Problems and Solutions for Bose, Spin and Fermi Systems

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Preface

The purpose of this book is to supply a collection of problems and solutions for Bose, spin and Fermi systems as well as coupled systems. So it covers essential parts of quantum theory and quantum field theory. For most of the problems the detailed solutions are provided which will prove to be valuable to graduate students as well as to research workers in these fields. Each chapter contains supplementary problems often with the solution provided. All the important concepts are provided either in the introduction or the problem and all relevant definitions are given. The topics range in difficulty from elementary to advanced. Almost all problems are solved in detail and most of the problems are self-contained. Students can learn important principles and strategies required for problem solving. Teachers will also find this text useful as a supplement, since important concepts and techniques are developed in the problems. The book can also be used as a text or a supplement for quantum theory, Hilbert space theory and linear and multilinear algebra or matrix theory. Computer algebra programs in SymbolicC++ and Maxima are also included. For Bose systems number states, coherent states and squeezed states are covered. Applications to nonlinear dynamical systems and linear optics are given. The spin chapter concentrates mostly on spin-$\frac{1}{2}$ and spin-1 systems, but also higher order spin's are included. The eigenvalue problem plays a central role. Exercises utilizing the spectral theorem and Cayley-Hamilton theorem are provided. For Fermi systems a special section on the Hubbard Hamilton operator is added. Chapter 4 is devoted to Lie algebras and their representation by Bose, Spin and Fermi operators. Superalgebras are also considered. Chapters 5 and 6 cover coupled Bose-Spin and coupled Bose-Fermi systems, respectively.

The material was tested in our lectures given around the world.

Any useful suggestions and comments are welcome.

The International School for Scientific Computing (ISSC) provides certificate courses for this subject. Please contact the first author if you want to do this course. More exercises can be found on the web page given below.

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Contents

Notation

$\emptyset$	empty set
$A \subset B$	subset A of set B
$A \cap B$	the intersection of the sets A and B
$A \cup B$	the union of the sets A and B
f, g	maps
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	natural numbers including 0
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	nonnegative real numbers
$\mathbb{C}$	complex numbers
$\mathbb{R}^n$	n-dimensional Euclidian space
$\mathbb{C}^n$	n-dimensional complex linear space
i	$:= \sqrt{-1}$
z	complex number
$\bar{z}$, z^*	complex conjugate of z
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\mathbf{v} \in \mathbb{C}^n$	element $\mathbf{v}$ of $\mathbb{C}^n$ (column vector)
$\mathbf{v}^*$	transpose and complex conjugate of $\mathbf{v}$
$\mathbf{v}^*\mathbf{v}$	scalar product
$\mathbf{0}$	zero vector (column vector)
t	time variable
ω	frequency
$\mathbf{x}$	space variable
$\mathbf{v}^T = (v_1, v_2, \ldots, v_n)$	vector of independent variables, ^{T} means transpose
$\|\,.\,\|$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^T\mathbf{y}$	scalar product (inner product) in vector space $\mathbb{R}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in vector space $\mathbb{R}^3$
det	determinant of a square matrix
tr	trace of a square matrix
0_n	$n \times n$ zero matrix
I_n	$n \times n$ unit matrix (identity matrix)
A^T	transpose of matrix A
A^*	transpose and complex conjugate of matrix A
I	identity operator
$[\,,\,]$	commutator
$[\,,\,]_+$	anticommutator
$\oplus$	direct sum
$\oplus$	XOR operation
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
$\epsilon_{jk\ell}$	total antisymmetric tensor $\epsilon_{123} = 1$

$\mathrm{sgn}(x)$	the sign of x, 1 if $x > 0$, -1 if $x < 0$, 0 if $x = 0$
λ	eigenvalue
ϵ	real parameter
$\otimes$	Kronecker product, Tensor product
$\wedge$	Grassmann product (exterior product, wedge product)
S_1, S_2, S_3	spin matrices for spin 1/2, 1, 3/2, 2, ...
σ_1, σ_2, σ_3	Pauli spin matrices
U	unitary operator, unitary matrix
Π	projection operator, projection matrix
ρ	density operator, density matrix
P	permutation matrix
$\mathbf{P}$	momentum operator
$\mathbf{Q}$	position operator
$\mathbf{k}$	wave vector $\mathbf{k} \cdot \mathbf{x} = k_1 x_1 + k_2 x_2 + k_3 x_3$
$b^\dagger$, b	Bose creation and annihilation operators
$\lvert n\rangle$	number states ($n = 0, 1, \ldots$)
$\lvert\beta\rangle$	coherent states ($\beta \in \mathbb{C}$)
$\lvert\zeta\rangle$	squeezed states ($\zeta \in \mathbb{C}$)
$c^\dagger$, c	Fermi creation and annihilation operators
ρ	density operator, density matrix $\rho \geq 0$, $\mathrm{tr}(\rho) = 1$
$\hat{N}$	number operator
L	Lagrange function
H	Hamilton function
$\mathcal{L}$	Lagrange density
$\hat{H}$	Hamilton operator
$\mathcal{H}$	Hilbert space
$L_2(\Omega)$	Hilbert space of square integrable functions
$\ell_2(S)$	Hilbert space of square-summable (infinite) sequences
$\langle\, ,\, \rangle$	scalar product in Hilbert space
δ	delta function
$\hbar$	Planck constant divided by 2π

The Pauli spin matrices σ_1, σ_2, σ_3 are used extensively in the book. They are given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They are hermitian and unitary matrices with eigenvalues $+1$ and -1. In some cases we also use σ_x, σ_y and σ_z to denote σ_1, σ_2 and σ_3. The matrices σ_+ and σ_- are defined by

$$\sigma_+ := \sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- := \sigma_1 - i\sigma_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

The spin matrices for spin-$\frac{1}{2}$ are defined as

$$S_1 = \frac{1}{2}\sigma_1, \quad S_2 = \frac{1}{2}\sigma_2, \quad S_3 = \frac{1}{2}\sigma_3$$

and

$$S_+ = S_1 + iS_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = S_1 - iS_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The spin-1 matrices are defined as

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with the eigenvalues $+1$, 0, -1. The matrices are hermitian. Then

$$S_+ = S_1 + iS_2 = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_- = S_1 - iS_2 = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The Kronecker product is extensively used in the book. Let $A := (a_{ij})_{ij}$ be an $m \times n$ matrix and B be an $r \times s$ matrix over $\mathbb{C}$. The *Kronecker product* of A and B is defined as the $(m \cdot r) \times (n \cdot s)$ matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}.$$

The Kronecker product is associative.

The spectral theorem for normal matrices will be utilized in the book:

Let $M_n(\mathbb{C})$ be the vector space of $n \times n$ matrices and $A \in M_n(\mathbb{C})$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A counted according to multiplicity. The following statements are equivalent
(i) A is normal
(ii) A is unitarily diagonalizable
(iii) $\sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk}|^2 = \sum_{k=1}^{n} |\lambda_k|^2$.
(iv) There exists an orthonormal set of n eigenvectors of A.

The Hermite and Laguerre polynomials are used in the book:

The functions

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \dots$$

are called Hermite polynomial. The first four Hermite polynomial are $H_0(x) = 1$, $H_0(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$.

The functions

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x})$$

are called Laguerre polynomials. The first four Laguerre polynomials are $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = x^2 - 4x + 2$, $L_3(x) = -x^3 + 9x^2 - 18x + 6$.

Chapter 1

Bose Systems

1.1 Commutators and Number States

1.2 Coherent States

1.3 Squeezed States

1.3.1 One-Mode Squeezed States

1.3.2 Two-Mode Squeezed States

Problem 1. Consider a parametric process in which photons from a pump laser decay into photons, one in mode 1 and one in mode 2 with energy and momentum preserving. So consider the operator

$$\hat{O}(s) := e^{s(b_1^\dagger b_2^\dagger - b_1 b_2)}$$

with the squeezing parameter s and the vacuum state $|00\rangle$. Find the state

$$|\psi\rangle = \hat{O}(s)|00\rangle$$

and the mean photon number.

Solution 1. We obtain

$$|\psi\rangle = \hat{O}(s)|00\rangle = (1 - tanh^2(s))^{1/2} e^{\tanh(s) b_1^\dagger b_2^\dagger}|00\rangle.$$

The mean photon number in both modes is

$$N = 2\mathrm{tr}(b_1^\dagger b_1 |\psi\rangle\langle\psi|) = \sinh^2(s).$$

1.4 Coherent Squeezed States

1.5 Hamilton Operators

1.6 Linear Optics

1.7 Classical Dynamical Systems

1.8 Supplementary Problems

Problem 1. Let $b^\dagger$, b be Bose creation and annihilation operators, $\hat{N} = b^\dagger b$ be the number operator, I be the identity operator and $z \in \mathbb{C}$.
(i) Show that

$$[b, \hat{N}^2] = (I + 2b^\dagger b)b, \qquad [b^\dagger, \hat{N}^2] = -b^\dagger(I + 2b^\dagger b).$$

(ii) Let $k \geq 1$. Show by induction that

$$b^\dagger(b^\dagger b)^k = (b^\dagger b - I)^k b^\dagger.$$

For $k = 1$ we have

$$b^\dagger b^\dagger b = b^\dagger(bb^\dagger - I) = b^\dagger bb^\dagger - b^\dagger = (b^\dagger b - I)b^\dagger.$$

(iii) Let $m \geq 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Show that

$$(b^\dagger)^m f(\hat{N}) = f(\hat{N} - mI)(b^\dagger)^m, \quad b^m f(\hat{N}) = f(\hat{N} + mI)b^m.$$

(iv) Show that

$$[b, (bb^\dagger)^k] = \sum_{\ell=1}^{k} \binom{k}{\ell} (bb^\dagger)^{k-\ell} b, \quad [b^\dagger, (bb^\dagger)^k] = \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^\ell (bb^\dagger)^{k-\ell} b^\dagger.$$

(v) Let k be a positive integer. Show that $[\hat{N}, b^k] = -kb^k$.

Problem 2. (i) Show that

$$\exp(\beta(b^\dagger + b^k)) = \exp(\beta b^\dagger) \exp\left(\sum_{j=0}^{k} \frac{1}{j+1} \binom{k}{j} \beta^{j+1} b^{k-j} \right).$$

(ii) Find the operator

$$\sinh(zb^\dagger b) - \sinh(zbb^\dagger).$$

(iii) Let $\epsilon \in \mathbb{R}$. Find the operator

$$f(\epsilon) = \cos(-\epsilon b^\dagger b) b \cos(\epsilon b^\dagger b)$$

using *parameter differentiation*. Note that $f(0) = b$.
(iv) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Show that

$$\exp(zb^\dagger b)(f(b, b^\dagger)) \exp(-zb^\dagger b) = f(be^{-z}, b^\dagger e^z).$$

Problem 3. Let $\mathbb{N}_0$ be the natural numbers including 0. Consider the separable Hilbert space $\ell_2(\mathbb{N}_0)$. Then the number states $|n\rangle$ $(n = 0, 1, 2, \ldots)$ form a basis in this Hilbert space. Consider the two sub Hilbert spaces

$$\mathcal{H}_0 := \{\, |2n\rangle \, : \, n = 0, 1, 2, \ldots \}$$
$$\mathcal{H}_1 := \{\, |2n+1\rangle \, : \, n = 0, 1, 2, \ldots \}.$$

The *projection operators* onto these Hilbert spaces are given by

$$\hat{\Pi}_0 = \sum_{n=0}^{\infty} |2n\rangle\langle 2n|, \qquad \hat{\Pi}_1 = \sum_{n=0}^{\infty} |2n+1\rangle\langle 2n+1|.$$

One has $\hat{\Pi}_0 + \hat{\Pi}_1 = I$ and $\hat{\Pi}_j \hat{\Pi}_k = \delta_{jk} \hat{\Pi}_j$, where I is the identity matrix.
(i) Show that the *parity operator* is given by

$$\hat{P} = \hat{\Pi}_0 - \hat{\Pi}_1 = \exp(i\pi b^\dagger b).$$

(ii) Show that $\hat{P} = I$, $\hat{P}b = -b\hat{P}$, $Pb^\dagger = -b^\dagger P$.
(iii) Let $|n\rangle$ be the number states ($n = 0, 1, \ldots$). Let $k = 0, 1, \ldots$. Define the operators

$$T_k := \sum_{n=0}^{\infty} |n\rangle\langle 2n + k|.$$

(iv) Show that $T_k T_{k'}^\dagger = \delta_{kk'} I$.
(v) Show that $T_k^\dagger T_k = P_k$ is a projection operator.
(vi) Show that $\sum_{k=0}^{\infty} P_k = I$.
(vii) Is the operator

$$\sum_{k=0}^{\infty} T_k \otimes T_k^\dagger$$

unitary?

Problem 4. Let $|n\rangle$ be the number states ($n = 0, 1, \ldots$). Let $z \in \mathbb{C}$ and $z = x + iy$. Consider the state

$$|z\rangle = M(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_n}} |n\rangle$$

where ($u \geq 0$)

$$M^2(u) = \frac{\sqrt{u}}{\sinh(\sqrt{u})}$$

and

$$\rho_n = \int_0^\infty u^n \rho(u) du, \qquad \rho(u) = \frac{1}{2} \exp(-\sqrt{u}).$$

Thus $\rho_n = (2n + 1)!$. Let $g(u)$ be defined by $\rho(u) = M^2(u) g(u)$. Show that

$$\langle z|z\rangle = 1, \qquad I = \frac{1}{\pi} \int_{\mathbb{C}} |z\rangle\langle z| g(|z|^2) dx dy.$$

Problem 5. Let $\hat{Q}$ and $\hat{P}$ be the self-adjoint operators acting in an appropriate subspace of the Hilbert space $L_2(\mathbb{R})$ with the canonical commutation relation

$$[\hat{P}, \hat{Q}] = -i\hbar I$$

where I is the identity operator.
(i) Show that

$$\exp(i(p'\hat{Q} - q'\hat{P})/\hbar) \exp((i(p\hat{Q} - q\hat{P})/\hbar) = \exp(i(p'q - pq')/(2\hbar)) \exp(i((p+p')\hat{Q} - (q+q')\hat{P})/\hbar).$$

(ii) Let n be a positive integer. Show that

$$[\hat{P}^n, \hat{Q}] = -i\hbar n \hat{P}^{n-1}, \qquad [\hat{P}, \hat{Q}^{n-1}] = -i\hbar n \hat{Q}^{n-1}.$$

(iii) Let n, m be positive integers. Show that

$$[\hat{P}^m, \hat{Q}^n] = \sum_{j=1}^{\min(m,n)} \binom{m}{j}\binom{n}{j}(-i\hbar)^j j! \hat{Q}^{n-j}\hat{P}^{m-j}.$$

(iv) Consider the Hamilton operator for the one-dimensional harmonic oscillator

$$\hat{H}(\hat{P}, \hat{Q}) = \frac{1}{2m}(\hat{P}^2 + m^2\omega^2\hat{Q}^2)$$

and

$$\hat{T} = \exp(-m\omega\hat{Q}^2/(2\hbar)).$$

Show that

$$\widetilde{H} = \hat{T}^{-1}\hat{H}\hat{T} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}i\omega(\hat{Q}\hat{P} + \hat{P}\hat{Q}).$$

Problem 6. Consider the *parity operator*

$$\hat{P} = \exp(i\pi b^\dagger b) \equiv \exp(i\pi\hat{N}).$$

(i) Find the spectrum of $\hat{P}$.
(ii) Find the commutators $[b^\dagger, \hat{P}]$, $[b^\dagger, \hat{P}]$, $[b^\dagger + b, \hat{P}]$.
(iii) Find the spectrum of the operator $b^\dagger + b$. Set

$$b = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}), \quad b^\dagger = \frac{1}{\sqrt{2}}(x - \frac{d}{dx}).$$

Problem 7. In the Hilbert space $\mathcal{H} = \ell_2(\mathbb{N}_0)$ Bose annihilation and creation operators denoted by b and $b^\dagger$ are defined as follows: They have a common domain

$$\mathcal{D}(b) = \mathcal{D}(b^\dagger) = \left\{ \xi = (x_0, x_1, x_2, \ldots)^T \, : \, \sum_{j=0}^{\infty} j|x_j|^2 < \infty \right\}.$$

Then $b\xi$ is given by

$$b(x_0, x_1, x_2, \ldots)^T = (x_1, \sqrt{2}x_2, \sqrt{3}x_3, \ldots)^T$$

and $b^\dagger\xi$ is given by

$$b^\dagger(x_0, x_1, x_2, \ldots) = (0, x_0, \sqrt{2}x_1, \sqrt{3}x_2, \ldots).$$

The infinite dimensional vectors

$$u_n = (0, 0, \ldots, 0, 1, 0, \ldots)^T$$

where the 1 is at the n position ($n = 0, 1, 2, \ldots$) form the standard basis in $\mathcal{H} = \ell_2(\mathbb{N}_0)$. Is

$$\xi = (1, 1/2, 1/3, \ldots, 1/n, \ldots)$$

an element of $\mathcal{D}(a)$? We have to check that

$$\sum_{j=0}^{\infty} j|x_j|^2 = \sum_{j=1}^{\infty} j|x_j|^2 = \sum_{j=1}^{\infty} j\left(\frac{1}{j+1}\right)^2 < \infty.$$

Problem 8. (i) Study the Hamilton operator

$$\hat{H}(t) = \hbar\omega_1 b^\dagger b + \hbar\omega_2(e^{i\omega t}b^\dagger + e^{-i\omega t}b).$$

(ii) Show that the Hamilton operator

$$\hat{H} = \hbar\omega_1 b^\dagger b + \hbar\omega_2((b^\dagger)^2 e^{-2i\omega t} + b^2 e^{2i\omega t})$$

generates squeezed states.
(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Let $|n\rangle$, $\beta\rangle$, $|\zeta\rangle$ be number states, coherent states and squeezed states, respectively and

$$\hat{H} = \hbar\omega_1 b^\dagger b + \hbar\omega_2(b + b^\dagger) + \hbar\omega_3 f(b^\dagger b).$$

Find $\langle n|\hat{H}|n\rangle$, $\langle\beta|\hat{H}|\beta\rangle$, $\langle\zeta|\hat{H}|\zeta\rangle$.
(iv) Consider the Hamilton operators

$$\hat{H}_0 = \hbar\omega b^\dagger b + \epsilon(b + b^\dagger), \quad \hat{H}_1 = \frac{\hbar\Omega}{2}\cos(\pi b^\dagger b).$$

Let $D(\beta)$ be the displacement operator. Find $D(\beta)\hat{H}_0 D^\dagger(\beta)$ and set $\beta = \epsilon/(\hbar\omega)$. Discuss. Show that $\hat{H}_1 D^\dagger(\beta) = D(\beta)\hat{H}_1$. Hint. Apply $b\cos(\pi b^\dagger b) = -\cos(\pi b^\dagger b)b$.

Problem 9. Let $|0\rangle$ be the vacuum states and $t \in \mathbb{R}$. Find

$$[e^{tb^2/2}, e^{t(b^\dagger)^2}]|0\rangle$$

utilizing the formula

$$[f(b), g(b^\dagger)] = \sum_{j=1}^{\infty} \frac{1}{j!}\frac{\partial^j}{\partial b^{\dagger j}} g(b^\dagger) \frac{\partial^j}{\partial b^j} f(b)$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are analytic functions and $b|0\rangle = 0|0\rangle$.

Problem 10. The squeezed coherent density operator ρ is defined as

$$\rho = S(\zeta)D(\beta)|0\rangle\langle 0|D^\dagger(\beta)S^\dagger(\zeta)$$

where

$$D(\eta) := \exp(\beta b^\dagger - \beta^* b), \qquad S(\zeta) := \exp(\frac{1}{2}(\zeta^* b^2 - \zeta(b^\dagger)^2)).$$

Calculate the expectation value $\langle 0|\rho|0\rangle$.

Problem 11. Let $\beta \in \mathbb{C}$ and $\zeta \in \mathbb{C}$. Consider the displacement operator $D(\beta)$ and squeeze operator $S(\zeta)$, respectively

$$D(\beta) := \exp(\beta b^\dagger - \bar{\beta} b), \qquad S(\zeta) := \exp(\frac{1}{2}\bar{\zeta} b^2 - \frac{1}{2}\zeta(b^\dagger)^2).$$

(i) Find the commutator

$$[\beta b^\dagger - \beta^* b, \zeta^* b^2 - \zeta(b^\dagger)^2].$$

(ii) Find the commutator $[D(\beta), S(\zeta)]$.
(iii) Can one find $\beta, \zeta \in \mathbb{C}$ such that the commutator $[D(\beta), S(\zeta)]$ is a unitary operator?

(iv) Show that

$$\langle\beta|b|\zeta\rangle = -\bar{\beta}e^{i\theta}\tanh(s)\langle\beta|\zeta\rangle.$$

Problem 12. (i) Coherent states $|\beta\rangle$ can be expressed using number states $|n\rangle$, i.e.

$$|\beta\rangle = e^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle.$$

Express number states $|n\rangle$ using coherent states $|\beta\rangle$.
(ii) Show that the projection on a coherent state $|\beta\rangle$ is given by

$$|\beta\rangle\langle\beta| =: e^{-(b-\beta I)^\dagger(b-\beta I)} :$$

where : : means normal ordering.
(ii) Let $|n\rangle$, $|m\rangle$ ($n, m = 0, 1, 2, \ldots$) be number states. Consider the squeezed displaced number states

$$|\beta, \zeta, n\rangle := D(\beta)S(\zeta)|n\rangle.$$

Find $\langle m|D(\beta)S(\zeta)|n\rangle$.

Problem 13. Let $\hat{N} := b^\dagger b$. The *displaced parity operator* is defined by

$$\Pi(\beta) := D(\beta)(-1)^{\hat{N}}D^\dagger(\beta)$$

where

$$(-1)^{\hat{N}} \equiv e^{i\pi\hat{N}}.$$

Consider the number states $|0\rangle$ and $|1\rangle$. Show that

$$D^\dagger(\beta)|0\rangle = e^{-(1/2)|\beta|^2}\sum_{m=0}^{\infty}(-\beta)^m\frac{1}{\sqrt{m!}}|m\rangle$$

and

$$D^\dagger(\beta)|1\rangle = e^{-(1/2)|\beta|^2}\left(\sum_{m=0}^{\infty}(-\beta)^m\beta^*\frac{1}{\sqrt{m!}}|m\rangle + \sum_{m=0}^{\infty}(-\beta)^m\frac{\sqrt{m+1}}{\sqrt{m!}}|m+1\rangle\right).$$

Then calculate $\Pi(\beta)|0\rangle$ and $\Pi(\beta)|1\rangle$.

Problem 14. Let $b_1^\dagger$, $b_2^\dagger$, b_1, b_2 be Bose creation and annihilation operators.
(i) Let

$$T = \frac{1}{2i}(b_1^\dagger b_2 - b_2^\dagger b_1)$$

and $\alpha \in \mathbb{R}$. Show that

$$e^{i\alpha T}\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix}e^{-i\alpha T} = \begin{pmatrix} e^{i\alpha T}b_1^\dagger e^{-i\alpha T} \\ e^{i\alpha T}b_2^\dagger e^{-i\alpha T} \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix}.$$

(ii) Let $\theta \in \mathbb{R}$. Show that

$$e^{-i\theta(b_1^\dagger b_2 + b_2^\dagger b_1)}b_1 e^{i\theta(b_1^\dagger b_2 + b_2^\dagger b_1)} = \cos(\theta)b_1 + i\sin(\theta)b_2$$
$$e^{-i\theta(b_1^\dagger b_2 + b_2^\dagger b_1)}b_2 e^{i\theta(b_1^\dagger b_2 + b_2^\dagger b_1)} = i\sin(\theta)b_1 + \cos(\theta)b_2.$$

Problem 15. Bose operators obey the commutation relations

$$[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0, \quad [b_j, k_k^\dagger] = \delta_{jk} I.$$

We have

$$[b_j, f(b_k^\dagger] = \delta_{jk} \frac{\partial}{\partial b_j^\dagger} f(b_j^\dagger)$$

where f is an analytic function.
(i) Do the operators

$$b_1^\dagger b_1, \quad b_2^\dagger b_2, \quad b_1^\dagger b_2, \quad b_2^\dagger b_1$$

commute with the operator

$$b_1^\dagger b_2 b_2^\dagger b_1 + b_2^\dagger b_1 b_1^\dagger b_2 \, ?$$

(ii) Show that

$$[b_j, \exp(b_j^\dagger B)] = B \exp(b_j^\dagger B)$$

where the operator B does not depend on $b_j^\dagger$.
(iii) Show that

$$b_j f(b_k^\dagger)|0\rangle = \left(\delta_{jk} \frac{d}{db_j^\dagger} f(b_j^\dagger + f(b_k^\dagger) b_j \right) |0\rangle = \delta_{jk} \frac{d}{db_j^\dagger} f(b_j^\dagger)|0\rangle.$$

(iv) Show that

$$b_j^\dagger \exp(b_j^\dagger B)|0\rangle = B^k |0\rangle$$

where B does depend on $b_j^\dagger$.

Problem 16. Let b_j, $b_j^\dagger$ $(j = 1, 2)$ be Bose annihilation and creation operators, respectively. Let $\hat{N}_j := b_j^\dagger b_j$ and the Hamilton operator

$$\hat{H} = \frac{1}{2} \sum_{j=1}^{2} [b_j^\dagger, b_j]_+ \equiv \hat{N}_1 + \hat{N}_2 + I.$$

We define (*Schwinger representation*)

$$J_+ := b_1^\dagger b_2, \quad J_- := b_2^\dagger b_1, \quad J_3 := \frac{1}{2}(\hat{N}_1 - \hat{N}_2).$$

(i) Find the commutators $[J_+, J_-]$, $[J_+, J_3]$, $[J_-, J_3]$.
(ii) Express $\hat{H}^2$ using the operators J_-, J_+, J_3.

Problem 17. Consider the Hamilton operator

$$\hat{H} = \hbar\omega_1 (b_1^\dagger b_1^\dagger b_1 b_1 + b_2^\dagger b_2^\dagger b_2 b_2 + b_3^\dagger b_3^\dagger b_3 b_3) + \hbar\omega_2 (b_1^\dagger b_2 + b_2^\dagger b_1 + b_3^\dagger b_2 + b_2^\dagger b_3).$$

(i) Show that the Hamilton operator commutes with the number operator

$$\hat{N} = b_1^\dagger b_1 + b_2^\dagger b_2 + b_3^\dagger b_3.$$

(ii) Show that the Heisenberg equations of motion are given by

$$\begin{aligned} i\frac{db_1}{dt} &= 2\omega_1 b_1^\dagger b_1 b_1 + \omega_2 b_2 \\ i\frac{db_2}{dt} &= 2\omega_1 b_2^\dagger b_2 b_2 + \omega_2 (b_1 + b_3) \\ i\frac{db_3}{dt} &= 2\omega_1 b_3^\dagger b_3 b_3 + \omega_2 b_2. \end{aligned}$$

Problem 18. Let $n \in \mathbb{N}_0$. Consider the normalized state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n\rangle \otimes |0\rangle + e^{in\theta}|0\rangle \otimes |n\rangle)$$

where $|0\rangle$ is the vacuum state and $|n\rangle$ is the number state. Show that the minimum uncertainty achievable by a suitable quantum measurement on these states is proportional to $1/n$.

Problem 19. Let $|\beta\rangle$ be a coherent state. Consider the four *Bell-cat states*

$$\begin{aligned} |B_{00}\rangle &= \frac{1}{\sqrt{2}}(|-\beta\rangle \otimes |-\beta\rangle + |\beta\rangle \otimes |\beta\rangle) \\ |B_{10}\rangle &= \frac{1}{\sqrt{2}}(|-\beta\rangle \otimes |-\beta\rangle - |\beta\rangle \otimes |\beta\rangle) \\ |B_{01}\rangle &= \frac{1}{\sqrt{2}}(|-\beta\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |-\beta\rangle) \\ |B_{11}\rangle &= \frac{1}{\sqrt{2}}(|-\beta\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |-\beta\rangle). \end{aligned}$$

Are the states orthogonal to each other? Note that $\langle\beta|\beta\rangle = 1$.

Problem 20. Consider the system of n canonical degrees of freedom representing n harmonic oscillators. We arrange these operators in vector form (column vector)

$$R := (X_1, P_1, X_2, P_2, \ldots, X_n, P_n)^T.$$

The canonical commutation relations are

$$[R_j, R_k] = i\sigma_{jk}, \qquad j,k = 1,2,\ldots,2n$$

where the *symplectic matrix* σ is defined by

$$\sigma := \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $\oplus$ denotes the direct sum. Density operators ρ can now be characterized by functions that are defined on phase space. Given a column vector $\boldsymbol{\xi}$ the *Weyl operator* (also called *Glauber operator*) is defined by

$$W(\boldsymbol{\xi}) := \exp(i\boldsymbol{\xi}^T \boldsymbol{\sigma} R).$$

These operators generate displacements in phase space. They are used to define the *characteristic function* $\chi_\rho(\boldsymbol{\xi})$ of the density operator ρ

$$\chi_\rho(\boldsymbol{\xi}) := \mathrm{tr}(\rho W(\boldsymbol{\xi})).$$

Show that this can be inverted to express ρ as an integral of $\chi(\boldsymbol{\xi})$, i.e. show that

$$\rho = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \chi_\rho(-\boldsymbol{\xi}) W(\boldsymbol{\xi}) d^{2n}\boldsymbol{\xi}.$$

Problem 21. Consider a quantum mechanical system where the Hamilton operator $\hat{H}(t)$ depends explicitly on t. Then the *Heisenberg equation of motion* for a linear operator $\hat{O}(t)$ takes the form

$$\frac{d}{dt}\hat{O}(t) = \frac{\partial}{\partial t}\hat{O}(t) + \frac{1}{i\hbar}[\hat{O}(t), \hat{H}(t)]$$

where d/dt is the total time derivative whereas $\partial/\partial t$ differentiates only the parametric time derivative. Consider the Hamilton operator

$$\hat{H}(t) = \begin{pmatrix} \hbar\omega_1 & 0 \\ 0 & \hbar\omega_2 \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \cos(\omega t)$$

where the w_{jk} are real and $w_{12} = w_{21}$ and $d\cos(\omega t)/dt = -\omega \sin(\omega t)$.

Problem 22. (i) Show that two-mode squeezed states can be generated either by entangling two independent single-mode squeezed states via a 50:50 beamsplitter or by employing the non-degenerate operation of a nonlinear medium in the presence of two incoming modes and the unitary operator describing two-mode squeezing is

$$U_{12}(\zeta) = \exp(-i(\zeta b_1 b_2 + \zeta^* b_1^\dagger b_2^\dagger)/2)$$

where $\zeta \in \mathbb{C}$ is the squeezing parameter ($\zeta = se^{i\theta}$).
(ii) Let $s \geq 0$ be the squeezing parameter. Show that

$$|\psi\rangle = e^{s(b_1^\dagger b_2^\dagger - b_1 b_2)}|00\rangle = (1 - \tanh^2(s))^{1/2} e^{\tanh(s) b_1^\dagger b_2^\dagger}|00\rangle.$$

Show that

$$\mathrm{tr}(b_1^\dagger b_1 |\psi\rangle\langle\psi|) = \sinh^2(s).$$

Problem 23. Some nonlinear optical processes can be described by a Hamilton operator $\hat{H}$ with two degrees of freedom and cubic terms in the creation and annihilation Bose operators

$$\hat{H} = \hbar\omega b_1^\dagger b_1 + 2\hbar\omega b_2^\dagger b_2 + g\hbar\omega(b_1^2 b_2^\dagger + (b_1^\dagger)^2 b_2).$$

Examples are harmonic generation, coherent spontaneous emission and down conversion. Consider the number operator

$$\hat{N} = b_1^\dagger b_1 + b_2^\dagger b_2.$$

(i) Find the commutator $[\hat{H}, \hat{N}]$.
(ii) Find the state

$$\hat{H}(|\beta_1\rangle|\beta_2\rangle) \equiv \hat{H}(|\beta\rangle \otimes |\beta\rangle).$$

Problem 24. A *beam splitter* is an optical device that splits a beam of light in two. In general a signal mode (index 1) is mixed with a reference mode (index 2) (the two inputs) at a beam splitter and measurements are performed on the two output modes. The two output modes are entangled. The action of a beam splitter is described by a unitary operator U connecting the input and output states

$$|\Psi_{out}\rangle = U|\Psi_{in}\rangle$$

where

$$U = \exp(i(\phi_T + \phi_R)L_3)\exp(2i\theta L_2)\exp(i(\phi_T - \phi_R)L_3)$$

and the operators L_1, L_2 and L_3 are given by

$$L_1 = \frac{1}{2}(b_1^\dagger b_2 + b_2^\dagger b_1), \quad L_2 = \frac{1}{2i}(b_1^\dagger b_2 - b_2^\dagger b_1), \quad L_3 = \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2).$$

Show that the operators satisfy $[L_j, L_k] = i\epsilon_{jk\ell}L_\ell$. The Levi-Civitá tensor $\epsilon_{jk\ell}$ is equal to $+1$ and -1 for even and odd permutations of its indices, respectively, and zero otherwise. The complex transmittance T and reflectance R of the beam splitter are defined by

$$T := e^{i\phi_T}\cos(\theta), \qquad R := e^{i\phi_R}\sin(\theta).$$

The input state is a product state

$$|\Psi_{in}\rangle = |\Psi_{in1}\rangle \otimes |\Psi_{in2}\rangle.$$

If $\phi_T = \phi_R = 0$, then the unitary operator U takes the form

$$U = \exp(2i\theta L_2) = \exp(\theta(b_1^\dagger b_2 - b_2^\dagger b_1).$$

For the measurement one assumes that $\Pi(l)$ is the positive operator-value measure that is realized by the measuring device with

$$\Pi(l) \geq 0, \qquad \sum_{l=1}^{n} \Pi(l) = I.$$

Consider the density operators $\Pi(l) := |\Psi_{out2}\rangle\langle\Psi_{out2}|$ and

$$\rho_{out1} = \frac{\mathrm{tr}_2(\rho_{out}\Pi(l))}{p}, \qquad \rho_{out} = U|\Psi_{in}\rangle\langle\Psi_{in}|U^*.$$

Show that

$$p(l) = \langle\Pi(l)\rangle = \mathrm{tr}_1(\mathrm{tr}_2(\rho_{out}\Pi(l)))$$

is the probability of obtaining the result l. Note that tr_1 and tr_2 denote the partial trace.

Problem 25. Consider n particles on a line, with the coordinates $x_1, x_2, \ldots, x_n$ and the Hamiltonian operator

$$\hat{H} = \hat{H}_0 + \hat{V}$$

where

$$\hat{H}_0 = -\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2n}\sum_{j<k}^{n}(x_j - x_k)^2, \qquad \hat{V} = g^2\sum_{j<k}^{n}(x_j - x_k)^{-2}.$$

(i) Show that if $x_i = \bar{x} + \xi_i$, where

$$\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x_i$$

then the variable $\bar{x}$ can be separated and discarded. This means that the motion of the centre of gravity of the system is trivial.
(ii) Show that these n variables are constrained, $\sum_{j=1}^{n} \xi_j = 0$.
(iii) Show that $\hat{H}_0$ and $\hat{V}$ takes the form

$$\hat{H}_0 = -\frac{1}{2}\sum_{j=1}^{n} \partial_j^2 + \frac{1}{2}\sum_{j=1}^{n} \xi_j^2, \qquad \hat{V} = \frac{g^2}{2}\sum_{j\neq k}^{n} (\xi_j - \xi_k)^{-2}$$

where

$$\partial_j := \frac{\partial}{\partial \xi_j} - \frac{1}{n}\sum_{k=1}^{n} \frac{\partial}{\partial \xi_k}.$$

(iv) Show that one can introduce operators $b_j^\dagger$ and b_j playing the role of creation and annihilation operators,

$$b_j^\dagger := \xi_j - \partial_j, \qquad b_j := \xi_j + \partial_j, \qquad \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} b_j^\dagger = 0$$

where

$$[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0, \qquad [b_j, b_k^\dagger] = \left(\delta_{jk} - \frac{1}{n}\right) I.$$

(v) Show that in terms of these operators the Hamilton operator takes the form

$$H_0 = \frac{1}{2}\left(\sum_{j=1}^{N} b_j^\dagger b_j + (n-1)I\right).$$

(vi) Let B and $B^\dagger$

$$B^\dagger := \frac{1}{2}\sum_{j=1}^{n} (b_j^\dagger)^2 - \hat{V}, \qquad B := \frac{1}{2}\sum_{j=1}^{n} (b_j)^2 - \hat{V}.$$

Show that $[\hat{H}, B^\dagger] = 2B^\dagger$, $[\hat{H}, B] = -2B$, $[B, B^\dagger] = 4\hat{H}$.

Problem 26. Consider the Hamilton function

$$H(\mathbf{q}, \mathbf{p}) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{q_1^2 q_2^2}{2}.$$

Quantize

$$p_1 \to -i\frac{\partial}{\partial q_1}, \qquad p_2 \to -i\frac{\partial}{\partial q_2}$$

and introduce

$$\hat{q}_j = \frac{1}{\sqrt{2}}(b_j^\dagger + b_j), \qquad \hat{p}_j = \frac{i}{\sqrt{2}}(b_j^\dagger - b_j)$$

where $\hbar = 1$ and $j = 1, 2$. Introduce the number basis

$$|n_1, n_2\rangle := \prod_{k=1}^{2} \frac{(b_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle$$

where $n_k = 0, 1, \ldots, \infty$. Recall that

$$b_1|n_1,n_2\rangle = \sqrt{n_1}|n_1-1,n_2\rangle, \quad b_1^\dagger|n_1,n_2\rangle = \sqrt{n_1+1}|n_1+1,n_2\rangle, \quad b_1^\dagger b_1|n_1,n_2\rangle = n_1|n_1,n_2\rangle$$

and analogously for b_2, $b_2^\dagger$. Calculate the Hamilton operator $\hat{H}$ in terms of $b_j^\dagger$, b_j. Then calculate the infinite dimensional matrix representation of $\hat{H}$ using the basis given above. Choose the order of the basis

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |02\rangle, \quad |11\rangle, \quad |20\rangle, \ldots$$

Note that $\langle m_2, m_1|n_1, n_2\rangle = \delta_{n_1,m_1}\delta_{n_2,m_2}$. Truncate the infinite dimensional matrix and calculate the eigenvalues numerically.

Problem 27. Let $\ell = 0, 1, 2, \ldots$. Consider the Hamilton operator

$$\hat{H}(\ell) = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{\ell(\ell+1)}{2}\frac{1}{x^2}.$$

Let

$$B_\ell = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x - \frac{\ell}{x}\right), \qquad B_\ell^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x - \frac{\ell}{x}\right).$$

Then $B_\ell + B_{-\ell}^\dagger = \sqrt{2}x$. Show that

$$[B_\ell, B_\ell^\dagger] = I + \frac{\ell}{x}I, \quad [B_\ell, B_{-\ell}] = -\frac{\ell}{x^2}I, \quad [B_{-\ell}, B_\ell^\dagger] = I$$

where I is the identity operator and

$$\hat{H}(\ell) = B_\ell B_\ell^\dagger + \ell I - \frac{1}{2}I.$$

Problem 28. Consider the operators

$$S_0 = b_H^\dagger b_H + b_V^\dagger b_V, \qquad S_1 = b_H^\dagger b_H - b_V^\dagger b_V,$$

$$S_2 = b_H^\dagger b_V e^{i\theta} + b_V^\dagger b_H e^{-i\theta}, \qquad S_3 = i b_V^\dagger b_H e^{-i\theta} - i b_H^\dagger b_V e^{i\theta}$$

where the subscripts H and V label the horizontal and vertical polarization modes, respectively; θ is the phase shift between these modes; and $b_{H,V}$ and $b_{H,V}^\dagger$ are the Bose annihilation and creation operators for the electromagnetic field in frequency space. Show that $[S_1, S_2] = 2iS_3$, $[S_2, S_3] = 2iS_1$, $[S_3, S_1] = 2iS_2$ and $[S_0, S_j] = 0$ for $j = 1, 2, 3$.

Problem 29. Let b_1, b_2 be Bose annihilation operators. Consider the Hamilton operator

$$\hat{H} = \hbar\omega b_1^\dagger b_1 + \hbar\omega_2 b_2^\dagger b_2 + \hbar(V b_1^\dagger b_2 + V^* b_1 b_2^\dagger)$$

where V is a complex coupling constant.
(i) Show that the Heisenberg equation of motion for b_1 and b_2 is given by

$$i\hbar\frac{db_1}{dt} = [b_1, \hat{H}](t), \qquad i\hbar\frac{db_2}{dt} = [b_2, \hat{H}](t).$$

(ii) Show that inserting the Hamilton operator yields

$$i\hbar\frac{db_1}{dt} = \hbar\omega_1 b_1 + \hbar V b_2, \qquad i\hbar\frac{db_2}{dt} = \hbar\omega_2 b_2 + \hbar V^* b_1.$$

(iii) Show that introducing the definitions

$$b_1 = \widetilde{b}_1 e^{-i\omega_1 t}, \qquad b_2 = \widetilde{b}_2 e^{-i\omega_2 t}$$

yields

$$i\frac{d\widetilde{b}_1}{dt} = V\widetilde{b}_2, \qquad i\frac{d\widetilde{b}_2}{dt} = V^*\widetilde{b}_1.$$

Problem 30. Show that the spectrum of the operator $b_1^\dagger b_2 + b_1 b_2^\dagger$ is discrete and coincides with the set $\mathbb{Z}$ of relative integers.

Problem 31. The *Rayleigh-Schrödinger perturbation theory* for the *anharmonic oscillator* with Hamilton operator ($\hat{p} = -i\hbar d/dq$)

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 + \widetilde{\gamma}\hat{q}^4$$

is divergent with the perturbation $\widetilde{\gamma}\hat{q}^4$. Let

$$\hat{q} =: \sqrt{\frac{\hbar}{2m\omega}}(b^\dagger + b), \qquad \hat{p} =: i\sqrt{\frac{\hbar m\omega}{2}}(b^\dagger - b).$$

Since

$$\hat{q}^2 = \frac{\hbar}{2m\omega}((b^\dagger)^2 + b^2 + 2b^\dagger b + I), \quad \hat{p}^2 = -\frac{\hbar m\omega}{2}((b^\dagger)^2 + b^2 - 2b^\dagger b - I)$$

and

$$\hat{q}^4 = \frac{\hbar^2}{4m^2\omega^2}((b^\dagger)^2 + b^2 + 2b^\dagger b + I)^2$$

we arrive for the Hamilton operator

$$\hat{H} = \hbar\omega(b^\dagger b + \frac{1}{2}I) + \frac{\hbar^2\widetilde{\gamma}}{4m^2\omega^2}((b^\dagger)^4 + b^4 + 6((b^\dagger)^2 + b^2) + 4(b^\dagger b^3 + (b^\dagger)^3 b) + 12b^\dagger b + 6(b^\dagger)^2 b^2 + 3I).$$

With $\hat{K} = \hat{H}/(\hbar\omega)$ and the dimensionless quantity $\gamma = (\hbar\widetilde{\gamma})/(4m^2\omega^3)$ we arrive at

$$\hat{K} = b^\dagger b + \frac{1}{2}I + \gamma((b^\dagger)^4 + b^4 + 6((b^\dagger)^2 + b^2) + 4(b^\dagger b^3 + (b^\dagger)^3 b) + 12b^\dagger b + 6(b^\dagger)^2 b^2 + 3I)$$

or

$$\hat{K} = b^\dagger b(1 + 12\gamma) + \left(\frac{1}{2} + 3\gamma\right) I + \gamma(b^4 + (b^\dagger)^4 + 6(b^2 + (b^\dagger)^2) + 4(b^\dagger b^3 + (b^\dagger)^3 b) + 6(b^\dagger)^2 b^2).$$

Let $\hat{K} = \hat{K}_0 + \hat{K}_1$ with

$$\hat{K}_0 = b^\dagger b(1 + 6\gamma) + \left(\frac{1}{2} + 3\gamma\right) I + 6\gamma(b^\dagger b)^2$$

and

$$\hat{K}_1 = \gamma(b^4 + (b^\dagger)^4 + 6(b^2 + (b^\dagger)^2) + 4(b^\dagger b^3 + (b^\dagger)^3 b)).$$

(i) Study the Rayleigh-Schrödinger perturbation theory for this Hamilton operator with $\hat{K}_1$ the perturbation. Take into account the invariance of the Hamilton operator under the transformation $b \mapsto -b$, $b^\dagger \mapsto -b^\dagger$.

(ii) Find the matrix representation of $\hat{K}$ using number states. Truncate the infinite dimensional matrix and find the eigenvalues as a function of γ.
(iii) Consider coherent states $|\beta\rangle$. Calculate the expectation value $\langle\beta|\hat{K}|\beta\rangle$. Discuss.
(iv) Consider squeezed states $|\zeta\rangle$. Calculate the expectation value $\langle\zeta|\hat{K}|\zeta\rangle$. Discuss.

Problem 32. Let $\zeta \in \mathbb{C}$. Calculate the commutator

$$[\zeta^* b^2, \zeta(b^\dagger)^2].$$

Problem 33. Show that

$$e^{-\alpha b^\dagger b} = e^\alpha \sum_{n=0}^{\infty} \frac{1}{n!}(1-e^\alpha)^n b^n (b^\dagger)^n.$$

Problem 34. Let $|\beta\rangle$ be a coherent state with $\beta = re^{i\phi}$ and $|\zeta\rangle$ a squeezed state with $\zeta = se^{i\theta}$. Find $\langle\beta|\zeta\rangle$.

Problem 35. Let $|\zeta\rangle$ be a squeezed state with $\zeta = se^{i\theta}$ and $|n\rangle$ $(n = 0, 1, 2, \ldots)$ be the number state. Find $\langle n|\zeta\rangle$.

Problem 36. Let $|\zeta, \beta\rangle = D(\beta)|\zeta\rangle$ be a displaced squeezed state. Let $|\zeta\rangle$ be a squeezed state with $\zeta = se^{i\theta}$, $|\beta\rangle$ and $|\gamma$ be coherent states. Show that

$$\langle\gamma|\zeta,\beta\rangle = \frac{1}{\sqrt{\cosh(s)}} e^{-i\Im(\gamma\beta^*) - \frac{1}{2}|\beta-\gamma|^2 - \frac{1}{2}(\beta^*-\gamma^*)^2 e^{i\theta}\tanh(s)}$$

Problem 37. Let

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = b^\dagger b \otimes I_2 + \alpha(b^\dagger + b) \otimes \sigma_1$$

with α dimensionless and real. Let $|n\rangle$ be the number state and $D(\beta)$ be the displacement operator. The normalized eigenvectors of the Pauli matrix σ_1 are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Are the states

$$D(\alpha)|n\rangle \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad D(-\alpha)|n\rangle \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenstates of the Hamilton operator $\hat{K}$?

Problem 38. Let $\zeta \in \mathbb{C}$ and $b^\dagger$ a Bose creation operator and b a Bose annihilation operator. Find the commutator

$$[\bar{\zeta} b^2, \zeta(b^\dagger)^2].$$

Problem 39. Let $b^\dagger$, b be Bose creation and annihilation operators, respectively. Then

$$[b + b^\dagger, b^\dagger b] = b - b^\dagger.$$

Find the commutator $[b+b^\dagger, \sqrt{b^\dagger b}]$. Since $b^\dagger b = \mathrm{diag}(0,1,2,3,\ldots)$ we have $\sqrt{b^\dagger b} = \mathrm{diag}(0,1,\sqrt{2},\sqrt{3},\ldots)$.

Problem 40. A homodyne tomography of a single field mode b consists of an ensemble of repeated measurements of the quadratures

$$\widetilde{b}_\phi = \frac{1}{2}(be^{-i\phi} + b^\dagger e^{i\phi})$$

for various phases ϕ relative to the local oscillator of the homodyne detector.

Problem 41. Squeezing can be produced by *Kerr effects* using optical fibers or cold atoms in an optical cavity or by type-I parametric interaction in a cavity.

Problem 42. (i) Consider the differential operators

$$\hat{Q} = q + i\hbar\frac{\partial}{\partial p}, \quad \hat{P} = -i\hbar\frac{\partial}{\partial q}.$$

Find the commutator $[\hat{Q},\hat{P}]f(p,q)$, where f is a smooth function.
(ii) Consider the differential operators

$$\hat{Q} = \frac{1}{2}q + i\hbar\frac{\partial}{\partial p}, \quad \hat{P} = -2i\hbar\frac{\partial}{\partial q}.$$

Find the commutator $[\hat{Q},\hat{P}]f(p,q)$, where f is a smooth function.
(iii) Consider the differential operators

$$\hat{Q} = \frac{1}{2}q + i\hbar\frac{\partial}{\partial p}, \quad \hat{P} = -i\hbar\frac{\partial}{\partial q} + \frac{1}{2}p.$$

Find the commutator $[\hat{Q},\hat{P}]f(p,q)$, where f is a smooth function.

Chapter 2

Spin Systems

2.1 Spin Matrices, Commutators and Anticommutators

2.2 Spin Matrices and Functions

2.3 Spin Hamilton Operators

2.4 Supplementary Problems

Problem 1. Let σ_1, σ_2, σ_3 be the Pauli spin matrices.
(i) Let A, B be two arbitrary 2×2 matrices over $\mathbb{C}$. Is

$$\frac{1}{2}\mathrm{tr}(AB) \equiv \sum_{j=1}^{3} \left(\frac{1}{2}\mathrm{tr}(\sigma_j A)\right)\left(\frac{1}{2}\mathrm{tr}(\sigma_j B)\right) ?$$

(ii) Let

$$S_+ := \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S_- := \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find the matrices

$$e^{i\pi\sigma_3/4} S_+ e^{-i\pi\sigma_3/4}, \qquad e^{i\pi\sigma_3/4} S_- e^{-i\pi\sigma_3/4}.$$

(iii) Consider the *Hadamard gate*

$$U_H = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Show that

$$U_H \sigma_1 U_H^{-1} = \sigma_3, \quad U_H \sigma_2 U_H^{-1} = -\sigma_2, \quad U_H \sigma_3 U_H^{-1} = \sigma_1.$$

Note that $U_H^{-1} = U_H$.
(iv) Find all 2×2 hermitian matrices such that $[H, \sigma_1] = 0_2$, $[H, \sigma_2] = 0_2$, $[H, \sigma_3] = 0_2$.

Problem 2. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $z \in \mathbb{C}$.
(i) Calculate

$$\sinh(z\sigma_1), \quad \sinh(z\sigma_2), \quad \sinh(z\sigma_3), \quad \cosh(z\sigma_1), \quad \cosh(z\sigma_2), \quad \cosh(z\sigma_3).$$

(ii) Let S_1, S_2, S_3 be the spin-1 matrices. Calculate

$$\sinh(zS_1), \quad \sinh(zS_2), \quad \sinh(zS_3), \quad \cosh(zS_1), \quad \cosh(zS_2), \quad \cosh(zS_3).$$

Problem 3. (i) Find all 2×2 matrices A such that

$$[A, A^*] = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Find all 3×3 matrices B such that

$$[B, B^*] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Problem 4. The *quaternions*

$$\mathbb{H} = \{\, a1 + bI + cJ + dK \;:\; a, b, c, d \in \mathbb{R} \,\}$$

form a real associative algebra with products specified by Hamilton's formula

$$I^2 = J^2 = K^2 = IJK = -1.$$

The conjugate of a quaternion $x = a1 + bI + cJ + dK$ is defined by $\bar{x} = a1 - bI - cJ - dK$, and its norm $|x|$ is defined by

$$|x|^2 = x\bar{x} = a^2 + b^2 + c^2 + d^2.$$

Show that

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I \mapsto -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\sigma_1,$$

$$J \mapsto -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\sigma_2, \quad K \mapsto -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\sigma_3$$

is a representation.

Problem 5. Let σ_0, σ_1, σ_2, σ_3 be the Pauli spin matrices with $\sigma_0 = I_2$ and 0_2 be the 2×2 zero matrix.
(i) Show that the three *alpha matrices*

$$\alpha_1 = \begin{pmatrix} 0_2 & \sigma_1 \\ \sigma_1 & 0_2 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0_2 & \sigma_2 \\ \sigma_2 & 0_2 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ \sigma_3 & 0_2 \end{pmatrix}$$

for the Dirac equation cannot be simultaneously diagonalized.
(ii) Consider the 4×4 *gamma matrices*

$$\gamma_1 = \begin{pmatrix} 0_2 & \sigma_1 \\ -\sigma_1 & 0_2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0_2 & \sigma_2 \\ -\sigma_2 & 0_2 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ -\sigma_3 & 0_2 \end{pmatrix}$$

and

$$\gamma_0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}.$$

Find the matrix $\gamma_1\gamma_2\gamma_3\gamma_0$ and $\mathrm{tr}(\gamma_1\gamma_2\gamma_3\gamma_0)$.
(iii) The gamma matrices are the four 4×4 matrices

$$\gamma_k = \begin{pmatrix} 0_2 & \sigma_k \\ -\sigma_k & 0_2 \end{pmatrix}, \quad k = 0, 1, 2, 3$$

where 0_2 is the 2×2 zero matrix. Are the matrices γ_k linearly independent?
(iv) Find the eigenvalues and eigenvectors of the γ_k's.
(v) Are the matrices γ_k invertible. Use the result from (iv). If so, find the inverse.
(vi) Find the commutators $[\gamma_k, \gamma_\ell]$ for $k, \ell = 0, 1, 2, 3$. Find the anticommutators $[\gamma_k, \gamma_\ell]_+$ for $k, \ell = 0, 1, 2, 3$.
(vii) Can the matrices γ_k be written as the Kronecker product of two 2×2 matrices?
(viii) Consider the 4×4 matrix

$$\rho = \frac{1}{4}\sum_{j,k=0}^{3} \alpha_{j,k}\sigma_j \otimes \sigma_k$$

with the real expansion coefficients α_{jk}. What is the conditions on the α_{jk}'s such that ρ is a density matrix, i.e. ρ is a positive semidefinite matrix with $\mathrm{tr}(\rho) = 1$? One can assume that $\alpha_{00} = 1$.

Problem 6. Let σ_1, σ_2, σ_3 be the Pauli spin matrices with the commutators

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \quad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2$$

and anticommutators $[\sigma_1,\sigma_2]_+ = [\sigma_2,\sigma_3]_+ = [\sigma_3,\sigma_1]_+ = 0_2$. Let A, B be 2×2 matrices. We define the *star operation* as

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

(i) Find the commutators $[\sigma_1\star\sigma_1,\sigma_2\star\sigma_2]$, $[\sigma_2\star\sigma_2,\sigma_3\star\sigma_3]$, $[\sigma_3\star\sigma_3,\sigma_1\star\sigma_1]$.
(ii) Find the anticommutators

$$[\sigma_1\star\sigma_1,\sigma_2\star\sigma_2]_+, \quad [\sigma_2\star\sigma_2,\sigma_3\star\sigma_3]_+, \quad [\sigma_3\star\sigma_3,\sigma_1\star\sigma_1]_+.$$

Discuss.

Problem 7. Let σ_1, σ_2, σ_3 be the Pauli spin matrices.
(i) Consider the unitary and hermitian 4×4 matrix

$$R := \sigma_1 \otimes \sigma_2.$$

Apply the *vec-operator* on R, i.e. find $\mathrm{vec}(R)$.
(ii) Calculate the discrete Fourier transform of the vector in $\mathbb{C}^4$. Then apply vec^{-1} to this vector.
(iii) Compare this 4×4 matrix with the 4×4 matrix R. Discuss.
(iv) Let

$$\hat{H} = \hbar\omega(\sigma_1\otimes\sigma_2 + \sigma_2\otimes\sigma_1).$$

Calculate $\exp(-i\hat{H}t/\hbar)$.

Problem 8. Let σ_1, σ_2, σ_3 be the Pauli spin matrices.
(i) Find all 2×2 hermitian matrices H such that $\sigma_1 H\sigma_1 = H$, $\sigma_2 H\sigma_2 = H$, $\sigma_3 H\sigma_3 = H$.
(ii) Find all 2×2 hermitian matrices K such that $\sigma_1 K\sigma_1 = K$, $\sigma_2 K\sigma_2 = K$.
(iii) Find all 4×4 hermitian matrices H such that

$$(\sigma_1\otimes\sigma_1)H(\sigma_1\otimes\sigma_1) = H, \ (\sigma_2\otimes\sigma_2)H(\sigma_2\otimes\sigma_2) = H, \ (\sigma_3\otimes\sigma_3)H(\sigma_3\otimes\sigma_3) = H.$$

(iv) Find all 4×4 hermitian matrices K such that

$$(\sigma_1\otimes\sigma_1)K(\sigma_1\otimes\sigma_1) = K, \quad (\sigma_2\otimes\sigma_2)K(\sigma_2\otimes\sigma_2) = K.$$

(v) Let $S_1 = \frac{1}{2}\sigma_1$, $S_2 = \frac{1}{2}\sigma_2$, $S_3 = \frac{1}{2}\sigma_3$ be the spin matrices. Let R be an $n\times n$ matrix over $\mathbb{C}$. Consider

$$V = S_3\otimes(I_n + R) + iS_2\otimes(I_n - R).$$

What is the condition on R such that V is unitary? Hint. Calculate VV^*.

Problem 9. Let S_1, S_2, S_3 be the 3×3 spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $S_+ := S_1 + iS_2$, $S_- := S_1 - iS_2$ with

$$S_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

(i) Let $z \in \mathbb{C}$. Find

$$\exp(zS_+ - \bar{z}S_2), \quad \exp(zS_+ - \bar{z}S_-)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Is the state normalized?
(ii) Find the commutators $[S_+, S_-]$, $[S_3, S_+]$, $[S_3, S_-]$.
(iii) Show that the commutation relations are preserved under the transformation

$$\widetilde{S}_+ = \cos(\theta)S_+ + \sin(\theta)S_- e^{i\pi S_3}, \quad \widetilde{S}_- = \cos(\theta)S_- + \sin(\theta)S_+ e^{-i\pi S_3}, \quad \widetilde{S}_3 = \frac{1}{2}[\widetilde{S}_+, \widetilde{S}_-].$$

Problem 10. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and $\sigma_0 = I_2$, where I_2 is the 2×2 unit matrix. The generators of the *Clifford algebra* $C\ell(2n, \mathbb{C}) \simeq M(2^n, \mathbb{C})$ in the *Jordan-Wigner representation* are given by the Kronecker product of the Pauli matrices

$$E_{2k-1} = i\underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{k-1} \otimes \sigma_1 \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k}$$
$$E_{2k} = i\underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{k-1} \otimes \sigma_2 \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k}$$

where $k = 1, \ldots, n$. Thus E_{2k-1} and E_{2k} are elements of the Pauli group.
(i) Find the eigenvalues and eigenvectors of E_{2k-1} and E_{2k}.
(ii) Consider the *gamma matrices*

$$\gamma_0 = -i\sigma_1 \otimes I_2, \quad \gamma_1 = \sigma_2 \otimes \sigma_1, \quad \gamma_2 = \sigma_2 \otimes \sigma_2, \quad \gamma_3 = \sigma_2 \otimes \sigma_3, \quad \gamma_5 = \sigma_3 \otimes I_2.$$

For the construction of the Clifford algebra $C\ell(4n, \mathbb{C}) \simeq M(4^n, \mathbb{C})$ one considers the $4^n \times 4^n$ matrix

$$E_j^{[k]} = \underbrace{\gamma_5 \otimes \cdots \otimes \gamma_5}_{k-1} \otimes \gamma_j \otimes \underbrace{I_4 \otimes \cdots \otimes I_4}_{n-k}$$

where $k = 1, \ldots, n$, $j = 0, \ldots, 3$ and I_4 is the 4×4 unit matrix with $I_4 = I_2 \otimes I_2$. Find the eigenvalues and eigenvectors of $E_j^{[k]}$.
(iii) Show that a representation of a four-dimensional Clifford algebra is given by

$$T_{+1} = \sigma_1 \otimes I_2, \quad T_{+2} = \sigma_3 \otimes \sigma_1, \quad T_{-1} = \sigma_2 \otimes I_2, \quad T_{-2} = \sigma_3 \otimes \sigma_2.$$

Problem 11. Consider the Pauli spin matrices to describe a spin-$\frac{1}{2}$ particle. In the square array of 4×4 matrices

$$\begin{matrix} I_2 \otimes \sigma_3 & \sigma_3 \otimes I_2 & \sigma_3 \otimes \sigma_3 \\ \sigma_1 \otimes I_2 & I_2 \otimes \sigma_1 & \sigma_1 \otimes \sigma_1 \\ \sigma_1 \otimes \sigma_3 & \sigma_3 \otimes \sigma_1 & \sigma_2 \otimes \sigma_2 \end{matrix}$$

each row and each column is a triad of commuting operators. Consider the hermitian 3×3 matrices to describe a particle with spin-1

$$S_1 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Is in the square array of 9×9 matrices

$$\begin{matrix} I_3 \otimes S_3 & S_3 \otimes I_3 & S_3 \otimes S_3 \\ S_1 \otimes I_3 & I_3 \otimes S_1 & S_1 \otimes S_1 \\ S_1 \otimes S_3 & S_3 \otimes S_1 & S_2 \otimes S_2 \end{matrix}$$

each row and each column a triad of commuting operators?

Problem 12. Let $|j\rangle$ ($j = 0, 1, \ldots, N$) be the standard basis in the Hilbert space $\mathbb{C}^{N+1}$, i.e.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \ |N\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

A *coherent spin state* can be written as

$$|\theta, \phi\rangle_N = \sum_{j=0}^{N} |j\rangle \binom{N}{j}^{1/2} (\cos(\theta/2))^{N-j} (\sin(\theta/2)e^{i\phi})^j$$

where θ, ϕ be the two angles in the spherical coordinate system. Write down the spin state for $N + 1$, $N = 2$, $N = 3$ and $N = 4$. Find the density matrix $\rho = |\theta, \phi\rangle\langle\phi, \theta|$.

Problem 13. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 4×4 matrix

$$R = a(\lambda, \mu)\sigma_1 \otimes \sigma_1 + b(\lambda, \mu)(\sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)$$

where

$$a(\lambda, \mu) = \frac{1}{4} \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2}, \qquad b(\lambda, \mu) = \frac{1}{2} \frac{\lambda\mu}{\lambda^2 - \mu^2}.$$

Does R satisfy the *braid like relation*

$$(R \otimes I_2)(I_2 \otimes R)(R \otimes I_2) = (I_2 \otimes R)(R \otimes I_2)(I_2 \otimes R)\,?$$

Problem 14. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$. Let $\langle A\rangle := \mathbf{v}^* A \mathbf{v}$ and $\langle B\rangle := \mathbf{v}^* B \mathbf{v}$. We have the identity

$$AB \equiv (A - \langle A\rangle I_n)(B - \langle B\rangle I_n) + A\langle B\rangle + B\langle A\rangle - \langle A\rangle\langle B\rangle I_n.$$

We approximate the $n \times n$ matrix AB as

$$AB \approx A\langle B\rangle + B\langle A\rangle - \langle A\rangle\langle B\rangle I_n.$$

Let $n = 2$ and

$$A = \sigma_1, \quad B = \sigma_2, \quad \mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find AB and $A\langle B\rangle + B\langle A\rangle - \langle A\rangle\langle B\rangle I_n$ and the distance (Frobenius norm) between the two matrices. Discuss.

Problem 15. Consider the three spin matrices S_1, S_2, S_3 for spin $s = 1/2$, $s = 1$, $s = 3/2$, $s = 2$, etc. For spin-1/2 we have the 2×2 matrices

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and for spin-1 we have the 4×4 matrices

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The spin-matrices S_1, S_2, S_3 satisfy the commutation relations $[S_1, S_2] = iS_3$, $[S_2, S_3] = iS_1$, $[S_3, S_1] = iS_2$. Consider the hierarchy of spin Hamilton operators

$$\begin{aligned}\hat{H} = & \hbar\omega_{11} S_1 \otimes I_{2s+1} \otimes I_{2s+1} + \hbar\omega_{12} I_{2s+1} \otimes S_2 \otimes I_{2s+1} + \hbar\omega_{13} I_{2s+1} \otimes I_{2s+1} \otimes S_3 \\ & + \hbar\omega_{21} S_1 \otimes S_2 \otimes I_{2s+1} + \hbar\omega_{22} S_1 \otimes I_{2s+1} \otimes S_3 + \hbar\omega_{23} I_{2s+1} \otimes S_2 \otimes S_3 + \hbar\omega_3 S_1 \otimes S_2 \otimes S_3\end{aligned}$$

where I_{2s+1} is the $(2s+1) \times (2s+1)$ identity matrix for the given spin s. Thus the Hamilton operator $\hat{H}$ is a hermitian $(2s+1)^3 \times (2s+1)^3$ matrix with trace equal to 0. Find the eigenvalues and normalized eigenvectors of $\hat{H}$. Then calculate the *partition function*

$$Z(\beta) := \frac{e^{-\beta\hat{H}}}{\operatorname{tr}(e^{-\beta\hat{H}})}.$$

Apply the following: Let A_1, A_2, A_3 be $n \times n$ matrices and I_n the $n \times n$ identity matrix. Let λ_1 be an eigenvalue of A_1 with normalized eigenvector $\mathbf{v}_1$, λ_2 an eigenvalue of A_2 with normalized eigenvector $\mathbf{v}_2$ and λ_3 an eigenvalue of A_3 with normalized eigenvector $\mathbf{v}_3$. Then

$$\mathbf{v} := \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3$$

is a normalized eigenvector of the $n^3 \times n^3$ matrix

$$\begin{aligned}\hat{K} = & c_{11} A_1 \otimes I_n \otimes I_n + c_{12} I_n \otimes A_2 \otimes I_n + c_{13} I_n \otimes I_n \otimes A_3 \\ & + c_{21} A_1 \otimes A_2 \otimes I_n + c_{22} A_1 \otimes I_n \otimes A_3 + c_{23} I_n \otimes A_2 \otimes A_3 + c_3 A_1 \otimes A_2 \otimes A_3\end{aligned}$$

with

$$\hat{K}\mathbf{v} = (c_{11}\lambda_1 + c_{12}\lambda_2 + c_{13}\lambda_3 + c_{21}\lambda_1\lambda_2 + c_{22}\lambda_1\lambda_3 + c_{23}\lambda_2\lambda_3 + c_3\lambda_1\lambda_2\lambda_3)\mathbf{v}.$$

Problem 16. Consider the vector space $\mathbb{R}^2$ and the standard basis $\{\, \mathbf{e}_1 \,, \mathbf{e}_2 \,\}$.
(i) Show that none of the four vectors

$$\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1, \quad \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1$$

can be written as the Kronecker product of two vectors in $\mathbb{R}^2$.
(ii) Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis in the vector space $\mathbb{C}^n$. Consider the vectors

$$\mathbf{e}_j \otimes \mathbf{e}_k - \mathbf{e}_k \otimes \mathbf{e}_j, \quad \mathbf{e}_j \otimes \mathbf{e}_k + \mathbf{e}_k \otimes \mathbf{e}_j$$

where $j \neq k$ and $j, k = 1, \ldots, n$ in the vector space $\mathbb{C}^{n^2}$. Show that none of these vectors, say $\mathbf{w}$ can be written as $\mathbf{w} = \mathbf{u} \otimes \mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$.

Problem 17. Consider the 2×2 matrices

$$Q := \sigma_3, \quad R := \sigma_2, \quad S := -\frac{1}{\sqrt{2}}(\sigma_2 + \sigma_3), \quad T := -\frac{1}{\sqrt{2}}(-\sigma_2 + \sigma_3)$$

and the entangled state (one of the Bell states)

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Show that

$$\langle\psi|Q \otimes S|\psi\rangle + \langle\psi|R \otimes S|\psi\rangle + \langle\psi|R \otimes T|\psi\rangle - \langle\psi|Q \otimes T|\psi\rangle = 2\sqrt{2}.$$

Problem 18. Can one find an invertible 8×8 matrix T such that

$$T(I_2 \otimes \sigma_1 \otimes I_2)T^{-1} = I_2 \otimes \sigma_3 \otimes I_2, \qquad T(I_2 \otimes \sigma_3 \otimes I_2)T^{-1} = \sigma_3 \otimes \sigma_1 \otimes \sigma_3.$$

Problem 19. Let S_1, S_2, S_3 be the spin matrices for spin $s = 1/2$, $s = 1$, $s = 3/2$, $s = 2$. Find the normalized state

$$\exp(-i\phi S_3)\exp(-i\theta S_2)\exp(-i\psi S_3)|0\rangle$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for spin $s = 1/2$, $s = 1$, $s = 3/2$, $s = 2$, respectively.

Problem 20. Let $s \in \{1/2, 1, 3/2, 2, \ldots\}$ be the spin and S_1, S_2, S_3 be the $(2s+1) \times (2s+1)$ spin matrices with the commutation relations

$$[S_1, S_2] = iS_3, \quad [S_2, S_3] = iS_1, \quad [S_3, S_1] = iS_2$$

and

$$S_1^2 + S_2^2 + S_3^2 = s(s+1)I_{2s+1}$$

where I_{2s+1} is the $(2s+1) \times (2s+1)$ identity matrix. The matrices act on the Hilbert space $\mathbb{C}^{2s+1}$. The Hilbert space $\mathbb{C}^{2s+1}$ is isomorphic to the Hilbert space $\ell_2(\mathbb{Z}_{2s+1})$ where $\mathbb{Z}_{2s+1}$ is the cyclic group of order $2s+1$. Let $m, n, k \in \{-s, -s+1, \ldots, 2s\}$. One defines the linear operators

$$(W_{m,n}\varphi)(k) := \exp\left(-4\frac{i\pi mn}{2s+1} + 4\frac{i\pi nk}{2s+1}\right)\varphi(k-2m)$$

where the operation $k - 2m$ is $\bmod\,(2s+1)$.
(i) Show that the linear operators $W_{m,n}$ are unitary and satisfy

$$W_{m,n}^* = W_{-m,-n}$$
$$W_{m,n}W_{m',n'} = \exp\left(\frac{4i\pi}{2s+1}(m'n - mn')\right)W_{m+m',n+n'}.$$

(ii) Let $s = 1/2$. Find the linear operators $W_{-1/2,-1/2}$, $W_{-1/2,1/2}$, $W_{1/2,-1/2}$, $W_{1/2,1/2}$.

Problem 21. Let σ_1, σ_2, σ_3 be the Pauli spin matrices, I_2 be the 2×2 identity matrix and 0_2 be the 2×2 zero matrix. Let α_1, α_2, α_3, β be the 4×4 matrices

$$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}$$

where $j = 1, 2, 3$.
(i) Show that these matrices satisfy

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \quad j, k = 1, 2, 3$$

$$\alpha_j \beta + \beta \alpha_j = 0_4, \quad j = 1, 2, 3$$

and $\beta^2 = I_4$.
(ii) Let

$$\Psi(t, \mathbf{x}) = \begin{pmatrix} \psi_1(t, \mathbf{x}) \\ \psi_2(t, \mathbf{x}) \\ \psi_3(t, \mathbf{x}) \\ \psi_4(t, \mathbf{x}) \end{pmatrix}.$$

The *Dirac equation* with rest mass m can be written as the 4×4 matrix-valued differential equation

$$i\hbar \frac{\partial}{\partial t} \Psi(t, \mathbf{x}) = \hat{H} \Psi(t, \mathbf{x})$$

where

$$\hat{H} = -i\hbar c \boldsymbol{\alpha} \cdot \nabla + mc^2 \beta \equiv \begin{pmatrix} mc^2 I_2 & -i\hbar c \boldsymbol{\sigma} \cdot \nabla \\ -i\hbar c \boldsymbol{\sigma} \cdot \nabla & -mc^2 I_2 \end{pmatrix}$$

with

$$\boldsymbol{\alpha} \cdot \nabla := \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}, \quad \boldsymbol{\sigma} \cdot \nabla := \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2} + \sigma_3 \frac{\partial}{\partial x_3}.$$

Find the time-evolution of $\Psi^* \Psi$, i.e. calculate

$$\frac{\partial}{\partial t}(\Psi^* \Psi) = \frac{\partial \Psi}{\partial t} \Psi + \Psi^* \frac{\partial \Psi}{\partial t}.$$

Problem 22. Study the spectrum of the Hamilton operator

$$\hat{H} = \hbar\omega(\sigma_1 \otimes \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3 \otimes \sigma_3).$$

First calculate the commutator with

$$K = \sigma_3 \otimes \sigma_3 \otimes I_2 + I_2 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes I_2 \otimes \sigma_3.$$

Discuss.

Problem 23. Let $\omega := \exp(2\pi i/4)$. Consider the 4×4 unitary matrices

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $c > 0$. The four-state *Potts quantum chain* is defined by the Hamilton operator

$$\hat{H} = -\frac{1}{\pi\sqrt{c}} \sum_{j=1}^{N} \left((\sigma_j + \sigma_j^2 + \sigma_j^3) + c(\Gamma_j \Gamma_{j+1}^3 + \Gamma_j^2 \Gamma_{j+1}^2 + \Gamma_j^3 \Gamma_{j+1})\right)$$

where N is the number of sites and one imposes cyclic boundary conditions $N + 1 \equiv 1$. Let $N = 2$. Find the eigenvalues and eigenvectors of $\hat{H}$. Obviously $\hat{H}$ is a 16×16 matrix.

Problem 24. (i) Consider the 16×16 hermitian matrices (Hamilton operators)

$$\begin{aligned} H = &\sigma_1 \otimes \sigma_1 \otimes I_2 \otimes I_2 + \sigma_2 \otimes \sigma_2 \otimes I_2 \otimes I_2 + \sigma_3 \otimes \sigma_3 \otimes I_2 \otimes I_2 \\ &+ I_2 \otimes \sigma_1 \otimes \sigma_1 \otimes I_2 + I_2 \otimes \sigma_2 \otimes \sigma_2 \otimes I_2 + I_2 \otimes \sigma_3 \otimes \sigma_3 \otimes I_2 \\ &+ I_2 \otimes I_2 \otimes \sigma_1 \otimes \sigma_1 + I_2 \otimes I_2 \otimes \sigma_2 \otimes \sigma_2 + I_2 \otimes I_2 \otimes \sigma_3 \otimes \sigma_3 \\ &+ \sigma_1 \otimes I_2 \otimes I_2 \otimes \sigma_1 + \sigma_2 \otimes I_2 \otimes I_2 \otimes \sigma_2 + \sigma_3 \otimes I_2 \otimes I_2 \otimes \sigma_3 \end{aligned}$$

and

$$\begin{aligned} K = &\sigma_1 \otimes I_2 \otimes \sigma_1 \otimes I_2 + \sigma_2 \otimes I_2 \otimes \sigma_2 \otimes I_2 + \sigma_3 \otimes I_2 \otimes \sigma_3 \otimes I_2 \\ &+ I_2 \otimes \sigma_1 \otimes I_2 \otimes \sigma_1 + I_2 \otimes \sigma_2 \otimes I_2 \otimes \sigma_2 + I_2 \otimes \sigma_3 \otimes I_2 \otimes \sigma_3 \\ &+ \sigma_1 \otimes I_2 \otimes \sigma_1 \otimes I_2 + \sigma_2 \otimes I_2 \otimes \sigma_2 \otimes I_2 + \sigma_3 \otimes I_2 \otimes \sigma_3 \otimes I_2. \end{aligned}$$

Is $[H, K] = 0_{16}$, i.e. do H and K commute? Find the eigenvalues of $\hat{H}$ and $\hat{K}$.
(ii) Find the eigenvalues and eigenvectors of the spin-Hamilton operator (32×32 hermitian matrix)

$$\begin{aligned} \hat{H} = &J(\sigma_1 \otimes \sigma_1 \otimes I_2 \otimes I_2 \otimes I_2 + \sigma_1 \otimes I_2 \otimes \sigma_1 \otimes I_2 \otimes I_2 \\ &+ \sigma_1 \otimes I_2 \otimes I_2 \otimes \sigma_1 \otimes I_2 + \sigma_1 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_1) \\ &+ h(\sigma_3 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_3 \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_3 \otimes I_2 \otimes I_2 \\ &+ I_2 \otimes I_2 \otimes I_2 \otimes \sigma_3 \otimes I_2 + I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_3). \end{aligned}$$

Problem 25. Find the eigenvalues and eigenvectors of the Hamilton operator

$$\hat{H} = a(\sigma_3 \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_3 \otimes I_2 + I_2 \otimes I_2 \sigma_3) + b(\sigma_1 \otimes \sigma_1 \otimes I_2 + I_2 \otimes \sigma_1 \otimes \sigma_1) + c(\sigma_2 \otimes I_2 \otimes \sigma_2).$$

Problem 26. The spin-1 matrices are given by the hermitian matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with the commutation relations $[S_1, S_2] = iS_3$, $[S_2, S_3] = iS_1$, $[S_3, S_1] = iS_2$. The five quadrupole matrices are given by the hermitian matrices

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad Q_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Show that $[U_1, U_2] = 2iS_3$ and $[V_1, V_2] = iS_3$. Show that the eight skew-hermitian matrices

$$iS_1,\ iS_2,\ iS_3,\ iU_1,\ iU_2,\ iV_1,\ iV_2,\ iQ_0$$

form a basis of the semisimple Lie algebra $su(3)$.
(ii) The Hamilton operator for the general quadrupolar interaction of two spin-1 nuclei can be expressed as the hermitian 9×9 matrix

$$\hat{H} = \hbar\omega_0(Q_0 \otimes Q_0) + \hbar\omega_1(V_1 \otimes V_1 + V_2 \otimes V_2) + \hbar\omega_2(U_1 \otimes U_1 + U_2 \otimes U_2).$$

Find the eigenvalues and normalized eigenvectors of $\hat{H}$. Study energy level crossing. Find the 9×9 permutation matrices P such that $P\hat{H}P = \hat{H}$. Study entanglement of the eigenvectors.

Problem 27. (i) Study the Hamilton operator for the one dimensional spin chain with $N+1$ lattice points (open end)

$$\hat{H} = -J\sum_{j=0}^{N-1} \sigma_{1,j}\sigma_{1,j+1} - \sum_{j=0}^{N-1} \sqrt{j}\sigma_{3,j}.$$

Show that applying the Jordan-Wigner transformation one obtains the Hamilton operator

$$\hat{H} = -J\sum_{j=0}^{N-1} (c_j^\dagger - c_j)(c_{j+1}^\dagger + c_{j+1}) - \sum_{j=0}^{N-1} \sqrt{j}(2c_j^\dagger c_j - I).$$

(ii) Find the spectrum of the Hamilton operator (open end boundary conditions)

$$\hat{H} = \frac{1}{2}\sum_{j=1}^{N-1} j(\sigma_{1,j}\sigma_{1,j+1} + \sigma_{2,j}\sigma_{2,j+1}).$$

Problem 28. (i) Let $N \geq 2$ and σ_1, σ_2, σ_3 be the Pauli spin matrices. The XY one-dimensional model with a transversal exterior field and open boundary conditions is given by the hermitian matrix

$$\hat{H} = \sum_{j=1}^{N-1} \mu_j((1+\gamma_j)\sigma_{1,j}\sigma_{1,j+1} + (1-\gamma_j)\sigma_{2,j}\sigma_{2,j+1}) + \sum_{j=1}^{N} \nu_j\sigma_{3,j}.$$

Thus $\hat{H}$ is a hermitian $2^N \times 2^N$ matrix with trace equal to 0. Solve the eigenvalue problem for $N = 2$ and $N = 3$. Extend to higher dimensions.
(ii) Consider the Hamilton operator

$$\hat{H} = J\sum_{k=1}^{N} \sigma_{k,1}\sigma_{k+1,1} + h\sum_{k=1}^{N} \sigma_{k,3}$$

with periodic boundary conditions, i.e. $\sigma_{N+1,3} \equiv \sigma_{1,3}$ and

$$\sigma_{k,1} = I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1 \otimes I_2 \otimes \cdots \otimes I_2$$

where σ_1 is at the k-th position. Thus $\sigma_{k,1}$ is a (hermitian) $2^N \times 2^N$ matrix. Thus the Hamilton operators is also a $2^N \times 2^N$ hermitian matrix. Solve the eigenvalue problem for $N = 3$ and $N = 4$.

(iii) Find the spectrum of the spin Hamilton operator

$$\hat{H} = -\sum_{j=0}^{N-1} (\sigma_{j,3}\sigma_{j+1,3} + \mu_1 \sigma_{j,1} + \mu_2 \sigma_{j,3})$$

for $N = 2$ and $N = 3$. Study entanglement of the eigenvectors as a function of μ_1 and μ_2.

Problem 29. Let S_1, S_2, S_3 be the spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with the commutation relations $[S_1, S_2] = iS_3$, $[S_2, S_3] = iS_1$, $[S_3, S_1] = iS_2$. Let $\theta \in [0, \pi/4]$. Consider the Hamilton operator

$$\hat{H}(\theta) = \cos(\theta) \sum_{k=1}^{N} (S_{k,1}S_{k+1} + S_{k,2}S_{k+1,2} + S_{k,3}S_{k+1,3})$$
$$+ \sin(\theta) \sum_{k=1}^{N} (S_{k,1}S_{k+1} + S_{k,2}S_{k+1,2} + S_{k,3}S_{k+1,3})^2$$

with periodic boundary conditions, i.e. $S_{N+1,1} \equiv S_{1,1}$, $S_{N+1,2} \equiv S_{1,2}$, $S_{N+1,3} \equiv S_{1,3}$ and

$$S_{k,1} := I_3 \otimes \cdots I_3 \otimes S_1 \otimes I_3 \otimes \cdots \otimes I_3$$

where the matrix S_1 is at the k-th position. Thus $S_{k,1}$ is a $3^N \times 3^N$ matrix. Let $N = 3$. Find the ground state of the Hamilton operator. Does the Hamilton operator $\hat{H}(\theta)$ commute with

$$\sum_{k=1}^{N} S_{k,3}\,?$$

Problem 30. (i) A triple spin coupling Hamilton operator is given by

$$\hat{H} = J(\sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes I_2 + I_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 + \sigma_1 \otimes I_2 \otimes \sigma_1 \otimes \sigma_1 + \sigma_1 \otimes \sigma_1 \otimes I_2 \otimes \sigma_1).$$

Find the eigenvalues and eigenvectors.
(ii) A Hamilton operator $\hat{H}$ with triple-spin coupling and a transverse field is given by

$$\hat{H} = a(\sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes I_2 + I_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1)$$
$$+ b(\sigma_3 \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_3 \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_3 \otimes I_2 + I_2 \otimes I_2 \otimes I_2 \otimes \sigma_3).$$

Find the eigenvalues and eigenvectors of $\hat{H}$.

Problem 31. We identify the states spin and spin down as follows

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We set $|\uparrow\downarrow\uparrow\rangle \equiv |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle$ etc. Are the two states

$$|\psi_1\rangle = \frac{1}{\sqrt{6}}(2|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{6}}(-2|\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)$$

orthonormal?

Problem 32. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $n \geq 1$ and $j = 1, \ldots, 2n$. The $2n$ matrices of size $2^n \times 2^n$ are defined recursively as

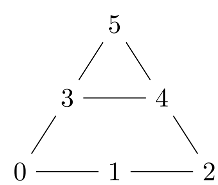
$$\gamma_j^{(n+1)} = \gamma_j^{(n)} \otimes \sigma_3 \quad j = 1, \ldots, 2n$$
$$\gamma_{2n+1}^{(n+1)} = I_{2^n} \otimes \sigma_1, \quad \gamma_{2n+2}^{(n+1)} = I_{2^n} \otimes \sigma_2$$

where $\gamma_1^{(1)} = \sigma_1$ and $\gamma_2^{(1)} = \sigma_2$ and I_{2^n} is the $2^n \times 2^n$ identity matrix. Find $\gamma_1^{(2)}$ and $\gamma_2^{(2)}$.

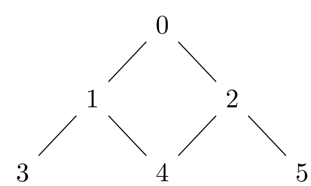
Problem 33. Let S_1, S_2, S_3 be the spin-$\frac{1}{2}$ matrices. Consider the Hamilton operator

$$\begin{aligned}\hat{H} = \sum_{j=1}^{2} \hbar\omega_j (& S_j \otimes S_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes S_j \otimes S_j \otimes I_2 \otimes I_2 \otimes I_2 \\ & + S_j \otimes I_2 \otimes I_2 \otimes S_j \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes S_j \otimes I_2 \otimes S_j \otimes I_2 \\ & + I_2 \otimes I_2 \otimes I_2 \otimes S_j \otimes S_j \otimes I_2 + I_2 \otimes I_2 \otimes I_2 \otimes S_j \otimes I_2 \otimes S_j + I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes S_j \otimes S_j)\end{aligned}$$

which describes nearest neighbour interaction for the lattice with six lattice points. Thus $\hat{H}$ is a hermitian 64×64 matrix with trace equal to 0.



Problem 34. Consider the XY-model for the lattice



Thus we have six lattice sites and the Hamilton operator is given by the hermitian $2^6 \times 2^6$ matrix

$$\begin{aligned}\hat{K} = \frac{\hat{H}}{\hbar\omega} = \sum_{j=1}^{2} (& \sigma_j \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 + \sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \\ & + I_2 \otimes \sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \\ & + I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes \sigma_j).\end{aligned}$$

Thus the eigenvalues are real. Actually the matrix is real symmetric. Find the eigenvalues. Note that $\mathrm{tr}(\hat{K}) = 0$, i.e. the sum of the eigenvalues of $\hat{K}$ is equal to 0. The matrix $\hat{K}$ is real symmetric. The *Jacobi method* finds the eigenvalues of a real symmetric matrix by applying a sequence of orthogonal transformations based on the matrix

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

to find the diagonal form of the real symmetric matrix and thus the eigenvalues. Apply the program below `eigenvaluesJacobi.cpp` to find the eigenvalues of $\hat{K}$. The program uses the `kron` operation (Kronecker product) of SymbolicC++ to find the real symmetric matrix $\hat{K}$ and then the function `rotate` finds the diagonal form. The program also tests whether the eigenvalues add up to 0.

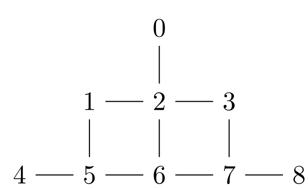
```
// eigenvaluesJacobi.cpp

#include <iostream>
#include <cmath>
#include <cstdio>
#include <cstdlib>
#include "symbolicc++.h"
using namespace std;

void rotate(Symbolic &M,Symbolic &D,int n)
{
  int count = 0;
  double eps = 1e-16;
  while(true)
  {
  double max = 0.0;
  int p = 0; int q = 0;
  for(int i=1;i<n;i++)
    for(int j=0;j<i;j++)
     {
     double h = fabs(double(M(i,j)));
     if(h > max) { max = h; p = i; q = j; }
     }
   if(max < eps) break;
   double theta = (M(q,q)-M(p,p))/(2.0*M(p,q));
   double t = 1.0;
   if(theta > 0.0) t = 1.0/(theta+sqrt(theta*theta+1.0));
   else t = 1.0/(theta-sqrt(theta*theta+1.0));
   double c = 1.0/sqrt(1.0+t*t); double s = t*c;
   M(p,p) = M(p,p)-M(q,p)*t; M(q,q) = M(q,q)+M(q,p)*t; M(p,q) = M(q,p) = 0.0;
   double r = s/(1.0+c);
   for(int j=0;j<n;j++)
   {
   if(j==p || j==q) continue;
   double h = M(p,j)-s*(M(q,j)+r*M(p,j));
   M(q,j) = M(q,j)+s*(M(p,j)-r*M(q,j)); M(p,j) = h;
   }
   for(int i=0;i<n;i++)
   {
   if(i==p || i==q) continue;
   double h = M(i,p)-s*(M(i,q)+r*M(i,p));
   M(i,q) = M(i,q)+s*(M(i,p)-r*M(i,q)); M(i,p) = h;
   }
   count++;
   if(count > 100*n*n*n) { cerr << "Iteration failed"; exit(0); }
   } // end while
   for(int i=0;i<n;i++) D(i) = M(i,i);
}  // end method rotate
```

```
int main(void)
{
  using SymbolicConstant::i;
  Symbolic I2("",2,2);
  I2(0,0) = 1.0; I2(0,1) = 0.0; I2(1,0) = 0.0; I2(1,1) = 1.0;
  Symbolic sig1("",2,2);
  sig1(0,0) = 0.0; sig1(0,1) = 1.0; sig1(1,0) = 1.0; sig1(1,1) = 0.0;
  Symbolic sig2("",2,2);
  sig2(0,0) = 0.0; sig2(0,1) = -i; sig2(1,0) = i; sig2(1,1) = 0.0;
  Symbolic K11 = kron(sig1,kron(sig1,kron(I2,kron(I2,kron(I2,I2)))));
  Symbolic K12 = kron(sig2,kron(sig2,kron(I2,kron(I2,kron(I2,I2)))));
  Symbolic K21 = kron(sig1,kron(I2,kron(sig1,kron(I2,kron(I2,I2)))));
  Symbolic K22 = kron(sig2,kron(I2,kron(sig2,kron(I2,kron(I2,I2)))));
  Symbolic K31 = kron(I2,kron(sig1,kron(I2,kron(sig1,kron(I2,I2)))));
  Symbolic K32 = kron(I2,kron(sig2,kron(I2,kron(sig2,kron(I2,I2)))));
  Symbolic K41 = kron(I2,kron(sig1,kron(I2,kron(I2,kron(sig1,I2)))));
  Symbolic K42 = kron(I2,kron(sig2,kron(I2,kron(I2,kron(sig2,I2)))));
  Symbolic K51 = kron(I2,kron(I2,kron(sig1,kron(I2,kron(sig1,I2)))));
  Symbolic K52 = kron(I2,kron(I2,kron(sig2,kron(I2,kron(sig2,I2)))));
  Symbolic K61 = kron(I2,kron(I2,kron(sig1,kron(I2,kron(I2,sig1)))));
  Symbolic K62 = kron(I2,kron(I2,kron(sig2,kron(I2,kron(I2,sig2)))));
  Symbolic K = K11+K12+K21+K22+K31+K32+K41+K42+K51+K52+K61+K62;
  int m = 64;
  Symbolic D("",m);
  rotate(K,D,m);
  cout << "D = " << D << endl; // eigenvalues
  double sumofeig = 0.0;
  for(int j=0;j<m;j++)
  sumofeig = D(j); // check whether sum of eigenvalues is 0
  cout << "sumofeig = " << sumofeig << endl;
  return 0;
}
```

Problem 35. Consider the XY-model for the lattice



with nearest neighbour interaction. There are nine lattice sites and thus the Hamilton operator $\hat{H}$ is a hermitian $2^9 \times 2^9$ matrix with trace equal to 0. Find the eigenvalues.

Problem 36. Let S_1, S_2, S_3 be the spin operators for spin-$\frac{1}{2}$. Consider the three spin Hamilton operators

$$\hat{H}_1 = \sum_{j=1}^{2} \hbar\omega_j (S_j \otimes S_j \otimes I_2 \otimes I_2 + S_j \otimes I_2 \otimes S_j \otimes I_2 + S_j \otimes I_2 \otimes I_2 \otimes S_j)$$

$$\hat{H}_2 = \sum_{j=1}^{2} \hbar\omega_j (S_j \otimes S_j \otimes I_2 \otimes I_2 + I_2 \otimes S_j \otimes S_j \otimes I_2 + I_2 \otimes I_2 \otimes S_j \otimes S_j)$$

$$\hat{H}_3 = \sum_{j=1}^{2} \hbar\omega_j (S_j \otimes S_j \otimes I_2 \otimes I_2 + I_2 \otimes S_j \otimes S_j \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes S_j \otimes S_j + S_j \otimes I_2 \otimes I_2 \otimes S_j)$$

with nearest neighbour interaction for the following lattices with four vertices

$$\begin{array}{ccc} & 1 & \\ & | & \\ 2 & — & 0 \\ & | & \\ & 3 & \end{array} \qquad 0 — 1 — 2 — 3 \qquad \begin{array}{ccc} 3 & — & 2 \\ | & & | \\ 0 & — & 1 \end{array}$$

Thus the Hamilton operators are 16×16 hermitian matrices with trace equal to 0. Find the spectrum of the Hamilton operators. Compare the ground states for the three different configurations. Study the entanglement of the eigenvectors of the ground states. Discuss.

Problem 37. Let A, H be $n \times n$ hermitian matrices, where H plays the role of the Hamilton operator. The *Heisenberg equation of motion* is given by

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[H, A(t)].$$

with $A = A(t=0) = A(0)$ and the solution of the initial value problem

$$A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar}.$$

Let E_j ($j = 1, 2, \ldots, n^2$) be an orthonormal basis in the *Hilbert space* $\mathcal{H}$ of the $n \times n$ matrices with scalar product

$$\langle X, Y \rangle := \mathrm{tr}(XY^*), \qquad X, Y \in \mathcal{H}.$$

Now $A(t)$ can be expanded using this orthonormal basis as

$$A(t) = \sum_{j=1}^{n^2} c_j(t) E_j$$

and H can be expanded as

$$H = \sum_{j=1}^{n^2} h_j E_j.$$

We find the time evolution for the coefficients $c_j(t)$, i.e. dc_j/dt, where $j = 1, 2, \ldots, n^2$. We have

$$\frac{dA(t)}{dt} = \sum_{j=1}^{n^2} \frac{dc_j}{dt} E_j.$$

Inserting this equation and the expansion for H into the Heisenberg equation of motion we arrive at

$$\sum_{j=1}^{n^2} \frac{dc_j}{dt} E_j = \frac{i}{\hbar} \sum_{k=1}^{n^2} \sum_{j=1}^{n^2} h_k c_j(t) [E_k, E_j] \, .$$

Taking the scalar product of the left and right-hand side of this equation with E_ℓ ($\ell = 1, \ldots, n^2$) gives

$$\sum_{j=1}^{n^2} \frac{dc_j(t)}{dt} \mathrm{tr}(E_j E_\ell^*) = \frac{i}{\hbar} \sum_{k=1}^{n^2} \sum_{j=1}^{n^2} h_k c_j(t) \mathrm{tr}(([E_k, E_j]) E_\ell^*)$$

where $\ell = 1, 2, \ldots, n^2$. Since $\mathrm{tr}(E_j E_\ell^*) = \delta_{j\ell}$ we obtain

$$\frac{dc_\ell}{dt} = \frac{i}{\hbar} \sum_{k=1}^{n^2} \sum_{j=1}^{n^2} h_k c_j(t) \mathrm{tr}(E_k E_j E_\ell^* - E_j E_k E_\ell^*)$$

where $\ell = 1, 2, \ldots, n^2$. Consider the Hamilton operator

$$\hat{H} = \hbar\omega(\sigma_1 \otimes \sigma_2 \otimes I_2 + I_2 \otimes \sigma_2 \otimes \sigma_3 + \sigma_1 \otimes I_2 \otimes \sigma_3$$

and

$$A = \sigma_1 \otimes \sigma_1 \otimes I_2 + I_2 \otimes \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes I_2 \otimes \sigma_3.$$

(i) Find the time evolution of the $c_j(t)$'s for the standard basis in the Hilbert space of the 8×8 matrices.
(ii) Let

$$X_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be an orthonormal basis in the Hilbert space of the 2×2 matrices. Find the time evolution of the $c_j(t)$'s for the basis given by

$$X_{j_0} \otimes X_{j_1} \otimes X_{j_2}, \qquad j_0, j_1, j_2 \in \{0, 1, 2, 3\}.$$

Apply computer algebra.

Problem 38. Study the XY-model on the *unit cube* with nearest neighbour interaction. Using the mapping

$$(0,0,0) \to 0, \quad (0,0,1) \to 1, \quad (0,1,0) \to 2, \quad (0,1,1) \to 3$$

$$(1,0,0) \to 4, \quad (1,0,1) \to 5, \quad (1,1,0) \to 6, \quad (1,1,1) \to 7$$

and the 12 interacting pairs

$$(0,1), \quad (0,2), \quad (0,4), \quad (1,3), \quad (1,5), \quad (2,3)$$

$$(2,6), \quad (3,7), \quad (4,5), \quad (4,6), \quad (5,7), \quad (6,7)$$

i.e.

$$((0,0,0),(0,0,1)), \quad ((0,0,0),(0,1,0)), \quad ((0,0,0),(1,0,0)), \quad ((0,0,1),(0,1,1)),$$

$$((0,0,1),(1,0,1)), \quad ((0,1,0),(0,1,1)), \quad ((0,1,0),(1,1,0)), \quad ((0,1,1),(1,1,1)),$$

$$((1,0,0),(1,0,1)), \quad ((1,0,0),(1,1,0)), \quad ((1,0,1),(1,1,1)), \quad ((1,1,0),(1,1,1))$$

the Hamilton operator takes the form

$$\begin{aligned}\hat{H} = \sum_{j=1}^{2} (&\sigma_j \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 + \sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \\
&+\sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \\
&+I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_j \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \\
&+I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 + I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \\
&+I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes \sigma_j \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 \\
&+I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes \sigma_j + I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes \sigma_j).\end{aligned}$$

Thus $\hat{H}$ is a $2^8 \times 2^8$ hermitian matrix with trace equal to 0.
(i) Does the Hamilton operator $\hat{H}$ commute with

$$\sum_{j=0}^{7} \sigma_{j,3}\,?$$

Apply computer algebra.
(ii) Does the Hamilton operator $\hat{H}$ commute with

$$\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3\,?$$

Apply computer algebra.

Problem 39. Consider the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \equiv (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3).$$

Show that the Hamilton operator is given by the hermitian 4×4 matrix

$$\hat{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with the eigenvalues -3 (1 times) and 1 (3 times) and the corresponding eigenvectors (for -3)

$$\begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}^T$$

and for 1

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^T, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Problem 40. (i) Find the eigenvalues of the 4×4 matrices

$$\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1, \quad \sigma_2 \otimes \sigma_3 - \sigma_3 \otimes \sigma_2, \quad \sigma_3 \otimes \sigma_1 - \sigma_1 \otimes \sigma_3.$$

(ii) Find the eigenvalues of the 4×4 matrices

$$\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1, \quad \sigma_2 \otimes \sigma_3 + \sigma_3 \otimes \sigma_2, \quad \sigma_3 \otimes \sigma_1 + \sigma_1 \otimes \sigma_3.$$

Problem 41. Consider the five point XXX model with cyclic boundary conditions

$$\hat{K} = \sum_{j=1}^{3} (\sigma_j \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_j \otimes \sigma_j \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_j \otimes \sigma_j \otimes I_2$$
$$+ I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes \sigma_j + \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j).$$

(i) Let

$$S_j = \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2$$
$$+ I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 + I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma_j$$

for $j = 1, 2, 3$. Show that K commutes with S_j ($j = 1, 2, 3$).
(ii) Let

$$\hat{C} = \sum_{j=1}^{3} (\sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_j \otimes I_2 \otimes \sigma_j \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 \otimes \sigma_j$$
$$+\sigma_j \otimes I_2 \otimes I_2 \otimes \sigma_j \otimes I_2 + I_2 \otimes \sigma_j \otimes I_2 \otimes I_2 \otimes \sigma_j).$$

In physics the 32×32 matrix is considered as a *non-local charge*. Show that $[\hat{K}, \hat{C}] = 0_{32}$. Apply computer algebra.

Problem 42. Consider the spin-$\frac{1}{2}$ matrices with $\mathbf{S} = (S_1, S_2, S_3)$ and the Hamilton operator with cyclic boundary condition and four lattice sites

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = \sum_{j=1}^{4} \mathbf{S}_j \cdot \mathbf{S}_{j+1}.$$

Thus

$$\hat{K} = \sum_{k=1}^{3} (S_k \otimes S_k \otimes I_2 \otimes I_2 + I_2 \otimes S_k \otimes S_k \otimes I_2 + I_2 \otimes I_2 \otimes S_k \otimes S_k + S_k \otimes I_2 \otimes I_2 \otimes S_k).$$

We have a hermitian 16×16 matrix with $\mathrm{tr}(\hat{K}) = 0$. Show that the eigenvalues and normalized eigenvectors are as follows. For the eigenvalue $+1$ we have the five normalized eigenvectors

$$(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$$
$$\frac{1}{2}(0,1,1,0,1,0,0,0,1,0,0,0,0,0,0,0)^T$$
$$\frac{1}{\sqrt{6}}(0,0,0,1,0,1,1,0,0,1,1,0,1,0,0,0)^T$$
$$\frac{1}{2}(0,0,0,0,0,0,0,1,0,0,0,1,0,1,1,0)^T$$
$$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)^T$$

For the eigenvalue -2 we have one eigenvector

$$\frac{1}{2\sqrt{3}}(0,0,0,1,0,-2,1,0,0,1,-2,0,1,0,0,0)^T$$

For the eigenvalue -1 we have three eigenvectors

$$\frac{1}{2}(0,1,-1,0,1,0,0,0,-1,0,0,0,0,0,0,0)^T$$
$$\frac{1}{\sqrt{2}}(0,0,0,0,0,1,0,0,0,0,-1,0,0,0,0,0)^T$$
$$\frac{1}{2}(0,0,0,0,0,0,0,1,0,0,0,-1,0,1,-1,0)^T$$

and for the eigenvalue 0 we have seven eigenvectors

$$\frac{1}{\sqrt{2}}(0,1,0,0,-1,0,0,0,0,0,0,0,0,0,0,0)^T$$

$$\frac{1}{\sqrt{2}}(0,0,1,0,0,0,0,0,-1,0,0,0,0,0,0,0)^T$$
$$\frac{1}{\sqrt{2}}(0,0,0,1,0,0,0,0,0,0,0,0,-1,0,0,0)^T$$
$$\frac{1}{\sqrt{2}}(0,0,0,0,0,0,1,0,0,0,0,0,-1,0,0,0)^T$$
$$\frac{1}{\sqrt{2}}(0,0,0,0,0,0,0,1,0,0,0,0,0,-1,0,0)^T$$
$$\frac{1}{\sqrt{2}}(0,0,0,0,0,0,0,0,0,1,0,0,-1,0,0,0)^T$$
$$\frac{1}{\sqrt{2}}(0,0,0,0,0,0,0,0,0,0,0,1,0,0,-1,0)^T.$$

Find the eigenvalues and eigenvectors of $\hat{K}$ with the algebraic Bethe ansatz. Discuss.

Problem 43. Consider the *triple spin operator* $K = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ and

$$T_1 = \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \quad T_2 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \quad T_3 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3.$$

Show that $[K, T_1] = [K, T_2] = [K, T_3] = 0_8$. Since $T_1^2 = I_8$ (analogously one can consider $T_2^2 = I_8$ and $T_3^2 = I_8$) we can construct the two projection operators

$$\Pi_1 = \frac{1}{2}(I_8 + T_1), \qquad \Pi_2 = \frac{1}{2}(I_8 - T_1).$$

They project into two four-dimensional subspaces. For Π_1 the basis is given by

$$\frac{1}{\sqrt{2}}(1,0,0,0,0,0,0,1)^T, \quad \frac{1}{\sqrt{2}}(0,1,0,0,0,0,1,0)^T,$$
$$\frac{1}{\sqrt{2}}(0,0,1,0,0,1,0,0)^T, \quad \frac{1}{\sqrt{2}}(0,0,0,1,1,0,0,0)^T.$$

For Π_2 the basis is given by

$$\frac{1}{\sqrt{2}}(1,0,0,0,0,0,0,-1)^T, \quad \frac{1}{\sqrt{2}}(0,1,0,0,0,0,-1,0)^T,$$
$$\frac{1}{\sqrt{2}}(0,0,1,0,0,-1,0,0)^T, \quad \frac{1}{\sqrt{2}}(0,0,0,1,-1,0,0,0)^T.$$

Show that the matrix representation in these two subspaces are given by

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

for Π_1 and

$$\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

for Π_2. Utilize the following SymbolicC++ program for the task.

```
// triplespin2.cpp

#include <iostream>
#include "symbolicc++.h"
using namespace std;

int main(void)
{
using SymbolicConstant::i;
Symbolic sqrt2 = sqrt(Symbolic(2)); // square root of 2
Symbolic I2("I2",2,2);
I2 = I2.identity();  // 2 times 2 identity matrix
Symbolic sig1("sig1",2,2);
sig1(0,0) = 0; sig1(0,1) = 1; sig1(1,0) = 1; sig1(1,1) = 0;
Symbolic sig2("sig2",2,2);
sig2(0,0) = 0; sig2(0,1) = -i; sig2(1,0) = i; sig2(1,1) = 0;
Symbolic sig3("sig3",2,2);
sig3(0,0) = 1; sig3(0,1) = 0; sig3(1,0) = 0; sig3(1,1) = -1;
Symbolic K = kron(sig1,kron(sig2,sig3));
Symbolic T1 = kron(sig1,kron(sig1,sig1));
Symbolic T2 = kron(sig2,kron(sig2,sig2));
Symbolic T3 = kron(sig3,kron(sig3,sig3));
Symbolic C1 = K*T1-T1*K; cout << "C1 = " << C1 << endl;
Symbolic C2 = K*T2-T2*K; cout << "C2 = " << C2 << endl;
Symbolic C3 = K*T3-T3*K; cout << "C3 = " << C3 << endl;
Symbolic v("v",4,8);
v(0,0) = 1/sqrt2; v(0,1) = 0; v(0,2) = 0; v(0,3) = 0;
v(0,4) = 0; v(0,5) = 0; v(0,6) = 0; v(0,7) = 1/sqrt2;
v(1,0) = 0; v(1,1) = 1/sqrt2; v(1,2) = 0; v(1,3) = 0;
v(1,4) = 0; v(1,5) = 0; v(1,6) = 1/sqrt2; v(1,7) = 0;
v(2,0) = 0; v(2,1) = 0; v(2,2) = 1/sqrt2; v(2,3) = 0;
v(2,4) = 0; v(2,5) = 1/sqrt2; v(2,6) = 0; v(2,7) = 0;
v(3,0) = 0; v(3,1) = 0; v(3,2) = 0; v(3,3) = 1/sqrt2;
v(3,4) = 1/sqrt2; v(3,5) = 0; v(3,6) = 0; v(3,7) = 0;
cout << v << endl;
Symbolic vT = v.transpose(); cout << vT << endl;
Symbolic e("e",4,4);
e = v*K*vT; cout << "e = " << e << endl;
Symbolic tr = e.trace(); cout << "tr = " << tr << endl;
Symbolic d = e.determinant(); cout << "d = " << d << endl;
return 0;
}
```

Problem 44. Let $s, n_0 \in \{1/2, 1, 3/2, 2, \ldots\}$ and $s = n_0$ and $|0\rangle$, $|1\rangle$, $\ldots$, $|n_0 + s\rangle$ be the standard basis in $\mathbb{C}^{n_0+s+1}$. Let $\gamma \in \mathbb{R}$. Given s. Is the state

$$|\gamma\rangle = \frac{1}{\sqrt{2s+1}} \sum_{n=n_0-s}^{n_0+s} \exp(in\gamma)|n\rangle$$

normalized?

Problem 45. Let U be a unitary operator on a Hilbert space $\mathcal{H}$. Let Π be the orthogonal projection onto $\{ v \in \mathcal{H} : Uv = v \}$. Then for any $w \in \mathcal{H}$ one has

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} U^j w = Pw$$

where the limit is with respect to the norm implied by the scalar product of the Hilbert space. This is *von Neumann's mean ergodic theorem.* Apply it to the Hilbert space $\mathbb{C}^2$ with $U = \sigma_1$ and

$$\sigma_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Apply it to the Hilbert space $\mathbb{C}^4$ with $U = \sigma_1 \otimes \sigma_1$ and

$$(\sigma_1 \otimes \sigma_1)(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Problem 46. Let $N > 2$. Consider a sequence of length N of binary variables (± 1) or Ising spins (with $+1 = \uparrow$ and $-1 = \downarrow$)

$$S = (s_0, s_1, \ldots, s_{N-1}).$$

Thus there are 2^N possible configurations. The autocorrelation function of a given S is defined as

$$C_k(S) := \sum_{j=0}^{N-1} s_j s_{j+k}$$

where all indices are taken modulo N. The *Bernasconi model* is the Hamilton function

$$H(S) = \sum_{k=1}^{N-1} C_k^2(S) \equiv \sum_{i,j=1}^{N-1} \sum_{k=1}^{N-1} s_i s_{i+k} s_j s_{j+k}.$$

So we have a long-range 4-spin interaction. Find the ground state for $N = 3$ and $N = 4$. Write a C++ program that finds the ground state for higher N's by running through all possible (2^N) configurations.

Chapter 3

Fermi Systems

3.1 States, Anticommutators, Commutators

3.2 Fermi Operators and Functions

3.3 Hamilton Operators

Problem 1. Study the spectrum of the operators

$$\sum_{j=0}^{N-1} e^{i\pi j} c_j^\dagger c_{j+1}, \qquad \sum_{j=0}^{N} e^{i\pi j} c_j^\dagger c_j.$$

Solution 1.

3.4 Hubbard Model

3.5 Supplementary Problems

Problem 1. Let $c_\uparrow^\dagger$, $c_\downarrow^\dagger$ be Fermi creation operators with spin-up and spin-down. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the four dimensional basis

$$c_\uparrow^\dagger c_\downarrow^\dagger|0\rangle, \quad c_\uparrow^\dagger|0\rangle, \quad c_\downarrow^\dagger|0\rangle, \quad |0\rangle.$$

(i) Find the matrix representation of

$$c_\uparrow^\dagger c_\downarrow, \quad c_\downarrow^\dagger c_\uparrow, \quad c_\uparrow^\dagger c_\uparrow, \quad c_\downarrow^\dagger c_\downarrow.$$

(ii) We define the operators

$$S_j := \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix} \sigma_j \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}, \quad j = 1, 2, 3.$$

Find the commutators $[S_1, S_2]$, $[S_2, S_3]$, $[S_3, S_1]$. Find the matrix representation with the basis given above.

Problem 2. Consider a two Fermion system with the matrix representation for the Fermi creation and annihilation operators

$$c_1^\dagger = \left(\frac{1}{2}\sigma_+\right) \otimes I_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c_1 = \left(\frac{1}{2}\sigma_-\right) \otimes I_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$c_2^\dagger = \sigma_3 \otimes \left(\frac{1}{2}\sigma_+\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c_2 = \sigma_3 \otimes \left(\frac{1}{2}\sigma_-\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

where

$$\frac{1}{2}\sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \frac{1}{2}\sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $\gamma_1, \gamma_2 \in \mathbb{C}$. Calculate the 4×4 matrices

$$\exp(\gamma_1 c_1^\dagger - \gamma_1^* c_1), \quad \exp(\gamma_2 c_2^\dagger - \gamma_2^* c_2), \quad \exp(\gamma_1 c_1^\dagger - \gamma_1^* c_1 + \gamma_2 c_2^\dagger - \gamma_2^* c_2).$$

Apply the spectral theorem. Note that $c_1^\dagger$, c_1, $c_2^\dagger$, c_2 are nonnormal matrices, but $\gamma_1 c_1^\dagger + \gamma_1^* c_1$ and $\gamma_2 c_2^\dagger - \gamma_2^* c_2$ are normal matrices.

Problem 3. (i) Let $c_j^\dagger$ $(j = 1, 2, 3)$ be Fermi creation operators. Consider the hermitian operators

$$T_{12} = c_1^\dagger c_2 + c_2^\dagger c_1, \qquad T_{23} = c_2^\dagger c_3 + c_3^\dagger c_2.$$

Find the commutators $[T_{12}, T_{23}]$ and $[[T_{12}, T_{23}], T_{12} + T_{23}]$.
(ii) Consider the operators

$$X_{01} := c_0^\dagger + c_1 + c_1^\dagger + c_0, \quad X_{12} := c_1^\dagger + c_2 + c_2^\dagger + c_1,$$

$$X_{23} := c_2^\dagger + c_3 + c_3^\dagger + c_2, \quad X_{30} := c_3^\dagger + c_0 + c_0^\dagger + c_3.$$

Find the commutators $[X_{01}, X_{12}]$, $[X_{12}, X_{23}]$, $[X_{23}, X_{30}]$, $[X_{30}, X_{01}]$. Do the operators commute with the number operator

$$\hat{N} = c_0^\dagger c_0 + c_1^\dagger c_1 + c_2^\dagger c_2 + c_3^\dagger c_3\,?$$

Problem 4. Let $\hat{N}_{j,\uparrow} = c_{j,\uparrow}^\dagger c_{j,\uparrow}$, $\hat{N}_{j,\downarrow} = c_{j,\downarrow}^\dagger c_{j,\downarrow}$. Show that

$$\Pi = \prod_{j=1}^{N} (I - \hat{N}_{j,\uparrow}\hat{N}_{j,\downarrow})$$

is a projection operator.

Problem 5. Consider the Fermi creation and annihilation operators $c_1^\dagger$, $c_2^\dagger$, c_1, c_2 and the basis

$$\{c_2^\dagger c_1^\dagger|0\rangle, c_2^\dagger|0\rangle, c_1^\dagger|0\rangle, |0\rangle\}.$$

(i) Find the matrix representation of the operators $c_1 + c_2 + c_1c_2$, $c_1^\dagger c_2^\dagger + c_2^\dagger c_1^\dagger$.
(ii) Find the matrix representation of the hermitian operator $c_1 + c_1^\dagger + c_2 + c_2^\dagger + c_1c_2 + c_2^\dagger c_1^\dagger$.
(iii) Find the eigenvalues of the Hamilton operator

$$\hat{H} = \hbar\omega_1 c_1^\dagger c_1 + \hbar\omega_2 c_2^\dagger c_2 + \hbar\kappa(c_1^\dagger c_2 + c_2^\dagger c_1)$$

utilizing the basis given above.
(iv) Let $\hat{N}_1 = c_1^\dagger c_1$ and $\hat{N}_2 = c_2^\dagger c_2$. Study the Hamilton operator

$$\hat{H} = \epsilon(\hat{N}_1 - \hat{N}_2) + t(c_1^\dagger c_2 + c_2^\dagger c_1) + \frac{1}{2}\kappa(\hat{N}_1 - \hat{N}_2)^2.$$

First show that the total number operator $\hat{N} = \hat{N}_1 + \hat{N}_2$ is a conserved quantity.

Problem 6. Let $c_{j,\sigma}^\dagger$ $(j = 1,2,3)$ be a Fermi creation operators of an electron with spin σ in the qubit j acting on the vacuum $|0\rangle$. Consider the states

$$\begin{aligned}
|S_{-3/2}\rangle &= c_{3,\downarrow}^\dagger c_{2,\downarrow}^\dagger c_{1,\downarrow}^\dagger|0\rangle \\
|S_{-1/2}\rangle &= (c_{3,\uparrow}^\dagger c_{2,\downarrow}^\dagger c_{1,\downarrow}^\dagger + c_{3,\downarrow}^\dagger c_{2,\uparrow}^\dagger c_{1,\downarrow}^\dagger + c_{3,\downarrow}^\dagger c_{2,\downarrow}^\dagger c_{1,\uparrow}^\dagger)|0\rangle \\
|S_{1/2}\rangle &= (c_{3,\downarrow}^\dagger c_{2,\uparrow}^\dagger c_{1,\uparrow}^\dagger + c_{3,\uparrow}^\dagger c_{2,\downarrow}^\dagger c_{1,\uparrow}^\dagger + c_{3,\uparrow}^\dagger c_{2,\uparrow}^\dagger c_{1,\downarrow}^\dagger)|0\rangle \\
|S_{3/2}\rangle &= c_{3,\uparrow}^\dagger c_{2,\uparrow}^\dagger c_{1,\uparrow}^\dagger|0\rangle.
\end{aligned}$$

Find the scalar products $\langle S_{-3/2}|S_{-1/2}\rangle$, $\langle S_{-3/2}|S_{1/2}\rangle$, $\langle S_{-3/2}|S_{3/2}\rangle$.

Problem 7. Consider the operator

$$K = c_1^\dagger c_2 + c_2^\dagger c_1 + c_2^\dagger c_3 + c_3^\dagger c_2 + c_3^\dagger c_1 + c_1^\dagger c_3.$$

Find $\exp(zK)c_j\exp(-zK)$ for $j = 1,2,3$. Note that

$$[K, c_1] = -c_2 - c_3, \quad [K, c_2] = -c_1 - c_3, \quad [K, c_3] = -c_1 - c_2.$$

Problem 8. Calculate

$$R(\epsilon_1, \epsilon_2, \epsilon_3) = \exp(\epsilon_1 c_1^\dagger c_2^\dagger + \epsilon_2 c_1 c_2 + \epsilon_3 (c_1^\dagger c_1 + c_2^\dagger c_2))$$

i.e. we want to disentangle the operator R.

Problem 9. Given the Hamilton operator

$$\hat{H} = t\sum_\sigma (c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma} + c_{2\sigma}^\dagger c_{3\sigma} + c_{3\sigma}^\dagger c_{2\sigma} + c_{3\sigma}^\dagger c_{4\sigma} + c_{4\sigma}^\dagger c_{3\sigma} + c_{4\sigma}^\dagger c_{1\sigma} + c_{1\sigma}^\dagger c_{4\sigma}) + U\sum_{j=1}^{4} n_{j\uparrow} n_{j\downarrow}$$

and the basis

$$\{ |1\rangle = c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle, \quad |2\rangle = c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle, \quad |3\rangle = c_{3\uparrow}^\dagger c_{3\downarrow}^\dagger|0\rangle, \quad |4\rangle = c_{4\uparrow}^\dagger c_{4\downarrow}^\dagger|0\rangle$$

$$|5\rangle = c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle, \quad |6\rangle = c_{1\uparrow}^\dagger c_{3\downarrow}^\dagger|0\rangle, \quad |7\rangle = c_{1\uparrow}^\dagger c_{4\downarrow}^\dagger|0\rangle, \quad |8\rangle = c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle,$$

$$|9\rangle = c_{1\downarrow}^\dagger c_{3\uparrow}^\dagger|0\rangle, \quad |10\rangle = c_{1\downarrow}^\dagger c_{4\uparrow}^\dagger|0\rangle, \quad |11\rangle = c_{2\uparrow}^\dagger c_{3\downarrow}^\dagger|0\rangle, \quad |12\rangle = c_{2\uparrow}^\dagger c_{4\downarrow}^\dagger|0\rangle,$$

$$|13\rangle = c_{2\downarrow}^\dagger c_{3\uparrow}^\dagger|0\rangle, \quad |14\rangle = c_{2\downarrow}^\dagger c_{4\uparrow}^\dagger|0\rangle, \quad |15\rangle = c_{3\uparrow}^\dagger c_{4\downarrow}^\dagger|0\rangle, \quad |16\rangle = c_{3\downarrow}^\dagger c_{4\uparrow}^\dagger|0\rangle \}.$$

(i) Calculate the matrix representation of $\hat{H}$ using this basis.
(ii) Find the eigenvalues of this 16×16 matrix.
(iii) Apply the Hamilton operator to the state

$$|\psi\rangle = \prod_{j=1}^{4} \frac{1}{\sqrt{2}}(c_{j\uparrow}^\dagger + c_{j\downarrow}^\dagger)|0\rangle.$$

Find the expectation value $\langle\psi|\hat{H}|\psi\rangle$.

Problem 10. Let $c_j^\dagger$, c_j $(j = 1, 2, 3)$ be Fermi creation and annihilation operators. Let $\hat{N}$ be the number operator

$$\hat{N} = c_1^\dagger c_1 + c_2^\dagger c_2 + c_3^\dagger c_3.$$

(i) Consider the Hamilton operator

$$\hat{H}_1 = t(c_1^\dagger c_2 + c_2^\dagger c_1 + c_2^\dagger c_3 + c_3^\dagger c_2 + c_1^\dagger c_3 + c_3^\dagger c_1) + k_1 c_1^\dagger c_1 + k_2 c_2^\dagger c_2 + k_3 c_3^\dagger c_3.$$

Show that $[\hat{H}, \hat{N}] = 0$. Given a basis with two Fermi particles

$$c_1^\dagger c_2^\dagger|0\rangle, \quad c_1^\dagger c_3^\dagger|0\rangle, \quad c_2^\dagger c_3^\dagger|0\rangle.$$

Find the matrix representation of $\hat{H}$ and $\hat{N}$. Given a basis with one Fermi particle

$$c_1^\dagger|0\rangle, \quad c_2^\dagger|0\rangle, \quad c_3^\dagger|0\rangle.$$

Find the matrix representation of $\hat{H}_1$ and $\hat{N}$.
(ii) Consider the Hamilton operator

$$\hat{H}_2 = t(c_1^\dagger c_2 + c_2^\dagger c_1 + c_2^\dagger c_1 + c_3^\dagger c_2) + k_1 c_1^\dagger c_1 + k_2 c_2^\dagger c_2 + k_3 c_3^\dagger c_3.$$

Compare the spectrum of $\hat{H}_1$ and $\hat{H}_2$.

(iii) Compare the spectrum of the two Hamilton operators

$$K_1 = t(c_1^\dagger c_2 + c_2^\dagger c_1 + c_2^\dagger c_3 + c_3^\dagger c_2 + c_3^\dagger c_1 + c_1^\dagger c_3)$$

and

$$K_2 = t(c_1^\dagger c_2 + c_2^\dagger c_1 + c_2^\dagger c_3 + c_3^\dagger c_2 + c_3^\dagger c_1^\dagger + c_1 c_3)$$

with the basis

$$c_3^\dagger c_2^\dagger c_1^\dagger|0\rangle,\ c_3^\dagger c_2^\dagger|0\rangle,\ c_3^\dagger c_1^\dagger|0\rangle,\ c_2^\dagger c_1^\dagger|0\rangle,\ c_3^\dagger|0\rangle,\ c_2^\dagger|0\rangle,\ c_1^\dagger|0\rangle,\ |0\rangle.$$

(iv) Study the Hamilton operator

$$\hat{H}_3 = \hbar\omega_1 c_1^\dagger c_1 + \hbar\omega_2 c_2^\dagger c_2 + \hbar\omega_3 c_3^\dagger c_3 + \hbar\kappa(c_1^\dagger c_2^\dagger c_3^\dagger + c_3 c_2 c_1).$$

Problem 11. (i) Study the spectrum of the hermitian operator

$$K = t_1(c_1^\dagger + c_1 + c_2^\dagger + c_2 + c_1^\dagger c_2^\dagger + c_2 c_1) + t_2(c_1^\dagger c_2 + c_2^\dagger c_1)$$

with the basis $\{c_1^\dagger c_2^\dagger|0\rangle,\ c_2^\dagger|0\rangle,\ c_1^\dagger|0\rangle,\ |0\rangle\}$.
(ii) Let $\hat{N}_j := c_j^\dagger c_j$ for $j = 1, 2$ be the number operators. Find the matrix representation of the Hamilton operator

$$\hat{H} = \hbar\omega_1(c_1^\dagger c_1 + c_2^\dagger c_2) + \hbar\omega_2(c_1^\dagger c_2 + c_2^\dagger c_1) + \hbar\omega_3 \hat{N}_1 \hat{N}_2$$

with the basis $\{c_2^\dagger c_1^\dagger|0\rangle,\ c_2^\dagger|0\rangle,\ c_1^\dagger|0\rangle,\ |0\rangle\}$ and then calculate the eigenvalues.

Problem 12. Let $c_j, c_j^\dagger$ be Fermi annihilation and creation operators ($j = 1, 2, 3$).
(i) Study the eigenvalue problem for the Hamilton operator

$$\hat{H} = \hbar\omega(c_1^\dagger c_1 + c_2^\dagger c_2 + c_3^\dagger c_3) + t_1(c_1^\dagger c_2 + c_2^\dagger c_1) + t_2(c_2^\dagger c_3 + c_3^\dagger c_2) + t_1(c_3^\dagger c_1 + c_1^\dagger c_3)$$

with the basis $\{\, c_2^\dagger c_1^\dagger|0\rangle,\ \ c_3^\dagger c_1^\dagger|0\rangle,\ \ c_3^\dagger c_2^\dagger|0\rangle \,\}$ and $t_1, t_2 > 0$ with $t_1 > t_2$.
(ii) Study the operator

$$\hat{K} = c_1^\dagger(c_2 - c_3) + c_2^\dagger(c_3 - c_1) + c_3^\dagger(c_1 - c_2).$$

Does the operator $\hat{K}$ commute with the number operator

$$\hat{N} = c_1^\dagger c_1 + c_2^\dagger c_2 + c_3^\dagger c_3\,?$$

Find the eigenvalues and eigenvectors of $\hat{K}$.

Problem 13. Consider the transfer operators

$$X_{12} := c_1^\dagger c_2 + c_2^\dagger c_1, \quad X_{23} := c_2^\dagger c_3 + c_3^\dagger c_2, \quad X_{34} := c_3^\dagger c_4 + c_4^\dagger c_3, \quad X_{41} := c_4^\dagger c_1 + c_1^\dagger c_4$$

and the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = X_{12} + X_{23} + X_{34}.$$

(i) Let

$$\hat{C} := [X_{12}, X_{23}] + [X_{23}, X_{34}].$$

Find the commutator $[\hat{K}, \hat{C}]$ and anticommutator $[\hat{K}, \hat{C}]_+$.
(ii) Let $\hat{D} := X_{12} + X_{23} + X_{34} + X_{41}$. Find the commutator $[\hat{K}, \hat{D}]$ and anticommutator $[\hat{K}, \hat{D}]_+$.

Problem 14. (i) Consider the Hamilton operator on a one-dimensional lattice

$$\hat{H} = t \sum_{j=0}^{N-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + h \sum_{j=0}^{N-1} c_j^\dagger c_j$$

with cyclic boundary conditions, i.e. $N \equiv 0$. Solve the eigenvalue problem for two particle states, i.e. the states are given by

$$c_j^\dagger c_k^\dagger |0\rangle, \qquad j, k = 0, 1, \ldots, N.$$

(ii) Let $N \geq 2$. Study the Hamilton operator (open ends)

$$\hat{H} = \sum_{j=2}^{N} (c_j^\dagger c_{j-1} + c_{j-1}^\dagger c_j + \frac{1}{2}(2\hat{N}_j - I)(2\hat{N}_{j-1} - I))$$

where $\hat{N}_j := c_j^\dagger c_j$.

Problem 15. Let $c_j^\dagger$, c_j ($j = 0, 1, \ldots, N$) be Fermi creation and annihilation operators and

$$\hat{N} = \sum_{j=0}^{N} c_j^\dagger c_j$$

be the number operator.
(i) Study the one-dimensional Hamilton operator (open ends)

$$\hat{H} = t_1 \sum_{j=0}^{N-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + t_2 \sum_{j=0}^{N-1} (c_j^\dagger c_{j+1}^\dagger + c_{j+1} c_j).$$

Does the Hamilton operator $\hat{H}$ commute with the number operator?
(ii) Study the Hamilton operator

$$\hat{H} = \hbar\omega \sum_{j=0}^{N} c_j^\dagger c_j + \gamma \sum_{j=0}^{N} (c_j^\dagger + c_j).$$

Problem 16. Consider the hermitian Hamilton operator

$$\hat{H} = t(c_N^\dagger c_{N-1}^\dagger \cdots c_2^\dagger c_1^\dagger + c_1 c_2 \cdots c_{N-1} c_N)$$

and the number operator

$$\hat{N} = \sum_{j=1}^{N} c_j^\dagger c_j.$$

Find the commutator $[\hat{H}, \hat{N}]$. Find the matrix representation of $\hat{H}$ and the eigenvalues and eigenvectors. Utilize

$$c_k^\dagger = \overbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \left(\frac{1}{2}\sigma_+\right) \otimes I_2 \otimes I_2 \otimes \cdots \otimes I_2}^{N\text{-times}}$$

$$c_k = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \underbrace{\left(\frac{1}{2}\sigma_-\right)}_{k\text{-th place}} \otimes I_2 \otimes I_2 \otimes \cdots \otimes I_2.$$

Consider first the cases $n = 1$ and $n = 2$.

Problem 17. Let $c_{j\sigma}^\dagger$, $c_{j\sigma}$ be Fermi creation and annihilation operators with spin σ ($\sigma \in \{\uparrow, \downarrow\}$) Consider the Hamilton operator

$$\hat{H} = t\sum_{j=1}^{2}(c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j+1\uparrow}^\dagger c_{j\uparrow} + c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j+1\uparrow}^\dagger c_{j\uparrow})$$

with periodic boundary conditions, i.e. $c_{3\sigma}^\dagger = c_{1,\sigma}$. Find the time-evolution of $c_{1,\uparrow}^\dagger$, $c_{2,\uparrow}$.

Problem 18. Let $c_{j\uparrow}^\dagger$, $c_{j\downarrow}^\dagger$, $c_{j\uparrow}$, $c_{j\downarrow}$ be Fermi creation and annihilation operators with spin up and spin down. Consider the Hamilton operator

$$\hat{K} = \lambda_1 \sum_{j=1}^{N} \hat{N}_{j\uparrow}\hat{N}_{j\uparrow} + \lambda_2 \sum_{j=1}^{N} \hat{N}_{j\uparrow} + \lambda_3 \sum_{j=1}^{N} \hat{N}_{j\downarrow}$$

where $\hat{N}_{j\sigma} := c_{j\sigma}^\dagger c_{j\sigma}$. Consider the density matrix

$$W = \frac{e^{-\beta\hat{K}}}{\text{tr}e^{-\beta\hat{K}}}.$$

Calculate the trace

$$\text{tr}(a_1 a_2 \dots a_n W)$$

where

$$a_i \in \{\, c_{j\uparrow}^+, c_{j\uparrow}, c_{j\downarrow}^+, c_{j\downarrow} \;:\; j = 1, 2, \dots, N\}.$$

Note that $\hat{K}$ commutes with the operators

$$\sum_{i=1}^{N} \hat{N}_{i\uparrow} \quad \text{and} \quad \sum_{i=1}^{N} \hat{N}_{i\downarrow}$$

the trace vanishes unless $(a_1 a_2 \dots a_n)$ contains an equal number of creation and annihilation operators with spin up and also an equal number of creation and annihilation operators with spin down. Terms like

$$\text{tr}(c_{j\uparrow}^+ c_{j\downarrow} W)$$

therefore vanish. In the expression

$$X := \text{tr}(a_1 a_2 \dots a_n W)$$

one now anticommutes a_1 successively to the right, each time extracting the anticommutator (either 0 or 1) from the trace. In particular show the prove the following three special cases

$$\operatorname{tr}\left(\hat{N}_{i\uparrow}W\right) = \frac{e^{\beta(\lambda_1+\lambda_3)}+1}{Z_0}\operatorname{tr}W$$
$$\operatorname{tr}\left(\hat{N}_{i\downarrow}W\right) = \frac{e^{\beta(\lambda_1+\lambda_2)}+1}{Z_0}\operatorname{tr}W$$
$$\operatorname{tr}\left(\hat{N}_{i\uparrow}\hat{N}_{i\downarrow}W\right) = \frac{1}{Z_0}\operatorname{tr}(W).$$

Problem 19. Consider the Hamilton operator

$$\hat{H} = t(c_{1\uparrow}^\dagger c_{2\uparrow} + c_{1\downarrow}^\dagger c_{2\downarrow} + c_{2\uparrow}^\dagger c_{1\uparrow} + c_{2\downarrow}^\dagger c_{1\downarrow}) + U_1 n_{1\uparrow}n_{1\downarrow} + U_2 n_{2\uparrow}n_{2\downarrow}.$$

Consider the six dimensional basis

$$c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle, \quad c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle, \quad c_{1\uparrow}^\dagger c_{2\uparrow}^\dagger|0\rangle, \quad c_{1\downarrow}^\dagger c_{2\downarrow}^\dagger|0\rangle, \quad c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle, \quad c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle.$$

Find the matrix representation of $\hat{H}$ and the eigenvalues of the matrix.

Problem 20. Study the Fermi Hamilton operator with three lattice sites and open ends

$$\hat{H} = \sum_{j=0}^{2}\sum_{\sigma\in\{\uparrow,\downarrow\}}(c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}) + V\sum_{j=0}^{2}(c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{j+1,\downarrow}c_{j+1,\uparrow} + c_{j+1,\uparrow}^\dagger c_{j+1,\downarrow}^\dagger c_{j,\downarrow}c_{j,\uparrow}).$$

Is the Hamilton operator hermitian? Does the Hamilton operator $\hat{H}$ commute with the number operator

$$\hat{N} = \sum_{j=0}^{2}\sum_{\sigma\in\{\uparrow,\downarrow\}} c_{j,\sigma}^\dagger c_{j,\sigma}\ ?$$

Problem 21. Consider the Fermi creation and annihilation operators

$$c_{j,\sigma}^\dagger \quad c_{j,\sigma}, \quad j=1,2 \quad \sigma\in\{\uparrow,\downarrow\}$$

and the basis with the 6 elements

$$c_{1,\uparrow}^\dagger c_{1,\downarrow}^\dagger|0\rangle,\ c_{2,\uparrow}^\dagger c_{2,\downarrow}^\dagger|0\rangle,\ c_{1,\uparrow}^\dagger c_{2,\uparrow}^\dagger|0\rangle,\ c_{1,\downarrow}^\dagger c_{2,\downarrow}^\dagger|0\rangle,\ c_{1,\uparrow}^\dagger c_{2,\downarrow}^\dagger|0\rangle,\ c_{1,\downarrow}^\dagger c_{2,\uparrow}^\dagger|0\rangle.$$

(i) Study the Hamilton operator with spin flip terms

$$\hat{H} = \hbar\omega(c_{1,\uparrow}^\dagger c_{2,\downarrow} + c_{1,\downarrow}^\dagger c_{2,\uparrow} + c_{2,\downarrow}^\dagger c_{1,\uparrow} + c_{2,\uparrow}^\dagger c_{1,\downarrow})(\hat{N}_{1,\uparrow} + \hat{N}_{1,\downarrow} + \hat{N}_{2,\uparrow} + \hat{N}_{2,\downarrow} - 2\hat{N}_{1,\uparrow}\hat{N}_{2,\downarrow}).$$

(ii) Study the Hamilton operator with spin flip terms

$$\hat{H}_1 = t_1(c_{1,\uparrow}^\dagger c_{2,\uparrow} + c_{2,\uparrow}^\dagger c_{1,\uparrow} + c_{1,\downarrow}c_{2,\downarrow} + c_{2,\downarrow}c_{1,\downarrow}) + t_2(c_{1,\uparrow}^\dagger c_{2,\downarrow} + c_{2,\downarrow}^\dagger c_{1,\uparrow}c_{1,\downarrow}^\dagger c_{2,\uparrow} + c_{2,\uparrow}^\dagger c_{1,\downarrow})$$

and

$$\hat{H}_2 = t_1(c_{1,\uparrow}^\dagger c_{2,\uparrow} + c_{2,\uparrow}^\dagger c_{1,\uparrow}) + c_{1,\downarrow}^\dagger c_{2,\downarrow} + c_{2,\downarrow}^\dagger c_{1,\downarrow} + t_2(c_{1,\uparrow}^\dagger c_{1,\downarrow}^\dagger + c_{2,\uparrow}^\dagger c_{2,\downarrow}).$$

(iii) Study the Hamilton operator

$$\hat{H} = t(c_{1,\uparrow}^\dagger c_{2,\uparrow} + c_{2,\uparrow}^\dagger c_{1,\uparrow} + c_{1,\downarrow}^\dagger c_{2,\downarrow} + c_{2,\downarrow}^\dagger c_{1,\downarrow}).$$

Problem 22. Let $\epsilon \in [0,1]$. Consider the one-dimensional Hamilton operator (open end boundary condition and L even)

$$\hat{H} = -\sum_{j=1}^{L-1}\left(\frac{1}{2}(1+\epsilon)\sigma_{1,j}\sigma_{1,j+1} + \frac{1}{2}(1-\epsilon)\sigma_{2,j}\sigma_{2,j+1}\right).$$

Show that under the *Jordan-Wigner transformation* one finds the Hamilton operator

$$\hat{H} = -\sum_{j=1}^{L-1}\left(\left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right) + \epsilon\left(c_j^\dagger c_{j+1}^\dagger + c_{j+1}c_j\right)\right)$$

where $c_j^\dagger$, c_j are Fermi creation and annihilation operators, respectively at lattice site j.

Problem 23. Let $\mathbb{Z}$ be the set of integers. Consider the Hamilton operator

$$\hat{H} = \sum_{j\in\mathbb{Z}} \epsilon_j c_j^\dagger c_j + \sum_{j\in\mathbb{Z}} V_{j+1,j}(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1})$$

where $c_j^\dagger$, c_j are Fermi creation and annihilation operators. and $V_{j,j+1}$ (real and positive) are the tunneling matrix connecting lattice site j to the lattice site $j+1$. Show that the corresponding eigenvalue equation is

$$\epsilon_j C_j + V_{j+1,j}C_{j+1} + V_{j,j-1}C_{j-1} = EC_j, \quad j \in \mathbb{Z}.$$

So we have a linear bounded operator in the Hilbert space $\ell_2(\mathbb{Z})$.

Problem 24. Let $\mathbb{Z}$ be the set of integers. Consider the Hamilton operator

$$\hat{H} = \sum_{n\in\mathbb{Z}}(\epsilon_n c_n^\dagger c_n + V_n(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}))$$

and the wave function

$$|\psi\rangle = \sum_{j\in\mathbb{Z}} f_j c_j^\dagger |0\rangle.$$

Find $\hat{H}|\psi\rangle$.

Problem 25. Let $j \in \mathbb{Z}$. Let $c_j^\dagger$, c_j be Fermi creation and annihilation operators at a lattice site j of a one-dimensional infinite chain. Let $W, t > 0$. The one-dimensional *Aubry Hamilton operator* is given by

$$\hat{H} = \sum_{j=-\infty}^{\infty}(W\cos(kn)c_j^\dagger c_j + t(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1})).$$

The operator commutes with the number operator $\hat{N}$.

(i) Consider the self-adjoint operators

$$X_{j,j+1} := c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}, \quad X_{j-1,j} := c_{j-1}^\dagger c_j + c_j^\dagger c_{j-1}.$$

Find the commutator $[X_{j,j+1}, X_{j-1,j}]$.
(ii) Consider the basis $\{c_j^\dagger|0\rangle\}$ ($j \in \mathbb{Z}$). Show that the matrix representation of $\hat{H}$ is

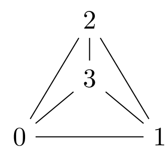
$$H_{jj} = W\cos(kn), \quad H_{j,j+1} = H_{j+1,j} = t, \quad H_{jk} = 0\,,\text{otherwise}$$

where $j = -\infty, \ldots, -1, 0, 1, \ldots, +\infty$.

Problem 26. Consider the hermitian Hamilton operator

$$\hat{H} = \hbar\omega(c_0^\dagger c_1 + c_1^\dagger c_0 + c_0^\dagger c_2 + c_2^\dagger c_0 + c_0^\dagger c_3 + c_3^\dagger c_0 + c_1^\dagger c_2 + c_2^\dagger c_1 + c_1^\dagger c_3 + c_3^\dagger c_1 + c_2^\dagger c_3 + c_3^\dagger c_2)$$

for the *tetrahedron*



The Hamilton operator $\hat{H}$ commutes with the number operator

$$\hat{N} = \sum_{j=0}^{3} c_j^\dagger c_j.$$

Consider the six-dimensional basis for two Fermi particles

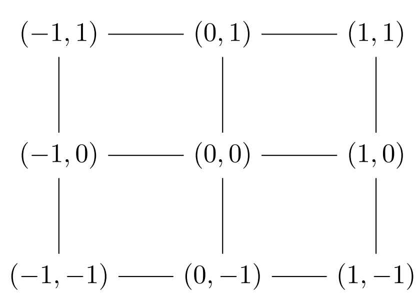
$$c_1^\dagger c_0^\dagger|0\rangle, \quad c_2^\dagger c_0^\dagger|0\rangle, \quad c_3^\dagger c_0^\dagger|0\rangle, \quad c_2^\dagger c_1^\dagger|0\rangle, \quad c_3^\dagger c_1^\dagger|0\rangle, \quad c_3^\dagger c_2^\dagger|0\rangle.$$

Find the matrix representation for the Hamilton operator $\hat{H}$. Find the eigenvalues and eigenvectors of the 6×6 matrix. Discuss the symmetry of the tetrahedron and these solutions.

Problem 27. Study the Hamilton operator

$$\begin{aligned}\hat{H} = \hbar\omega(&c_{-1,1}^\dagger c_{0,1} + c_{0,1}^\dagger c_{-1,1} + c_{0,1}^\dagger c_{1,1} + c_{1,1}^\dagger c_{0,1}\\ &+c_{-1,0}^\dagger c_{0,0} + c_{0,0}^\dagger c_{-1,0} + c_{0,0}^\dagger c_{1,0} + c_{1,0}^\dagger c_{0,0}\\ &+c_{-1,-1}^\dagger c_{-1,0} + c_{-1,0}^\dagger c_{-1,-1} + c_{-1,0}^\dagger c_{-1,1} + c_{-1,1}^\dagger c_{-1,0}\\ &+c_{-1,1}^\dagger c_{-1,0} + c_{-1,0}^\dagger c_{-1,1} + c_{-1,0}^\dagger c_{-1,-1} + c_{-1,-1}^\dagger c_{-1,0}\\ &+c_{0,1}^\dagger c_{0,0} + c_{0,0}^\dagger c_{0,1} + c_{0,0}^\dagger c_{-1,0} + c_{-1,0}^\dagger c_{0,0}\\ &+c_{1,1}^\dagger c_{1,0} + c_{1,0}^\dagger c_{1,1} + c_{1,0}^\dagger c_{-1,1} + c_{-1,1}^\dagger c_{1,0})\end{aligned}$$

for the lattice



Problem 28. Let $n \geq 2$. Consider the *tridiagonal matrix*

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & & \\ c_1 & a_2 & b_2 & & \\ 0 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix}.$$

We set $D_0 = 1$ and $D_1 = a_1$. For $n \geq 2$ we set $D_n = \det(A_n)$, i.e. the determinant of A_n. Then the determinant D_n ($n \geq 2$) satisfies the recurrence relation

$$D_n = a_n D_{n-1} - c_{n-1} b_{n-1} D_{n-2}, \quad n = 2, 3, \ldots$$

If we set $a_1 = a_2 = \cdots = a_n = -\lambda$ (λ will be the eigenvalue) then we obtain the characteristic polynomial for the matrix

$$M_n = \begin{pmatrix} 0 & b_1 & 0 & & \\ c_1 & 0 & b_2 & & \\ 0 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & 0 \end{pmatrix}.$$

The matrix ($b_j = c_j = 1$)

$$K = \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & & \\ 0 & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}$$

appears for the matrix representation of Fermi systems of the form

$$\sum_{j=1}^{N-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j).$$

Solve the characteristic equation and thus find the eigenvalues of K.

Problem 29. Let $n \geq 3$. Study the eigenvalue problem for the Hamilton operator of the one-dimensional chain (open ends)

$$\hat{H} = t \sum_{j=1}^{n-1} c_j^\dagger c_{j+1} + U c_1^\dagger c_1 c_2^\dagger c_2 \cdots c_n^\dagger c_n.$$

First show that the Hamilton operator $\hat{H}$ commutes with the number operator

$$\hat{N} = \sum_{j=1}^{n} c_j^\dagger c_j$$

and therefore we can consider the subspaces with a fixed number of Fermi operators.

Problem 30. Given the Hubbard Hamilton operator

$$\hat{H} = t \sum_{\sigma \in \{\uparrow,\downarrow\}} (c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma} + c_{2\sigma}^\dagger c_{3\sigma} + c_{3\sigma}^\dagger c_{2\sigma} + c_{3\sigma}^\dagger c_{4\sigma} + c_{4\sigma}^\dagger c_{3\sigma} + c_{4\sigma}^\dagger c_{1\sigma} + c_{1\sigma}^\dagger c_{4\sigma})$$
$$+U \sum_{j=1}^{4} \hat{N}_{j\uparrow}\hat{N}_{j\downarrow}$$

where $\hat{N}_{j\sigma} := c_{j\sigma}^\dagger c_{j\sigma}$ with $\sigma \in \{\uparrow, \downarrow\}$.
(i) Show that $[\hat{H}, \hat{N}_e] = 0$ and $[\hat{H}, \hat{S}_3] = 0$.
(ii) Given the basis

$$\{\, |1\rangle = c_{1\uparrow}^\dagger c_{2\uparrow}^\dagger |0\rangle, \quad |2\rangle = c_{1\uparrow}^\dagger c_{3\uparrow}^\dagger |0\rangle, \quad |3\rangle = c_{1\uparrow}^\dagger c_{4\uparrow}^\dagger |0\rangle,$$

$$|4\rangle = c_{2\uparrow}^\dagger c_{3\uparrow}^\dagger |0\rangle, \quad |5\rangle = c_{2\uparrow}^\dagger c_{4\uparrow}^\dagger |0\rangle, \quad |6\rangle = c_{3\uparrow}^\dagger c_{4\uparrow}^\dagger |0\rangle \,\}.$$

Calculate the matrix representation of $\hat{H}$ using this basis, i.e. $\langle j|\hat{H}|k\rangle$. Find the eigenvalues of this 6×6 matrix.

Problem 31. Let L be the number of lattice sites counting from $j = 0$ to $j = L - 1$. Consider the Hamilton operator (one-dimensional Hubbard model)

$$\hat{H} = t \sum_{j=0}^{L-1} \sum_{\sigma \in \{\uparrow,\downarrow\}} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}) + U \sum_{j=0}^{L-1} \hat{N}_{j,\uparrow}\hat{N}_{j,\downarrow}$$

with cyclic boundary conditions, i.e. $c_L = c_0$ and $c_L^\dagger = c_0^\dagger$.
(i) Consider the operators

$$\hat{R}_- := \sum_{j=0}^{L-1} (-1)^{j+1} c_{j\uparrow} c_{j\downarrow}, \quad \hat{R}_+ := \sum_{j=0}^{L-1} (-1)^{j+1} c_{j\downarrow}^\dagger c_{j\uparrow}^\dagger$$

and

$$\hat{R}_3 := \frac{1}{2} \sum_{j=0}^{L-1} (\hat{N}_{j\uparrow} + \hat{N}_{j\downarrow} - I).$$

Find the commutators $[R_+, \hat{H}]$, $[R_-, \hat{H}]$, $[R_3, \hat{H}]$. Discuss.
(ii) Consider the operators

$$\hat{S}_- := \sum_{j=0}^{L-1} c_{j\uparrow}^\dagger c_{j\downarrow}, \quad \hat{S}_+ := \sum_{j=0}^{L-1} c_{j\downarrow}^\dagger c_{j\uparrow}, \quad \hat{S}_3 = \frac{1}{2} \sum_{j=0}^{L-1} (n_{j\uparrow} + n_{j\downarrow} - I).$$

Find the commutators $[S_+, \hat{H}]$, $[S_-, \hat{H}]$, $[S_3, \hat{H}]$. Discuss.
(iii) The *particle hole operation* is given by

$$C \begin{pmatrix} c_{j\uparrow} \\ c_{j\downarrow} \end{pmatrix} C^{-1} = (-1)^j \begin{pmatrix} c_{j\uparrow}^\dagger \\ c_{j\downarrow}^\dagger \end{pmatrix}.$$

Apply it to the one-dimensional Hubbard model.

Problem 32. Consider the 2×2 density matrix (pure state)

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Let $c_1^\dagger$, $c_2^\dagger$ be Fermi creation operators and c_1, c_2 be Fermi annihilation operators. Consider the operator

$$\hat{W} = \begin{pmatrix} c_1^\dagger & c_2^\dagger \end{pmatrix} \rho \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

(i) Show that the operator is hermitian.
(ii) Find $\hat{W}^2$.
(iii) Is the operator $\hat{W}$ a density operator?

Problem 33. Let $c_1^\dagger$, $c_2^\dagger$, c_1, c_2 be Fermi creation and annihilation operators, respectively. Find the matrix representation of the hermitian operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = c_2^\dagger c_1^\dagger + c_1 c_2$$

with the four dimensional basis $c_2^\dagger c_1^\dagger|0\rangle$, $c_2^\dagger|0\rangle$, $|0\rangle$. Then find the eigenvalues and eigenvectors of the hermitian 4×4 matrix.

Solution 33. We have

$$\begin{aligned} \hat{K}c_2^\dagger c_1^\dagger|0\rangle &= |0\rangle \\ \hat{K}c_2^\dagger|0\rangle &= 0|0\rangle \\ \hat{K}c_1^\dagger|0\rangle &= 0|0\rangle \\ \hat{K}|0\rangle &= c_2^\dagger c_1^\dagger|0\rangle. \end{aligned}$$

Hence the matrix representation is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

with the eigenvalues $+1$, -1, 0 (twice).

Problem 34. Let $c_1^\dagger$, $c_2^\dagger$, c_1, c_2 be Fermi creation and annihilation operators, respectively. Find the Lie algebra generated by the operators $A := c_1^\dagger c_2$, $B := c_2^\dagger c_1$. Discuss.

Solution 34. We obtain

$$[A, B] = c_1^\dagger c_1 - c_2^\dagger c_2 = C.$$

It follows that

$$[A, C] = -2c_2^\dagger c_1 = -2A$$

$$[B, C] = 2c_2^\dagger c_1 = 2B.$$

Thus we have a three-dimensional Lie algebra with the basis $\{A, B, C\}$ and the commutation relations

$$[A, B] = C, \quad [A, C] = -2A, \quad [B, C] = 2B.$$

The Lie algebra is simple.

Problem 35. Consider the spin-$\frac{1}{2}$ matrices

$$S_1 = \frac{1}{2}\sigma_1, \quad S_2 = \frac{1}{2}\sigma_2, \quad S_3 = \frac{1}{2}\sigma_3.$$

Find the operators

$$\hat{S}_1 = \begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix} S_1 \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}, \quad \hat{S}_2 = \begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix} S_2 \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}, \quad \hat{S}_3 = \begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix} S_3 \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}.$$

Solution 35. We obtain

$$\hat{S}_1 = \frac{1}{2}(c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow)$$
$$\hat{S}_2 = \frac{i}{2}(-c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow$$
$$\hat{S}_3 = \frac{1}{2}(c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow.$$

Chapter 4

Lie Algebras

4.1 Lie Algebras and Bose Operators

Problem 1. Let $b^\dagger$, b be Bose creation and annihilation operators. Calculate the commutator

$$[\sqrt{b^\dagger b}, b^\dagger + b].$$

Find the Lie algebra generated by

$$b^\dagger b, \quad \sqrt{b^\dagger b}, \quad b^\dagger + b.$$

Solution 1.

4.2 Lie Algebras and Spin Operators

4.3 Lie Algebras and Fermi Operators

Problem 2. Find the Lie algebra generated by the operators

$$c_1^\dagger c_2, \quad c_2^\dagger c_1, \quad c_1^\dagger c_1 c_2^\dagger c_2.$$

Is the Lie algebra semi-simple? Find the adjoint representation.

Solution 2.

4.4 Lie Superalgebra

4.5 Supplementary Problems

Problem 1. Let $b^\dagger$, b be Bose creation and annihilation operators and $\hat{N} = b^\dagger b$ be the number operator.
(i) Show that the generators of the Lie algebra $so(2,1)$ expressed in Bose creation and annihilation operators are given by

$$K_+ = -\frac{i}{2}b^\dagger b^\dagger, \quad K_- = \frac{i}{2}bb, \quad K_0 = \frac{1}{2}(b^\dagger b + \frac{1}{2}I).$$

(ii) The Lie algebra $su(1,1)$ consists of the three basis elements $\{K_0, K_+, K_-\}$ which satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0.$$

The *Casimir invariant* is

$$C = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+).$$

Let $b^\dagger$, b be Bose creation and annihilation operators. Let $k = 1, 2, \ldots$. Show that

$$K_0 = b^\dagger b + kI, \quad K_+ = b^\dagger(2kI + b^\dagger b)^{1/2}, \quad K_- = (2kI + b^\dagger b)^{1/2}b$$

is a representation. Show that the eigenvalues of C are $k(k-1)$.
(iii) Let $\hat{N} = b^\dagger b$ be the number operator. Do the matrices

$$K_+ = \hat{N}^{1/2}b^\dagger, \quad K_- = b\hat{N}^{1/2}, \quad K_0 = \hat{N} + \frac{1}{2}I$$

form a basis of a Lie algebras? This means calculate the commutators

$$[K_+, K_-], \quad [K_+, K_0, \quad [K_-, K_0].$$

Show that ($\beta \in \mathbb{C}$)

$$\exp(\beta K_+ - \bar{\beta}K_-)|0\rangle = (1 - |z|^2)^{1/2} \sum_{n=0}^{\infty} z^n |n\rangle$$

where

$$z := \frac{\beta}{|\beta|}\tanh(|\beta|), \quad |z| < 1.$$

Problem 2. (i) Let $b_1, b_2, \ldots, b_n$ be the Bose annihilation operators. Let

$$\hat{N}_j := b_j^\dagger b_j, \quad j = 1, 2, \ldots, n.$$

Consider the operators

$$L_1 := \frac{1}{2}(b_1^\dagger b_2^\dagger \cdots b_{n-1}^\dagger b_n + b_1 b_2 \cdots b_{n-1} b_n^\dagger)$$
$$L_2 := \frac{1}{2i}(b_1^\dagger b_2^\dagger \cdots b_{n-1}^\dagger b_n - b_1 b_2 \cdots b_{n-1} b_n^\dagger)$$
$$L_3 := \frac{1}{2}((\hat{N}_1 + I)\prod_{j=1}^{n-1}\hat{N}_j - \hat{N}_n \prod_{j=1}^{n-1}(\hat{N}_j + I)).$$

Find the commutation relations $[L_1, L_2]$, $[L_2, L_3]$, $[L_3, L_1]$.

(ii) Let $b_1, b_2, \ldots, b_n$ be the Bose annihilation operators. Let

$$\hat{N}_j := b_j^\dagger b_j, \quad j = 1, 2, \ldots, n.$$

Consider the operators

$$H_1 := \frac{1}{2}(b_1^\dagger b_2^\dagger \cdots b_n^\dagger + b_1 b_2 \cdots b_n)$$
$$H_2 := \frac{1}{2i}(b_1^\dagger b_2^\dagger \cdots b_n^\dagger - b_1 b_2 \cdots b_n)$$
$$H_3 := \frac{1}{2}\left(\prod_{j=1}^{n} \hat{N}_j - \prod_{j=1}^{n}(\hat{N}_j + I)\right).$$

Find the commutation relations $[H_1, H_2]$, $[H_2, H_3]$, $[H_3, H_1]$. Discuss.

Problem 3. Show that the ten operators

$$\hat{O}_1 = b_1^\dagger b_1^\dagger + b_1^\dagger b_1 + b_1 b_1^\dagger + b_1 b_1$$
$$\hat{O}_2 = b_2^\dagger b_2^\dagger + b_2^\dagger b_2 + b_2 b_2^\dagger + b_2 b_2$$
$$\hat{O}_3 = b_1^\dagger b_1^\dagger - b_1^\dagger b_1 - b_1 b_1^\dagger + b_1 b_1$$
$$\hat{O}_4 = b_2^\dagger b_2^\dagger - b_2^\dagger b_2 - b_2 b_2^\dagger + b_2 b_2$$
$$\hat{O}_5 = b_1^\dagger b_2^\dagger + b_1^\dagger b_2 + b_1 b_2^\dagger + b_1 b_2$$
$$\hat{O}_6 = b_1^\dagger b_2^\dagger - b_1^\dagger b_2 - b_1 b_2^\dagger + b_1 b_2$$
$$\hat{O}_7 = i(b_1^\dagger b_1^\dagger - b_1 b_1)$$
$$\hat{O}_8 = i(b_2^\dagger b_2^\dagger - b_2^\dagger b_2)$$
$$\hat{O}_9 = i(b_1^\dagger b_2^\dagger - b_1^\dagger b_2 + b_1 b_2^\dagger - b_1 b_2)$$
$$\hat{O}_{10} = i(b_1^\dagger b_2^\dagger + b_1^\dagger b_2 - b_1 b_2^\dagger - b_1 b_2)$$

form a basis of the Lie algebra $o(3, 2)$.

Problem 4. Let $j, k \in \{1, 2\}$ and $s \in \{1, 2, \ldots, n\}$. Let $b_{js}^\dagger$, b_{js} be Bose creation and annihilation operators and I the identity operator. Consider the operators

$$C_{jk} := \sum_{s=1}^{n} b_{js}^\dagger b_{ks} + \frac{1}{2} n \delta_{jk} I, \qquad B_{jk} := \sum_{s=1}^{n} b_{js} b_{ks}$$

and thus

$$B_{jk}^\dagger = \sum_{s=1}^{n} b_{js}^\dagger b_{ks}^\dagger.$$

Show that the operators C_{jk}, B_{jk} and $B_{jk}^\dagger$ form a basis of the non-compact symplectic Lie algebra $sp(4, \mathbb{R})$. Note that $B_{jk} = B_{kj}$ and $B_{jk}^\dagger = B_{kj}^\dagger$ with $(j \neq k)$. Thus the dimension of the Lie algebra is $n(2n + 1) = 10$ where $n = 2$.

Problem 5. Consider the operators

$$\hat{K}_- = b_1 b_2, \quad \hat{K}_+ = b_1^\dagger b_2^\dagger, \quad \hat{K}_0 = \frac{1}{2}(b_1^\dagger b_1 + b_2^\dagger b_2 + I)$$

and the Hamilton operator

$$\hat{H} = \hbar\omega_1(b_1^\dagger b_1 + b_2^\dagger b_2) + \hbar\omega_2 b_1^\dagger b_1 b_2^\dagger b_2 + \hbar\omega_3(b_1^\dagger b_1^\dagger b_1 b_1 + b_2^\dagger b_2^\dagger b_2 b_2).$$

Show that the Hamilton operator can be expressed as

$$\hat{H} = (2\hbar\omega_1 - \hbar\omega_3)I + (2\hbar\omega_1 - 6\hbar\omega_3)\hat{K}_0 + 4\hbar\omega_3\hat{K}_0^2 + (\hbar\omega_2 - 2\hbar\omega_3)\hat{K}_+\hat{K}_-.$$

Problem 6. Let b_j, $b_j^\dagger$ be Bose annihilation and creation operators.
(i) Show that a basis of the Lie algebra $sp(2n,\mathbb{R})$ is given by the $n(2n+1)$ bilinear operators

$$\begin{aligned} H_j &= \frac{1}{2}(b_j^\dagger b_j + b_j b_j^\dagger) \quad j = 1,\ldots,n \\ C_{jk} &= b_j^\dagger b_k \quad j \neq k, \quad j,k = 1,\ldots,n \\ & b_j^\dagger b_k^\dagger, \;\; b_j b_k \quad j \leq k, j = 1,\ldots,n. \end{aligned}$$

(ii) Show that the n^2 operators H_j and C_{jk} form a basis of the Lie algebra $u(n)$ which is a sub Lie algebra of $sp(2n,\mathbb{R})$.

Problem 7. Let $b_{jn}^\dagger$, b_{km} $(j,k = 1,2,3; m,n = 1,\ldots,N)$ be Bose creation and annihilation operators, respectively with the commutation relation $[b_{km}, b_{jn}] = \delta_{jk}\delta_{mn}I$, where I is the identity operator. Show that a realization of the non-compact symplectic Lie algebra $sp(2n,\mathbb{R})$ $(n = 3)$ is given by

$$A_{jk} = \sum_{n=1}^N b_{jn}^\dagger b_{kn}^\dagger, \quad B_{jk} = \sum_{n=1}^N b_{jn} b_{kn}, \quad C_{jk} = \frac{1}{2}\sum_{n=1}^N (b_{jn}^\dagger b_{kn} + b_{kn} b_{jn}^\dagger).$$

Problem 8. Let $b_k^\dagger$, b_k be Bose creation and annihilation operators. Consider the operators

$$P_+ := -\frac{1}{2}\sum_{k=1}^n b_k^\dagger b_k^\dagger, \quad P_- := \frac{1}{2}\sum_{k=1}^n b_k b_k, \quad P_3 := \frac{1}{4}\sum_{k=1}^n (b_k^\dagger b_k + b_k b_k^\dagger).$$

Find the commutators. Discuss.

Problem 9. Show that the 10 operators

$$\begin{aligned} J_0 &= \frac{1}{2}(b_1^\dagger b_1 + b_2^\dagger b_2 + I) \\ J_1 &= \frac{1}{2}(b_1^\dagger b_2 + b_2^\dagger b_1) \\ J_2 &= \frac{i}{2}(b_2^\dagger b_1 - b_1^\dagger b_2) \\ J_3 &= \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2) \\ K_1 &= \frac{1}{4}((b_1^\dagger)^2 - b_1^2 + (b_2^\dagger)^2 - (b_2)^2) \\ K_2 &= -\frac{i}{4}((b_1^\dagger)^2 - b_1^2 + (b_2^\dagger)^2 - b_2^2) \end{aligned}$$

$$K_3 = -\frac{1}{2}(b_1^\dagger b_2^\dagger + b_1 b_2)$$
$$L_1 = \frac{i}{4}((b_1^\dagger)^2 - b_1^2 - (b_2^\dagger)^2 + b_2^2)$$
$$L_2 = \frac{1}{4}((b_1^\dagger)^2 + b_1^2 + (b_2^\dagger)^2 + b_2^2)$$
$$L_3 = -\frac{i}{2}(b_1^\dagger b_2^\dagger - b_1 b_2)$$

form a basis of a Lie algebra. Find all Lie sub-algebras.

Problem 10. Consider the eigenvalue problem $\hat{H}u(x) = Eu(x)$ with the Hamilton operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 + \frac{g^2}{x^2}$$

(i) Show that introducing the dimensionless variables

$$\widetilde{u}(\widetilde{x}(x)) = u(x), \quad \widetilde{x}(x) = \left(\frac{m\omega}{\hbar}\right)^{1/2} x, \quad k^2 = \frac{mg^2}{\hbar^2}$$

we obtain the second order linear differential equation

$$\left(-\hbar\omega\frac{d^2}{d\widetilde{x}^2} + \frac{1}{2}\hbar\omega\widetilde{x}^2 + \hbar\omega\frac{k^2}{\widetilde{x}^2}\right)\widetilde{u}(\widetilde{x}) = E\widetilde{u}(\widetilde{x})$$

or

$$\left(-\frac{d^2}{d\widetilde{x}^2} + \frac{1}{2}\widetilde{x}^2 + \frac{k^2}{\widetilde{x}^2}\right)\widetilde{u}(\widetilde{x}) = \widetilde{E}\widetilde{u}(\widetilde{x})$$

where $\widetilde{E} = E/(\hbar\omega)$.
(ii) Define the differential operators

$$b = \frac{1}{\sqrt{2}}\left(\widetilde{x} + \frac{d}{d\widetilde{x}}\right), \qquad b^\dagger = \frac{1}{\sqrt{2}}\left(\widetilde{x} - \frac{d}{d\widetilde{x}}\right)$$

with $[b, b^\dagger] = I$ and

$$K_- = \frac{1}{2}\left(b^2 - \frac{k^2}{\widetilde{x}^2}\right), \quad K_+ = \frac{1}{2}\left((b^\dagger)^2 - \frac{k^2}{\widetilde{x}^2}\right), \quad K_0 = \frac{1}{2}\left(b^\dagger b + \frac{1}{2}I + \frac{k^2}{\widetilde{x}^2}\right).$$

Show that $[K_0, K_\pm] = \pm K_\pm$ and $[K_-, K_+] = 2K_0$.

Problem 11. Let $b_1^\dagger$, $b_2^\dagger$, $b_3^\dagger$ be Bose creation operators and $\hat{N}_1 := b_1^\dagger b_1$, $\hat{N}_2 := b_2^\dagger b_2$, $\hat{N}_3 := b_3^\dagger b_3$.
(i) Consider the operators

$$L_x := \frac{1}{2}(b_1^\dagger b_2^\dagger b_3 + b_1 b_2 b_3^\dagger), \quad L_y := \frac{1}{2i}(b_1^\dagger b_2^\dagger b_3 - b_1 b_2 b_3^\dagger)$$

and

$$L_z := \frac{1}{2}(\hat{N}_1\hat{N}_2(\hat{N}_3 + I) - (\hat{N}_1 + I)(\hat{N}_2 + I)\hat{N}_3).$$

Find the commutators $[L_x, L_y]$, $[L_z, L_x]$, $[L_y, L_z]$.

(ii) Consider the operators

$$H_x = \frac{1}{2}(b_1^\dagger b_2^\dagger b_3^\dagger + b_1 b_2 b_3), \quad H_y = \frac{1}{2i}(b_1^\dagger b_2^\dagger b_3^\dagger - b_1 b_2 b_3)$$

and

$$H_z = \frac{1}{2}(\hat{N}_1\hat{N}_2\hat{N}_3 - (\hat{N}_1 + I)(\hat{N}_2 + I)(\hat{N}_3 + I)).$$

Find the commutators $[H_x, H_y]$, $[H_z, H_x]$, $[H_y, H_z]$.

Problem 12. (i) Consider the operators

$$K_+ = \frac{1}{2}(b_1^\dagger b_1^\dagger + b_2^\dagger b_2^\dagger + b_3^\dagger b_3^\dagger), \quad K_- = \frac{1}{2}(b_1 b_1 + b_2 b_2 + b_3 b_3)$$

and

$$K_3 = \frac{1}{2}\left(b_1^\dagger b_1 + b_2^\dagger b_2 + b_3^\dagger b_3 + \frac{3}{2}I\right).$$

Show that $[K_3, K_\pm] = \pm K_\pm$, $[K_+, K_-] = -2K_3$ and hence we have a basis of the Lie algebra $su(1,1)$.
(ii) What Lie algebra is generated by the operators

$$\hat{K} = (b_1^\dagger)^2 b_2, \qquad \hat{K}^\dagger = b_2^\dagger b_1^2\ ?$$

Problem 13. Consider the Lie algebra $s\ell(2,\mathbb{R})$. Consider the operators $\hat{P}$, $\hat{Q}$ and the Lie algebra generated by the elements $\hat{P}^2$, $\hat{Q}^2$ and $\hat{P}\hat{Q}+\hat{Q}\hat{P}$. Show that the isomorphism between $s\ell(2,\mathbb{R})$ and the Lie algebra generated by the elements $\hat{P}^2$, $\hat{Q}^2$ and $\hat{P}\hat{Q}+\hat{Q}\hat{P}$ has the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \leftrightarrow \frac{1}{2}(a(\hat{P}\hat{Q}+\hat{Q}\hat{P}) + b\hat{Q}^2 - c\hat{P}^2).$$

Problem 14. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Show that the irreducible matrix representation of the *Clifford algebra* $C(5,0)$, modulo overall sign, is unique up to the unitary transformation and can be given by the 4×4 matrices

$$\alpha_j = I_2 \otimes \sigma_j, \quad j = 1, 3$$

$$\alpha_2 = \sigma_2 \otimes \sigma_2, \quad \alpha_4 = \sigma_1 \otimes \sigma_2, \quad \alpha_5 = \sigma_3 \otimes \sigma_2.$$

Find the anticommutator $[\alpha_4, \alpha_5]_+$.

Problem 15. The real Lie algebra $s\ell(2,\mathbb{C})$ is a six-dimensional Lie algebra with the basis

$$J_\pm, \quad J_0, \quad K_\pm, \quad K_0.$$

The non-vanishing commutators are

$$[J_0, J_+] = J_+, \quad [J_0, K_+] = K_+, \quad [K_0, K_+] = -J_+, \quad [K_0, J_+] = K_+,$$

$$[J_0, J_-] = -J_-, \quad [J_0, K_-] = -K_-, \quad [K_0, K_-] = J_-, \quad [K_0, J_-] = -K_-,$$

$$[J_+, J_-] = 2J_0, \quad [J_+, K_-] = 2K_0, \quad [K_+, K_-] = -2J_0, \quad [J_-, K_+] = 2K_0.$$

Show that the two quadratic Casimir operators are given by

$$C_1 = J_0^2 + \frac{1}{2}(J_+J_- + J_-J_+) - K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+)$$
$$C_2 = J_0K_0 + \frac{1}{2}(J_+K_- + J_-K_+) + K_0J_0 + \frac{1}{2}(K_+J_- + K_-J_+).$$

Let σ_1, σ_2, σ_3 be the Pauli spin matrices, and $\sigma_\pm := \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and

$$\mathbf{z} = \begin{pmatrix} z_1 & z_2 \end{pmatrix}, \quad \bar{\mathbf{z}} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}, \quad \frac{\partial}{\partial \mathbf{z}} = \begin{pmatrix} \partial/\partial z_1 \\ \partial/\partial z_2 \end{pmatrix}, \quad \frac{\partial}{\partial \bar{\mathbf{z}}} = \begin{pmatrix} \partial/\partial \bar{z}_1 & \partial/\partial \bar{z}_2 \end{pmatrix}.$$

A realization for J_+, J_-, K_+, K_- is given by

$$J_+ = \mathbf{z}\sigma_+\frac{\partial}{\partial \mathbf{z}} - \frac{\partial}{\partial \bar{\mathbf{z}}}\sigma_+\bar{\mathbf{z}}, \qquad J_- = \mathbf{z}\sigma_-\frac{\partial}{\partial \mathbf{z}} - \frac{\partial}{\partial \bar{\mathbf{z}}}\sigma_-\bar{\mathbf{z}}$$

$$K_+ = i(\mathbf{z}\sigma_+\frac{\partial}{\partial \mathbf{z}} + \frac{\partial}{\partial \bar{\mathbf{z}}}\sigma_+\bar{\mathbf{z}}), \qquad K_- = i(\mathbf{z}\sigma_-\frac{\partial}{\partial \mathbf{z}} + \frac{\partial}{\partial \bar{\mathbf{z}}}\sigma_-\bar{\mathbf{z}}).$$

Calculate the other operators from the commutation relations.

Problem 16. Let $c_1^\dagger$, $c_2^\dagger$, c_1, c_2 be Fermi creation and annihilation operators. Let $\hat{N}_1 = c_1^\dagger c_1$, $\hat{N}_2 = c_2^\dagger c_2$.
(i) Construct the Lie algebra generated by the operators

$$c_1^\dagger c_1, \quad c_2^\dagger c_2, \quad c_1^\dagger c_2, \quad c_2^\dagger c_1.$$

(ii) Construct the Lie algebra generated by the operators

$$c_1^\dagger c_1, \quad c_2^\dagger c_2, \quad c_1^\dagger c_2, \quad c_2^\dagger c_1, \quad \hat{N}_1\hat{N}_2.$$

(iii) Construct the Lie algebra generated by the operators

$$c_{1,\uparrow}^\dagger c_{1,\uparrow}, \quad c_{2,\uparrow}^\dagger c_{2,\uparrow}, \quad c_{1,\downarrow}^\dagger c_{1,\downarrow}, \quad c_{2,\downarrow}^\dagger c_{2,\downarrow}, \quad c_{1,\uparrow}^\dagger c_{2,\uparrow}, \quad c_{2,\uparrow}^\dagger c_{1,\uparrow}, \quad c_{1,\downarrow}^\dagger c_{2,\downarrow}, \quad c_{2,\downarrow}^\dagger c_{1,\downarrow}.$$

Problem 17. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $N > 2$ and $j = 1, \ldots, N-1$. Let $\sigma_{1,j}$, $\sigma_{2,j}$, $\sigma_{3,j}$ be the Pauli matrices acting on site j, i.e.

$$\sigma_{1,j} = I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1 \otimes I_2 \otimes \cdots \otimes I_2$$

where σ_1 is at j-th position and we have N terms. Thus $\sigma_{1,j}$ is a $2^N \times 2^N$ matrix. Analogously we define $\sigma_{2,j}$ and $\sigma_{3,j}$. Let

$$E_j := -\frac{1}{2}(\sigma_{1,j}\sigma_{1,j+1} + \sigma_{2,j}\sigma_{2,j+1} + \frac{1}{2}(q+q^{-1})(\sigma_{3,j}\sigma_{3,j+1} - I) + \frac{1}{2}(q-q^{-1})(\sigma_{3,j} - \sigma_{3,j+1})).$$

Show that

$$E_jE_j = (q+q^{-1})E_j$$
$$E_jE_{j\pm 1}E_j = E_j$$
$$E_jE_k = E_kE_j \quad (k \neq j \pm 1).$$

The $2^N \times 2^N$ E_j $(j = 1, \ldots, N-1)$ matrices are generators of a *Temperley-Lieb algebra.*

Problem 18. Let $c_{j\sigma}^\dagger$, $c_{j\sigma}$ be Fermi creation and annihilation operators with spin σ.
(i) Consider operators

$$S = c_{1\uparrow}^\dagger c_{1\downarrow} + c_{2\uparrow}^\dagger c_{2\downarrow}, \qquad S^\dagger = c_{1\downarrow}^\dagger c_{1\uparrow} + c_{2\downarrow}^\dagger c_{2\uparrow}$$

and

$$S_3 = \frac{1}{2}(c_{1\downarrow}^\dagger c_{1\downarrow} + c_{2\downarrow}^\dagger c_{2\downarrow} - c_{1\uparrow}^\dagger c_{1\uparrow} - c_{2\uparrow}^\dagger c_{2\uparrow}).$$

Find the commutators $[S, S^\dagger]$, $[S, S_3]$, $[S^\dagger, S_3]$. Discuss.
(iii) Consider the Hamilton operator

$$\hat{H} = -t \sum_{\sigma \in \{\uparrow,\downarrow\}} (c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma}).$$

Find the commutators $[\hat{H}, S]$, $[\hat{H}, S^\dagger]$, $[\hat{H}, S_3]$. Discuss.
(iv) Let $c_{j\sigma}^\dagger$, $c_{j\sigma}$ be Fermi creation and annihilation operators with spin σ. Consider operators

$$R_+ = -c_{1\uparrow} c_{1\downarrow} + c_{2\uparrow} c_{2\downarrow}, \qquad R_- = -c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger + c_{2\downarrow}^\dagger c_{2\uparrow}^\dagger$$

and

$$R_3 = \frac{1}{2}(c_{1\uparrow}^\dagger c_{1\uparrow} + c_{2\uparrow}^\dagger c_{2\uparrow} + c_{1\downarrow}^\dagger c_{1\downarrow} + c_{2\downarrow}^\dagger c_{2\downarrow} - I).$$

Find the commutators $[R_+, R_-]$, $[R_+, R_3]$, $[R_-, R_3]$. Discuss.
(v) Consider the Hamilton operator

$$\hat{H} = -t \sum_{\sigma \in \{\uparrow,\downarrow\}} (c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma}).$$

Find the commutators $[\hat{H}, R_+]$, $[\hat{H}, R_-]$, $[\hat{H}, R_3]$. Discuss.

Problem 19. Let $c_1^\dagger$, $c_2^\dagger$, $c_3^\dagger$ be Fermi creation operators.
(i) Consider the two operators

$$L_1 = \frac{1}{2}(c_1^\dagger c_2^\dagger c_3 + c_1 c_2 c_3^\dagger), \qquad L_2 = \frac{1}{2i}(c_1^\dagger c_2^\dagger c_3 - c_1 c_2 c_3^\dagger).$$

Find the Lie algebra generated by L_1 and L_2. Set $[L_1, L_2] = iL_3$.
(ii) Consider the two operators

$$L_1 = \frac{1}{2}(c_1^\dagger c_2^\dagger c_3^\dagger + c_1 c_2 c_3), \qquad L_2 = \frac{1}{2i}(c_1^\dagger c_2^\dagger c_3^\dagger - c_1 c_2 c_3).$$

Find the Lie algebra generated by L_1 and L_2. Set $[L_1, L_2] = iL_3$.

Problem 20. Let $c_1^\dagger$, $c_2^\dagger$, c_1, c_2 be Fermi creation and annihilation operators. Let I be the unit operator.
(i) Find the Lie algebra generated by the operators

$$c_1^\dagger c_1, \quad c_2^\dagger c_2, \quad c_1^\dagger + c_1, \quad c_2^\dagger + c_2.$$

(ii) Do the operators

$$\{\, c_1^\dagger c_2^\dagger, c_1^\dagger c_2, c_1 c_2^\dagger, c_1^\dagger c_1, c_2^\dagger c_2, c_1 c_2, I \,\}$$

form a basis of a Lie algebra under the commutator? For example we have

$$[c_1^\dagger c_2^\dagger, c_1 c_2] = I - c_1^\dagger c_1 - c_2^\dagger c_2.$$

(iii) Consider the operators

$$S_0 = c_1^\dagger c_1 + c_2^\dagger c_2, \quad S_1 = c_1^\dagger c_1 - c_2^\dagger c_2,$$

$$S_2 = c_1^\dagger c_1 e^{i\phi} + c_2^\dagger c_2 e^{-i\phi}, \quad S_3 = ic_1^\dagger c_1 e^{-i\phi} - ic_2^\dagger c_2 e^{i\phi}.$$

Find the commutators. Discuss.
(iv) Find the Lie algebra generated by the operators

$$c_1^\dagger c_1, \;\; c_2^\dagger c_2, \;\; c_1^\dagger + c_1, \;\; c_2^\dagger + c_2, \;\; c_1^\dagger + c_2, \;\; c_1 + c_2^\dagger.$$

Problem 21. Consider the Pauli spin matrices σ_0, σ_1, σ_2, σ_3, where $\sigma_0 = I_2$. One has $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_3\sigma_1 = i\sigma_2$. Let $\mu, \nu = 0, 1, 2, 3$. Consider the sixteen 4×4 matrices

$$M_{\mu\nu} = \sigma_\mu \otimes \sigma_\nu$$

where $\otimes$ is the Kronecker product.
(i) Find the commutators $[M_{\mu\nu}, M_{\alpha\beta}]$. Discuss.
(ii) Find the anticommutators $[M_{\mu\nu}, M_{\alpha\beta}]_+$. Discuss.

Problem 22. The $osp(1|2)$ superalgebra has the five generators

$$J_0, \;\; J_+, \;\; J_-, \;\; V_+, \;\; V_-$$

with the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0, \quad [J_0, V_\pm] = \pm\frac{1}{2}V_\pm, \quad [J_\pm, V_\pm] = 0, \quad [J_\pm, V_\mp] = V_\pm$$

and anticommutation relations

$$[V_\pm, V_\pm]_+ = \pm\frac{1}{2}J_\pm, \qquad [V_+, V_-]_+ = -\frac{1}{2}J_0.$$

(i) Let $b^\dagger$, b be Bose creation and annihilation operators and I the identity operator. Show that

$$J_+ = -\frac{1}{2}(b^\dagger)^2, \quad J_- = \frac{1}{2}b^2, \quad J_0 = \frac{1}{2}b^\dagger b + \frac{1}{4}I$$

$$V_+ = \frac{i}{2\sqrt{2}}b^\dagger, \qquad V_- = \frac{i}{2\sqrt{2}}b.$$

is a realization of the superalgebra.
(ii) Let $E_{(jk)}$ be a 3×3 matrix having 1 at the position of the jth row and k column and 0 otherwise. Show that a representation using these matrices and linear combinations of it is given by

$$H = \frac{1}{2}(E_{(11)} - E_{(33)}), \qquad J_+ = E_{(13)}, \qquad J_- = E_{(31)},$$

$$V_+ = \frac{1}{2}(E_{(12)} + E_{(23)}), \qquad V_- = \frac{1}{2}(-E_{(21)} + E_{(32)}).$$

Problem 23. The superalgebra $u(1|1)$ is defined as follows: The bosonic and fermionic bilinear combinations $b^\dagger b$ and $c^\dagger c$ generate the Lie algebras of $u_B(1)$ and $u_F(1)$, respectively. The Bose-Fermi bilinears

$$b \otimes c^\dagger, \qquad b^\dagger \otimes c$$

close with the set $\{b^\dagger b, c^\dagger c\}$ under the anticommutations

$$[b \otimes c^\dagger, b^\dagger \otimes c]_+ = b^\dagger b \otimes I_F + I_B \otimes c^\dagger c, \quad [b \otimes c^\dagger, b \otimes c^\dagger]_+ = [b \otimes c, b \otimes c]_+ = 0_B \otimes 0_F.$$

Note that

$$[b^\dagger b \otimes I_F, I_B \otimes c^\dagger c] = 0_B \otimes 0_F.$$

Hence, the bilinear combinations $b \otimes c^\dagger$ and $b^\dagger \otimes c$ are the odd generators and $b^\dagger b \otimes I_F$, $I_B \otimes c^\dagger c$ are the even generators of the Lie superalgebra $u(1|1)$. Find the states

$$(b \otimes c^\dagger)(|n\rangle \otimes |0\rangle_F), \quad (b \otimes c^\dagger)(|n\rangle \otimes c^\dagger|0\rangle_F), \quad (b^\dagger \otimes c)(|n\rangle \otimes |0\rangle_F), \quad (b^\dagger \otimes c)(|n\rangle \otimes c^\dagger|0\rangle_F).$$

Chapter 5

Bose-Spin Systems

5.1 Solved Problems

5.2 Supplementary Problems

Problem 1. Let

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Study the spectrum of the Hamilton operator

$$\hat{K}_1 = \frac{\hat{H}_1}{\hbar\omega} = b \otimes S_- + b^\dagger \otimes S_+.$$

(ii) Study the spectrum of the Hamilton operator

$$\hat{K}_2 = \frac{\hat{H}_2}{\hbar\omega} = b^\dagger \otimes S_- + b \otimes S_+.$$

(iii) Study the spectrum of the Hamilton operator

$$\hat{H} = \hbar\omega b^\dagger b \otimes I_2 + \gamma_1(b + b^\dagger) \otimes S_1 + \gamma_2 I_B \otimes S_3.$$

Problem 2. Consider the Hamilton operator

$$\hat{H} = \hbar\omega_1 b^\dagger b \otimes I_2 + \hbar\omega_2 I_B \otimes \sigma_3 + +\kappa(b \otimes S_+ + b^\dagger \otimes S_-).$$

(i) Find the matrix representation using the basis

$$|n\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |n\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $|n\rangle$ $(n = 0, 1, \ldots)$ are the number states.
(ii) Let $|\beta\rangle$ be a coherent state. Consider the basis

$$|\beta\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\beta\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find

$$(\langle\widetilde{\beta}| \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix})\hat{H}(|\beta\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}), \quad (\langle\widetilde{\beta}| \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix})\hat{H}(|\beta\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}),$$
$$(\langle\widetilde{\beta}| \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix})\hat{H}(|\beta\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \quad (\langle\widetilde{\beta}| \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix})\hat{H}(|\beta\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

Problem 3. Let $b^\dagger$, b Bose creation and annihilation operators, respectively. Study the spectrum of the Hamilton operator

$$\hat{H} = I_B \otimes \mu(B_1\sigma_1 + B_2\sigma_2 + B_3\sigma_3) + \hbar\omega b^\dagger b \otimes I_2 + \rho(b \otimes S_+ - b^\dagger \otimes S_-).$$

Problem 4. (i) The Hamilton operator $\hat{H}$ for the *Jaynes-Cummings model* neglecting the so-called counter-rotating terms is given by

$$\hat{H} = \hbar\omega b^\dagger b \otimes I_2 + \frac{1}{2} I_B \otimes \hbar\omega_0\sigma_3 + \gamma(b \otimes \sigma_+ + b^\dagger \otimes \sigma_-).$$

Let $|\beta\rangle$ be a coherent state and

$$|\mathbf{v}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the expectation value $(\langle\beta|\otimes\langle\mathbf{v}|)\hat{H}(|\beta\rangle\otimes|\mathbf{v}\rangle)$.
Let $|\zeta\rangle$ be a squeezed state and

$$|\mathbf{v}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the expectation value $(\langle\zeta|\otimes\langle\mathbf{v}|)\hat{H}(|\zeta\rangle\otimes|\mathbf{v}\rangle)$.
(ii) The Hamilton operator $\hat{H}$ for the two-photon Jaynes-Cummings model is given by

$$\hat{H} = \hbar\omega b^\dagger b \otimes I_2 + \frac{1}{2}I_B \otimes \hbar\omega_0\sigma_3 + \gamma(b^2 \otimes S_+ + (b^\dagger)^2 \otimes S_-).$$

Let $|\beta\rangle$ be a coherent state, i.e. $b|\beta\rangle = \beta|\beta\rangle$ and

$$|\mathbf{v}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the expectation value

$$(\langle\beta|\otimes\langle\mathbf{v}|)\hat{H}(|\beta\rangle\otimes|\mathbf{v}\rangle).$$

Let $|\zeta\rangle$ be a squeezed state and

$$|\mathbf{v}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the expectation value

$$(\langle\zeta|\otimes\langle\mathbf{v}|)\hat{H}(|\zeta\rangle\otimes|\mathbf{v}\rangle).$$

Problem 5. Let $z \in \mathbb{C}$ and

$$S_+ = \frac{1}{2}\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S_- = \frac{1}{2}\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(i) Find the commutator $[b^\dagger \otimes S_-, b \otimes S_+]$, anticommutator $[b^\dagger \otimes S_-, b \otimes S_+]_+$ and

$$\exp(z(b^\dagger \otimes S_- + b \otimes S_+))$$

(ii) Find the commutator $[b^\dagger \otimes S_+, b \otimes S_-]$, anticommutator $[b^\dagger \otimes S_+, b \otimes S_-]_+$ and

$$\exp(z(b^\dagger \otimes S_+ + b \otimes S_-)).$$

Problem 6. Let $b^\dagger$, b be Bose creation and annihilation operators with $[b, b^\dagger] = I_B$ and $\sigma_0 = I_2$, σ_1, σ_2, σ_3 be the Pauli spin matrices. The *Rabi model* is given by the Hamilton operator

$$\hat{H} = \hbar\omega b^\dagger b \otimes I_2 + \hbar\Omega I_B \otimes \sigma_3 + \gamma(b^\dagger + b) \otimes \sigma_1$$

where $2\hbar\Omega$ is the energy difference between the two levels and γ is a coupling constant.
(i) Let $|\beta\rangle$ be a coherent state. Find the expectation value

$$(\langle\beta| \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix})\hat{H}(|\beta\rangle \otimes (\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix})).$$

(ii) Let $|\zeta\rangle$ be a squeezed state. Find the expectation value

$$(\langle\beta| \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix})\hat{H}(|\beta\rangle \otimes (\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}).$$

Discuss.

Problem 7. Consider the Hamilton operator with two Bose operators $b_1 = b \otimes I_B$, $b_2 = I_B \otimes b$

$$\hat{H} = \hbar\omega_1(b^\dagger b + \frac{1}{2}I_B) \otimes I_B \otimes I_2 + \hbar\omega_1 I_B \otimes (b^\dagger b + \frac{1}{2}I_B) \otimes I_2 + \hbar\omega_2 I_B \otimes I_B \otimes \sigma_3$$
$$+\hbar\omega_3((b \otimes I_B + b^\dagger \otimes I_B) \otimes \sigma_+ + (I_B \otimes b + I_B \otimes b^\dagger) \otimes \sigma_-).$$

Let

$$\hat{K} = b^\dagger b \otimes I_B \otimes I_B - I_B \otimes b^\dagger b \otimes I_2 + I_B \otimes I_B \otimes \sigma_3.$$

Find the commutator $[\hat{H}, \hat{K}]$.

Problem 8. Let

$$U(\epsilon) = \exp(\epsilon(b - b^\dagger) \otimes \sigma_1).$$

Find

$$U(\epsilon)\left(|n\rangle \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right), \qquad U(\epsilon)\left(|n\rangle \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right).$$

Problem 9. Let

$$S_+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \quad S_- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \quad S_3 = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

Does the Hamilton operator

$$\hat{H} = \hbar\omega_1 b^\dagger b + \hbar\omega_2 I_B \otimes S_3 + \hbar\omega_3(b^\dagger \otimes S_- + b \otimes S_+) + \hbar\omega_4 b^\dagger b^\dagger bb \otimes I_2 + \hbar\omega_5 I_B \otimes (S_3)^2$$

commute with the number operator $\hat{N} = I_B \otimes S_3 + b^\dagger b \otimes I_2$?

Problem 10. Let $|n\rangle$ be the number states, $D(\beta)$ be the displacement operator and σ_1 be the first Pauli spin matrix. The normalized eigenvectors of σ_1 are given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

with eigenvalues $+1$ and -1. Consider the Hamilton operator

$$\hat{K} = b^\dagger b \otimes I_2 + \alpha(b^\dagger + b) \otimes \sigma_1$$

where α is a dimensionless real quantity. Are the product states

$$D(\alpha)|n\rangle \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \qquad D(-\alpha)|n\rangle \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

eigenstates of the Hamilton operator $\hat{H}$?

Problem 11. Consider the Hilbert space $\mathcal{H}_S$ with a (self-adjoint) Hamilton operator $\hat{H}_S$ (S stands for system). The identity operator in this Hilbert space is denoted by I_S. Let

$\mathcal{H}_E$ be a Hilbert space with a (self-adjoint) Hamilton operator $\hat{H}_E$ (S stands for environment sometimes called bath). The identity operator in this Hilbert space is denoted by I_E. Let $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ be the product Hilbert space. Consider the Hamilton operator

$$\hat{K} = \hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E + \hat{V}$$

where the operator $\hat{V}$ acts in the product Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$. The *von Neumann equation* for the density matrix of the Hamilton operator $\hat{K}$ is given by

$$\frac{d}{dt}\rho_{SE}(t) = -\frac{i}{\hbar}(\hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E + \hat{V})(t).$$

Consider the operator (switch to the interaction picture)

$$\widetilde{V}(t) = e^{i(\hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E)t/\hbar}\hat{V}e^{-i(\hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E)t/\hbar}$$

and the density matrix

$$\widetilde{\rho}(t) = e^{i(\hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E)t/\hbar}\hat{\rho}_{SE}(t)e^{-i(\hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E)t/\hbar}.$$

Therefore

$$\hat{\rho}_{SE}(t) = e^{-i(\hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E)t/\hbar}\widetilde{\rho}(t)e^{i(\hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E)t/\hbar}.$$

We apply a tilde to indicate operators in the interaction picture. Thus we obtain

$$i\hbar\frac{d\widetilde{\rho}}{dt} = [\widetilde{V}(t), \widetilde{\rho}(t)].$$

The perturbation expansion (*Dyson series*) yields

$$\widetilde{\rho}(t) = \sum_{j \geq 0} \int_0^t dt \cdots \int_0 dt_j \left(\frac{1}{i\hbar}\right)^j [\widetilde{V}(t_1), \ldots, [\widetilde{V}(t_n), \widetilde{\rho}(0)] \ldots].$$

To find the Born-Markov master equation one computes the time evolution up to second order and perform the trace over the Hilbert space $\mathcal{H}_E$ (environment, bath), i.e. the *partial trace*. One obtains

$$\frac{d\widetilde{\rho}}{dt} = \frac{1}{i\hbar}\mathrm{tr}_E[\widetilde{V}(t), \rho(0)] - \frac{1}{\hbar^2}\int_{t_1=0}^t dt_1 \mathrm{tr}_E([\widetilde{V}(t), [\widetilde{V}(t_1), \rho(0)]]).$$

One normally assumes that at $t = 0$ the density operator is a tensor product of the form $\rho(0) = \rho_S(0) \otimes \rho_E(0)$. Consider the Hilbert spaces $\mathcal{H}_S = \mathbb{C}^2$ and $\mathcal{H}_E = \ell_2(\mathbb{N}_0)$ with

$$\hat{K} = \frac{1}{2}\hbar\omega_1\sigma_3 \otimes I_B + \hbar\omega_2(S_+ \otimes b + S_- \otimes b^\dagger) + I_2 \otimes \hbar\omega_3 b^\dagger b$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $I_B = \mathrm{diag}(1, 1, 1, \ldots)$. Find $\widetilde{\rho}(t)$ with

$$\rho_S(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \rho_E(0) = \mathrm{diag}(1, 0, 0, \ldots).$$

Problem 12. Let $\mathbf{S} = (S_1, S_2, S_3)$

$$S_1 = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_3 = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the spin-$\frac{1}{2}$ matrices, $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$

$$\hat{p}_1 = -i\hbar\frac{\partial}{\partial x_1}, \quad \hat{p}_2 = -i\hbar\frac{\partial}{\partial x_2}, \quad \hat{p}_3 = -i\hbar\frac{\partial}{\partial x_3}$$

and $\mathbf{x} = (x_1, x_2, x_3)$. Let $\times$ be the vector product. Then we have

$$\mathbf{x} \times \hat{\mathbf{p}} = \begin{pmatrix} x_2\hat{p}_3 - x_3\hat{p}_2 \\ x_3\hat{p}_1 - x_1\hat{p}_3 \\ x_1\hat{p}_2 - x_2\hat{p}_1 \end{pmatrix}.$$

Let $\cdot$ be the scalar product. Then since $\hat{p}_j\hat{p}_k = \hat{p}_k\hat{p}_j$ we have

$$(\mathbf{x} \times \hat{\mathbf{p}}) \cdot \hat{\mathbf{p}} = (x_2\hat{p}_3 - x_3\hat{p}_2)\hat{p}_1 + (x_3\hat{p}_1 - x_1 p_3)\hat{p}_2 + (x_1\hat{p}_2 - x_2\hat{p}_1)\hat{p}_3 = 0.$$

Let I_2 be the 2×2 identity matrix and I be the identity operator. Consider the operator in the product space

$$\hat{\mathbf{J}} := (\mathbf{x} \times \hat{\mathbf{p}}) \otimes I_2 + I \otimes \mathbf{S}.$$

Find the operator

$$\hat{\mathbf{J}} \cdot (\hat{\mathbf{p}} \otimes I_2).$$

Chapter 6

Bose-Fermi Systems

6.1 Solved Problems

6.2 Supplementary Problems

Problem 1. Let $c^\dagger$, c be Fermi creation and annihilation operators and I_F the identity operator. Let $b^\dagger$, b be Bose creation and annihilation operators and I_B the identity operator. Consider the operators

$$b^\dagger b \otimes I_F, \quad b^\dagger b \otimes c^\dagger c, \quad I_B \otimes c^\dagger c$$

and

$$V = b^\dagger \otimes c + b \otimes c^\dagger.$$

(i) Find the commutators between these operators.
(ii) Find the anticommutator between these operators.

Problem 2. Consider the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = (b + b^\dagger) \otimes c^\dagger c.$$

(i) Find the matrix representation of $\hat{K}$ using the basis $|n\rangle \otimes |0\rangle$, $|n\rangle \otimes c^\dagger |0\rangle$.
(ii) Let $|\beta\rangle$ be a coherent state. Find

$$(\langle\beta| \otimes \langle 0|)\hat{K}(|\beta\rangle \otimes |0\rangle), \qquad (\langle\beta| \otimes \langle 0|)\hat{K}(|\beta\rangle \otimes c^\dagger|0\rangle),$$

$$(\langle\beta| \otimes \langle 0|c)\hat{K}(|\beta\rangle \otimes |0\rangle), \qquad (\langle\beta| \otimes \langle 0|c)\hat{K}(|\beta\rangle \otimes c^\dagger|0\rangle).$$

(iii) Let $|\zeta\rangle$ be a squeezed state. Find

$$(\langle\zeta| \otimes \langle 0|)\hat{K}(|\zeta\rangle \otimes |0\rangle), \qquad (\langle\zeta| \otimes \langle 0|)\hat{K}(|\zeta\rangle \otimes c^\dagger|0\rangle),$$

$$(\langle\zeta| \otimes \langle 0|c)\hat{K}(|\zeta\rangle \otimes |0\rangle), \qquad (\langle\zeta| \otimes \langle 0|c^\dagger)\hat{K}(|\zeta\rangle \otimes c|0\rangle).$$

Problem 3. (i) Let $z_1, z_2 \in \mathbb{C}$. Find the spectrum of the Hamilton operator

$$\hat{H} = z_1 b \otimes c^\dagger \otimes b + z_2 b^\dagger \otimes c \otimes b + z_1^* b^\dagger \otimes c \otimes b^\dagger + z_2^* b \otimes c^\dagger \otimes b^\dagger$$

where z_1, z_2 are complex numbers.
(ii) Let $z \in \mathbb{C}$. Study the spectrum of the Hamilton operator

$$\hat{H} = z(b \otimes c^\dagger \otimes b) + z^*(b^\dagger \otimes c \otimes b^\dagger).$$

(iii) Let $z \in \mathbb{C}$. Study the spectrum of the Hamilton operator

$$\hat{H} = z(c \otimes b^\dagger \otimes c) + z^*(c^\dagger \otimes b \otimes c^\dagger).$$

Problem 4. Study the spectrum of the Hamilton operator

$$\hat{H} = \hbar\omega_1 b^\dagger b \otimes I_F + \hbar\omega_2 I_B \otimes c^\dagger c + \gamma e^{i\phi} b^\dagger \otimes c + \gamma e^{-i\phi} b \otimes c^\dagger\,.$$

Problem 5. Let $|0\rangle_B$, $|0\rangle_F$ be the vacuum state for Bose and Fermi, respectively, i.e. $b|0\rangle_B = 0|0\rangle_B$, $c|0\rangle_F = 0|0\rangle_F$. In the product Hilbert space we have the basis

$$\frac{1}{\sqrt{j!}}(b^\dagger)^j |0\rangle_B \otimes (c^\dagger)^k |0\rangle_F$$

where $j = 0, 1, 2, \ldots$ and $k = 0, 1$. Apply the operators $b^\dagger b \otimes c^\dagger c$, $(b^\dagger + b) \otimes (c^\dagger + c)$ to this basis, i.e. find the matrix representation of these operators.

Problem 6. (i) Study the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = b^\dagger \otimes c \otimes b + b \otimes c^\dagger \otimes b^\dagger.$$

Does the Hamilton operator commute with $b^\dagger b \otimes c^\dagger c \otimes b^\dagger b$?
(ii) Study the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = c^\dagger \otimes b \otimes c + c \otimes b^\dagger \otimes c^\dagger.$$

Does the Hamilton operator commute with $c^\dagger c \otimes b^\dagger b \otimes c^\dagger c$?
(iii) Study the Hamilton operators

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = b^\dagger \otimes c \otimes b^\dagger + b \otimes c^\dagger \otimes b.$$

Does the Hamilton operator commute with $b^\dagger b \otimes c^\dagger c \otimes b^\dagger b$?
(iv) Study the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = c^\dagger \otimes b \otimes c^\dagger + c \otimes b^\dagger \otimes c.$$

Does the Hamilton operator commute with $c^\dagger c \otimes b^\dagger b \otimes c^\dagger c$? Does the Hamilton operator commute with the number operator

$$c^\dagger c \otimes I_B \otimes I_F + I_F \otimes b^\dagger b \otimes I_F + I_F \otimes I_B \otimes c^\dagger c \ ?$$

(v) Study the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = c \otimes b^\dagger \otimes c^\dagger + c^\dagger \otimes b \otimes c.$$

Does the Hamilton operator commute with $c^\dagger c \otimes b^\dagger b \otimes c^\dagger c$?

Problem 7. Let $c_1^\dagger$, $c_2^\dagger$, c_1, c_2 be Fermi creation and annihilation operators and $b^\dagger$, b be Bose creation and annihilation operators.
(i) Consider the operators

$$b^\dagger b \otimes I_F, \quad I_B \otimes c_1^\dagger c_2, \quad I_B \otimes c_2^\dagger c_2, \quad I_B \otimes c_1^\dagger c_2, \quad I_B \otimes c_2^\dagger c_1.$$

Find the commutators between these operators.
(ii) Consider the operators

$$b^\dagger \otimes c_1, \quad b \otimes c_1^\dagger, \quad b^\dagger \otimes c_2, \quad b \otimes c_2^\dagger.$$

Find the commutators and anticommutators between these operators. Find the commutators and anticommutators between these operators and the operators from (i).
(iii) Study the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = (b + b^\dagger) \otimes (c_1^\dagger c_2 + c_2^\dagger c_1).$$

Find the matrix representation of $\hat{K}$ using the basis $|n\rangle \otimes |0\rangle$, $|n\rangle \otimes c^\dagger|0\rangle$.

Problem 8. Let $c^\dagger$, c be Fermi creation and annihilation operators. Study the algebra generated by the elements

$$b_1^\dagger b_1 \otimes I_F, \;\; b_2^\dagger b_2 \otimes I_F, \;\; b_1^\dagger b_2 \otimes I_F, \;\; b_2^\dagger b_1 \otimes I_B \otimes c^\dagger c.$$

Apply the operators to the states

$$|n_1, n_2\rangle_B \otimes |0\rangle_F, \qquad |n_1, n_2\rangle_B \otimes c^\dagger|0\rangle$$

where $n_1, n_2 = 0, 1, \ldots$.

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