Trapezium Rule - Solutions

1. We must evaluate $\int_1^{1.8} f(x)dx$ using the Composite Trapezium Rule, given the data

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.543	1.668	1.811	1.971	2.151	2.352	2.577	2.828	3.107

.

(a) We use $h = 0.1$. So $x_0 = 1, x_1 = 1.1, x_2 = 1.2$ and so on up to $x_N = 1.8$ ($N = 8$ is the number of subintervals into which x has been divided, as given in the data). For the Composite Trapezium Rule we have

$$\begin{aligned}\int_1^{1.8} f(x)dx &\simeq \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{N-1}) + f(x_N)\right] \\ &= \frac{0.1}{2}\left[1.543 + 3.107 + 2\left(1.668 + 1.811 + ... + 2.828\right)\right] \\ &= 1.7683\end{aligned}$$

(b) Now we use $h = 0.2$. So $x_0 = 1, x_1 = 1.2, x_2 = 1.4, x_3 = 1.6$ and $x_4 = x_N = 1.8$ ($N = 4$ is the number of subintervals into which x has been divided, using $h = 0.2$). For the Composite Trapezium Rule we have

$$\begin{aligned}\int_1^{1.8} f(x)dx &\simeq \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{N-1}) + f(x_N)\right] \\ &= \frac{0.2}{2}\left[1.543 + 3.107 + 2\left(1.811 + 2.151 + 2.577\right)\right] \\ &= 1.7728\end{aligned}$$

(c) Finally, we use $h = 0.4$. So $x_0 = 1, x_1 = 1.4$ and $x_2 = x_N = 1.8$ ($N = 2$ is the number of subintervals into which x has been divided, using

$h = 0.4$). For the Composite Trapezium Rule we have

$$\begin{aligned}
\int_1^{1.8} f(x)dx &\simeq \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{N-1}) + f(x_N)\right] \\
&= \frac{0.4}{2}\left[1.543 + 3.107 + 2\,(2.151)\right] \\
&= 1.7904
\end{aligned}$$

2. Determine $\int_0^{\pi/2} f(x)dx$ where $f(x) = \frac{\cos x}{1+x}$ to an accuracy $\varepsilon = 10^{-2}$.

   Firstly we note that the expression for the maximum error in the Composite Trapezium Rule

   $$|\Delta| = \frac{h^2}{12}\left(\frac{\pi}{2} - 0\right)M$$

   contains the term

   $$M = \max_{0 \leqslant x \leqslant \pi/2}\left|\frac{d^2}{dx^2}\left(\frac{\cos x}{1+x}\right)\right|.$$

   We find that

   $$\frac{d^2}{dx^2}\left(\frac{\cos x}{1+x}\right) = \left(\frac{1}{1+x}\right)\left(-\cos x + \frac{2\sin x}{1+x} + \frac{2\cos x}{(1+x)^2}\right).$$

   This equals 1 when $x = 0$ and 0.3026 when $x = \frac{\pi}{2}$. We need to determine if this function has a maximum/minimum between 0 and $\frac{\pi}{2}$, because if it does, then the absolute value of that maximum/minimum might be larger than 1. This requires setting the derivative of this function (which would be the 3rd derivative of $f(x)$) equal to zero, and solving the resulting nonlinear equation, using a method such as Newton's Method. If we do this we find that there is a turning point at $x = 1.0287$, but $\frac{d^2}{dx^2}\left(\frac{\cos x}{1+x}\right)$ at this value of x has the value 0.2856, which is less than 1. Hence, $\frac{d^2}{dx^2}\left(\frac{\cos x}{1+x}\right)$ has its maximum value at $x = 0$, and so $M = 1$.

   We now find an upper bound for h

   $$\begin{aligned}
   |\Delta| &= \frac{h^2}{12}\left(\frac{\pi}{2} - 0\right)1 \leqslant \varepsilon = 10^{-2} \\
   &\Rightarrow h \leqslant 0.276395
   \end{aligned}$$

This gives

$$\begin{aligned} h &= \frac{\pi/2 - 0}{N} \leqslant 0.276395 \\ &\Rightarrow N \geqslant 5.68 \Rightarrow N = 6 \end{aligned}$$

and this gives the actual value of h that must be used

$$h = \frac{\pi/2 - 0}{6} = 0.261799$$

We thus have the discrete data

i	0	1	2	3	4	5	$6 = N$
x_i	0	0.2618	0.5236	0.7854	1.0472	1.3090	$1.5708 = \frac{\pi}{2}$
$f(x_i)$	1	0.7655	0.5684	0.3960	0.2442	0.1121	0

The Composite Trapezium Rule gives

$$\begin{aligned} \int_0^{\pi/2} \frac{\cos x}{1+x} dx &\simeq \frac{h}{2}\left[f(x_0) + f(x_N) + 2\sum_{i=1}^{N-1} f(x_i) \right] \\ &= 0.6771 \end{aligned}$$

If we use $N = 1000000$ (representing an accuracy of about 10^{-11}, and done using a computer, of course) we obtain $\int_0^{\pi/2} \frac{\cos x}{1+x} dx = 0.6736$. Given the high degree of accuracy we may regard this as the true value. We see then that

$$|0.6736 - 0.6771| = 0.0035 < \varepsilon = 0.01$$

and the integral has been evaluated to the required degree of accuracy. It is not necessary to perform this check; we have included it simply to verify that our error analysis does indeed work.

3. In this problem we must find

$$P = \int_{100}^{200} \frac{dx}{\ln x}$$

where P is the number of prime numbers between 100 and 200. Now, we expect P to be an integer so we may impose an accuracy of 0.5 on the problem. We then have

$$|\Delta| = \frac{h^2}{12}(200-100)\, M \leqslant \varepsilon = 0.5$$

We deliberately chose the error to be 0.5, which is the uncertainty in any "measured" integer value. Here, M is the maximum value of the absolute value of the second derivative of $\frac{1}{\ln x}$ on $[100, 200]$. We find

$$\frac{d^2}{dx^2}\left(\frac{1}{\ln x}\right) = \frac{2+\ln x}{x^2(\ln x)^3} = \frac{2}{x^2(\ln x)^3} + \frac{1}{x^2(\ln x)^2}.$$

At the endpoints of the interval $[100, 200]$ this function has the values 6.7631×10^{-6} and 1.2267×10^{-6}. Since x^2 and $\ln x$ are both strictly increasing functions of x on $[100, 200]$, both terms on the RHS of the above equation are strictly decreasing. Thus, $\frac{d^2}{dx^2}\left(\frac{1}{\ln x}\right)$ takes its maximum value at $x = 100$, and so $M = 6.7631\times 10^{-6}$. All this gives

$$\begin{aligned} h &\leqslant 94.1895 \\ &\Rightarrow N \geqslant \frac{200-100}{94.1895} = 1.06 \\ &\Rightarrow N = 2 \\ &\Rightarrow h = \frac{100}{2} = 50. \end{aligned}$$

We thus have the discrete data

i	0	1	$2 = N$
x_i	100	150	200
$\frac{1}{\ln x_i}$	0.2171	0.1996	0.1887

Applying the Composite Trapezium Rule

$$\begin{aligned} \int_{100}^{200} \frac{1}{\ln x}dx &\simeq \frac{h}{2}[0.2171 + 0.1887 + 2(0.1996)] \\ &= \frac{50}{2}[.8050] \\ &= 20.126 \end{aligned}$$

Now the accuracy imposed was $\varepsilon = 0.5$ and so we have

$$|P - 20.126| \leqslant 0.5 \Rightarrow 19.626 \leqslant P \leqslant 20.626$$

which suggests $P = 20$. There are, in fact, 21 prime numbers on the interval $[100, 200]$.

4. To find the area enclosed by $y = e^x$ and $y = 20 \ln x$, we need to evaluate the integral

$$\left| \int_a^b (20 \ln x - e^x)\, dx \right|$$

where a and b are the x-values of the points of intersection. We find these intersection points using Newton's Method, and they are

$$(1.1759, 3.2410) \quad \text{and} \quad (3.1268, 22.8009)\,.$$

So we need to evaluate the integral

$$\left| \int_{1.1759}^{3.1268} (20 \ln x - e^x)\, dx \right|.$$

We impose an accuracy of $\varepsilon = 0.1$. Before using the error term to determine the number of subintervals N, we must first determine

$$M = \max_{[1.1759, 3.1268]} \left| \frac{d^2}{dx^2} (20 \ln x - e^x) \right|.$$

Now

$$\left| \frac{d^2}{dx^2} (20 \ln x - e^x) \right| = \left| -\frac{20}{x^2} - e^x \right|.$$

At the endpoints of the interval $[1.1759, 3.1268]$ this function has the values 17.7051 and 24.8465. We must check to see if there is a larger value between 1.1759 and 3.1268. We differentiate one more time and put the resulting function equal to zero. In other words, we are looking for a turning point of the function $\frac{20}{x^2} + e^x$. Using Newton's Method we find a turning point at 1.8474, but the function has a value of only 12.2034 at this x-value. So we conclude that $\frac{20}{x^2} + e^x$ has its maximum on $[1.1759, 3.1268]$ at the point $x = 3.1268$, and so $M = 24.8465$. We now have for the error

$$|\Delta| = \frac{h^2}{12}(3.1268 - 1.1759)\,24.8465 \leqslant \varepsilon = 0.1$$

and so

$$\begin{aligned} h &\leqslant 0.1573 \\ &\Rightarrow N \geqslant \frac{3.1268 - 1.1759}{0.1573} = 12.4 \\ &\Rightarrow N = 13 \\ &\Rightarrow h = \frac{3.1268 - 1.1759}{13} = 0.150069 \end{aligned}$$

We thus obtain the discrete points ($f(x_i) \equiv 20\ln x - e^x$)

i	0	1	2	3	4	5	6
x_i	1.1759	1.3259	1.4760	1.6261	1.7761	1.9262	2.0763
$f(x_i)$	0	1.8770	3.4116	4.6397	5.5820	6.2477	6.6368

i	7	8	9	10	11	12	13
x_i	2.2263	2.3764	2.5265	2.6765	2.8266	2.9767	3.1268
$f(x_i)$	6.7412	6.5455	6.0269	5.1554	3.8929	2.1929	0

The Composite Trapezium Rule gives

$$\begin{aligned} \int_{1.1759}^{3.1268} (20\ln x - e^x)\,dx &\simeq \frac{h}{2}\left[f(x_0) + f(x_N) + 2\sum_{i=1}^{N-1} f(x_i)\right] \\ &= 8.8465 \end{aligned}$$

The result obtained using $N = 1000000$ (which represents an accuracy of about 10^{-10} so that this result will be regarded as the true value) is 8.9031 and so

$$|8.9031 - 8.8465| = 0.0566 < \varepsilon = 0.1$$

which confirms the accuracy of the result.

5. Obviously we integrate

$$f(t) \equiv \frac{2}{\sqrt{\pi}} e^{-t^2}$$

between 0 and 1. We have

$$f''(t) = \frac{4}{\sqrt{\pi}}(2t^2 - 1)e^{-t^2}.$$

To find the maximum we find the extrema of $f''(t)$, i.e.

$$f'''(t) = \frac{8}{\sqrt{\pi}}t(3 - 2t^2)e^{-t^2} = 0$$

which yields the extrema at $t = \sqrt{3/2}$ on $(0, 1)$ of $\frac{8}{\sqrt{\pi}}e^{-\frac{3}{2}} \approx 1.0071$. Furthermore $|f''(t = 0)| = 4/\sqrt{\pi} \approx 2.25676$ and $|f''(t = 1)| = \frac{4}{\sqrt{\pi}}e^{-1} \approx 0.83021$. Consequently

$$(1-0)\frac{h^2}{12}2.25676 < 10^{-3} \Rightarrow h < 0.07292$$

$$\Rightarrow n = \left\lceil \frac{1-0}{h} \right\rceil = 14 \text{ and } h = \frac{1}{14}.$$

where $\lceil z \rceil$ is z rounded **up** to the nearest integer (the so-called *ceiling* function).

Now,

t	$f(t)$
$\frac{0}{14}$	1.1283791670955126
$\frac{1}{14}$	1.1226367919015658
$\frac{2}{14}$	1.1055844115243310
$\frac{3}{14}$	1.0777374288127760
$\frac{4}{14}$	1.0399260289083296
$\frac{5}{14}$	0.9932540738376424
$\frac{6}{14}$	0.9390456064795604
$\frac{7}{14}$	0.8787825789354449

t	$f(t)$
$\frac{8}{14}$	0.8140378831146369
$\frac{9}{14}$	0.7464078827435117
$\frac{10}{14}$	0.6774484321740555
$\frac{11}{14}$	0.6086178568166220
$\frac{12}{14}$	0.5412296275979184
$\frac{13}{14}$	0.4764165700409094
$\frac{14}{14}$	0.4151074974205948

and

$$\int_0^1 \frac{2}{\sqrt{\pi}}e^{-t^2}dt \approx \frac{h}{2}\left[f(t_0) + 2f(t_1) + 2f(t_2) + ... + 2f(t_{13}) + f(t_{14})\right]$$

Thus we find $\text{erf}(1) \approx 0.8424$.

6. We have

$$\begin{aligned} f(x) &= e^{-x^2} \\ f'(x) &= -2xe^{-x^2} \\ f''(x) &= 2(2x^2-1)e^{-x^2} \\ f'''(x) &= 4x(3-2x^2)e^{-x^2}. \end{aligned}$$

To evaluate $\max\limits_{[-1,1]} |f''(x)|$ we solve $f'''(x) = 0$ which yields $x = 0, x = \pm\sqrt{\frac{3}{2}} \notin [-1, 1]$ with $|f''(x = 0)| = 2$. At the boundaries we have $|f''(x = -1)| = |f''(x = 1)| = \frac{2}{e} < 2$. Thus the maximum is $M = 2$, and so we must satisfy

$$\frac{h^2}{12}(b-a) \cdot M = \frac{h^2}{12}(1-(-1)) \cdot 2 \leqslant 0.015$$

with

$$h \leqslant \sqrt{3 \cdot 0.015} = 0.212132034, \qquad n = \frac{b-a}{h} \geqslant \frac{b-a}{0.212132034}.$$

Consequently

$$n = \left\lceil \frac{1-(-1)}{0.212132034} \right\rceil = 10.$$

and

$$h = \frac{b-a}{n} = 0.2$$

7. We have

$$\begin{aligned} f(x) &\equiv xe^x \\ f'(x) &= xe^x + e^x \\ f''(x) &= xe^x + 2e^x \\ f'''(x) &= xe^x + 3e^x. \end{aligned}$$

To evaluate $\max\limits_{[-1,1]} |f''(x)|$ we solve $f'''(x) = 0$ which yields $x = -3 \notin [-1, 1]$. At the boundaries we have $|f''(x = -1)| = \frac{1}{e}$ and $|f''(x = 1)| = 3e$. Thus the maximum is $M = 3e$, and we must satisfy

$$\frac{h^2}{12}(b-a) \cdot M = \frac{h^2}{12}(1-(-1)) \cdot 3e \leqslant 0.05$$

with

$$h \leqslant \sqrt{\frac{0.1}{e}}, \qquad n = \frac{b-a}{h} \geqslant \frac{b-a}{\sqrt{\frac{0.1}{e}}}.$$

Consequently,

$$n = \left\lceil \frac{1-(-1)}{\sqrt{\frac{0.1}{e}}} \right\rceil = 11.$$

and

$$h = \frac{b-a}{n} = \frac{2}{11}.$$

Now,

j	x_j	$factor$	$factor \times f(x_j)$
0	-1	1	-0.36787944
1	$-1+\frac{2}{11}$	2	-0.72201791
2	$-1+\frac{4}{11}$	2	-0.67354425
3	$-1+\frac{6}{11}$	2	-0.57703311
4	$-1+\frac{8}{11}$	2	-0.41525476
5	$-1+\frac{10}{11}$	2	-0.16601832
6	$-1+\frac{12}{11}$	2	0.19912170
7	$-1+\frac{14}{11}$	2	0.71647741
8	$-1+\frac{16}{11}$	2	1.43223370
9	$-1+\frac{18}{11}$	2	2.40494173
10	$-1+\frac{20}{11}$	2	3.70861423
11	1	1	2.71828183
		$\sum$	8.257922815

where $factor$ indicates the coefficients in the Composite Trapezium Rule

$$\int_{-1}^{1} xe^x dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{10}) + f(x_{11})\right].$$

Hence,

$$\int_{-1}^{1} xe^x dx \approx \left(\frac{1}{11}\right) 8.257922815 = 0.750720255.$$

Comparing to the analytical result

$$\int_{-1}^{1} xe^x = \frac{2}{e}$$

we find

$$\left|\frac{2}{e}-0.750720255\right|\approx 0.015$$

which is more accurate than required.

8. The error in the Trapezium Rule on the interval I satisfies

$$\Delta \leqslant \frac{h^3}{12}\max_I |f''(x)|\,.$$

If $f(x)=ax+b$, then

$$f''(x)=0$$

and so

$$\Delta = 0$$

which means that the Trapezium Rule is exact when $f(x)=ax+b$.

9. Firstly we note that the expression for the maximum error in the Composite Trapezium Rule

$$|\Delta| = \frac{h^2}{12}\left(\frac{\pi}{2}-0\right)M$$

contains the term

$$M = \max_{\left[0,\frac{\pi}{2}\right]}\left|\frac{d^2}{dx^2}\left(\frac{\sin x}{1+x^2}\right)\right|.$$

We find that

$$\frac{d^2}{dx^2}\left(\frac{\sin x}{1+x^2}\right) = \left(-\frac{\sin x}{(1+x^2)} - \frac{4x\cos x}{(1+x^2)^2} - \frac{2\sin x}{(1+x^2)^2} + \frac{8x^2\sin x}{(1+x^2)^3}\right).$$

This equals 0 when $x=0$ and 0.0187 when $x=\frac{\pi}{2}$. We need to determine if this function has a maximum/minimum between 0 and $\frac{\pi}{2}$, because if it does, then the absolute value of that maximum/minimum might be larger than 1. This requires setting the derivative of this function (which would be the 3rd derivative of $f(x)$) equal to zero, and solving the resulting nonlinear equation, using a method such as Newton's Method. If we do this we find that there is a turning point at $x=0.4114$. Furthermore, $\left|\frac{d^2}{dx^2}\left(\frac{\sin x}{1+x^2}\right)\right|$ at this value of x has the value 1.6915, which is greater than 0.0187. Hence, $\left|\frac{d^2}{dx^2}\left(\frac{\sin x}{1+x}\right)\right|$ has its maximum value at $x=0.4114$, and so $M=1.6915$.

We now find an upper bound for h

$$\begin{aligned} |\Delta| &= \frac{h^2}{12}\left(\frac{\pi}{2}-0\right)1.6915 \leqslant \varepsilon = 2\times 10^{-2} \\ &\Rightarrow h \leqslant 0.3005 \end{aligned}$$

This gives

$$\begin{aligned} h &= \frac{\pi/2-0}{N} \leqslant 0.3005 \\ &\Rightarrow N \geqslant 5.23 \Rightarrow N = 6 \end{aligned}$$

and this gives the actual value of h that must be used

$$h = \frac{\pi/2-0}{6} = 0.261799$$

We thus have the discrete data

i	0	1	2	3	4	5	$6 = N$
x_i	0	0.2618	0.5236	0.7854	1.0472	1.3090	$1.5708 = \frac{\pi}{2}$
$f(x_i)$	1	0.9040	0.6800	0.4373	0.2385	0.0954	0

The Composite Trapezium Rule gives

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin x}{1+x^2}dx &\simeq \frac{h}{2}\left[f(x_0)+f(x_N)+2\sum_{i=1}^{N-1} f(x_i)\right] \\ &= 0.5197. \end{aligned}$$

If we use $N = 1000000$ (representing an accuracy of about 10^{-12}, and done using a computer, of course) we obtain $\int_0^{\pi/2} \frac{\sin x}{1+x^2}dx = 0.5270$. Given the high degree of accuracy we may regard this as the true value. We see then that

$$|0.5197 - 0.5270| = 0.0073 < \varepsilon = 0.02$$

and the integral has been evaluated to the required degree of accuracy.

10.