we can write the map (1) as difference equation

$$x_{t+1} = \rho + c_2(x_t \cos(\tau_t) - y\sin(\tau_t) \qquad (2a)$$

$$y_{t+1} = c_2 x_t \sin(\tau_t) + y\cos(\tau_t) \qquad (2b)$$

where

$$\tau_t := c_1 - \frac{c_2}{1 + x_t^2 + y_t^2}$$

and $t = 0, 1, 2, \ldots$. Find the fixed points. Study the stability of the fixed points.

Problem 13. Consider the coupled system of difference equations

$$x_{t+1} = b(3y_t + 1)x_t(1 - x_t), \qquad y_{t+1} = b(3x_t + 1)y_t(1 - y_t) \qquad (1)$$

where b is a bifurcation parameter and $t = 0, 1, 2, \ldots$. Find the fixed points and study their stability.

Problem 14. The *generalized Baker's map* is an analytically treatable but nontrivial sample for a *multifractal* . It is defined by

$$x_{t+1} = \begin{cases} \lambda_a x_t & y_t < a \\ \frac{1}{2} + \lambda_b x_t & y_t > a \end{cases} \qquad (1a)$$

$$y_{t+1} = \begin{cases} \frac{1}{a} y_t & y_t < a \\ \frac{1}{1-a}(y_t - a) & y_t > a \end{cases} \qquad (1b)$$

where

$$x_t \in [0, 1], \qquad y_t \in [0, 1]. \qquad (2)$$

(i) Show that an expression for the $f(\alpha)$-spectrum of its attractor is given by

$$f(\alpha) = 1 + \frac{(1-\kappa)\log(1-\kappa) + \kappa \log \kappa}{(1-\kappa)\log \lambda_a + \kappa \log \lambda_b}$$

$$\kappa = \frac{\log a - (\alpha - 1)\log \lambda_a}{(\alpha - 1)\log(\lambda_b/\lambda_a) + \log(a/b)}$$

where $b = 1 - a$.
(ii) Let

$$\lambda_a = \lambda_b = \frac{1}{5}, \qquad a = \frac{2}{5}, \qquad b = \frac{3}{5}.$$

Show that

$$f(\alpha) = \frac{(1-\kappa)\log(1-\kappa) + \kappa \log \kappa}{\log \frac{2}{3}}$$

$$\kappa = \frac{\log \kappa - \log(1-\kappa)}{\log \frac{1}{5}}.$$

(iii) Show that by a Legendre transformation

$$(1 - q)D_4 = q\alpha_q - f(\alpha_q)$$

with

$$q = \frac{\partial f(\alpha)}{\partial \alpha}$$

$$\alpha_q = \alpha|_{\partial f(\alpha)/\partial \alpha = q}$$

we get the Rényi dimensions. Show that

$$D_{-1} = 1.4434, \qquad D_0 = 1.4307, \qquad D_2 = 1.4063$$

with the parameters given above.

Problem 15. Consider a one parameter set of diffeomorphisms Φ_μ. Assume that

(a) The origin is a fixed point of Φ_μ.
(b) For $\mu < 0$ the spectrum of Φ_μ at the origin is contained in

$$\{ z \in \mathbb{C} : |x| < 1 \}.$$

(c) For $\mu = 0$ resp. $\mu > 0$ the spectrum of Φ_μ at the origin has two resp.

$$|\lambda(\mu)| > 1.$$

The remaining part of the spectrum is contained in $\{ z \in \mathbb{C} : |z| < 1 \}$.

We consider now a one parameter family $\Phi_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ of diffeomorphisms satisfying (a), (b) and (c) and such that

$$\frac{d}{d\mu}(|\lambda(\mu)|)_{\mu=0} > 0. \tag{2}$$

(i) Show that Φ_μ, by coordinate transformations, into a simple form: We change the μ coordinate in order to obtain (d)

$$|\lambda(\mu)| = 1 + \mu.$$

After an appropriate (μ dependent) coordinate change of $\mathbb{R}^2$ we then have

$$\Phi(r, \varphi, \mu) = ((1+\mu)r, \varphi + f(\mu), \mu) + \quad \text{terms of order} \quad r^2 \tag{8}$$

where

$$x_1 := r\cos\varphi, \qquad x_2 := r\sin\varphi. \tag{9}$$

Here

$$\Phi = \Phi' + \quad \text{terms of order} \quad r^l$$

means that the derivatives of Φ and Φ' up to order $l-1$ with respect to (x_1, x_2) agree for $(x_1, x_2) = (0, 0)$. We now put in one extra condition (e)

$$f(0) \pm \frac{d}{l} 2\pi \quad \text{for all} \quad k, l \leq 5. \tag{11}$$

(iii) Prove the following. Suppose Φ_μ satisfies (2), (3), (4), (5) and (6) and is C^k, $k \geq 5$. Then for μ near 0, by a μ dependent coordinate change in $\mathbb{R}^2$, one can bring Φ_μ in the form

$$\Phi_\mu(r, \varphi) = ((1+\mu)r - f_1(\mu)r^3, \varphi + f_2(\mu) + f_3(\mu)r^2) + \quad \text{terms of order} \quad r^5. \tag{10}$$

For each μ, the coordinate transformation of $\mathbb{R}^2$ is C^∞. The induced coordinate transformation on $\mathbb{R}^2 \times \mathbb{R}$ is only C^{k-4}.

Problem 16. Consider the following two-dimensional non-invertible map

$$\Theta_{t+1} = 2\Theta_t \mod 2\pi, \qquad z_{t+1} = \lambda z_t + \cos\Theta_t \tag{1}$$

where

$$2 > \lambda > 1, \qquad 0 < \Theta < 2\pi. \tag{2}$$

Find the *fractal basin boundary.*

Problem 17. Consider the two-dimensional map $S_c : [0,1] \times [0,1] \to [0,1] \times [0,1]$

$$S_c(x, y) = (T_c(x), U_c(x, y)) \tag{1}$$

wherein the parametrized maps $T_c, U_c : [0,1] \to [0,1]$ are given by

$$T_c(x) = \begin{cases} \dfrac{1-c}{c}x + c & 0 \leq x \leq c \\ \dfrac{1}{1-c} - \dfrac{1}{1-c}x & c \leq x \leq 1 \end{cases} \tag{2}$$

and

$$U_c(x, y) = \begin{cases} \alpha y & 0 \leq x \leq c \\ \alpha + (1-\alpha)y & c < x \leq 1 \end{cases} \tag{3}$$

respectively, where α and c in (2) and (3) satisfy $\alpha, c \in (0,1)$.
(i) Show that the map is invertible.
(ii) Find the Frobenius-Perron operator.

Problem 18. Consider the Hénon area-preserving map ($\mathbb{R}^2 \to \mathbb{R}^2$)

$$x_1 = x\cos\alpha - y\sin\alpha + x^2\sin\alpha$$
$$y_1 = x\sin\alpha + y\cos\alpha - x^2\cos\alpha.$$

(i) Show that the two fixed points lie on the line $y = x\tan(\alpha/2)$.
(ii) Verify that this map is conjugate to

$$X_1 = X\cos\alpha - Y\sin\alpha + (X\cos(\alpha/2) - Y\sin(\alpha/2))^2\sin(\alpha/2)$$

$$Y_1 = X \sin\alpha + Y \cos\alpha - (X\cos(\alpha/2) - Y\sin(\alpha/2)))^2 \cos(\alpha/2)$$

by a rotation through the angle $\alpha/2$.
(iii) Prove that it is conjugate to its own inverse by using the transformation $(X, Y) \to (X, -Y)$.

Problem 19. In matrix form, the cat map is given by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{mod } 1$$

which transforms the column vector (x_{1n}, x_{2n}) into the column vector (x_{1n+1}, x_{2n+1}), where all x_1 and x_2 are taken modulo 1.
(i) Given the recursion for the Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2}, \qquad F_1 = F_2 = 1$$

find $T^{(n)}$ by induction.
(ii) Show that the determinant of $T^{(n)}$ is equals 1.
(iii) Find the eigenvalues of $T^{(n)}$.
(iv) Show that fixed points of $T^{(n)}$ correspond to orbits of period length n and any divisors of n for $n = 2$.

Problem 20. Find the fixed points of the two-dimensional *Tinkerbell map* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}(x_1, x_2) = (x_1^2 - x_2^2 + c_1 x_1 + c_2 x_2, 2x_1 x_2 + c_3 x_1 + c_4 x_2)$$

and study their stability, where $c_1 = -0.3$, $c_2 = -0.6$, $c_3 = 2$, $c_4 = 0.5$.

Problem 21. Let a, b, c, d be real parameters. The *Tinkerbell map* is $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f_1(x_1, x_2) = x_1^2 - x_2^2 + ax_1 + bx_2, \qquad f_2(x_1, x_2) = 2x_1 x_2 + cx_1 + dx_2$$

or written as difference equation

$$x_{1,t+1} = x_{1,t} - x_{2,t} + ax_{1,t} + bx_{2,t}, \qquad x_{2,t+1} = 2x_{1,t}x_{2,t} + cx_{1,t} + dx_{2,t}$$

where $t = 0, 1, \ldots$. Is $dx_1 \wedge dx_2$ invariant under the map? Find the fixed points and study their stability. Calculate the first iterate of the map and their fixed points, i.e. find periodic points. Study the stability of these periodic points.

Problem 22. Consider the two-dimensional map $\mathbf{f}(x, y) : [0, 1]^2 \to [0, 1]^2$ given by

$$\mathbf{f}_\alpha(x, y) = \begin{cases} (2x, \alpha y) & \text{if } 0 \le x \le 1/2 \\ (2x - 1, \alpha y + (1 - \alpha)) & \text{if } 1/2 \le x \le 1 \end{cases}$$

where $1/2 \leq \alpha < 1$. Discuss the behaviour of the map for $\alpha \geq 1/2$.

Problem 23. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}(x, y) = ((4/\pi)\arctan(x), y/2)$$

i.e. $f_1(x,y) = \frac{4}{\pi}\arctan(x)$, $f_2(x,y) = y/2$. (i) Find the fixed points. Discuss.
(ii) Find the stable manifold of $(0,0)$. Find the unstable manifold of $(0,0)$. Find the basins of attraction.

Problem 24. Let

$$\mathbf{f}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \bmod 1.$$

Then $\mathbf{f}$ is defined on the unit square in $\mathbb{R}^2$ (or on the torus). Find the Ljapunov exponents of any orbit of the map.

Problem 25. The *Chebyshev polynomials* $T_k(x)$ may be defined by the expression

$$T_k(x) := \cos(k \cos^{-1} x) \qquad (1)$$

where k is a positive integer or zero and $|x| \leq 1$.
(i) Show that

$$T_{n+m}(x) + T_{n-m}(x) = 2T_n(x)T_m(x), \qquad n \leq m.$$

(ii) Show that

$$T_{k+1}(x) - xT_k(x) + \frac{1}{4}T_{k-1}(x) = 0.$$

Problem 26. Let a sequence of functions $r_k(x)$ be defined as follows. The zeroth function if defined to be $r_0(x) = (x)$, the first function to be $r_1(x) = a_1/(b_1 + x)$, and the kth function is obtained from the preceeding function $r_{k-1}(x)$ by replacing x by $a_k/(b_k + x)$, where $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ are constants.
(i) Show that

$$r_0(x) = x, \qquad r_1(x) = \frac{a_1}{b_1 + x}, \qquad r_2(x) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + x}}, \cdots \qquad (1)$$

and, in general,

$$r_k(x) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{\cdot}{\cdot + \cfrac{a_k}{b_k + x}}}}}.$$

The expression $r_k \equiv r_k(0)$, obtained by setting $x = 0$ in $r_k(x)$, is called a continued fraction of k stages.
(ii) Show that the result of clearing fractions in the expression for $r_k(x)$ is the ratio of two linear functions of x, of the form

$$r_k(x) = \frac{A_k + C_k x}{B_k + D_k x}, \qquad k = 0, 1, 2, \cdots,$$

(iii) Deduce that

$$\frac{A_k + C_k x}{B_k + D_k x} \equiv \frac{(b_k A_{k-1} + a_k C_{k-1}) + A_{k-1}x}{(b_k B_{k-1} + a_k D_{k-1}) + B_{k-1}x}, \qquad k = 1, 2, \ldots,$$

for all values of x for which $r_k(x)$ is defined, and that also

$$A_1 = Ca_1, \qquad C_1 = 0, \qquad B_1 = Cb_1, \qquad D_1 = C$$

where C is an arbitrary nonzero constant of proportionality. The right-hand member of the identity is the result of replacing k by $k-1$ and x by $a_k/(b_k + x)$ in the left-hand member.

Problem 27. The *trace map* is given by

$$u_{t+1} = 1 + 4u_{t-1}^2(u_t - 1), \qquad t = 1, 2, \ldots \tag{1}$$

with u_0, u_1 given. Show that

$$u_{t+1} = 2u_t^2 - 1, \quad t = 0, 1, 2, \ldots \tag{2}$$

is an *invariant* of the trace map.

Definition. Equation (2) is an invariant of (1) if (2) is satisfied for the pair (u_t, u_{t+1}) then (1) implies that (u_{t+1}, u_{t+2}) also satisfies (2).

Problem 28. Consider the 18-parameter family of mappings of the plane given by

$$x' = \frac{f_1(y) - xf_2(y)}{f_2(y) - xf_3(y)}, \qquad y' = \frac{g_1(x') - yg_2(x')}{g_2(x') - yg_3(x')} \tag{1}$$

with

$$\mathbf{f}(x) := (\mathbf{A}_0\mathbf{X}) \times (\mathbf{A}_1\mathbf{X}), \qquad \mathbf{g}(x) := (\mathbf{A}_0^T\mathbf{X}) \times (\mathbf{A}_1^T\mathbf{X}) \qquad (2a)$$

where $\times$ denotes the cross product and

$$\mathbf{X} := \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \qquad \mathbf{A}_i := \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \xi_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}, \quad i = 0, 1. \qquad (2b)$$

(i) Show that each member of this family possesses a 1-parameter family of invariant curves that fills the plane

$$(\alpha_0+K\alpha_1)x^2y^2+(\beta_0+K\beta_1)x^2y+(\gamma_0+K\gamma_i)x^2+(\delta_0+K\delta_1)xy^2+(\epsilon_0+K\epsilon_1)xy$$

$$+(\xi_0+K\xi_1)x+(\kappa_0+K\kappa_1)y^2+(\lambda_0+K\lambda_1)y+(\mu_0+K\mu_1)=0 \qquad (4)$$

where the integration constant K is invariant on each curve (ii) Show that the biquadratic equation (3) can be parametrized in terms of elliptic functions yielding the second integration constant. For these reasons map (1) is called integrable.

Problem 29. Show that the more general iteration

$$x_{n+1} = a + b(x_n + x_{n-1}) + cx_nx_{n-1} - x_{n-2} \qquad (1)$$

has the invariant

$$I(x_{n-1}, x_n, x_{n+1}) =$$

$$x_{n-1}^2x_n^2+x_{n-1}^2-a(x_{n-1}+x_n+x_{n+1})-b(x_{n-1}x_n+x_{n-1}x_{n+1}+x_nx_{n+1})-cx_{n-1}x_nx_{n+1} \qquad (2)$$

Problem 30. Consider the system of difference equations

$$x_{1n+1} = x_{2n} + \mu x_{1n}(1 - x_{1n}^2 - x_{2n}^2)$$

$$x_{2n+1} = -x_{1n} + \mu x_{2n}(1 - x_{1n}^2 - x_{2n}^2)$$

where μ is a real bifurcation parameter and $n = 0, 1, 2, \ldots$.
(i) Show that for $\mu = 0$ the system admits the first integral.
(ii) Show that for $\mu \neq 0$ the system admits the one-dimensional integral manifold

$$x_{1n}^2 + x_{2n}^2 - 1 = 0.$$

Problem 31. Consider the skew-tent map $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} x/a & \text{for } 0 \le x \le a \\ (1-x)/(1-a) & \text{for } a < x \le 1 \end{cases}$$

where $0.5 \leq a < 1$. Consider the coupled system

$$x_{t+1} = f(x_t + \delta(y_t - x_t)), \qquad y_{t+1} = f(y_t + \epsilon(x_t - y_t))$$

where $t = 0, 1, \ldots$.
(i) Let $\delta = 0$, $\epsilon = 1$ and $x_0 = 0.2$, $y_0 = 0.3$ (master-slave system). Does the system synchronizes?
(ii) Let $\delta = 0.5$, $\epsilon = 0.5$ and $x_0 = 0.2$, $y_0 = 0.3$. Does the system synchonizes?

Problem 32. Consider the following two-dimensional non-invertible map

$$\Theta_{n+1} = 2\Theta_n \mod 2\pi, \qquad z_{n+1} = \lambda z_n + \cos\Theta_n \qquad (1)$$

where

$$2 > \lambda > 1, \qquad 0 < \Theta < 2\pi. \qquad (2)$$

Find the *fractal basin boundary.*

Problem 33. Consider the *Fibonacci trace map*

$$x_{t+1} = 2x_t x_{t-1} - x_{t-2} \qquad (1)$$

where $t = 2, 3, \ldots$. Map (1) is a discrete dynamical system with various physical applications. Show that the Fibonacci trace map is reversible and possesses the invariant

$$\tilde{I}(x_{t-1}, x_t, x_{t+1}) = x_{t-1}^2 + x_t^2 + x_{t+1}^2 - 2x_{t-1}x_n x_{n+1} - 1. \qquad (2)$$

If $I = 0$ and $|x_i| \leq 1$, $i = 1, 2, 3$, we are in a region of homogeneous chaos. This follows from a semi-conjugacy to a hyperbolic automorphism of the torus which makes the system pseudo-Anosov

$$x_t = \cos(2\pi\theta_t), \qquad \theta_{t+1} = \theta_t + \theta_{t-1}. \qquad (3)$$

Problem 34. Consider the *Fibonacci trace map*

$$x_{t+1} = 2x_t x_{t-1} - x_{t-2} \qquad (1)$$

where $t = 2, 3, \ldots$. Map (1) is a discrete dynamical system with various physical applications. Show that the Fibonacci trace map is reversible and possesses the invariant

$$\tilde{I}(x_{t-1}, x_t, x_{t+1}) = x_{t-1}^2 + x_t^2 + x_{t+1}^2 - 2x_{t-1}x_t x_{t+1} - 1. \qquad (2)$$

If $I = 0$ and $|x_i| \leq 1$, $i = 1, 2, 3$, we are in a region of homogeneous chaos. This follows from a semi-conjugacy to a hyperbolic automorphism of the torus which makes the system pseudo-Anosov

$$x_t = \cos(2\pi\theta_t), \qquad \theta_{t+1} = \theta_t + \theta_{t-1}. \tag{3}$$

Problem 35. Show that the more general iteration

$$x_{n+1} = a + b(x_n + x_{n-1}) + cx_n x_{n-1} - x_{n-2} \tag{1}$$

has the invariant

$$I(x_{n-1}, x_n, x_{n+1}) =$$

$$x_{n-1}^2 x_n^2 + x_{n-1}^2 - a(x_{n-1} + x_n + x_{n+1}) - b(x_{n-1}x_n + x_{n-1}x_{n+1} + x_n x_{n+1}) - cx_{n-1}x_n x_{n+1} \tag{2}$$

The task now is to identify the original Fibonacci system out of this three-parameter class of dynamical systems.

Problem 36. The *Anosov map* is defined as follows: $\Omega = [0, 1)^2$,

$$\boldsymbol{\phi}(x, y) = (x + y, x + 2y).$$

In matrix form we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \bmod 1.$$

Thus $\boldsymbol{\phi}$ maps Ω $1 - 1$ onto itself.
(i) Show that the map preserves Lebesgue measure.
(ii) Show that $\boldsymbol{\phi}$ is invertible. Show that the entire sequence can be recovered from one term.
(iii) Show that $\boldsymbol{\phi}$ is mixing.

Problem 37. Show that the more general iteration

$$x_{t+1} = a + b(x_t + x_{t-1}) + cx_t x_{t-1} - x_{t-2} \tag{1}$$

has the invariant

$$I(x_{t-1}, x_t, x_{t+1}) =$$

$$x_{t-1}^2 x_t^2 + x_{t-1}^2 - a(x_{t-1} + x_t + x_{t+1}) - b(x_{t-1}x_t + x_{t-1}x_{t+1} + x_t x_{t+1}) - cx_{t-1}x_t x_{t+1}. \tag{2}$$

The task now is to identify the original Fibonacci system out of this three-parameter class of dynamical systems.

Problem 38. In matrix form, the *cat map* is given by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{mod } 1$$

which transforms the column vector (x_{1n}, x_{2n}) into the column vector (x_{1n+1}, x_{2n+1}), where all x_1 and x_2 are taken modulo 1.
(i) Given the recursion for the Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2}, \qquad F_1 = F_2 = 1$$

find $T^{(n)}$ by induction.
(ii) Show that the determinant of $T^{(n)}$ is equals 1.
(iii) Find the eigenvalues of $T^{(n)}$.
(iv) Show that fixed points of $T^{(n)}$ correspond to orbits of period length n and any divisors of n for $n = 2$.

Problem 39. The family of invertible transformations $\mathbf{f}_{\alpha,p}$, acts on the torus $\mathbb{T}^2$, where $\mathbb{T}$ denotes the circle $[0, 2\pi]$, $\alpha \in \mathbb{T}$, and p is any nonzero integer. It is defined by

$$\mathbf{f}_{\alpha,p}(x_1, x_2) = (x_1 + \alpha, px_1 + x_2).$$

(i) Show that the normalized Lebesgue measure

$$d\mu(x_1, x_2) = \frac{1}{(2\pi)^2} dx_1 dx_2$$

is invariant under the map $\mathbf{f}_{\alpha,p}$.
(ii) Find $\mathbf{f}_{\alpha,p}^{(n)}$, i.e. find the n-th iterate of $\mathbf{f}_{\alpha,p}$. Calculate

$$\mathbf{f}_{\alpha,p}(x_1, x_2) - \mathbf{f}_{\alpha,p}(y_1, y_2).$$

(iii) Discuss the map for $\alpha = 0$.
(iv) Let $L_2([0, 2\pi] \times [0, 2\pi])$ be the Hilbert space of the square integrable functions in the Lebesgue sense. Let $\langle \, , \, \rangle$ be the scalar product in this Hilbert space. We can define a unitary operator U associated with the invertible map $\mathbf{f}_{\alpha,p}$ by

$$Ug(x_1, x_2) := g(\mathbf{f}_{\alpha,p}(x_1, x_2)), \qquad g \in L_2([0, 2\pi] \times [0, 2\pi]).$$

Consider the orthonormal basis

$$\phi_{m,n}(x_1, x_2) = \frac{1}{2\pi} \exp(imx_1) \exp(inx_2), \qquad m, n \in \mathbb{Z}$$

in this Hilbert space. Find the matrix representation of U. Find the spectrum of the unitary operator U.

(v) Show that the map $\mathbf{f}_{\alpha,p}$ is not chaotic.
(vi) Is the map $\mathbf{f}_{\alpha,p}$ mixing?

Problem 40. Consider the one-parameter family of maps $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f} : \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} & \text{for } l(x_n, y_n) \leq 0 \\ \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + (1-\eta) \begin{pmatrix} 1 \\ -\eta^{-1} \end{pmatrix} & \text{for } l(x_n, y_n) > 0 \end{cases}$$

where the $l(x, y)$ defined as

$$C: \; l(x, y) := \eta x + y - \frac{1+\eta}{2}$$

separates the two branches of f. After an appropriate linear coordinate transformation, f corresponds to the area-preserving case of the map

$$v_{n+1} = a v_n - \text{sign}(v_n) + b z_n$$
$$z_{n+1} = v_n$$

with parameters $a = (\eta^2 + 1)/\eta$ and $b = -1$. Discuss the behaviour of the map.

Problem 41. Show that the substitution

$$y_t = a \tan(u_t), \qquad t = 0, 1, 2, \ldots \tag{1}$$

transforms the second order difference equation

$$y_{t+2} y_{t+1} y_t = a^2 (y_{t+2} + y_{t+1} + y_t) \tag{2}$$

to the equation

$$\tan(u_{t+2} + u_{t+1} + u_t) = 0$$

and hence obtain the general solution in the form

$$y_t = a \tan\left(c_1 \cos(\frac{2t\pi}{3}) + c_2 \sin(\frac{2t\pi}{3}) + \frac{n\pi}{2} \right)$$

where c_1 and c_2 are arbitrary constants and n is an arbitrary integer.

Problem 42. Find the exact solution of the initial value problem for the system of difference equations

$$x_{1t+1} = 2x_{1t}^2 - 2x_{2t}^2 - 1, \quad x_{2t+1} = 4x_{1t} x_{2t}, \quad t = 0, 1, 2, \ldots.$$

Problem 43. Find the exact solution of the initial value problem for the system of difference equations

$$x_{1t+1} = 4x_{1t}(1 - x_{1t}) + 4x_{2t}^2, \quad x_{2t+1} = 4x_{2t}(1 - 2x_{1t}), \quad t = 0, 1, 2, \ldots .$$

Problem 44. Find the exact solution of the initial value problem for the system of difference equations

$$x_{1t+1} = (2x_{1t} - 2x_{2t} - 1)(2x_{1t} + 2x_{2t} - 1), \quad x_{2t+1} = 4x_{2t}(2x_{1t} - 1), \quad t = 0, 1, 2, \ldots .$$

Problem 45. Let $\mathbf{f}$ be an invertible map of $\mathbb{R}^n$ with $n \geq 2$. Let $\mathbf{p}$ be a fixed point saddle. A point that is in both the stable and unstable manifold of $\mathbf{p}$ and that is distinct from $\mathbf{p}$ is called a *homoclinic point*. If $\mathbf{x}$ is a homoclinic point, then

$$\mathbf{f}^{(n)}(\mathbf{x}) \to \mathbf{p} \quad \text{and} \quad \mathbf{f}^{(-n)}(\mathbf{x}) \to \mathbf{p}$$

as $n \to \infty$. The orbit of a homoclinic point is called a *homoclinic orbit*. A point in the stable manifold of a fixed point $\mathbf{p}$ and in the unstable manifold of a different fixed point $\mathbf{q}$ is called a *heteroclinic point*. The orbit of a heteroclinic point is called a *heteroclinic orbit*.
(i) Give an example of an invertible map in $\mathbb{R}^2$ which has a homoclinic orbit.
(ii) Give an example of an invertibale map in $\mathbb{R}^2$ which has a heteroclinic orbit.

Problem 46. In a saddle-node bifurcation a pair of periodic orbits are created "out of nothing". One of the periodic orbits is always unstable (the saddle) while the other periodic orbit is always stable (the node). Give a two-dimensional map that shows a saddle-node bifurcation.

Problem 47. Let $n \geq 2$. An invertible integer matrix, $A \in GL_n(\mathbb{Z})$, generates a toral automorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ via the formula

$$f \circ \pi = \pi \circ A, \qquad \pi : \mathbb{R}^n \to \mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n.$$

The set of fixed points of f is given by

$$\mathrm{Fix}(f) := \{\, x^* \in \mathbb{T}^n \,:\, f(x^*) = x^* \,\}.$$

Let $\sharp\mathrm{Fix}(f)$ be the number of fixed points of f. Now we have: if $\det(I_n - A) \neq 0$, then

$$\sharp\mathrm{Fix}(f) = |\det(I_n - A)|.$$

Let $n = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that $\det(I_2 - A) \neq 0$ and find $\sharp\mathrm{Fix}(f)$.

Problem 48. Consider the map

$$x_{t+1} = x_t - k\sin(2\pi y_{t+1}), \quad y_{t+1} = x_t + y_t$$

where $t = 0, 1, 2, \ldots$. Show that the standard map is an area-preserving map.

Problem 49. Consider the *standard map* given by

$$r_{t+1} = r_t - \frac{k}{2\pi}\sin(2\pi\theta_t)$$
$$\theta_{t+1} = \theta_t + r_{t+1} \mod 1.$$

Find the fixed points in dependence of k and study their stability.

Problem 50. Give a *Markov partion* of the *skinny Baker map*. The skinny Baker map is given by $B(x, y) : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$

$$B(x, y) := \begin{cases} (x/3, 2y) & \text{if } 0 \leq y \leq 1/2 \\ (x/3 + 2/3, 2y - 1) & \text{if } 1/2 < y \leq 1. \end{cases}$$

Problem 51. Let

$$\mathbf{f}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mod 1.$$

Then $\mathbf{f}$ is defined on the unit square in $\mathbb{R}^2$ (or on the torus). Find the Ljapunov exponents of any orbit of the map.

Problem 52. Find the condition on the bifurcation parameter b such that the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = bx_2 + 1 - ax_1^2, \qquad f_2(x_1, x_2) = x_1$$

is area-preserving.

Problem 53. A one-dimensional map f is called an *invariant* of a two-dimensional map g if

$$g(x, f(x)) = f(f(x)).$$

Let

$$f(x) = 2x^2 - 1.$$

Show that f is an invariant for

$$g(x, y) = y - 2x^2 + 2y^2 + d(1 + y - 2x^2).$$

Problem 54. The *beam-beam map* defined on $\mathbb{R}^2$ is given by

$$x' = x\cos(2\pi\omega) + (y + 1 - e^{-x^2})\sin(2\pi\omega)$$
$$y' = -x\sin(2\pi\omega) + (y + 1 - e^{-x^2})\cos(2\pi\omega)$$

where ω is a bifurcation parameter. Find the fixed points and study their stability.

Problem 55. Consider the family of mappings

$$\mathbf{f}_\mu ; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y + g_\mu(x) \\ x - g_\mu(x') \end{pmatrix}$$

where

$$g_\mu(x) = \mu x - (1 - \mu)x^2.$$

(i) Show that these mappings can be written as a product of two involutions, $\mathbf{f}_\mu = \mathbf{I}_2\mathbf{I}_1$, where

$$\mathbf{I}_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$
$$\mathbf{I}_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y + g_\mu(x) \\ x - g_\mu(x') \end{pmatrix}.$$

(ii) Show that the line $y = 0$ is invariant under $\mathbf{I}_1$.

Problem 56. Study the coupled system of first order difference equations

$$m(t+1) = \tanh\left(\frac{p}{T}m(t) - \frac{1}{T^2}q(t)\right)$$
$$q(t+1) = m(t)\text{sech}^2\left(\frac{p}{T}m(t) - \frac{1}{T^2}q(t)\right)$$

where $= 0, 1, 2, \ldots$ and the bifurcation parameter are in the range $-3 < p < 3$, $0 < T < 2$. Find the fixed points. Find periodic orbits. Does the system show chaotic behaviour?

Problem 57. (i) The logistic map

$$x_{t+1} = 4x_t(1 - x_t), \quad t = 0, 1, 2, \ldots$$

and $x_0 \in [0,1]$ is probably the most studied map with chaotic behaviour. Study the two-dimensional map

$$x_{1,t+1} = 4x_{1,t}(1-x_{1,t})$$
$$x_{2,t+1} = 4x_{1,t}x_{2,t}(1-x_{2,t})$$

where $x_{1,0}, x_{2,0} \in [0,1]$. This means the second equation is modulated by the solution of the logistic map.
(iii) Study the higher-dimensional case

$$x_{1,t+1} = 4x_{1,t}(1-x_{1,t})$$
$$x_{2,t+1} = 4x_{1,t}x_{2,t}(1-x_{2,t})$$
$$x_{3,t+1} = 4x_{1,t}x_{2,t}x_{3,t}(1-x_{3,t})$$

where $x_{1,0}, x_{2,0}, x_{3,0} \in [0,1]$. Extend to n dimensions.

Problem 58. Let $x_0, x_1 \in [0,1]$. Study the map

$$x_{t+2} = x_{t+1}x_t, \qquad t = 0,1,2,\ldots.$$

Problem 59. Study the coupled circle maps

$$\theta(t+1) = \theta(t) + \omega - \frac{a}{2\pi}\sin(2\pi\theta(t)) + \frac{\epsilon}{4\pi}\sin(2\pi(\theta(t)-\phi(t)))$$
$$\phi(t+1) = \phi(t) + \omega - \frac{a}{2\pi}\sin(2\pi\phi(t)) + \frac{\epsilon}{4\pi}\sin(2\pi(\phi(t)-\theta(t))).$$

Problem 60. Solve the difference equation

$$x_{t+1} = x_t x_{t-1}\cdots x_1 x_0 + \sum_{t=0}^{t} x_t, \quad t = 0,1,2,\ldots.$$

Problem 61. Let $\mu = 3/4$. Study the two-dimensional difference equation

$$x_{t+1} = \frac{1}{2}x_t + \frac{x_t y_t}{4(1-\mu)x_t + (4\mu-2)y_t}$$
$$y_{t+1} = \frac{1}{2}y_t + \frac{x_t y_t}{4(1-\mu)y_t + (4\mu-2)x_t}$$

where $t = 0,1,\ldots$.

Problem 62. Let $a, b > 0$. The *Duffing map* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f_1(x_1, x_2) = x_2, \quad f_2(x_1, x_2) = -bx_1 + ax_2 - x_2^3$$

or written as difference equation ($t = 0, 1, \ldots$)

$$x_{1,t+1} = x_{2,t}, \quad x_{2,t+1} = -bx_{1,t} + ax_{2,t} - x_{2,t}^3.$$

Find the fixed points of the map and study their stability.

Problem 63. Let $r > 0$. Find the fixed points and their stability of the two-dimensional map

$$x_{t+1} = r(3y_t + 1)x_t(1 - x_t)$$
$$y_{t+1} = r(3x_t + 1)y_t(1 - y_t).$$

Study the transition to chaos in the range $[1.0, 1.08]$.

Problem 64. (i) Consider the differential two-form in $\mathbb{R}^2$

$$\omega = dx_1 \wedge dx_2.$$

Find all smooth maps $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ which leaves ω invariant.
(ii) Consider the differential one-form in $\mathbb{R}^2$

$$\alpha = x_1 dx_2 - x_2 dx_1.$$

Find all smooth maps $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ which leaves α invariant.

Problem 65. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function which admits at least one root, i.e. $f(x) = 0$ admits a solution. The *regular falsi method* for finding the roots of f is given by the second order nonlinear difference equation

$$x_{t+2} = \frac{x_t f(x_{t+1}) - x_{t+1} f(x_t)}{f(x_{t+1}) - f(x_t)}, \quad t = 0, 1, \ldots$$

(i) Show that with $y_{t+1} = x_{t+2}$ the second order difference equation can be written as

$$x_{t+1} = y_t$$
$$y_{t+1} = \frac{x_t f(y_t) - y_t f(x_t)}{f(y_t) - f(x_t)}.$$

Study these equations for the logistic map $f(x) = 4x(1 - x)$. Select the initial values $(x_0 = 1/2, x_1 = 1/4)$ and $(x_0 = 1/2, x_1 = 3/4)$.

Problem 66. Consider the nonlinear coupled system of difference equations

$$x_{1,t+1} = x_{1,t}^2 - x_{2,t}^2$$
$$x_{2,t+1} = 2x_{1,t}x_{2,t}.$$

(i) Show that the fixed points are given by $(0,0)$ and $(1,0$. Study the stability of the fixed points.
(ii) Define r_t and ϕ_t via $x_{1,t} = r_t\cos(\phi_t)$, $x_{2,t} = r_t\sin(\phi_t)$. Show that r_t and ϕ_t satisfy the difference equations

$$r_{t+1} = r_t^2, \qquad \phi_{t+1} = 2\phi_t.$$

Discuss.
(iii) Let $V(x_{1,t}, x_{2,t}) = x_{1,t}^2 + x_{2,t}^2$. Study $\Delta V := V(x_{1,t+1}, x_{2,t+1}) - V(x_{1,t}, x_{2,t})$. Discuss.

Problem 67. Consider the nonlinear coupled system of difference equations

$$x_{1,t+1} = \frac{x_{2,t}}{1 + x_{1,t}^2}$$
$$x_{2,t+1} = \frac{x_{1,t}}{1 + x_{2,t}^2}.$$

Find the fixed points and study their stability.

Problem 68. Let $\mu_1, \mu_2 > 0$ be the bifuration parameters. Consider the linearly coupled system of nonlinear difference equations

$$x_{1,t+1} = 1 - \mu_1 x_{1,t}^2 + \mu_2(x_{2,t} - x_{1,t}, \quad x_{2,t+1} = 1 - \mu_1 x_{2,t}^2 + \mu_2(x_{1,t} - x_{2,t}).$$

(i) Find the fixed points and study their stability.
(ii) Find period-2 orbits by studying the second iterate.

Problem 69. Let $k > 0$ be the bifurcation parameter. The *whisker map* is given by

$$f_1(x,\theta) = x + 4k\sin(\theta), \qquad f_2(x,\theta) = \theta - \ln(|1 + x + 4k\sin(\theta)|) \mod 2\pi.$$

(i) Show that the fixed points are given by

$$(0,0), \quad (0,\pi), \quad (0,-\pi)$$

and

$$(-2,0), \quad (-2,\pi), \quad (-2,-\pi).$$

Study the stability of the fixed point.
Show that there is a period-doubling sequence around $(0,0)$, $(-2,\pi)$, $(-2,-\pi)$.

Problem 70.)(i) Let $k > 0$. Consider the autonomous system of differential equations

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = ku_1(u_1 - 1).$$

Find the fixed points and study their stability.
(ii) Motivated by the Lie series expansion for the solution of the system of differential equations and truncation we replace the system of differential equations by the two-dimensional map

$$f_1(x_1, x_2) = x_1 + x_2 + kx_1(x_1 - 1), \qquad f_2(x_1, x_2) = x_2 + kx_1(x_1 - 1).$$

Study this map.

Problem 71. Let the map $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on some open set containing $\mathbf{p}$. Suppose that $\det(J\mathbf{f}(\mathbf{p})) \neq 0$. Then there is an open set V containing $\mathbf{p}$ and an open set W containing $\mathbf{f}(\mathbf{p})$ such that $\mathbf{f} : V \to W$ has a continuous inverse $\mathbf{f}^{-1} : W \to V$ which is differentiable for all $\mathbf{y} \in W$ (*inverse function theorem*).. Apply the inverse function theorem to the map $(n = 2)$

$$f_1(x_1, x_2) = x_1 + x_2 + kx_1(x_1 - 1), \quad f_2(x_1, x_2) = x_2 + kx_1(x_1 - 1)$$

where $k > 0$.

Problem 72. Study the two-dimensional map with $r \geq 0$

$$x_{1,t+1} = f(x_{1,t}) + rx_{1,t}x_{2,t}, \qquad x_{2,t+1} = f(x_{2,t}) + rx_{1,t}x_{2,t}$$

where $t = 0, 1, \ldots$ and $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 1 - 2x^2$. Note that the map $f : [-1,1] \to [-1,1]$, $f(x) = 1 - 2x^2$ shows fully developed chaos.

Problem 73. Let H be the Heaviside step function, i.e. $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$. Consider the two-dimensional map (butcher's map)

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} (1/2 + \epsilon)H(2y_t - 1) \\ 0 \end{pmatrix} \mod 1.$$

Discuss the behaviour of the map.

Problem 74. Note that the map $f : [-1,1] \to [-1,1]$ $f(x) = 1 - 2x^2$ is fully chaotic. Let $\mu_1 \in (0,2], \mu_2 > 0$ be the bifurcation parameters. Consider the smooth map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_{1,\mu_1,\mu_2}(x_1,x_2) = 1 - \mu_1 x_1^2 + \mu_2(x_2 - x_1)$$
$$f_{2,\mu_1,\mu_2}(x_1,x_2) = 1 - \mu_1 x_2^2 + \mu_2(x_1 - x_2).$$

(i) Find the fixed points of the map and study their stability.
(ii) The second iterate is given by

$$f_{1,\mu_1,\mu_2}(f_{1,\mu_1,\mu_2}(x_1,x_2), f_{2,\mu_1,\mu_2}(x_1,x_2))$$
$$= 1 - \mu_1(1 - \mu_1 x_1^2 + \mu_2(x_2 - x_1))^2 + \mu_2(x_1 - x_2)(\mu_1(x_1 + x_2) + 2\mu_2)$$
$$f_{2,\mu_1,\mu_2}(f_{1,\mu_1,\mu_2}(x_1,x_2), f_{2,\mu_1,\mu_2}(x_1,x_2))$$
$$= 1 - \mu_1(1 - \mu_1 x_2^2 + \mu_2(x_1 - x_2))^2 + \mu_2(x_2 - x_1)(\mu_1(x_1 + x_2) + 2\mu_2).$$

Find the fixed points of the second iterate (and thus periodic orbits) and study their stability.

Problem 75. Let $k \geq 0$ be the bifurcation parameter. Discuss the behaviour of the two-dimensional map

$$\theta_{t+1} = \theta_t + r_{t+1} \mod 1$$
$$r_{t+1} = r_t - \frac{k}{2\pi}\sin(2\pi\theta_t)$$

in dependence of the bifurcation parameter k, where $t = 0, 1, \ldots$.

Problem 76. The *Lozi map* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f_1(x_1,x_2) = 1 + x_2 - a|x_1|, \qquad f_2(x_1,x_2) = bx_1.$$

Show that if $b \in (0,1)$, $a > 0$, $2a + b < 4$, $b < (a^2 - a)?(2a + 1)$ and $\sqrt{2}a > b + 2$ then there is a hyperbolic fixed point of saddle type.

Problem 77. Let n be a natural number with $n \geq 2$. We set

$$x_0 = 0, \quad y_0 = x_1 = 1, \quad y_1 = n$$

and

$$x_{t+2} = \left\lfloor \frac{y_t + n}{y_{t+1}} \right\rfloor - x_t, \quad y_{t+2} = \left\lfloor \frac{y_t + n}{y_{t+1}} \right\rfloor - y_t$$

where $\lfloor a \rfloor$ denotes the greatest integer not greater than a. The ratio x_t/y_t is called the Farey fraction and

$$\frac{x_0}{y_0}, \; \frac{x_1}{y_1}, \; \frac{x_2}{y_2}, \; \ldots$$

is called the Farey sequence. Let $n = 5$. Find the sequence.

Problem 78. Study the second order difference equation

$$x_{t+1} + x_{t-1} = \frac{\mu x_t}{1 + x_t^2}, \quad t = 1, 2, \ldots$$

with given $x_1, x_0 > 0$ (initial value problem).

Problem 79. The recurrence relation

$$x_{t+2} = x_{t+1} + x_t, \qquad t = 0, 1, \ldots$$

with $x_0 = x_1 = 1$ provides the Fibonacci sequence. Study the recurrence relations

$$x_{t+2} = e^{i\pi t} x_{t+1} + x_t$$
$$x_{t+2} = e^{i\pi x_t} x_{t+1} + x_t$$

with $t = 0, 1, \ldots$ and $x_0 = x_1 = 1$.

Problem 80. Consider the two-dimensional map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = -x_2 + g(x_1), \qquad f_2(x_1, x_2) = x_1$$

or written as a system of difference equations

$$x_{1,t+1} = -x_{2,t} + g(x_{1,t}), \qquad x_{2,t+1} = x_{1,t}$$

where $t = 0, 1, \ldots$ and $g : \mathbb{R} \to \mathbb{R}$ is the non-invertible map $g(x) = 4x(1-x)$. Find the fixed points of the map and study their stability. Find the first iterate and study the stability of the periodic points.

Problem 81. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a two-dimensional analytic map.
(i) Find the condition on $\mathbf{f}$ such that $dx_1 \wedge dx_2$ is invariant, i.e. $\mathbf{f}$ should be area preserving.
(ii) Find the condition on $\mathbf{f}$ such that $x_1 dx_1 + x_2 dx_2$ is invariant.
(iii) Find the condition on $\mathbf{f}$ such that $x_1 dx_1 - x_2 dx_2$ is invariant.
(iv) Find the condition on $\mathbf{f}$ such that $x_1 dx_2 + x_2 dx_1$ is invariant.
(v) Find the condition on $\mathbf{f}$ such that $x_1 dx_2 - x_2 dx_1$ is invariant.

Problem 82. Show that the *McMillan maps*

$$f_1(x_1, x_2) = x_2, \quad f_2(x_1, x_2) = -x_1 - \frac{\beta x_2^2 + \epsilon x_2 + \xi}{\alpha x_2^2 + \beta x_2 + \gamma}$$

are a family of area-preserving rational maps preserving the biquadratic foliation

$$\alpha x_1^2 x_2^2 + \beta(x_1^2 x_2 + x_1 x_2^2) + \gamma(x_1^2 + x_2^2) + \epsilon x_1 x_2 + \xi(x_1 + x_2) + K = 0$$

where K is the parameter which parametrizes each invariant curve in the plane.

Problem 83. Let $a, b > 0$. The *Duffing map* is $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f_1(x_1, x_2) = x_2, \qquad f_2(x_1, x_2) = -bx_1 + ax_2 - x_2^3$$

or written as difference equation

$$x_{1,t+1} = x_{2,t}, \qquad x_{2,t+1} = -bx_{1,t} + ax_{2,t} - x_{2,t}^3$$

where $t = 0, 1, \ldots$. Is $dx_1 \wedge dx_2$ invariant under the map? Find the fixed points and study their stability. Calculate the first iterate of the map and their fixed points, i.e. find periodic points. Study the stability of these periodic points.

Problem 84. Let $a, d > 0$. Consider the coupled logistic map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = 1 - ax_1^2 + d(x_2 - x_1), \qquad f_2(x_1, x_2) = 1 - ax_2^2 + d(x_1 - x_2)$$

or written as difference equation

$$x_{1,t+1} = 1 - ax_{1,t}^2 + d(x_{2,t} - x_{1,t}), \qquad x_{2,t+1} = 1 - ax_{2,t}^2 + d(x_{1,t} - x_{2,t}).$$

(i) Is $dx_1 \wedge dx_2$ invariant under the map?
(ii) Find the fixed points of the map and study their stability.
(iii) Calculate the first iterate of the map and their fixed points, i.e. find periodic points. Study the stability of these periodic points.
(iv) Apply the transformation

$$v_{1,t} = \frac{1}{2}(x_{1,t} + x_{2,t}), \qquad v_{2,t} = \frac{1}{2}(x_{1,t} - x_{2,t})$$

and show that $v_{2,t}$ is given by

$$v_{2,t+1} = \prod_{\tau=0}^{t+1}(-2(av_{1,\tau} + d))v_{2,0}.$$

(v) Show that the stability of the $x_{1,t} = x_{2,t} = v_{1,t}$ is given by

$$\left|\prod_{\tau=1}^{p}(-2(ax_{1,t} + d))\right| < 1.$$

(v) Let $d = 0.1$. Show that *Hopf bifurcation* occurs at $a \geq 1$.

Problem 85. Study the two-dimensional map ($t = 0, 1, \ldots$)

$$\theta_{t+1} = \theta_t + \xi \mod 1$$
$$x_{t+1} = x_t f(\theta_t)$$

where

$$f(\theta) := \begin{cases} -1 \text{ for } 0 \leq \theta < 0.5 \\ \;\;1 \;\text{ for } 0.5 \leq \theta < 1 \end{cases}$$

and the number ξ is chosen irrational. This is a skew-product dynamical system, where a variable (θ) satisfies a self-contained difference equation and the variable is utilized to force a second difference equation.

Problem 86. Study the two-dimensional map

$$\theta_{t+1} = \theta_t + c \mod 1$$
$$x_{t+1} = f(x_t) + \epsilon \sin(2\pi\theta_t)$$

where $f(x) = 4x(1-x)$.

Problem 87. A predator-prey model is described by the two-dimensional map

$$x_{1,t+1} = x_{1,t} \exp(b(1 - x_{1,t}/K) - a x_{2,t}), \quad x_{2,t+1} = x_{1,t}(1 - \exp(-a x_{2,t}))$$

where $K, a, b > 0$.
(i) Study first the one-dimensional case with $x_{2,t} = 0$. Discuss the stability of the fixed point $x_1^* = K$ as a function of the parameter b with $0 < b < 3$. What happens at $b \sim 2.692$?
(ii) For the two-dimensional case set $K = 10$ and define the new parameter $q := x_1^*/K$, where (x_1^*, x_2^*) is the nontrivial fixed point of the two-dimensional map. Show that for $q = 0.4$, $a = 0.15b(1 - \exp(-06b))^{-1}$ must hold.
(iii) Keeping $q = 0.4$ fixed, plot the attractor for different values $0 < b < 3$. At which b_c does the nontrivial fixed point lose stability? Find b_c numerically with the help of linearization. What happens when $b = 2.2$?

Problem 88. Let $x_0, y_0 \in \{0, 1\}$. Solve the boolen equations

$$x_{t+1} = x_t \oplus y_t, \qquad y_{t+1} = x_t \cdot y_t$$

where $x_0 = 1$, $y_0 = 1$ and $t = 0, 1, \ldots$. Find the fixed points. Does the sequence x_t, y_t tend to a fixed point?

Problem 89. Let $r > 0$ be the bifurcation parameter. Study the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = -x_2 + x_1(rx_1^2 + 2 - r), \quad f_2(x_1, x_2) = x_1$$

or written as difference equations

$$x_{1,t+1} = -x_{2,t} + x_{1,t}(rx_{1,t}^2 + 2 - r), \quad x_{2,t+1} = x_{1,t}.$$

Problem 90. Let $x_{1,0}, x_{2,0} \in [-1, 1]$. Study the system of difference equations

$$x_{1,t+1} = 1 - 2(|x_{2,t}|)^{1/2}, \qquad x_{2,t+1} = 1 - 2(|x_{1,t}|)^{1/2}.$$

Problem 91. A delayed version of the circle map is given by

$$\theta_{t+1} = \theta_t + \mu_1 \sin(2\pi\phi_t) + \mu_2, \quad \phi_{t+1} = \theta_t$$

where $t = 0, 1, \ldots$ and μ_1, μ_2 are positive bifurcation parameters. Show that at its parameter plane (μ_1, μ_2) the map has a symmetry line $\mu_2 = 0.5$ and various Arnold tongues.

Problem 92. Study the system of difference equations

$$x_{1,t+1} = (2 - ax_{2,t})x_{1,t}, \qquad x_{2,t+1} = x_{2,t} - b(x_{2,t})^2$$

$(t = 0, 1, \ldots)$ where a and b are positive constants.

Problem 93. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = x_1 + x_2, \qquad f_2(x_1, x_2) = x_1 x_2$$

or written as a system of difference equations

$$x_{1,t+1} = x_{1,t} + x_{2,t}, \quad x_{2,t+1} = x_{1,t}x_{2,t}, \quad t = 0, 1, \ldots.$$

(i) Find the fixed points of $\mathbf{f}$ and study their stability.
(ii) Find the fixed points of $\mathbf{f}(\mathbf{f})$ and study their stability.

Problem 94. Study the second order difference equation

$$x_{t+1} + x_{t-1} = -x_t + \frac{at + b}{x_t} + c$$

where $t = 0, 1, \ldots$ and $x_{-1} = 0$, $x_0 = 1/2$, $a = 1$, $b = 1$, $c = 1$.

Problem 95. Study the two-dimensional map

$$x_{t+1} = e^{y_t} x_t, \qquad y_{t+1} = e^{-x_t} y_t$$

with $t = 0, 1, \ldots$ and $x_0 = 1$, $y_0 = 1$. The fixed point of the map is $(0, 0)$.

Problem 96. Let n be a natural number. The recursive relation used to determine the *Farey fraction* x_k/y_k is given by

$$x_{k+2} = \left\lfloor \frac{y_k + n}{y_{k+1}} \right\rfloor x_{k+1} - x_k, \quad y_{k+2} = \left\lfloor \frac{y_k + n}{y_{k+1}} \right\rfloor y_{k+1} - y_k$$

where the initial conditions are $x_0 = 0$, $y_0 = x_1 = 1$ and $y_1 = n$. The sequence of x_k/y_k is called the *Farey sequence*. The floor of a denoted by $\lfloor a \rfloor$ is the greatest integer which is not greater than a, for example $\lfloor 5.3 \rfloor = 5$. Write a C++ program to determine the Farey sequence for given n. Determine the first 11 elements of the sequence for $n = 5$.

Problem 97. Study the recursion

$$\theta_{t+1} = 2\theta_t + \theta_{t-1} \mod 2 \quad t = 1, 2, \ldots$$

with the initial conditions (i) $\theta_0 = 0$, $\theta_1 = 0$; (ii) $\theta_0 = 1$, $\theta_1 = 1$; (iii) $\theta_0 = 0$, $\theta_1 = 1$; (iv) $\theta_0 = 1$, $\theta_1 = 1$.

Problem 98. Let a, b, c be positive real numbers and $t = 0, 1, 2 \ldots$. Study the two-dimensional map

$$x_1(t+1) = \frac{c + ax_1(t) + bx_2(t)}{a + bx_1(t) + cx_2(t)}, \quad x_2(t+1) = \frac{b + cx_1(t) + ax_2(t)}{a + bx_1(t) + cx_2(t)}.$$

First show that it admits the fixed point $(x_1^*, x_2^*) = (1, 1)$ and study the stability of this fixed point.

Problem 99. Let $a, b > 0$. Study the delayed logistic map

$$x_{t+1} = ax_t + dx_{t-1}(1 - x_{t-1})$$

with $t = 1, 2, \ldots$ and the initial conditions x_0, x_1. With $y_t = x_{t-1}$ (i.e. $y_{t+1} = x_t$) we can write the map as

$$\begin{aligned} x_{t+1} &= ax_t + dy_t(1 - y_t) \\ y_{t+1} &= x_t \end{aligned}$$

with the initial conditions x_0 and y_0. Show that the map shows the transitions

fixed point $\mapsto$ Hopf bifurcation $\mapsto$ torus $\mapsto$ locking $\mapsto$ chaos $\mapsto$ hyperchaos with increasing d.

Problem 100. Let $r_1, r_2 > 1$ and $a_1, a_2 > 0$. Study the prey-predator model

$$x_{1,t+1} = r_1 x_{1,t}(1 - x_{1,t})(1 - a_1 x_{2,t})$$
$$x_{2,t+1} = r_2 x_{2,t}(1 - x_{2,t})(1 - a_2(1 - x_{1,t})).$$

Problem 101. Study the two-dimensional map

$$x_{t+1} = \frac{1}{2}\ln(\cosh(4y_t)), \quad y_{t+1} = \frac{1}{2}\ln(\cosh(4x_t))$$

with $t = 0, 1, \ldots$ and the initial conditions $x_0 = 1/2$ and $y_0 = 1$.

Problem 102. Study the two-dimensional map

$$x_{t+1} = 4x_t(1 - x_t)4y_t(1 - y_t), \qquad y_{t+1} = 4y_t(1 - y_t)4x_t(1 - x_t)$$

with $t = 0, 1, \ldots$ and $x_0 = 1/4$ and $y_0 = 3/4$.

Problem 103. Study the two-dimensional map

$$x_{t+1} = \sin(x_t + y_t), \qquad y_{t+1} = \cos(x_t - y_t).$$

Problem 104. Let $a, r > 0$. Consider the *delayed logistic map* ($t = 1, 2, \ldots$)

$$x_{t+1} = ax_t + rx_t + rx_{t-1}(1 - x_{t-1}$$

with the initial values x_0 and x_1. Setting $y_t = x_{t-1}$ (i.e. $y_{t+1} = x_t$) we can write it as a first order system

$$x_{t+1} = ax_t + ry_t(1 - y_t), \quad y_{t+1} = x_t, \quad t = 1, 2, \ldots$$

In other words we have the map

$$f_1(x, y) = ax + ry(1 - y), \quad f_2(x, y) = x.$$

(i) Find $df_1 \wedge df_2$.
(ii) Show that $(0, 0)$ is a fixed point.
(iii) Let $a = 0.5$ and $r \in [1, 4]$. Show that we have a transition

fixed point, Hopf bifurcation, torus, locking, chaos, hyperchaos

Problem 105. (i) Study the two-dimensional map ($t = 0, 1, 2, \ldots$)

$$x_{t+1} = x_t + \sin(\pi x_t)\sin(\pi y_t), \quad y_{t+1} = y_t + \sin(\pi x_t)\sin(\pi y_t)$$

with the initial values x_0 and y_0.
(ii) Study the two-dimensional map ($t = 0, 1, 2, \ldots$)

$$x_{t+1} = x_t + \sin(\pi x_t)\sin(\pi y_t), \quad y_{t+1} = y_t - \sin(\pi x_t)\sin(\pi y_t)$$

with the initial values x_0 and y_0.

Problem 106. Consider the Hénon map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = x_2 + 1 - a x_1^2, \quad f_2(x_1, x_2) = b x_1$$

where $a > 0$ and $b > 0$.
(i) Let $\omega = dx_1 \wedge dx_2$. Find $\mathbf{f}^*(dx_1 \wedge dx_2)$.
(ii) Let $\alpha_1 = x_1 dx_2 + x_2 dx_1$. Find $\mathbf{f}^*\alpha_1$.
(iii) Let $\alpha_2 = x_1 dx_2 - x_2 dx_1$. Find $\mathbf{f}^*\alpha_2$.
(iv) Let $\alpha_3 = x_1 dx_1 + x_2 dx_2$. Find $\mathbf{f}^*\alpha_3$.

Problem 107. Study the two-dimensional map

$$x_{t+1} = 4x_t(1 - x_t), \quad y_{t+1} = x_t y_t, \quad t = 0, 1, 2, \ldots$$

Problem 108. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 4x(1-x)$. Then $df/dx = 4 - 8x$. Study the two-dimensional map

$$x_{t+1} = 4x_t(1 - x_t), \quad y_{t+1} = |(4 - 8x_t)| y_t$$

where $x_0, y_0 \in (0, 1)$.

Problem 109. Study the modulated circle map

$$\theta_{t+1} = \theta_t + r_1 \sin(2\pi\theta_t) + r_2 \sin(2\pi\phi_t) + r \mod 1$$
$$\phi_{t+1} = \phi_t + b \mod 1$$

Problem 110. Let $a > 0$ and $1 > b > 0$. Consider the system of nonlinear two-dimensional difference equations

$$x_{t+1} = x_t + a(e^{y_t - x_t} - 1), \quad y_{t+1} = y_t + b(x_t - y_t), \quad t = 0, 1, 2, \ldots.$$

(i) Find the fixed points. Find the variational equation and study the stability of the fixed points.

(ii) Find the second iterate.
(iii) Show that if $y_t > x_t$, then x_{t+1} increases and y_{t+1} decreases.
(iv) Show that

$$y_t = b\sum_{j=1}^{\infty}(1-b)^{j-1}x_{t-j}.$$

Problem 111. Let $\mu_1 > 0$ and $\mu_2 > 0$. Study the two-dimensional map

$$x_{t+1} = \mu_1 x_t(1 - x_t - y_t), \qquad y_{t+1} = \mu_2 x_t y_t.$$

First find the fixed points and study their stability.

Problem 112. Study the two-dimensional map $(t = 0, 1, \ldots)$

$$x_{t+1} = x_t + y_{t+1} \equiv x_t + y_t + 2(x_t^2 - r), \quad y_{t+1} = y_t + 2(x_t^2 - r)$$

where $r \in [0, 2]$ is the bifurcation parameter. First find the fixed points and study their stability.

Problem 113. Study the second order difference equation

$$x_{t+1} - 2x_t + x_{t-1} = x_t(1 - x_t), \quad t = 1, 2, \ldots$$

with given $x_0 > 0$ and $x_1 > 0$.

Problem 114. Study the second order map

$$x_{t+1} - 2x_t + x_{t-1} = x_t(1 - x_t), \quad t = 1, 2, \ldots.$$

Problem 115. The difference equation

$$x_{t+2} = x_{t+1} + x_t, \quad t = 0, 1, \ldots$$

with $x_0 = 0$, $x_1 = 1$ provides the Fibonacci sequence. Study the case $(t = 0, 1, \ldots)$

$$x_{t+2} = x_{t+1} + x_t \mod 2$$

with $x_0 = 0$, $x_1 = 1$. Is the sequence eventually periodic?

Problem 116. Study the difference equation $(t = 0, 1, 2, \ldots)$

$$\det\begin{pmatrix} x_{t+2} & x_t \\ x_t & x_{t+1} \end{pmatrix} = 0$$

with $x_0 = 1$, $x_1 = 1$.

Problem 117. Let $\mu, \mu_1, \mu_2, \mu_3 > 0$. Study the two-dimensional map $(t = 0, 1, \ldots)$

$$x_{t+1} = x_t + \mu x_t(\mu_1 - \mu_2(2x_t + y_t) - 2\mu_3 x_t)$$

$$y_{t+1} = y_t + \mu y_t(\mu_1 - \mu_2(x_t + 2y_t) - 2\mu_3 y_t).$$

First find the fixed points.

Problem 118. Let $a > 0$ and $\sigma \geq 0$. Consider the potential

$$U(q) = (q-a)^2(q+a)^2 \equiv q^4 - 2a^2q^2 + a^4.$$

Then

$$\frac{dU(q)}{dq} = 4q^3 - 4a^2 q.$$

Conider the two-dimensional map

$$q_{t+1} = q_t + \sigma p_{t+1}$$
$$p_{t+1} = p_t - \sigma \frac{dU(q = q_t)}{dq} = p_t - 4\sigma q_t(q_t^2 - a^2).$$

(i) Find the Jacobian matrix

$$J = \begin{pmatrix} \partial q_{t+1}/\partial q_t & \partial q_{t+1}/\partial p_t \\ \partial p_{t+1}/\partial q_t & \partial p_{t+1}/\partial p_t \end{pmatrix}.$$

Then find $\det(J)$.
(ii) Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Find $M^T J M$, where ^{T} denotes the transpose.
(iii) Find the fixed points of the map and study their stability.

Problem 119. Study the initial value problem of the difference equation

$$\det \begin{pmatrix} x_{t+2} & x_{t+1} \\ x_{t+1} & x_t \end{pmatrix} = 0$$

where $t = 0, 1, 2, \ldots$ and $x_0 = 1$, $x_1 = 1/2$.

Problem 120. Study the initial value problem of the difference equation

$$x_{t+2} = \frac{x_{t+1} + x_t}{1 + x_{t+1}x_{t+1}}, \quad t = 0, 1, 2, \ldots$$

where $x_0 = 1$, $x_1 = 1/2$.

Problem 121. Study the initial value problem of the difference equation

$$\det \begin{pmatrix} x_{t+3} & x_{t+2} & x_{t+1} \\ x_{t+2} & x_{t+1} & x_t \\ x_{t+1} & x_t & 0 \end{pmatrix} = 0$$

where $t = 0, 1, 2, \ldots$ and $x_0 = 1$, $x_1 = 1/2$, $x_2 = 1/3$.

Problem 122. Let $\{\,,\,\}$ be the *Poisson bracket*. It is defined by

$$\{Q(p,q), P(p,q)\} := \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}.$$

Consider the transformation

$$Q(p,q) = f(q+yp) + aq + bq, \quad P(p,q) = xf(q+yp) + cq + dp$$

with $ad - bc = 1$, f is a differentiable function of $q + yp$ and x, y are real numbers. Show that the transformation is canonical if and only if x, y are connected by

$$y = \frac{bx - d}{ax - c}.$$

Problem 123. Let (open disc)

$$D^2 := \{\, (x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < 1 \,\}.$$

Show that D^2 is homeomorphic to $\mathbb{R}^2$.

Problem 124. Show that the two-dimensional invertible map

$$x_{1,t+1} = \frac{3}{2}x_1 + 2x_1^2 - \frac{1}{2}x_{2,t} + r, \quad x_{2,t+1} = x_{1,t}, \qquad t = 0, 1, 2, \ldots$$

admits a saddle-node bifurcation if the bifurcation parameter r approaches 0 from above.

Problem 125. Let r_1, r_2 be the bifurcation parameters. Study the two-dimensional map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = 1 - r_1 x_1^2 + r_2(x_2 - x_1)$$

$$f_2(x_1, x_2) = 1 - r_1 x_2^2 + r_2(x_1 - x_2)$$

or written as system of difference equations

$$x_{1,t+1} = 1 - r_1 x_{1,t}^2 + r_2(x_{2,t} - x_{1,t}), \quad x_{2,t+1} = 1 - r_1 x_{2,t}^2 + r_2(x_{1,t} - x_{2,t}).$$

Find the fixed points and study their stability. Find the second iterate and the fixed points of it and thus find periodic orbits. Study their stability.

Problem 126. Study the map

$$f_1(x,\phi) = \frac{1}{2}(e^{2x}\sin^2(\phi) + e^{-2x}\cos^2(\phi))$$
$$f_2(x,\phi) = \frac{1}{2}(e^{2x}\cos^2(\phi) + e^{-2x}\sin^2(\phi).$$

First find the fixed points and calculate

$$\det\begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial \phi \\ \partial f_2/\partial x & \partial f_2/\partial \phi \end{pmatrix}.$$

Note that

$$f_1(x,\phi) + f_2(x,\phi) = \cosh(2x).$$

Problem 127. Give an interpretation of the *Arnold cat map*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mod 1.$$

Problem 128. Find the inverse of the *Arnold cat map*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mod 1.$$

Note that the determinant of the matrix is $+1$.

Problem 129. Let $a, b > 0$. Consider the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1,x_2) = a - bx_1 + x_1^2 x_2, \quad f_2(x_1,x_2) = bx_1 + x_2 - x_1^2 x_2.$$

Let $a = 1$ and consider b as the bifurcation parameter. Show that the map admits Hopf bifurcation. First find the fixed points.

Problem 130. Let $a, b, c, d \in \mathbb{R}$. Consider the transformation

$$Q(p,q) = f(q+yp) + aq + bp, \quad P(p,q) = xf(q+yp) + cq + dp$$

with $ad - bc = 1$, f is a differentiable function and x, y are real numbers. Show that the transformation is canonical if and only if x, y are connected by

$$y = \frac{bx - d}{ax - c}.$$

Problem 131. Let $t = 0, 1, 2, \ldots$. Consider the two-dimensional map ($\alpha \in \mathbb{R}$ and $\sin(\alpha) \neq 0$)

$$x(t+1) = (x(t) + y^3(t))\cos(\alpha) - y(t)\sin(\alpha)$$
$$y(t+1) = (x(t) + y^3(t))\sin(\alpha) + y(t)\cos(\alpha).$$

Find the inverse of the map if it exists.

Problem 132. (i) Consider the differential one-form in $\mathbb{R}^2$

$$\alpha = x_1 dx_2 + x_2 dx_1.$$

Find the maps $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ that leave α invariant.
(ii) Consider the differential one-form in $\mathbb{R}^2$

$$\alpha = x_1 dx_2 - x_2 dx_1.$$

Find the maps $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ that leave α invariant.

Problem 133. Let $r > 0$. Study the two-dimensional map

$$x_{t+1} = r y_t^2, \qquad y_{t+1} = \frac{1}{r} x_t^2$$

for $r = 2$. First find the fixed points. Are the fixed points stable? Study the four cases with the initial values (i) $x_0 = 1$, $y_0 = 1$; (ii) $x_0 = 1/2$, $y_0 = 1$; (iii) $x_0 = 1$, $y_0 = 1/2$; (iv) $x_0 = 1/2$, $y_0 = 1/2$.

2.3 Complex Maps

Let $\mathbb{C}$ be the complex numbers. A complex map is a map $f : \mathbb{C} \to \mathbb{C}$. In particular important are analytic maps such as $f(z) = 2z + z^2$, $f(z) = \exp(z)$, $f(z) = sin(2z)$ or $f(z) = i + iz$. Sometimes on has to work with the extended complex plane $\mathbb{C} \cup \{\infty\}$ (also called the Riemann sphere) and adopt the convention that $1/0 = \infty$ and $1/\infty = 0$.

2.3.1 Solved Problems

Problem 134. Study the behaviour of the fixed points of the complex map $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z^2 \tag{1}$$

i.e. find the fixed point from $f(z^*) = z^*$ and study their stability.

Problem 135. Let

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \tag{1}$$

be a *Möbius transformation.*
(i) Show that f is defined and continuous on the extended complex plane.
(ii) Show that

$$f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \tag{2}$$

is one-to-one and onto.
(iii) Show that f is a homeomorphism of the extended complex plane.

Problem 136. The complex map $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = az + b$, where $a, b \in \mathbb{C}$.
(i) Find for which values of a and b there exists a fixed point of f.
(ii) Show that if $a \neq 1$, then f is topologically conjugate to function of the form $g(z) = cz$.
(iii) Describe the stable set of the fixed point of f (when it exists). (iv) For the cases where f does not have a fixed point, describe the dynamics of f.

Problem 137. Consider the complex analytic map $f_c : \mathbb{C} \to \mathbb{C}$

$$f_c(z) = z^2 + c$$

where $z, c \in \mathbb{C}$ and c fixed.
(i) Find the fixed points of f_c.
(ii) Find the fixed points of $f_c(f_c)$, i.e. periodic points of f_c.

Problem 138. Consider the complex map $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^3$.
(i) Find the fixed points of f and determine if they are attracting, repelling or non-hyperbolic.
(ii) Find all periodic points of f. Determine if they are attracting, repelling or non-hyperbolic.

Problem 139. Show that all complex quadratic polynomials are topologically conjugate to a polynomial of the form $q_c(z) = z^2 + c$, where $c \in \mathbb{C}$.

Problem 140. Consider the complex map $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = az$$

where a is a complex number with $|a| \neq 1$. Find the fixed points of this map. Find $f^{(n)}$ and discuss the cases $|a| < 1$ and $|a| > 1$.

Problem 141. Let $f : \mathbb{C} \to \mathbb{C}$ be a differentiable complex function. Let z^* be a fixed point of f. If $|f'(z^*)| < 1$, then the stable set of z^* contains a neighbourhood of z^*. If $|f'(z^*)| > 1$, then there is a neighbourhood of z^* all of whose points must leave the neighbourhood under iteration of f. Apply this statement to the function

$$f(z) = 2\sin(z) + z.$$

Problem 142. Let $\lambda = \exp(2\pi i\gamma)$ and $\gamma = \frac{1}{2}(\sqrt{5}-1)$. Study the complex map

$$f(z) = \frac{\lambda}{2}z^2 + 1 - \frac{\lambda}{2}.$$

Problem 143. The complex map $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = az + b$, where $a, b \in \mathbb{C}$.
(i) Find for which values of a and b there exists a fixed point of f.
(ii) Show that if $a \neq 1$, then f is topologically conjugate to function of the form $g(z) = cz$.
(iii) Describe the stable set of the fixed point of f (when it exists).
(iv) For the cases where f does not have a fixed point, describe the dynamics of f.

Problem 144. Let $z \in \mathbb{C}$. Consider the map

$$f_{\mu,\gamma}(z) = \frac{\mu z(1-\gamma z)}{1+\mu(1-\gamma)z}, \qquad \mu > 1, \ \ 0 \leq \gamma \leq 1.$$

Consider the invertible map

$$\phi(z) = z + 1 - \frac{1}{\mu}.$$

Find ϕ^{-1} and

$$F_{\mu,\gamma}(z) = (\phi^{-1} \circ f_{\mu,\gamma} \circ \phi)(z).$$

What happens to the fixed points under the map?

Problem 145. The complex map $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = az + b$, where $a, b \in \mathbb{C}$.
(i) Find for which values of a and b there exists a fixed point of f.
(ii) Show that if $a \neq 1$, then f is topologically conjugate to function of the form $g(z) = cz$.
(iii) Describe the stable set of the fixed point of f (when it exists).
(iv) For the cases where f does not have a fixed point, describe the dynamics of f.

Problem 146. Show that all complex quadratic polynomials are topologically conjugate to a polynomial of the form $q_c(z) = z^2 + c$, where $c \in \mathbb{C}$.

Problem 147. Let $P(z)$ be a polynomial of degree $n \geq 2$ with distinct zeros $\zeta_1, \ldots, \zeta_n$. Show that

$$\sum_{j=1}^{n} \frac{1}{P'(\zeta_j)} = 0$$

where $'$ denotes the derivative, i.e. $P'(\zeta) \equiv dP(z = \zeta)/dz$.

Problem 148. Consider the functions $f(z) = z^3$ and $h(z) = z + 1/z$. Find a function p such that

$$h(f(z)) = p(h(z)). \qquad (1)$$

Problem 149. Show the following. Let $P(z)$ be a polynomial. Then either

1. $P(z)$ has a fixed point q with $P'(q) = 1$,
2. $P(z)$ has a fixed point q with $|P'(q)| > 1$.

Problem 150. Let $z \in \mathbb{C}$ and consider the analytic map

$$f(z) = \exp(z).$$

Find the solutions (fixed points) of the equation

$$z = f(z).$$

We set $z = x + iy$ ($x, y \in \mathbb{R}$. Then

$$x + iy = \exp(x + iy) \equiv e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

Thus we have to solve

$$e^x \cos(y) - x = 0, \qquad e^x \sin(y) - y = 0.$$

Problem 151. Given any four complex numbers z_1, z_2, z_3, z_4 one defines the cross ratio $[z_1, z_2, z_3, z_4]$ by

$$[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$

Show that

$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]$$

for all *Möbius transformations*

$$f(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$.

Problem 152. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then the *Ikeda laser map* is given by

$$z \mapsto r + c_1 z \exp\left(i\left(c_1 - \frac{c_3}{1 + |z|^2}\right)\right)$$

where r, c_1, c_2, c_3 are real bifurcation parameters. Show that the map can show chaotic behaviour for certain parameter values.

Problem 153. In the study of the Potts model the following complex map appears

$$z_{t+1} = \frac{abz_t - 1/2}{az_t + b - 3/2}, \quad t = 0, 1, \ldots$$

where $a, b \in \mathbb{R}$ and $a, b > 0$. Find the fixed points and study their stability.

Problem 154. The properties of the logistic map $x_{t+1} = 4x_t(1 - x_t)$ ($x_0 \in [0, 1], t = 0, 1, 2, \ldots$) are well-known. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Study the map

$$z_{t+1} = 4z_t(1 - z_t).$$

With $z_t = x_t + iy_t$ we can write

$$x_{t+1} = 4(x_t - x_t^2 + y_t^2), \qquad y_{t+1} = 4y_t(1 - x_t).$$

With $z = re^{i\phi}$ ($r \geq 0$) we could also write

$$r_{t+1}e^{\phi_{t+1}} = 4r_t e^{i\phi_t}(1 - r_t e^{i\phi_t}).$$

With $\phi_t = 0$ for all t we end up at the logistic map. First find the fixed points and study their stability.

Problem 155. Study the map $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = \frac{az+b}{cz+d}$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$. First find the fixed points.

Problem 156. Find the solution of the map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = 4z(1-z)$ or written as difference equation

$$z_{t+1} = 4z_t(1-z_t), \qquad t = 0, 1, \ldots$$

Problem 157. Let $\mathbb{R}^+$ be the nonnegative real numbers. Consider $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \sqrt{x}$. Then $x^* = 0$ and $x^* = 1$ are fixed points of the map, i.e solutions of $\sqrt{x^*} = x^*$. The fixed point $x^* = 0$ is unstable and the fixed point $x^* = 1$ is stable. Find the fixed points of the map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \sqrt{z}$. Set $z = re^{i\phi}$ with $r \geq 0$ and $\phi \in [0, 2\pi)$. Study the stability of the fixed points. Iterate the map f, i.e. find $f(f(z))$ and the fixed points of $f(f(z))$.

Problem 158. Consider the complex map

$$z_{t+1} = z_t - \frac{z_t^3 - 1}{3z_t^2}, \quad t = 0, 1, 2, \ldots$$

(i) Find the fixed points.
(ii) Study the initial value

$$z_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \equiv e^{i\pi/3}.$$

Problem 159. Find the fixed points of

$$f(z) = \frac{az+b}{cz+d}$$

with $ad - bc = 1$ and $c \neq 0$.

Problem 160. Let

$$\mathbb{D} := \{\, z \,:\, |z| \leq 1 \}.$$

(i) Consider the automorphism $f : \mathbb{D} \to \mathbb{D}$

$$f(z) = \frac{z + 1/2}{1 + z/2}.$$

Find the fixed points.
(ii) Iterate $1/2$ and $i/2$.

2.4 Higher Dimensional Maps

2.4.1 Solved Problems

Problem 161. (i) Let $s_1(0), s_2(0), s_3(0) \in \{+1, -1\}$. Study the time-evolution ($t = 0, 1, 2, \ldots$) of the coupled system of equations

$$s_1(t+1) = s_2(t)s_3(t)$$
$$s_2(t+1) = s_1(t)s_3(t)$$
$$s_3(t+1) = s_1(t)s_2(t)$$

for the eight possible initial conditions, i.e. (i) $s_1(0) = s_2(0) = s_3(0) = 1$, (ii) $s_1(0) = 1$, $s_2(0) = 1$, $s_3(0) = -1$, (iii) $s_1(0) = 1$, $s_2(0) = -1$, $s_3(0) = 1$, (iv) $s_1(0) = -1$, $s_2(0) = 1$, $s_3(0) = 1$, (v) $s_1(0) = 1$, $s_2(0) = -1$, $s_3(0) = -1$, (vi) $s_1(0) = -1$, $s_2(0) = 1$, $s_3(0) = -1$, (vii) $s_1(0) = -1$, $s_2(0) = -1$, $s_3(0) = 1$, (viii) $s_1(0) = -1$, $s_2(0) = -1$, $s_3(0) = -1$. Which of these initial conditions are fixed points?
(ii) Let $s_1(0), s_2(0), s_3(0) \in \{+1, -1\}$. Study the time-evolution ($t = 01, 2, \ldots$) of the coupled system of equations

$$s_1(t+1) = s_2(t)s_3(t)$$
$$s_2(t+1) = s_1(t)s_2(t)s_3(t)$$
$$s_3(t+1) = s_1(t)s_2(t)$$

for the eight possible initial conditions, i.e. (i) $s_1(0) = s_2(0) = s_3(0) = 1$, (ii) $s_1(0) = 1$, $s_2(0) = 1$, $s_3(0) = -1$, (iii) $s_1(0) = 1$, $s_2(0) = -1$, $s_3(0) = 1$, (iv) $s_1(0) = -1$, $s_2(0) = 1$, $s_3(0) = 1$, (v) $s_1(0) = 1$, $s_2(0) = -1$, $s_3(0) = -1$, (vi) $s_1(0) = -1$, $s_2(0) = 1$, $s_3(0) = -1$, (vii) $s_1(0) = -1$, $s_2(0) = -1$, $s_3(0) = 1$, (viii) $s_1(0) = -1$, $s_2(0) = -1$, $s_3(0) = -1$. Which of these initial conditions are fixed points?

Problem 162. Study the fixed points of the three-dimensional map $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$

$$f_1(x,y,z) = xz\left(\frac{(xz+2y^2)(1+xy^2+zy^2)^2}{(x^2z^2+y^2z^2+x^2y^2)^2(1+2x^2y^2z^2)}\right)^{1/3}$$
$$f_2(x,y,z) = y\left(\frac{(x^2y^2+y^2z^2+z^2x^2)(xz+2y^2)}{(1+xy^2+zy^2)(1+2x^2y^2z^2)}\right)^{1/3}$$
$$f_3(x,y,z) = \left(\frac{(x^2y^2+y^2z^2+z^2x^2)(1+xy^2+zy^2)^2}{(xz+2y^2)^2(1+2x^2y^2z^2)}\right)^{1/3}$$

which appears at renormalization group transformation.

Problem 163. Consider the recursion

$$x_{t+1} = x_t^2 + y_t^2 z_t$$

$$y_{t+1} = x_t y_t + x_t y_t z_t$$
$$z_{t+1} = x_t^2 z_t + z_t^2$$

where $t = 0, 1, 2, \ldots$ and x_0, y_0, z_0 are the initial conditions. Find the fixed points and study their stability.

Problem 164. Let $N \geq 2$ and $j = 0, 1, \ldots, N-1$. Let $c \in (0, 1)$. Study the coupled difference equations

$$x_j(t+1) = f((1-c)x_j(t) + \frac{c}{N-1} \sum_{j=0}^{N-1} x_j(t)), \quad t = 0, 1, \ldots$$

where $f(y) = 4y(1-y)$.

Problem 165. (i) Calculate the *weight matrix* W for the Hopfield network which stores the two patterns

$$\mathbf{x}_0 = (1, 1, 1, 1, 1, 1, 1, 1)^T, \qquad \mathbf{x}_1 = (1, 1, -1, -1, -1, -1, 1, 1)^T$$

(ii) Which of these two vectors are fixed points under iteration of the network?
(iii) Consider the vector

$$\mathbf{s}(t = 0) = (-1, 1, 1, 1, 1, 1, 1, -1)^T.$$

Calculate the evolution of this vector under synchronous evolution. Does it approaches a fixed point?

Problem 166. Let $a, b > 0$. Study the three dimensional map $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$

$$f_1(x_1, x_2, x_3) = a - x_2^2 - bx_3, \quad f_2(x_1, x_2, x_3) = x_1, \quad f_3(x_1, x_2, x_3) = x_2$$

or written as difference equations

$$x_{1,t+1} = a - x_{2,t}^2 - bx_{3,t}, \quad x_{2,t+1} = x_{1,t}, \quad x_{3,t+1} = x_{2,t}$$

where $t = 0, 1, \ldots$. Show that $dx_1 \wedge dx_2 \wedge dx_3$ is invariant under the map. Are

$$x_1 dx_2 + x_2 dx_3 + x_3 dx_1, \qquad x_1 dx_2 - x_2 dx_3 + x_3 dx_1$$

invariant under the map $\mathbf{f}$?

Problem 167. The *Arnold cat map* given by

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} \mod 1$$

has been studied by many authors. The map shows chaotic behaviour. Let $a, b \in \mathbb{N}$, i.e. a, b are positive integers. Consider the matrix

$$M(a,b) = \begin{pmatrix} 1 & a \\ b & ab+1 \end{pmatrix}.$$

It contains the matrix given above with $a = b = 1$.
(i) Find the determinant of $M(a,b)$ and thus show that the matrix is invertible. Find the inverse matrix.
(ii) What are the conditions on $a, b \in \mathbb{N}$ such that $M(a,b)$ is a normal matrix?
(iii) Find the eigenvalues and normalized eigenvectors of $M(a,b)$.
(iv) Find the two one-dimensional Liapunov exponents for the map ($t = 0, 1, 2, \ldots$)

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ b & ab+1 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} \mod 1.$$

(v) Let $\otimes$ be the Kronecker product. Calculate $M(a,b) \otimes M(c,d)$ with $c, d \in \mathbb{N}$. Find the eigenvalues and normalized eigenvectors of $M(a,b) \otimes M(c,d)$. Utilize the results from (iii).
(vi) Find the four one-dimensional Liapunov exponents for the map ($t = 0, 1, 2, \ldots$)

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \\ x_{4,t+1} \end{pmatrix} = (M(a,b) \otimes M(c,d)) \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \end{pmatrix}.$$

(vii) The star product of the matrices $M(a,b)$ and $M(c,d)$ is defined as

$$M(a,b) \star M(c,d) = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & a & 0 \\ 0 & b & ab+1 & 0 \\ d & 0 & 0 & cd+1 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of $M(a,b) \star M(c,d)$. Utilize the result from (iii).
(viii) Find the four one-dimensional Liapunov exponents for the map

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \\ x_{4,t+1} \end{pmatrix} = (M(a,b) \star M(c,d)) \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \end{pmatrix}.$$

Problem 168. (i) Calculate the weight matrix W for the Hopfield network which stores the two patterns (3)

$$\mathbf{x}_0 = (1,1,1,1,1,1,1,1)^T, \quad \mathbf{x}_1 = (1,1,-1,-1,-1,-1,1,1,)^T.$$

(ii) Which of these vectors are fixed points under iteration of the network? (4)
(iii) Consider the vector

$$\mathbf{s}(t=0) = (-1,1,1,1,1,1,1,-1)^T.$$

Calculate the evolution of this vector under synchronous evolution. Does it approach a fixed point? (3)

Problem 169. Let $t = 0,1,2,\ldots$, $K > 0$ and $N > 1$. Consider the map

$$p_j(t+1) = p_j(t) + \frac{K}{2\pi}(\sin(2\pi(x_{j+1}(t) - x_j(t)) - \sin(2\pi(x_j(t) - x_{j-1}(t)))$$
$$x_j(t+1) = x_j(t) + p_j(t+1)$$

where $j = 1,2,\ldots,N$ and $x_{j+N} \equiv x_j$, $p_{j+N} \equiv p_j$ (periodic boundary conditions). Find

$$\sum_{j=1}^{N} dx_j(t+1) \wedge dp_j(t+1)$$

where $\wedge$ denotes the exterior product (also called wegde or Grassmann product).

2.5 Bitwise Problems

Problem 170. Let $x_t, y_t \in \{0,1\}$ and $t = 0,1,2,\ldots$. We denote by $\oplus$ the XOR-operation and by $+$ the OR-operation. Solve the iteration

$$x_{t+1} = x_t + y_t, \qquad y_{t+1} = x_t \oplus y_t$$

with $x_0 = 1$, $y_0 = 1$. First find the fixed points of the map, i.e. solve the set of equations

$$x + y = x, \qquad x \oplus y = y.$$

Does (x_t, y_t) tend to a fixed point for $t \to \infty$?

Problem 171. Let $x_1(0), x_2(0), x_3(0) \in \{0,1\}$ and let $\oplus$ be the XOR-operation. Study the time-evolution ($t = 01,2,\ldots$) of the coupled system of equations

$$x_1(t+1) = x_2(t) \oplus x_3(t)$$
$$x_2(t+1) = x_1(t) \oplus x_3(t)$$
$$x_3(t+1) = x_1(t) \oplus x_2(t)$$

for the eight possible initial conditions, i.e. (i) $x_1(0) = x_2(0) = x_3(0) = 0$, (ii) $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 1$, (iii) $x_1(0) = 0$, $x_2(0) = 1$, $x_3(0) = 0$,

(iv) $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 0$, (v) $x_1(0) = 0$, $x_2(0) = 1$, $x_3(0) = 1$, (vi) $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 1$, (vii) $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 0$, (viii) $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 1$. Which of these initial conditions are fixed points?

Problem 172. Let $t = 0, 1, 2, \ldots$ and $x_t, y_t, z_t \in \{0, 1\}$ and the map

$$x_{t+1} = y_t \cdot z_t, \quad y_{t+1} = z_t + x_t, \quad z_{t+1} = x_t \oplus y_t$$

where $\cdot$ denotes the AND operation, $+$ the OR operation and $\oplus$ the XOR operation.
(i) Find the fixed points of the map.
(ii) Solve the map with the initial condition $x_0 = 1$, $y_0 = 1$, $z_0 = 1$. Does the solution tend to a fixed point?

Problem 173. The *Hopf bifurcation theorem* for maps in the plane $\mathbf{f}_r : \mathbb{R}^2 \to \mathbb{R}^2$, where r is the bifurcation parameter, is as follows.

Theorem. (Hopf bifurcation theorem) Let $\mathbf{f}(r, \mathbf{x})$ be a one-parameter family of maps in the plane satisfying:
a) An isolated fixed point $\mathbf{x}^*(r)$ exists.
b) The map $\mathbf{f}_r$ is C^k $(k \geq 3)$ in the neighbourhood of $(\mathbf{x}^*(r_0); r_0)$.
c) The Jacobian matrix $D_\mathbf{x}\mathbf{f}(\mathbf{x}^*(r); r)$ possesses a pair of complex, simple eigenvalues

$$\lambda(r) = e^{\alpha(r) + i\omega(r)}$$

and $\bar{\lambda}(r)$, such that the critical value $r = r_0$

$$|\lambda(r_0)| = 1, \quad (\lambda(r_0))^3 \neq 1, \quad (\lambda(r_0))^4 \neq 1, \quad \frac{d|\lambda(r)|}{dr}(r = r_0) > 0.$$

(Existence) Then there exists a real number $\epsilon_0 > 0$ and a C^{k-1} function such that

$$r(\epsilon) = r_0 + r_1\epsilon + r_3\epsilon^3 + O(\epsilon^4)$$

such that for each $\epsilon \in (0, \epsilon_0]$ the map $\mathbf{f}_r$ has an invariant manifold $H(r)$, i.e. $\mathbf{f}(H(r); r) = H(r)$. The manifold $H(r)$ is C^r diffeomorphic to a circle and consists of points at a distance $O(|r|^{1/2})$ of $\mathbf{x}^*(r)$, for $r = r(\epsilon)$.

(Uniqueness) Each compact invariant manifold close to $\mathbf{x}^*(r)$ for $r = r(\epsilon)$ is contained in $H(r) \cup \{0\}$.

(Stability) If $r_3 < 0$ (respectively $r_3 > 0$) then for $r > 0$ (respectively $r > 0$), the fixed point $\mathbf{x}^*(r(\epsilon))$ is stable (respectively unstable) and for $r > 0$ (respectively $r < 0$) the fixed point $\mathbf{x}^*(r(\epsilon))$ is unstable (respectively stable) and the surrounding manifold $H(r(\epsilon))$ is attracting (respectively repelling). When $r_3 < 0$ (respectively $r_3 > 0$) the bifurcation at $r = r(\epsilon)$ is said to be *supercritical* (respectively *subcritical*).

Study Hopf bifurcation for the two-dimensional map

$$f_1(x_1, x_2) = rx_1(3x_2 + 1)(1 - x_1), \qquad f_2(x_1, x_2) = rx_2(3x_1 + 1)(1 - x_2)$$

and $r \in \mathbb{R}$.

Problem 174. The *Denman-Beavers iteration* for the square root of an $n \times n$ matrix A with no eigenvalues on $\mathbb{R}^-$ is

$$Y_{k+1} = \frac{1}{2}(Y_k + Z_k^{-1})$$
$$Z_{k+1} = \frac{1}{2}(Z_k + Y_k^{-1})$$

with $k = 0, 1, 2, \ldots$ and $Z_0 = I_n$ and $Y_0 = A$. The iteration has the properties that

$$\lim_{k\to\infty} Y_k = A^{1/2}, \qquad \lim_{k\to\infty} Z_k = A^{-1/2}$$

and, for all k,

$$Y_k = AZ_k, \quad Y_k Z_k = Z_k Y_k, \quad Y_{k+1} = \frac{1}{2}(Y_k + AY_k^{-1}).$$

(i) Can the Denman-Beavers iteration be applied to the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}?$$

(ii) Find Y_1 and Z_1.

2.6 Supplementary Problems

Problem 175. Let r be the bifurcation parameter. Study the noninvertible 2-dimensional map

$$x_{1,t+1} = r(3x_{2,t}+1)x_{1,t}(1-x_{1,t}), \qquad x_{2,t+1} = r(3x_{1,t}+1)x_{2,t}(1-x_{2,t}).$$

Problem 176. Study the two-dimensional map

$$\begin{aligned} x_{t+1} &= x_t \cos(\omega\tau) + y_t \sin(\omega\tau) + \mu \sin(2kx_t)\sin(\omega\tau) \\ y_{t+1} &= y_t \cos(\omega\tau) - x_t \sin(\omega\tau) + \mu \sin(2kx_t)\cos(\omega\tau) \end{aligned}$$

for fixed $\omega\tau$ and k and the bifurcation parameter $\mu \in [0,1]$.

Problem 177. Study the map

$$x_{t+1} = 4x_t(1-x_t), \qquad y_{t+1} = 2x_t y_t \bmod 1$$

where $t = 0, 1, 2, \ldots$ and $x_0, y_0 \in [0,1]$.

Problem 178. Study the map

$$x_{t+1} = 4y_t(1-y_t), \qquad y_{t+1} = 4x_t(1-x_t)$$

where $t = 0, 1, 2, \ldots$ and $x_0, y_0 \in [0,1]$ with $x_0 \neq y_0$.

Problem 179. Consider the two equations

$$f_1(x_1, x_2) = 0, \qquad f_2(x_1, x_2) = 0$$

with $f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$ and $f_2(x_1, x_2) = x_1 + x_2$. Study the coupled system of difference equations

$$\begin{aligned} x_{1,t+1} &= x_{1,t} - \frac{1}{f_{1,1}(x_{1,t}, x_{2,t})} f_1(x_{1,t}, x_{2,t}) \\ x_{2,t+1} &= x_{2,t} - \frac{1}{2}\frac{1}{f_{2,2}(x_{1,t+1}, x_{2,t})} f_2(x_{1,t+1}, x_{2,t}) \end{aligned}$$

where $t = 0, 1, 2, \ldots$, $f_{1,1}$ is the partial derivative of f_1 with respect to x_1 and $f_{2,2}$ is the partial derivative of f_2 with respect to x_2. Consider the initial values $x_{1,0} = 0.2$, $x_{2,0} = 0.8$.

Problem 180. Consider the difference equation $z_{t+1} = 4z_t(1-z_t)$ with $t = 0, 1, 2, \ldots$ and $z_0 \in \mathbb{C}$. Show that with $z_t = x_t + iy_t$ and x_t, $y_t \in \mathbb{R}$ we can write

$$\begin{aligned} x_{t+1} &= 4x_t(1-x_t) + 4y_t^2 \\ 1 - x_{t+1} &= x_t^2 - 2x_t(1-x_t) + (1-x_t)^2 - 4y_t^2 \\ y_{t+1} &= 4y_y - 8x_t y_t \equiv 4(1-x_t)y_t - 4x_t y_t \end{aligned}$$

or in matrix form

$$\begin{pmatrix} x_{t+1} \\ 1-x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 4 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & -4 \\ 0 & 0 & -2 & 0 & 0 & 2 & -2 & 2 & 0 \end{pmatrix} \left(\begin{pmatrix} x_t \\ 1-x_t \\ y_t \end{pmatrix} \otimes \begin{pmatrix} x_t \\ 1-x_t \\ y_t \end{pmatrix} \right).$$

Problem 181. Let $a > 0$ and $r > 0$. Consider the delayed logistic map

$$x_{t+1} = ax_t + rx_{t-1}(1-x_{t-1}), \quad t = 1, 2, \ldots$$

with the two initial values x_0 and x_1. Setting $y_t = x_{t-1}$ (i.e. $y_{t+1} = x_t$) we can write the map as a first order system

$$x_{t+1} = ax_t + ry_t(1-y_t), \quad y_{t+1} = x_t, \quad t = 1, 2, \ldots$$

with the initial values $y_1 = x_0$, $y_2 = x_1$. Thus we can write the first order system as the map

$$f_1(x,y) = ax + ry(1-y), \qquad f_2(x,y) = x.$$

Let $a = 0.5$. The fixed points are given by the solutions of the equations

$$ax + ry(1-y) = x, \qquad x = y.$$

Thus $(0,0)$ is a fixed point.
(i) Show that with increasing the bifurcation parameter r one has transitions

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fixed point -> Hopf bifurcation -> torus -> locking -> chaos -> hyperchaos
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(ii) Calculate

$$df_1(x,y) \wedge df_2(x,y).$$

Problem 182. Study the modulated circle map

$$\begin{aligned} \theta_{t+1} &= \theta_t + a_1 \sin(2\pi\theta_t) + a_2 \sin(2\pi\phi_t) + r, \quad \bmod 1 \\ \phi_{t+1} &= \phi_t + b \end{aligned}$$

where $b = (\sqrt{5}-1)/2$, $a_1 = 0.15$, $a_2 = 0.01$ and r is the bifurcation parameter $r \in (0,1)$.

Problem 183. Study the coupled circle map

$$\theta_{t+1} = \theta_t + a_1 \sin(2\pi\theta_t) + a_2 \sin(2\pi\phi_t) + r_1, \quad \bmod 1$$
$$\phi_{t+1} = \phi_t + b_1 \sin(2\pi\phi_t) + b_2 \sin(2\pi\theta_1) + r_2, \quad \bmod r_2$$

with $a_1 = a_2 = b_1 = b_2 = 1/2$ and $r_1, r_2 \in (0,1)$ be the bifurcation parameters.

Problem 184. Study the map

$$x_{t+1} = 4x_t(1 - x_t)$$
$$y_{t+1} = x_t y_t$$

where $t = 0, 1, \ldots$ and $x_0, y_0 \in (0,1)$.

Problem 185. Study the map

$$x_{t+1} = 4x_t(1 - x_t)$$
$$y_{t+1} = |(4 - 8x_t)|y_t$$

where $x_0, y_0 \in (0,1)$.

Problem 186. Let $\epsilon \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$. Study the globally coupled map

$$x_{0,t+1} = (1-\epsilon) f(x_{0,t}) + \frac{\epsilon}{3} \sum_{j=0}^{2} f(x_{j,t})$$

$$x_{1,t+1} = (1-\epsilon) f(x_{1,t}) + \frac{\epsilon}{3} \sum_{j=0}^{2} f(x_{j,t})$$

$$x_{2,t+1} = (1-\epsilon) f(x_{2,t}) + \frac{\epsilon}{3} \sum_{j=0}^{2} f(x_{j,t})$$

with the initial values $x_{0,0}$, $x_{1,0}$, $x_{2,0}$.

Problem 187. Let $\mathbb{Z}$ be the set of integers. Consider the two-dimensional lattice $\mathbb{Z} \times \mathbb{Z}$ and $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. Let $t = 0, 1, 2, \ldots$. Consider the two-dimensional cellular automata ($s_{ij}(t) \in \{0,1\}$)

$$s_{ij}(t+1) = s_{i,j+1}(t) \oplus s_{i-1,j}(t) \oplus s_{i,j}(t) \oplus s_{i+1,j}(t) \oplus s_{i,j-1}(t)$$

where $\oplus$ is the XOR-operation and at $t = 0$ we have $s_{0,0}(t = 0) = 1$ and 0 otherwise for all other lattice sites. Calculate $s_{i,j}(t = 1)$ and $s_{i,j}(t = 2)$. The four nearest neighbours around $(0,0)$ are

$$(1,0), \quad (0,1), \quad (-1,0), \quad (0,-1).$$

Problem 188. Is the function $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = \frac{e^z - 1}{e^z + 1}$$

analytic? Find the fixed points.

Problem 189. Consider $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z + 2z^2 + 3z^3.$$

(i) Find the fixed points of f.
(ii) Find the fixed points of df/dz.
(iii) Find the fixed points of d^2f/dz^2.

Problem 190. Schwarz lemma. Let $\mathbb{D} := \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ centered at the orign. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(0) = 0$ (i.e. 0 is a fixed point). Then

$$|f(z)| \leq |z| \text{ for all } z \text{ in } \mathbb{D} \text{ and } |df(z=0)/dz| \leq 1.$$

Apply the lemma to

$$f(z) = \frac{1}{2}(\sin(z) + z).$$

Problem 191. Let $\mathbb{D} := \{\, z \,:\, |z| < 1 \,\}$. Let $f : \mathbb{D} \to \mathbb{D}$ be analytic and assume that the map f is not an elliptic Möbius transformation nor the identity. Then there is an $\tilde{z} \in \bar{\mathbb{D}}$ such that $f^{(n)}(z) \to \tilde{z}$ for all $z \in \mathbb{D}$. Consider the map

$$f(z) = \frac{z + 1/2}{1 + z/2}.$$

(i) Find $f(0)$, $f(f(0))$, $f(f(f(0)))$, $\ldots$.
(ii) Find $f(-1/2)$, $f(f(-1/2))$, $f(f(f(-1/2)))$, $\ldots$.
(iii) Find the fixed points of $f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$.

Problem 192. Let n be a positive integer with $n \geq 2$. Let c be any real number. We define x_t recursively by $x_0 = 0$, $x_1 = 1$ and for $t \geq 0$,

$$x_{t+2} = \frac{cx_{t+1} - (n-t)x_t}{t+1}.$$

Fix n and then take c to be the largest value for which $x_{t+1} = 0$. Find x_t in terms of n and t, $2 \leq t \leq n$.

Problem 193. Consider the difference equation given by

$$\det\begin{pmatrix} x_{t+1} & x_t \\ x_t & x_{t-1} \end{pmatrix} = 1$$

with $t = 1, 2, \ldots$. Find the solution with $x_0 = x_1 = 1$. Find the solution with $x_0 = x_1 = 1$ and $x_t \bmod 2$.

Problem 194. Study the map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x, y) = e^{x^2 - y^2}\cos(2xy), \qquad f_2(x, y) = e^{x^2 - y^2}\sin(2xy).$$

First find the fixed points if there any. Is the map invertible?

Problem 195. Consider the map $: \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ given by

$$f_1(n_1, n_2) = \frac{1}{2}((n_1 + n_2)^2 + n_1 + 3n_2)$$
$$f_2(n_1, n_2) = \frac{1}{2}((n_1 + n_2)^2 + 3n_1 + n_2).$$

Show that $(0, 0)$ is a fixed point. Are there other fixed points? Is the map invertible?

Problem 196. Let $r > 0$ be the bifurcation parameter. Study the coupled system of maps

$$x_{1,t+1} = r\left(x_{1,t} - \frac{1}{4}(x_{1,t} + x_{2,t})^2\right) \quad x_{2,t+1} = \frac{1}{r}\left(x_{2,t} + \frac{1}{4}(x_{1,t} + x_{2,t})^2\right).$$

Obviously $(0, 0)$ is a fixed point. Is the fixed point stable?

Problem 197. Study the map

$$x_{t+1} = (x_t + x_{t-1}) \bmod p \quad t = 1, 2, \ldots$$

for $p = 7$ and $x_0 = 1$, $x_1 = 2$.

Problem 198. Consider the three-dimensional map

$$x_{1,t+1} = r_1 - x_{2,t}^2 - r_2 x_{3,t}, \quad x_{2,t+1} = x_{1,t}, \quad x_{3,t+1} = x_{2,t}$$

where $t = 0, 1, 2, \ldots$ and $r_1 > 0$, $r_2 > 0$ are the bifurcation parameters. Show that the map can show hyperchaotic behaviour (depending on r_1 and r_2) i.e. two one-dimensional Liapunov exponents can be positive.

Problem 199. Let $k = 0, 1, 2, \ldots$ and

$$y_k = \int_0^1 \frac{x^k}{1 + x + x^2} dx$$

with

$$y_0 = \int_0^1 \frac{1}{1 + x + x^2} dx = \frac{2}{\sqrt{3}} \arctan \left(\frac{2x + 1}{\sqrt{3}} \right), \quad y_1 = \int_0^1 \frac{x}{1 + x + x^2} dx = \frac{1}{2} \ln(1+x+x^2) - \frac{1}{\sqrt{3}} \arctan(\left(\frac{2x+1}{\sqrt{3}} \right.$$

Show that

$$y_{k+2} + y_{k+1} + y_k = \frac{1}{k+1}, \quad k = 0, 1, 2, \ldots$$

Chapter 3

Fractals

In this chapter we consider problems involving fractals. These include the Cantor set, the Mandelbrot set, the Julia set and iterated function systems. To investigate fractals one needs the fractal dimensions. Fractal dimensions are the capacity, the Hausdorff dimension,

Definition. Let X be a subset of $\mathbb{R}^n$. Let $N(\epsilon)$ be the number of n-dimensional cubes (boxes) of side ϵ to cover the set X. The *capacity* D of X is defined as

$$D := \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln\left(\frac{1}{\epsilon}\right)}. \tag{1}$$

Definition. Let X be a subset of $\mathbb{R}^n$. A cover of X is a (possibly infinite) collection of balls the union of which contains X. The diameter of a cover $\mathcal{A}$ is the maximum diameter of the balls in $\mathcal{A}$. For $d, \epsilon > 0$, we define

$$\alpha(d,\epsilon) := \inf_{\substack{\mathcal{A} = \text{ coverof X} \\ ;\ \text{diam}\,\mathcal{A} \le \epsilon}} \sum_{A \in \mathcal{A}} (\text{diam} A)^d \tag{2}$$

and

$$\alpha(d) := \lim_{\epsilon \to 0} \alpha(d,\epsilon). \tag{3}$$

There is a unique d_0 such that

$$d < d_0 \Rightarrow \alpha(d) = \infty$$

$$d > d_0 \Rightarrow \alpha(d) = 0. \tag{4}$$

This d_0 is defined to be the *Hausdorff dimension* of X, written $HD(X)$.

Definition. A hyperbolic) iterated function system consists of a complete metric space (X, d) together with a finite set of contraction mappings

$$w_n : X \to X$$

with respective contractivity factors s_n, for $n = 1, 2, \ldots, N$. The notation for the iterated function system is $\{X;\ w_n,\ n = 1, 2, \ldots, N\}$ and its contractivity factor is

$$s = \max\{s_n :\ n = 1, 2, \ldots, N\}.$$

The following theorem summarizes the main facts about a hyperbolic iterated function system.

Theorem 1. Let $\{X;\ w_n,\ n = 1, 2, \ldots, N\}$ be a hyperbolic iterated function system with contractivity factor s. Then the transformation

$$W :\ \mathcal{H}(X) \to \mathcal{H}(X)$$

defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B)$$

for all $B \in \mathcal{H}(X)$, is a contraction mapping on the complete metric space $(\mathcal{H}(X),\ h(d))$ with contractivity factor s. That is

$$h(W(B),\ W(C)) \leq s \cdot h(B,\ C)$$

for all $B,\ C \in \mathcal{H}(X)$. Its unique fixed point, $A \in \mathcal{H}(X)$, obeys

$$A = W(A) = \bigcup_{n=1}^{N} w_n(A),$$

and is given by

$$A = \lim_{n \to \infty} W^{\circ n}(B)$$

for any $B \in \mathcal{H}(X)$.

The fixed point $A \in \mathcal{H}(X)$ described in the theorem is called the attractor of the iterated function system.

An iterated function system consisting of contractive similarity mappins has a unique attractor $A \subset \mathbb{R}^n$ which is invariant under the action of the system.

Let (X, d) be a compact metric space. In the following, it will be always assumed that $X = [0, 1]$ and d is the Euclidean distance. Let $w_i : X \to X,\ i = 1, \ldots, N$ be a set of contracting maps with contracting factors $\lambda_i,\ 0 < \lambda_i < 1$,

$$d(w_i(x), w_i(y)) \leq \lambda_i d(x, y), \qquad i = 1, \ldots, N. \tag{1}$$

Let $\mathbf{p} = (p_1, \ldots, p_N)$ be a set of positive weights $p_i > 0$, $i = 1, \ldots, N$, and $\Sigma_1^N p_i = 1$. Then

$$\{X, w_i, p_i, ik = 1, \ldots, N\} \equiv \{\mathbf{X}, \mathbf{w}, \mathbf{p}\}$$

is called a hyperbolic IFS. An IFS is said to be linear when all the maps w_i are affine and a linear IFS is said to be homogeneous when all the scales $\lambda_i, 0 < |\lambda_i| < 1$ are equal: $\lambda_1 = \cdots = \lambda_N = \lambda$.

If one is interested in the attractor and in the p-balanced measure, it is not restrictive to consider only positive scales $\lambda > 0$ while dealing with homogeneous IFS. Given the IFS $\{\mathbf{X}, \mathbf{w}, \mathbf{p}\}$ there exists a unique attractor A, characterized as the closure of the set of fixed points and a unique attractive invariant p-balanced measure μ with support A.

Problem 1. The *Cantor set* is constructed as follows. We set

$$\begin{aligned} E_0 &= [0,1] \\ E_1 &= \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right] \\ E_2 &= \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right] \\ ... &= ... \end{aligned} \qquad (1)$$

In other words. Delete the middle third of the line segment $[0,1]$ and then the middle third from all the resulting segments and so on ad infinitum. The set defined by

$$C := \bigcap_{k=0}^{\infty} E_k \qquad (2)$$

is called the standard Cantor set (or Cantor ternary set).
(i) Show that the Cantor set is of Lebesgue measure zero.
(ii) Show that there is an bijective mapping $f : C \mapsto [0,1]$.
(iii) Find the capacity of the interval $I = [0,2]$. Find the capacity of the standard Cantor set C.
(iv) Show that for the standard Cantor set $HD(C) \leq (\ln 2)/(\ln 3)$.

Problem 2. Consider the unit interval $[0,1]$. The construction of the *Smith-Volterra-Cantor set* starts with the removal of the middle $1/4 \equiv 1/2^2$ from $[0,1]$, i.e. we obtain the set

$$[0,3/8] \cup [5/8,1].$$

Next we remove the subintervals of length $1/16 \equiv 1/2^4$ from the middle of each of the two remaining intervals $[0,3/8]$ and $[5/8,1]$. We arrive at the four intervals

$$[0,5/32] \cup [7/32,3/8] \cup [5/8,25/32] \cup [27/32,1].$$

Thus we remove subintervals of length $1/2^{2n}$ from the middle of each of the 2^{n-1} remaining intervals. Repeating this process to infinity we obtain the Smith-Volterra-Cantor set. Find the total length of the intervals removed from $[0,1]$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 3x & \text{for } x \leq 1/2 \\ 3-3x & \text{for } x \geq 1/2 \end{cases} \qquad (1)$$

(i) Show that the set

$$\Lambda := \{x \in [0,1] \ : \ f^{(n)}(x) \text{ is in } [0,1] \text{ for all } n\} \qquad (2)$$

Problem 4. What is the box-counting dimension of the countable set S given by the infinite sequence

$$S = \{\, 0,\, 1/2,\, 1/3,\, 1/4, \ldots \}.$$

Problem 5. Consider the equilateral triangle with the vertices

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}.$$

(i) Find the area of the triangle.
(ii) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Consider the three contracting maps

$$f_1(\mathbf{x}) = \frac{1}{2}\mathbf{x}, \quad f_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad f_3(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix}.$$

Find

$$\begin{matrix} f_1(\mathbf{x}_1), & f_2(\mathbf{x}_1), & f_3(\mathbf{x}_1) \\ f_1(\mathbf{x}_2), & f_2(\mathbf{x}_2), & f_3(\mathbf{x}_2) \\ f_1(\mathbf{x}_3), & f_2(\mathbf{x}_3), & f_3(\mathbf{x}_3). \end{matrix}$$

Show that we obtain the vertices given above and three new vertices that describe an inscribed equilateral triangle. Find the area of this triangle.

Problem 6. Let $\mathbb{C}$ be the complex plane. Let $c \in \mathbb{C}$. The Mandelbrot set M is defined as follows

$$M := \{c \in \mathbb{C} \,:\, c, c^2 + c, (c^2 + c)^2 + c, \ldots \not\to \infty\}. \qquad (1)$$

(i) Show that to find the Mandelbrot set one has to study the recursion relation

$$z_{t+1} = z_t^2 + c \qquad (2)$$

where $t = 0, 1, 2, \ldots$ and $z_0 = 0$.
(ii) Write the recursion relation in real and imaginary part. For a given $c \in \mathbb{C}$ (or $(c_1, c_2) \in \mathbb{R}^2$) we can now study whether or not c belongs to M.
(iii) Show that $(c_1, c_2) = (0, 0)$ belongs to M.
(iv) Show that the Mandelbrot set lies within $|c| < 2$.
(v) Show that if $|z| > 2$ the sequence diverges.

Problem 7. A ratio list is a finite list of positive numbers, $(r_1, r_2, \cdots, r_n)$. An *iterated function system* realizing a ratio list $(r_1, r_2, \cdots, r_n)$ in a metric space S is a list $(f_1, f_2, \cdots, f_n)$ iff

$$K = f_1[K] \cup f_2[K] \cup \cdots \cup f_n[K].$$

(ii) Show that the triadic Cantor set is an invariant set for an iterated function system realizing the ratio list (1/3,1/3).
(ii) Show that the Sierpinski gasket is an invariant set for an iterated function system realizing the ratio list (1/2,1/2,1/2).
(iii) The dimension associated with a ratio list $(r_1, r_2, \cdots, r_n)$ is the positive number s such that

$$r_1^s + r_2^s + \cdots + r_n^s = 1.$$

Let $(r_1, r_2, \cdots, r_n)$ be a ratio list. Suppose each $r_i < 1$. Show that there is a unique nonnegative number s satisfying

$$\sum_{i=1}^{n} r_i^s = 1.$$

The number s is 0 if and only if $n = 1$.

Problem 8. A ratio list $(r_1, r_2, \cdots, r_n)$ is called contracting (or hyperbolic) iff $r_i < 1$ for all i. The number s is called the *similarity dimension* of a (nonempty compact) set K iff there is a finite "decomposition" of K

$$K = \bigcup_{i=1}^{n} f_i[K]$$

where $(f_1, f_2, \cdots, f_n)$ is an iterated function system of similarities realizing a ratio list with dimension s. Show that the similarity dimension of the triadic Cantor set is given by $s = \ln 2$. Show that the similarity dimension of the Koch curve is given by $s = \ln(4)/\ln(3)$.

Problem 9. Consider an iterated function system. Show that from the invariance property it immediately follows that

$$\int_A f(x) d\mu(x) = \sum_{i=1}^{N} p_i \int_A f \circ w_i(x) d\mu(x) \tag{2}$$

where f is any simple continuous on X. Moreover it can be shown that $\{X, w_i, p_i,\ i = 1, \ldots, N\}$ is equivalent to

$$\{X, w_i \circ w_j, p_i p_j; i, j = 1, \ldots, N\} \equiv \{\mathbf{X}, \mathbf{w}^{\circ 2}, \mathbf{p}^2\}$$

in the sense that two IFS have the same attractor and equivalent p-balanced measures: in fact they have the same fixed points, so their closure A, which gives the attractor, is also the same, and

$$\int_A f(x)d\mu(x) = \sum_{i=1}^{N} p_i \int_A f \circ w_i(x)d\mu(x) = \sum_{i,j=1}^{N} p_i p_j \int_A f \circ w_i \circ w_j(x)d\mu(x)$$

so that μ is a p-balanced measure for the second IFS. From the uniqueness of such a measure, the assertion follows.

Problem 10. The *Sierpinski carpet* is constructed as follows: Consider the unit square $[0,1] \times [0,1]$. Partition the unit square into nine equal squares and removing the interior of the middle one. This process is then repeated in each of the remaining eight squares. The first three steps in this construction are displayed in the figure. Show that the fractal dimension of the Sierpinski carpet is $\log_3(8)$.

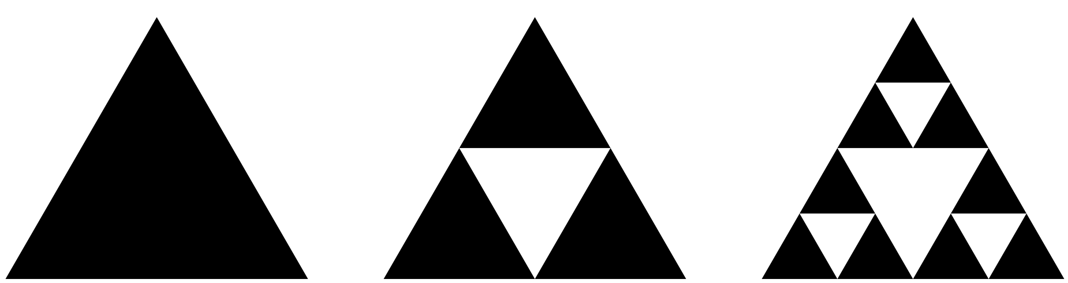
Problem 11. Find the area of the *Sierpinski carpet* starting from the unit square $[0,1]^2$.

Problem 12. Consider the iterated function system (*Sierpinski triangle*)

$$\mathbf{f}_1(x_1,x_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\mathbf{f}_2(x_1,x_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$
$$\mathbf{f}_3(x_1,x_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

Apply the maps to the vertices of the unit square $[0,1]^2$. Discuss.

Problem 13. Starting from an equilateral triangle the first two steps in construction of the triangular *Sierpinski gasket* are given in the figure. Find the fractal dimension (capacity).



Problem 14. The *Sierpinski gasket* in d dimensional Euclidean space ($d \geq 2$) is constructed as follows: One starts with a d-dimensional hypertetrahedron. The midpoints of the edges are connected, creating $(d+1)$ smaller hypertetrahedra. The set at the centre (bounded by the faces of these new tetrahedra) is then removed. This procedure is repeated for the $(d+1)$ new tetrahedra and so forth. Show that the capacity is given by

$$C = \frac{\ln(d+1)}{\ln(2)}.$$

Problem 15. Consider the iterated function system (*Sierpinski carpet*)

$$\mathbf{f}_1(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\mathbf{f}_2(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$
$$\mathbf{f}_3(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$$
$$\mathbf{f}_4(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$
$$\mathbf{f}_5(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$
$$\mathbf{f}_6(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2/3 \end{pmatrix}$$
$$\mathbf{f}_7(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$
$$\mathbf{f}_8(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}.$$

Apply the maps to the vertices of the unit square $[0,1]^2$. Discuss.

Problem 16. Consider the logistic map $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = rx(1-x), \qquad r > 4 \tag{1}$$

or

$$x_{t+1} = rx_t(1-x_t), \qquad r > 4, \quad t = 0,1,2,\ldots \tag{2}$$

Assume that $x_0 \in [0,1]$. Discuss the solutions.

Problem 17. Consider the set C obtained from the unit interval $[0,1]$ by first removing the middle third of the interval and then removing the middle fifths of the two remaining intervals. Now iterate this process, first removing the middle thirds, then removing middle fifths. The set C is what remains when this process is repeated infinitely. Is C a fractal? If so, what is its fractal dimension?

Problem 18. Can a fractal that is totally disconnected (topological dimension 0) have a fractal dimension larger than 1?

Problem 19. Give the matrices to perform any one of the following operations

- Rotation around the z-axis and the other two axis
- Scaling
- Shearing of the x by the z-coordinate
- Translation

in any order using only matrix multiplications. Use the homogeneous form of a point: $(x, y, z, 1)^T$ and 4×4 matrices to solve the problem.

Problem 20. Compute exactly the area of the Koch snowflake.

Problem 21. Describe the filled Julia set for the map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^3$.

Problem 22. Consider the map

$$g_\lambda(z) = \lambda(z - z^3), \quad \lambda > 0.$$

(i) Find the fixed points.
(ii) Show that the points

$$p_+(\lambda) = \sqrt{\frac{\lambda+1}{\lambda}}, \qquad p_-(\lambda) = -\sqrt{\frac{\lambda+1}{\lambda}}$$

lie on a cycle of period 2.

Problem 23. Let s be a positive number. Study the *Julia set* of the rational map

$$f(z) = \left(\frac{z^2 + s - 1}{2z + s - 2}\right)^2.$$

Problem 24. (i) The observed volume $V(\varepsilon)$ of a fractal in dimensions d when covered with N d-dimensional cubes has the form

$$V(\varepsilon) = \varepsilon^d N(\varepsilon).$$

Since $N(\varepsilon) \sim \varepsilon^{D_0}$, this implies

$$V(\varepsilon) = \varepsilon^{d-D_0}$$

where $d - D_0$ is called the co-dimension of the fractal. If $d - D_0 > 0$, the observed volume decreases with resolution ε. This defines the category of *thin* fractals, for which $V(\varepsilon)$ vanishes for $\varepsilon \to 0$.
If $d - D_0 = 0$, this does not imply that we have a traditional geometric object. We can, nonetheless, have a ramified structure. Such objects converge to a nonzero volume, having, e.g. ramified structures attached to it. Such objects are called *fat fractals.* Fat fractals can be distinguished by measuring the difference between observed and the real volume by means of an exponent α as

$$V(\varepsilon) - V \sim \varepsilon^{\alpha}.$$

For fat fractals, we have $0 < \alpha < 1$, whereas for traditional objects its value is typically unity. Rather than as a dimension, α should be seen as a codimension.
(ii) Remove instead of the middle third from an interval the proportions $1/3, 1/9, 1/27, \ldots$. What is the dimension of the emergent fat fractal?

Problem 25. Each $x \in [0,1]$ can be written as

$$x = \sum_{j=1}^{\infty} \frac{\epsilon_j}{2^j}$$

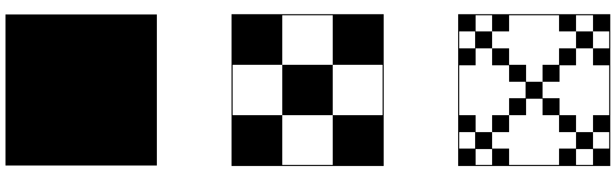
with $\epsilon_j = 0$ or $\epsilon_j = 1$. Define the function $f : [0,1] \to [0,1)$ as

$$f(x) = \sum_{j=1}^{\infty} \frac{2\epsilon_j}{3^j}.$$

The function f is known as Cantor function. Let $x = 1/8$. Find $f(x)$.

Problem 26. The *Koch curve* is self-similar on each length scale. Each time the length of the unit is reduced by a factor of 3, the number of segments is increased by a factor of 4. Find the box dimension D.

Problem 27. Starting from the unit square the figure shows the successive stages of generating a fractal. Find the fractal dimension (capacity).



Problem 28. Consider a triangle in the Euclidean plane $\mathbb{R}^2$ with the vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) ordered counterclockwise.
(i) Consider the three midpoints for each side of the triangle and thus construct a midpoint triangle. Find the area of this triangle compared to the original triangle. Find the perimeter of this triangle compared the the original triangle.
(ii) The area A_1 of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$A_1 = \frac{1}{2} \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \frac{1}{2}(x_2 y_3 - x_3 y_2 + x_1 y_2 - x_2 y_1 + x_3 y_1 - x_1 y_3).$$

Let $s_1, s_2 \in (0, 1)$ and $s_1 + s_2 = 1$. Find the area of the triangle with the vertices

$$(s_1 x_1 + s_2 x_2, s_1 y_1 + s_2 y_2), \quad (s_1 x_2 + s_2 x_3, s_1 y_2 + s_2 y_3), \quad (s_1 x_3 + s_2 x_1, s_1 y_3 + s_2 y_1).$$

Problem 29. Consider the 3×3 binary matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

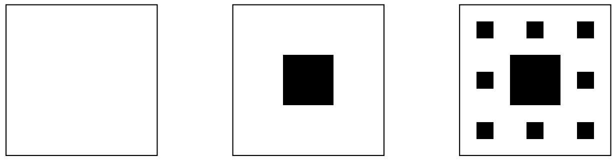
where 1 is identified with a black pixel and 0 with a white pixel. Let $\otimes$ be the Kronecker product. What fractal is generated by

$$A, \quad A \otimes A, \quad A \otimes A \otimes A,$$

Give the fractal dimension. Note that

$$A \otimes A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Problem 30. Starting from the unit square the figure shows the successive stages of generating a fractal, i.e. we remove iteratively a square of relative area $\frac{1}{9}$ from the center of a square. Find the box dimension (capacity).

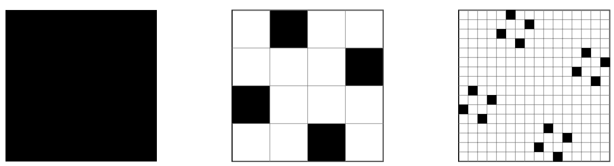


Problem 31. Consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let $\otimes$ be the Kronecker product. What fractal is generated by $A \otimes A$, $A \otimes A \otimes A$, etc.

Problem 32. Find the box-counting dimension of the fractal generated by the process displayed in the figure

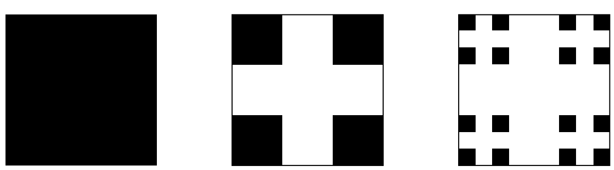


Problem 33. Consider the 4×4 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

What fractals is generated by $P \otimes P$, $P \otimes P \otimes P$, etc.

Problem 34. What is the box-counting dimension of the fractal generated by the process displayed in the figure.



Problem 35. Study the four similitudes

$$\mathbf{f}_1(x_1,x_2) = \frac{3}{4}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\mathbf{f}_1(x_1,x_2) = \frac{3}{4}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$
$$\mathbf{f}_1(x_1,x_2) = \frac{3}{4}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}$$
$$\mathbf{f}_1(x_1,x_2) = \frac{3}{4}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}$$

for the unit square $[0,1]^2$ with the vertices $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.

Problem 36. Let $k \geq 2$ and $\cup$ be the union of sets. Consider a closed and bounded subset S of the Euclidean space $\mathbb{R}^2$. It is said to be self-similar if it can be expressed in the form

$$S = S_1 \cup S_2 \cup \cdots \cup S_k$$

where the sets S_1, S_2, $\ldots$, S_k are nonoverlapping sets, each of which is congruent to S scaled by the same factor s with $0 < s < 1$ (scaling factor). The box dimension of a self-similar set S is denoted by $d_B(S)$ and defined by

$$d_B(S) = \frac{\ln(k)}{\ln(1/s)}.$$

(i) Consider the unit square $[0,1]^2$, $k = 4$ and $s = 1/2$. Find $d_B(S)$.
(ii) For the Sierpinksi carpet one has $k = 8$ and $s = 1/3$. Find $d_B(S)$.

Problem 37. The *rotation matrix*

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

is an element of the Lie group $SO(2,\mathbb{R})$ with the inverse matrix given by

$$R^{-1}(\theta) = R(-\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Consider the Euclidean space $\mathbb{R}^2$. A *similitude* with a scaling factor $0 < s < 1$ is a map of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$\mathbf{f}(x_1,x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} sR(\theta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

where $\theta, t_1, t_2 \in \mathbb{R}$. Thus the map consists of a scaling by a factor s, a rotation about the origin $(0,0)$ and a translation in the x_1 direction by t_1 and in the x_2 direction by t_2. Let $\theta = \pi/4$, $s = 1/2$, $t_1 = 1$, $t_2 = -1$. Find

$$\mathbf{f}(0,0), \quad \mathbf{f}(0,1), \quad \mathbf{f}(1,0), \quad \mathbf{f}(1,1).$$

Problem 38. Consider the 3×3 binary matrix

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

What fractals is generated by $B \otimes B$, $B \otimes B \otimes B$, etc.

Problem 39. The construction of a two-dimensional Cantor set is as follows: We start from the unit square $[0,1] \times [0,1]$. In the first step we construct four subsets of the unit square given by

1) $[0,1/3]\times[0,1/3]$, 2) $[2/3,1]\times[0,1/3]$, 3) $[0,1/3]\times[2/3,1]$, 4) $[2/3,1]\times[2/3,1]$.

Within each of these four subsets we construct again four subsets with the scaling factors $1/3$. Thus for the first subset $[0,1/3] \times [0,1/3]$ we obtain the four subsets

$[0,1/9]\times[0,1/9]$, $[2/9,1/3]\times[0,1/9]$, $[0,1/9]\times[2/9,1/3]$, $[2/9,1/3]\times[2/9,1/3]$.

Analogously we do the construction for the other three subsets (squares). We repeat this process now up to infinity. Find the Hausdorff dimension and the capacity of this set.

Problem 40. Let $I = [0,1]$ and $0 < x_0 < x_1 < 1$. A *cookie-cutter map* is a mapping

$$f : [0,x_0] \cup [x_1,1] \mapsto I$$

with the properties that (i) $f|_{[0,x_0]}$ and $f|_{[x_1,1]}$ are $1-1$ maps onto I, and (ii) f is $C^{1+\gamma}$ differentiable, i.e. differentiable with a Hölder continuous derivative Df satisfying $|Df(x) - Df(y)| < c|x-y|^\gamma$ for some $c > 0$ and $|Df(x)| > 1$ for all $x \in [0,x_0] \cup [x_1,1]$. Then the *cookie-cutter set* associated with the map f is the set

$$S := \{\, x \in [0,x_0] \cup [x_1,1] \ : \ f^{(n)} \in [0,x_0] \cup [x_1,1] \text{ for all } n \geq 0\}.$$

Let $x_0 = 1/3$, $x_1 = 2/3$ and $f(x) = 3x \bmod 1$. Find S.

Problem 41. Let $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ be an iterated function system given by

$$F_1(x) = \frac{1}{4}x, \qquad F_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

Describe the attractor of $\{F_1, F_2\}$ and find its Hausdorff dimension and capacity.

Problem 42. Consider the unit square $[0,1]^2 = [0,1] \times [0,1]$. The Sierpinksi carpet is constructed from the eight maps

$$\mathbf{f}_1(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\mathbf{f}_2(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$
$$\mathbf{f}_3(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}$$
$$\mathbf{f}_4(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$
$$\mathbf{f}_5(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/3 & 2/3 \end{pmatrix}$$
$$\mathbf{f}_6(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$$
$$\mathbf{f}_7(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2/3 & 1/3 \end{pmatrix}$$
$$\mathbf{f}_8(x_1,x_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}.$$

Apply the maps to the four vertices of the unit square

$$\mathbf{v}_1 = (x_1,x_2) = (0,0), \quad \mathbf{v}_2 = (x_1,x_2) = (0,1), \quad \mathbf{v}_3 = (x_1,x_2) = (1,0), \quad \mathbf{v}_4 = (x_1,x_2) = (1,1).$$

Discuss.

Problem 43. The *Appolonian gasket* is a subset of the Euclidean space $\mathbb{R}^2$. Find the fractal dimension of the Appolonian gasket.

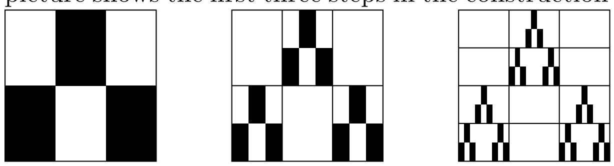
Problem 44. Consider Pascal's triangle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
```

Apply the mod 2 operation. Disucss.

Problem 45. Is the Cantor function (devil staircase) continuous?

Problem 46. Find the fractal dimension of the *Hironaka curve*. The picture shows the first three steps in the construction of the Hironaka curve.



Problem 47. Show that the Kronecker product $\otimes$ and the 2×3 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

can be used to construct the Hironaka curve.

Problem 48. Find the Hausdorff dimension for the *Koch curve*.

Problem 49. Consider the Euclidean space $\mathbb{R}^3$. The *Menger sponge* is constructed as follows. Starting point is the unit cube $[0,1]^3 \equiv [0,1]\times[0,1]\times[0,1]$ with 8 vertices, 12 edges and volume 1. One subdivides this cube into $27 = 3^3$ smaller equal cubes by trisecting the edges (which have length 1). Thus the scaling factor is $1/3$. The trema to remove consists of the center cube and the 6 cubes in the centers of the faces of the cube. Hence 20 cubes remain each with volume $1/27 = 1/3 \cdot 1/3 \cdot 1/3$. The boundary of these 20 cubes must also remain in order the set to be compact. Next we apply the same approach to the remaining 20 cubes and obtain $20^2 = 400$ cubes each with length $1/9$ and volume $1/9 \cdot 1/9 \cdot 1/9$. The Menger sponge is the set of points which remain if one applies this process infinitely often. Find the fractal dimension of the Menger sponge. Each face of the Menger sponge is a Sierpinski carpet.

Problem 50. Consider the Euclidean space $\mathbb{R}^3$ and the unit cube $[0,1]^3$. The *Menger sponge* is constructed by the 20 maps ($j = 1, \ldots, 20$)

$$\mathbf{f}_j(x_1, x_2, x_3) = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathbf{t}_j$$

where $\mathbf{t}_j = (\, t_{j,1} \quad t_{j,2} \quad t_{j,3}\,)^T$ for $t_{j,1}, t_{j,2}, t_{j,3} \in \{0, 1/3, 2/3\}$, except for for the six case when exactly two coordinates are $1/3$ and the case when all three coordinates are 1/3. This leads to the cases

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1/3\end{pmatrix}, \quad \begin{pmatrix}0\\0\\2/3\end{pmatrix}, \quad \begin{pmatrix}0\\1/3\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1/3\\2/3\end{pmatrix},$$

$$\begin{pmatrix}0\\2/3\\0\end{pmatrix}, \quad \begin{pmatrix}0\\2/3\\1/3\end{pmatrix}, \quad \begin{pmatrix}0\\2/3\\2/3\end{pmatrix}, \quad \begin{pmatrix}1/3\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1/3\\0\\2/3\end{pmatrix},$$

$$\begin{pmatrix}1/3\\2/3\\0\end{pmatrix}, \quad \begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}, \quad \begin{pmatrix}2/3\\0\\0\end{pmatrix}, \quad \begin{pmatrix}2/3\\0\\1/3\end{pmatrix}, \quad \begin{pmatrix}2/3\\0\\2/3\end{pmatrix},$$

$$\begin{pmatrix}2/3\\1/3\\0\end{pmatrix}, \quad \begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}, \quad \begin{pmatrix}2/3\\2/3\\0\end{pmatrix}, \quad \begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix}, \quad \begin{pmatrix}2/3\\2/3\\2/3\end{pmatrix}.$$

Consider $\mathbf{f}_{20}$

$$\mathbf{f}_{20}(\mathbf{x}) = \frac{1}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} + \begin{pmatrix}2/3\\2/3\\2/3\end{pmatrix}.$$

Apply it to the vertices

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Problem 51. Let $x \in [0,1]$. Consider the logistic map $f_r(x) = rx(1-x)$ with $r = r_\infty \approx 3.570....$ Show that the corresponding invariant set $A \subset [0,1]$ has both Hausdorff and box dimensions equal to ≈ 0.538.

Problem 52. The complex plane and the Riemann sphere (or extended complex plane) is denoted by $\mathbb{C}$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, respectively. For any complex-valued rational function f on the Riemann sphere $\hat{\mathbb{C}}$ such taht the point ∞ is an attracting fixed point of f, one defines

$$\begin{aligned}\mathcal{L}(f) &= \text{the basin attraction of} \infty \text{for the map} f\\ \mathcal{K}(f) &= \hat{\mathbb{C}} \setminus \mathcal{L}(f)\\ \mathcal{J}(f) &= \partial(\mathbb{K}(f))\end{aligned}$$

where $\partial(S)$ denotes the boundary of a set S. If the function f is a polynomial, then $\mathcal{J}(f)$ is the Julia set of f, $\mathcal{K}(f)$ is the filled Julia set with $\mathcal{K}(f)$ given by

$$\mathcal{K}(f) = \{\, z \in \hat{\mathbb{C}} \,:\, f^{(n)}(z) \nrightarrow \text{ as} \, n \to \infty \,\}.$$

Find the filled Julia set and the Julia set for

$$f(z) = z^2.$$

Problem 53. Consider the map

$$f_c(z) = z^2 + c$$

with $c = 1/4$.
(i) Is $z = 1/2$ an element of the filled Julia set?
(ii) Is $z = -1/2$ an element of the filled Julia set?
(iii) Is $z = i/2$ an element of the filled Julia set?
(iv) Is $z = -i/2$ an element of the filled Julia set?

Problem 54. Consider the map

$$f_c(z) = z^2 + c$$

with $c = 1/4$. Is $z = 3/5$ an element of the filled Julia set.

Problem 55. The *Riemann-Liouville definition* for the *fractional derivative* of a function f is given by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{\tau=0}^{\tau=t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

where $\Gamma(.)$ is the gamma function and the integer n is given by $n-1 \leq \alpha < n$. Let $f(t) = t^2$. Find the fractional derivative of f with $\alpha = 1/2$.

Problem 56. The *Weierstrass function* $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n^2 x)$$

is continuous but nowhere differentiable. Find the derivative in the sense of generalized functions.

Problem 57. Study the iterated function system $F = \{[0,1] : f_1, f_2\}$, where the metric on the unit interval $[0,1]$ is the Euclidean metric and

$$f_1(x) = \frac{1}{2}x, \qquad f_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

First study $f_1(0)$, $f_1(f_1(0))$, $f_1(f_1(f_1(0)))$ etc and $f_2(0)$, $f_2(f_2(0))$, $f_2(f_2(f_2(0)))$ etc.

Problem 58. Study the iterated function system $F = \{[0,1] : f_1, f_2\}$, where the metric on the unit interval $[0,1]$ is the Euclidean metric and

$$f_1(x) = \frac{2}{3}x, \qquad f_2(x) = \frac{2}{3}x + \frac{1}{3}.$$

First study $f_1(0)$, $f_1(f_1(0))$, $f_1(f_1(f_1(0)))$ etc and $f_2(0)$, $f_2(f_2(0))$, $f_2(f_2(f_2(0)))$ etc.

Problem 59. Let $x \in [0,1]$. Consider the linear functions

$$f_L(x) = \frac{x}{3}, \qquad f_R(x) = \frac{1}{3}(2+x).$$

(i) Find the fixed points of f_L. Find the fixed points of f_R.
(ii) Let $x = 1$. Find $f_L(x)$, $f_L(f_L(x))$, $f_L(f_L(f_L(x)))$, $\ldots$.
(iii) Let $x = 1$. Find $f_R(x)$, $f_R(f_R(x))$, $f_R(f_R(f_R(x)))$, $\ldots$.
(iv) Let $x = 1/2$. Find $f_L(x)$, $f_L(f_L(x))$, $f_L(f_L(f_L(x)))$, ... and $f_R(x)$, $f_R(f_R(x))$, $f_R(f_R(f_R(x)))$, $\ldots$.

Problem 60. Let $c, z \in \mathbb{C}$, $|c| < 1$ and $|1-c| < 1$. These conditions are satisfied by $c = \frac{1}{2}(1+i)$. Study the maps

$$f_0(z) = cz, \qquad f_1(z) = c + (1-c)z.$$

First find the fixed points for f_0 and f_1.

Problem 61. Let $c, z \in \mathbb{C}$, $|c| < 1$ and $|1-c| < 1$. These conditions are satisfied by $c = \frac{1}{2}(1 + i\frac{\sqrt{3}}{2})$. Study the maps

$$f_0(z) = c\bar{z}, \qquad f_1(z) = c + (1-c)\bar{z}.$$

First find the fixed points for f_0 and f_1.

Problem 62. Study the iterated function system with the maps

$$\mathbf{f}_1(x_1,x_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{f}_2(x_1,x_2) = \frac{1}{2}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}.$$

Problem 63. Study the difference equation

$$x_{t+1} = \sqrt{x_t + 2}, \quad t = 0, 1, 2, \ldots$$

with $x_0 = 0$. First find the fixed points.

Problem 64. Consider the map $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z^2 + i.$$

(i) Find $f(1)$, $f(f(1))$, $f(f(f(1)))$, $\dots$. Discuss.
(ii) Find $f(-1)$, $f(f(-1))$, $f(f(f(-1)))$, $\dots$. Discuss.
(iii) Find $f(i)$, $f(f(i))$, $f(f(f(i)))$, $\dots$. Discuss.
(iv) Find $f(-i)$, $f(f(-i))$, $f(f(f(-i)))$, $\dots$. Discuss.

Problem 65. Let $c \in \mathbb{C}$. Consider the function

$$f_c(z) = z^2 + c.$$

Show that the Mandelbrot set M is contained in the disk of radius 2 in the complex plane, i.e.

$$M \subset \{\, c \in \mathbb{C} \,:\, |c| \leq 2 \,\}.$$

Problem 66. Let $c \in \mathbb{C}$. Consider the function $f : \mathbb{C} \to \mathbb{C}$

$$f_c(z) = z^2 + c.$$

A point c is called a *Misiurewicz point* if 0 is strictly preperiodic for f_c. One calls $c \in \mathbb{C}$ a Misiurewicz point of type (m, n) if $m \geq 1$ is the smallest integer such that $f_c^{(n)}(0)$ is periodic and n is the primitive period of $f_c^{(m)}(0)$. Give examples for Misiurewicz points.

Problem 67. Consider the Mandelbrot set

$$M = \{\, c \in \mathbb{C} \,:\, \text{for all}\, n \geq 1,\; |f_c^{(n)}(0) \leq 2 \,\} \quad \text{where} \quad f_c(z) = z^2 + c.$$

M is a subset of the complex plane.
(i) Is $c = 1$ an element of the Mandelbrot set?
(ii) Is $c = -1$ an element of the Mandelbrot set?
(iii) Is $c = i$ an element of the Mandelbrot set?
(iv) Is $c = -i$ an element of the Mandelbrot set?

Problem 68. Let $d = 2, 3, \dots$. Consider the function $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z^d.$$

Find $f^{(n)}(z)$ and $\lim_{n\to\infty} f^{(n)}(z)$.

Problem 69. Let $0 < r < 1$ and $\phi \in \mathbb{R}$. Consider the two maps

$$f_0(z) = re^{i\phi} z, \qquad f_1(z) = re^{i\phi} + (1 - re^{i\phi})z.$$

(i) Find the fixed points of the two maps f_0 and f_1.
(ii) Show that $f_0(1) = f_1(0)$.

Problem 70. Let $\alpha > 0$. Show that the box dimension of the countable set

$$\left\{ \frac{1}{n^\alpha} \, : \, n \in \mathbb{N} \right\}$$

is equal to $1/(\alpha+1)$.

Problem 71. Study the fractal generated by the 3×3 matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and the Kronecker product $\otimes$. Note that

$$M \otimes M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Problem 72. Study the tree linear maps $f_j : \mathbb{R}^2 \to \mathbb{R}$

$$f_1(x_1, x_2) = \frac{1}{2}(x_1, x_2)$$
$$f_2(x_1, x_2) = \frac{1}{2}(x_1 - 1, x_2) + (1, 0)$$
$$f_3(x_1, x_2) = \frac{1}{2}(x_1 - \frac{1}{2}, x_2 - \frac{\sqrt{3}}{2}) + (\frac{1}{2}, \frac{\sqrt{3}}{2}).$$

Problem 73. Consider the unit square $[0,1]^2$ with the four vertices $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. Study the four maps

$$\mathbf{f}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{f}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},$$

$$\mathbf{f}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \qquad \mathbf{f}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Problem 74. Study the iterated functions system

$$\mathbf{f}_1(x_1, x_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{f}_2(x_1, x_2) = \frac{1}{2}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}.$$

Apply it to the vertices $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ of the unit square $[0,1]^2$.

Problem 75. Study the pair of maps

$$f_1(z) = \frac{1}{1+z}, \qquad f_2(z) = \frac{z}{1+z}.$$

Bibliography

Books

Antimirov M. Ya., Kolyshkin A. A. and Vaillancourt R.
Complex Variables, Academic Press, 1998

Arrowsmith D. K. and Place C. M.
An Introduction to Dynamical Systems, Cambridge University Press, 1990

Arrowsmith D. K. and Place C. M.
Dynamical Systems: Differential equations, maps and chaotic behaviour, Chapman and Hall, 1992

Baker G. L. and Gollub J. P.
Chaotic Dynamics: An Introduction, Cambridge University Press, 1990

Barnsley M. F.
Fractals Everywhere, Academic Press. Boston (1988)

Beardon A. F.
Iteration of Rational Functions, Springer (1991)

Beck C. and Schlögl F.
Thermodynamics of Chaotic Systems, Cambridge University Press, 1993

Berge P., Pomeau Y. and Vidal C.
Order within Chaos, Wiley, New York, 1984

Birkhoff G. D.
Collected Mathematical Papers, Vols. 1,2,3 Am. Math. Soc., Providence, 1950

Bowen R.
Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Mathematics **470**, Springer, Berlin, 1975

Carleson L. and Gamelin T. G. W.
Complex Dynamics, Springer, 1993

Cvitanovic P.
Universality in Chaos, second edition, Adam Hilger (1989)

Choe Geon Ho

Computational Ergodic Theory, Algorithms and Computation in Mathematics, Vol. 13, Springer Verlag, 2005

Collet P. and Eckmann J.-P.
Iterated Maps on the Intervals as Dynamical Systems, Birckhäuser, Boston, 1980

Devaney R. L.
An Introduction to Chaotic Dynamical Systems, second edition, Addison-Wesley, 1989

Edgar G. A.
Measure, Topology and Fractal Geometry, Springer, New York, 1990

Elaydi S. N.
An Introduction to Difference Equations, second edition, Springer, 1999

Falconer K.
Techniques in Fractal Geometry, John Wiley, Chichester, 1997

Feder J.
Fractals, Plenum Press, New York, 1988

Fröberg C. E.
Numerical Mathematics: Theory and Computer Applications, Benjamin-Cummings, 1985

Gulick D.
Encounters with Chaos and Fractals, second edition, CRC Press, 2012

Gutzwiller M. C.
Chaos in Classical and Quantum Mechanics, Springer-Verlag, 1990

Holmgren R. A.
A First Course in Discrete Dynamical Systems, second edition. Springer, New York, 1996

Katok A. and Hasselblatt B.
Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1995

Krause U. and Nesemann T.
Differenzengleichungen und diskrete dynamische Systeme, Teubner, Stuttgart-Leipzig 1999

Lang S.
Complex Analysis, second edition, Springer, New York, 1985

Lynch S.
Dynamical Systems with Applications Using Maple, second edition, Springer, new York, 2014

Mandelbrot B.
The Fractal Geometry of Nature, W. H. Freeman, 1983

Menger K.
Dimensionstheorie, B. G. Teubner, 1928

Nakamura K.
Quantum Chaos - A New Paradigm of Nonlinear Dynamics, Cambridge University Press, Cambridge, 1993

Ott E.
Chaos in Dynamical Systems, Cambridge University Press, 1993

Peitgen H.-O., Jürgens H. and Saupe D.
Fractals for the Classroom, Part II, Springer-Verlag (1992)

Peitgen H.-O. and Richter P. H.
The Beauty of Fractals, Springer-Verlag, 1986

Ruelle D.
Thermodynamic Formalism, New York, Addison-Wesley, 1978

Schuster H. G.
Deterministic Chaos, VCH, Weinheim, 1988

Silverman J. .
The Arithmetic of Dynamical Systems, Springer, 2007

Spiegel M. R. *Calculus of Finite Differences and Difference Equations*, Schaum's Outline Series, McGraw-Hill, 1971

Steeb W.-H. *A Handbook of Terms Used in Chaos and Quantum Chaos*, BI Wissenschaftsverlag, Mannheim, 1991

Steeb W.-H.
The Nonlinear Workbook, sixth edition, World Scientific Publishing, Singa-

pore, 2015

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume I: Introductory Level
World Scientific Publishing, Singapore, 2009

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume II: Adanced Level
World Scientific Publishing, Singapore, 2009

Steeb W.-H.
Problems and Solutions in Scientific Computing with C++ and Java Simulations, World Scientific Publishing, Singapore, 2004

Strogatz S.
Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering, Addison-Wesley, 1994

Walters P.
An Introduction to Ergodic Theory, Graduate Texts in Mathematics 79, Springer, New York, 1982

Papers

Adachi S., Toda M. and Ikeda K., "Recovery of Liouville dynamics in quantum mechanically suppressed chaotic behaviour", J. Phys. A: Math. Gen. **22**, 3291-3306 (1989)

Aizawa Y., "Symbolic Dynamics Approach to the Two-D Chaos in Area-Preserving Maps", Prog. Theor. Phys. **71**, 1419-1421 (1984)

Aizawa Y. and Murakami C., "Generalization of Baker's Transformation", Prog. Theor. Phys. **69**, 1416-1426 (1983)

Aizawa Y., Murakami C. and Kohyama T., "Statistical Mechanics of of Intermittent Chaos", Prog. Theor. Phys. Supp. **79**, 96-124 (1984)

Anosov D. V., "Ergodic Properties of Geodesic Flows on Closed Riemannian Manifolds of Negative Curvature", Sov. Math. Dokl. **4**, 1153-1156 (1963)

Antoniou I. and Tasaki S., "Spectral decomposition of the Renyi map", J. Phys. A: Math. Gen. **26**, 73-94 (1993)

Aubrey S., "The Devil's Staircase Transformation in Incommensurate Lattices", Lect. Notes in Math. **925**, 221-245 (1982)

Aubrey S., "The Twist Map, the extended Frenkel-Kontorova Model and the Devil's Staircase", Physica **7D**, 240-258 (1983)

Banks J. and Dragan V., "Smale's Horseshoe Map via Ternary Numbers", SIAM Review, **36**, 265-271 (1994)

Barlow M. T. and Taylor S. J., "Fractional dimension of sets in discrete spaces", J. Phys. A: Math. Gen. **22**, 2621-2626 (1989)

Beddington J. R., Free C. A. and Lawton J. H., "Dynamic Complexity in Predator-Prey Models Framed in Difference Equations", Nature **255**, 58-60 (1975)

Birkhoff G. D., "Surface Transformations and Their Dynamical Applications", Acta Math. **43**, 1-119 (1920)

Bowen R., "Topological entropy for noncompact sets", Trans. Amer. Math. Soc. **184**, 125-138 (1973)

Bullett S., "Invariant Circles for the Piecewise-Linear Standard Map", Comm. Math. Phys. **107**, 241-262 (1986)

Burton R. and Easton R. W., "Ergodicity of Linked Twist Maps", Springer Lecture Notes in Mathematics, **819** 35-49 (1980)

Chirikov B. V., "A universal instability of many-dimensional oscillator systems", Phys. Rep. **52** 265-379 (1979)

Chirikov B. V., Lieberman M. A., Shepelyansky D. L. and Vivaldi F. M., "A Theory of Modulational Diffusion", Physica D **14**, 289-304 (1984)

Collet P., Eckmann J.-P. and Koch H., "On Universality for Area-Preserving Maps of the Plane", Physica D **3**, 457-467 (1981)

Crawford J. D. and Cary J. R., "Decay of Correlations in a Chaotic Measure-Preserving Transformation", Physica D **6**, 223-232 (1983)

Daido H., "Onset of Intermittency from Torus", Prog. Theor. Phys. **71**, 402-405 (1984)

Das N. and Dutta T. K., "Hopf Bifurcations on a Nonlinear Chaotic Discrete Model", Int. J. Sci. Eng. Appl. **1**, 127-134 (2012)

Dana I. and Fishman S., "Diffusion in the Standard Map", Physica D **17**, 63-74 (1985)

Derrida B., Gervois A. and Pomeau Y., "Universal metric properties of bifurcations of endomorphisms", J. Phys. A: Math. Gen. **12** 269-296 (1979)

Devaney R., "Reversible Diffeomorphisms and Flows", Trans. Am. Math. Soc. **218**, 89-113 (1976)

Devaney R., "A Piecewise Linear Model for the Zones of Instabilty of an Area-Preserving Map", Physica D **10**, 367-393 (1984)

Devaney R. and Nitecki Z., "Shift Automorphisms in the Hénon Mapping", Comm. in Math. Phys. **67**, 137-146 (1979)

Ebeling W. and Nicolis G. "Word frequency and entropy of symbolic sequences: a dynamical perspective", Chaos, Solitons and Fractals, **2**, 635-650 (1992)

Eckmann J.-P., Koch H. and Wittwer P., "A Computer-Assisted Proof of Universality for Area-Preserving Maps", Mem. Am. Math. Soc. **47:289**, 1-122 (1984)

Falconer K. J., "A subadditive thermodynamic formalism for mixing repellers", J. Phys. A: Math. Gen. **21**, L737-L742 (1988)

Feigenbaum M. J., "Quantitative Universality for a Class of Non-Linear Transformations", J. Stat. Phys. **19** 25-52 (1978)

Feigenbaum M. J., "The universal metric properties of nonlinear transformations", J. Stat. Phys. **21** 669-706 (1979)

Fermi E., Pasta J. and Ulam S., "Studies of Nonlinear Problems. I", Lect. App. Math. **15**, 143-156 (1974)

Feudel U., Pikovsky A. and Politi A., "Renormalization of correlations and spectra of a strange non-chaotic attractor", J. Phys. A: Math. Gen. **29**, 5297-5311 (1996)

Fournier-Prunaret D. and Lopez-Ruiz R., "Basin bifurcations in a two-dimensional logistic map", arXiv:nlin/0304059v1

Froeschle C., "Numerical Study of a Four Dimensional Mapping, Astron. and Astrophs. **16**, 172-189 (1972)

Fujisaka H., Kamifukumoto H. and Inoue M., "Intermittency Associated with the Breakdown of the Chaos Symmetry", Prog. Theor. Phys. **69**, 333-337 (1983)

Fujisaka H., "Theory of Diffusion and Intermittency in Chaotic Systems", Prog. Theor. Phys. **71**, 513-523 (1984)

Fujiksaka H., Ishii H., Inoue M. and Yamad T., "Intermittency Caused by Chaotic Modulation. II", Prog. Theor. Phys. **76**, 1198-1209 (1986)

Gefen Y., Aharony A., Shapir Y. and Mandelbrot B. B., "Phase transitions on fractals: II. Sierpinski gaskets", J. Phys. A: Math. Gen. **17**, 435-444 (1984)

Goroff D. L., "Hyperbolic Sets for Twist Maps", Ergod. Th. Dyn. Sys. **5**, 337-339 (1985)

Grassberger P. and Procaccia I., "Measuring the Strangeness of Strange

Attractors", Physica D **9**, 189-208 (1983)

Greene J. M., "Two-Dimensional Area Preserving Mappings", J. Math. Phys. **9**, 760-768 (1968)

Greene J. M., "A method for determining a stochastic transition", J. Math. Phys. **20**, 1183-1201 (1979)

Greene J. M., MacKay R. S., Vivaldi F. and Feigenbaum M. J., "Universal Behaviour in Families of Area-Preserving Maps", Physica **3D**, 468-486 (1981)

Greene J. M. and Percival I. C., "Hamiltonian Maps in the Complex Plane", Physica D **3**, 530-548 (1981)

Grossmann S. and Thomae S., "Invariant distributions and stationary correlation functions of one-dimensional discrete processes", Z. Naturf. **32a**, 1353-1363 (1977)

Gunaratne G. H., Jensen M. H. and Procaccia I. "Universal strange attractors on wrinkled tori", Nonlinearity **1**, 157-180 (1988)

Hammel S., Yorke J. A. and Grebogi C., "Do Numerical Orbits of Chaotic Dynamical Processes Represent True Orbits?" J. Complexity **3**, 136-145 (1987)

Harayama T. and Aizawa Y., "Poincaré-Birkhoff Chains in the Critical Regime between Chaos and Torus", Prog. Theor. Phys. **84**, 23-27 (1990)

Hardy Y. and Sabatta D., "Encoding, symbolic dynamics, cryptography and C++ implementations", Phys. Lett. A **366**, 575-584 (2007)

Hemmer P. C., "The exact invariant density for a cusp-shaped return map", J. Phys. A: Math. Gen. **17**, L247-L249 (1984)

Horita T., Hata H., Mori H. and Tomita K., "Dynamics on Critical Tori at the Onset of Chaos and Critical KAM Tori", Prog. Theor. Phys. **81**, 1073-1078 (1989)

Hutchinson J. E., "Fractals and Self Similarity", Indiana University Mathematics Journal **30**, 713-747 (1981)

Kaizoji T., "Non-Stationary Chaos", arXiv:1007.3638v1

Kaneko K., "Transition from Torus to Chaos Accompanied by Frequency Lockings with Symmetry Breaking", Prog. Theor. Phys. **69**, 1427-1442 (1983)

Kaneko K., "Fates of Three-Torus. I", Prog. Theor. Phys. **71**, 282-294 (1984)

Kaneko K., "Spatiotemporal Chaos on One- and Two-Dimensional Coupled Map Lattices", Physica D **37**, 60-82 (1989)

Kaneko K. "Fractalization of Torus", Prog. Theor. Phys. **71**, 1112-1114 (1984)

Kaneko K. "Partition complexity in a network of chaotic elements", J. Phys. A: Math. Gen. **24**, 2107-2119 (1991)

Kaneko K. and Bagley R. J., "Arnold Diffusion, Ergodicity and Intermittency in a Coupled Standard Mapping", Phys. Lett. A **110**, 435-440 (1985)

Karney C. F. F. "Long-Time Corrleations in the Stochastic Regime", Physica 8D, 360-380 (1983)

Katok A., "Periodic and Quasi-Periodic Orbits for Twist Maps", Springer Lecture Notes in Physics", **179**, 47-65 (1983)

Katok A., "Bernoulli Diffeomorphisms on Surface", Ann. Math. **110**, 529-547 (1979)

Kiyoshi Sogo, "Inverse problem in chaotic map theory", Chaos, Solitons and Fractals, **41**, 1817-1822 (2009)

Konishi T. and Kaneko K., "Diffusion in Hamiltonian chaos and its size dependence", J. Phys. A: Math. Gen. **23**, L715-L720 (1990)

Li W. and Bak P., "Fractal Dimension of Cantori", Phys. Rev. Lett. **57**, 655-658 (1986)

Lindsey K. A., "Shapes of Polynomial Julia Sets", arXiv:1209.0143v2

Lopez-Ruiz R. and Fournier-Prunaret, "Complex pattern on the plane: Different types of basin fractalization in a 2-D mapping", Int. J. Bifurcation Chaos **13**, 287 (2003)

Loskutov A. Yu., Rybalko S. D., Feudel U. and Kurths J., "Suprression of chaos by cyclic parametric excitation in two-dimensional maps", J. Phys. A: Math. Gen. **29**, 5759-5771 (1996)

Maeda S., "The Similarity Method for Difference Equations", IMA Journal of Applied Mathematics, **38**, 129-134 (1987)

MacKay R. S., "A Renormalization Approach to Invariant Circles in Area-Preserving Maps", Physica D **7**, 283-300 (1983)

Mather J. N., "Non-existence of invariant circles", Ergod. Th. and Dyn. Sys. **4**, 301-309 (1984)

May R. M., "Simple mathematical models with very complicated dynamics", Nature **261**, 459-467 (1976)

McMullen C., "The Hausdorff Dimension of General Sierpinski Carpets", Nagoya Math. J. **96**, 1-9 (1984)

Meiss J. D. and Ott E., "Markov-Tree Model Transport in Area Preserving Maps", Physica D **20**, 387-402 (1986)

Mori H., So B.-C. and Ose T., "Time-Correlation Functions of One-Dimensional Transformations", Prog. Theor. Phys. **66**, 1266-1283 (1981)

Mori H., Okamoto H. and Ogasawara M., "Self-Similar Cascades of Band Splittings of Linear Mod 1 Maps", Prog. Theor. Phys. **71**, 499-512 (1984)

Morse M. and Hedlund G. A., "Symbolic dynamics I", Am. J. Math. **60**, 815-866 (1938)

Moser J. K., "The Analytic Invariants of an Area Preserving near a Hyperbolic Fixed Point", Comm. Pure and Appl. Math. **9**, 673-692 (1956)

Moser J. K., "On the Integrability of Area-preserving Cremona Mappins near an elliptic fixed point", Bol. Soc. Mat. Mexicana, 176-180 (1961)

Nakamura K. and Hamada M., "Asymptotic expansion of homoclinic structures in a symplectic mapping", J. Phys. A: Math. Gen. **29**, 7315-7327 (1996)

Nguyen H. D., "A Digital Binomial Theorem", arXiv:1412.3181v1

Oseledec V. I., "A Multiplicative Ergodic Theorem: Lyapunov Characteristic Numbers for Dynamical Systems", Trans. Moscow Math. Soc. **19**, 197-231 (1968)

Ott E., Withers W. D. and Yorke J. A., "Is the Dimension of Chaotic Attractors Invariant under Coordinate Changes?", Journal of Statistical Physics, **36**, 687-697 (1984)

Percival I. C., "Variational Principles for Invariant Tori and Cantori", AIP Conference Proceeding, **57**, 302-310 (1979)

Percival I. C., "Chaotic Boundary of a Hamiltonian Map", Physica D **6**, 67-77 (1982)

Pesin Ja. B., "Ljapunov Characteristic Exponents and Ergodic Properties of Smooth Dynamical Systems with an Invariant Measure", Soviet Mathematics-Doklady **17**, 196-199 (1976)

Ping P., Xu F. and Wang Z.-J., "Color Image encryption based on Two-Dimensional Cellular Automata", **24**, 1350071-1-14 (2013)

Pomeau Y. and Manneville P., "Intermittent transition to turbulence in dissipative dynamical systems", Commun. Math. Phys. **74**, 189-197 (1980)

Radons G. and Stoop R., "Superpositions of Multifractals: Generators of Phase Transitions in the Generalized Thermodynamic Formalism" J. Stat. Phys. **82**, 1063-1080 (1996)

Rimmer R., "Symmetry and Bifurcation of Fixed Points of Area-Preserving Maps", J. Diff. Eqns. **29**, 329-344 (1978)

Schäfer A. and Müller B., "Bounds for the fractal dimension of space", J. Phys. A: Math. Gen. **19**, 3891-3902 (1986)

Schmidt G., "Stochasticity and Fixed Point Transitions", Phys. Rev. A **22**, 2849-2854 (1980)

Schmidt G. and Bialek J., "Fractal Diagrams for Hamiltonian Stochasticity", Physica **5D**, 397-404 (1982)

Schröder E., "Ueber iterirte Functionen", Math. Ann. **3**, 296-322 (1870)

Schwegler H. and Mackey M. C., "A simple model for the approach of entropy to thermodynamic equilibrium", J. Phys. A: Math. Gen. **27**,1939-1951 (1994)

Sinai Ya. G., "The Construction of Markov Partitions", Funct. Ana. Appl. **2**, 70-80 (1968)

Sinai Ya. G., "Gibbs Measures in Ergodic Theory", Russ. Math. Surv. **27**, 21-69 (1972)

Steeb W.-H., Solms F., Tan Kiat Shi and Stoop R., "Cubic Map, Complexity and Ljapunov Exponent", Physica Scripta **55**, 520-522 (1997)

Takahashi Y. and Oono Y., "Towards the Statistical Mechanics of Chaos", Prog. Theor. Phys. **71**, 851-853 (1984)

Theiler J., Mayer-Kress G. and Kadtke J., "Chaotic attractors of a locally conservative hyperbolic map with overlap", Physica D **48**, 425-444 (1991)

Tsuda I., "On the Abnormality of Period Doubling Bifurcations", Prog. Theor. Phys. **68**, 1985-1994 (1981)

Umberger D. K. and Farmer J. D., "Fat Fractals on the Energy Surface", Phys. Rev. Lett. **55**, 661-664 (1985)

Widom M. and Kadanoff L. P., "Renormalization Group Analysis of Bifurcations in Area Preserving Maps", Physica D **5**, 287-292 (1982)

Wojtkowski M., "A Model Problem with the Coexistence of Stochastic and Integrable Behaviour", Comm. in Math. Phys. **80**, 453-464 (1981)

van Wyk M. and Steeb W.-H., "Stochastic Analysis of the ZigZag Map", Int. J. Mod. Phys. C **14**, 397-403 (2003)

Yalcinkaya T. and Lai Y.-C., "Chaotic Scattering", Computers in Physics, **9**, 511-539 (1995)

Yamaguchi Y., "Renormalization Group Approach to Universal Metric Properties of Whisker Mapping", Prog. Theor. Phys. **72**, 694-709 (1984)

Yoshida T. and Tomita K. "Characteristic Structures of Power Spectra in Periodic Chaos", Prog. Theor. Phys. **76**, 752-767 (1986)

Young L. S., "Dimension, entropy and Lyapunov exponents, Ergodic Theory Dynamical Systems **2**, 109-124 (1982)

Zisook A. B. and Shenker S. J., "Renormalization Group for Intermittency in Area-Preserving Mappings", Phys. Rev. A **25**, 2824-2826 (1982)

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