# Taylor series - Solutions

1.

$$\begin{aligned} f(x)=\sin(x) &= \sin(0)+x\cos(0)+\frac{x^2}{2!}(-\sin(0))+\frac{x^3}{3!}(-\cos(0))+\ldots \\ &\quad +\frac{x^4}{4!}(\sin(0))+\frac{x^5}{5!}(\cos(0))+\frac{x^6}{6!}(-\sin(0))+\ldots \\ &= 0+x+0-\frac{x^3}{3!}+0+\frac{x^5}{5!}+0+\ldots \\ &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}. \end{aligned}$$

Convergence:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\frac{(2n-1)!}{(-1)^{n-1} x^{2n-1}}\right|=\lim_{n\to\infty}\left|\frac{(-1)\,x^2}{(2n+1)\,(2n)}\right| \\ &= 0 \text{ for all } x. \end{aligned}$$

Interval of convergence is thus $-\infty<x<\infty$.

$$\begin{aligned} f(x) &= \cos(x)=\cos(0)+x\,(-\sin(0))+\frac{x^2}{2!}(-\cos(0))+\frac{x^3}{3!}\sin(0)+\ldots \\ &\quad +\frac{x^4}{4!}(\cos(0))+\frac{x^5}{5!}(-\sin(0))+\frac{x^6}{6!}(-\cos(0))+\ldots \\ &= 1+0-\frac{x^2}{2!}+0+\frac{x^4}{4!}-0-\frac{x^6}{6!}+\ldots \\ &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}. \end{aligned}$$

Convergence:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n}}{(2n)!}\frac{(2n-2)!}{(-1)^{n-1} x^{2n-2}}\right|=\lim_{n\to\infty}\left|\frac{(-1)\,x^2}{(2n)\,(2n-1)}\right| \\ &= 0 \text{ for all } x. \end{aligned}$$

Interval of convergence is thus $-\infty<x<\infty$.

2.

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) \\
&= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots\right) \\
&= \frac{1}{2}\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \ldots\right) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.
\end{aligned}$$

Convergence:

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n}}{(2n)!}\frac{(2n-2)!}{x^{2n-2}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n)(2n-1)}\right| = 0 \text{ for all } x.$$

So interval of convergence is $-\infty < x < \infty$

$$\begin{aligned}
\sinh(x) &= \frac{1}{2}\left(e^x - e^{-x}\right) \\
&= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots - 1 - (-x) - \frac{(-x)^2}{2!} - \frac{(-x)^3}{3!} + \ldots\right) \\
&= \frac{1}{2}\left(2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots\right) \\
&= 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Convergence:

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{(2n+1)!}\frac{(2n-1)!}{x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n)}\right| = 0 \text{ for all } x.$$

So interval of convergence is $-\infty < x < \infty$.

3.

$$f^{(1)}(x) = \frac{1}{1+x} \qquad f^{(2)}(x) = \frac{-1}{(1+x)^2} \qquad f^{(3)}(x) = \frac{2}{(1+x)^3}$$
$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \qquad f^{(5)}(x) = \frac{24}{(1+x)^5} \qquad \text{etc.}$$

and so

$f^{(1)}(0) = 1 \quad f^{(2)}(0) = -1 \quad f^{(3)}(0) = 2 \quad f^{(4)}(0) = -6 \quad f^{(5)}(0) = 24$ etc.

So for $n \geqslant 1$ we see that

$$f^{(n)}(0) = (-1)^{n+1}(n-1)!.$$

Now we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x-0)^n f^{(n)}(0)}{n!} \\ &= \frac{(x-0)^0 f^{(0)}(0)}{0!} + \sum_{n=1}^{\infty} \frac{(x-0)^n f^{(n)}(0)}{n!} \end{aligned}$$

where we have explicitly written the first term ($n = 0$). So for $f(x) = \ln(1+x)$ we have

$$\begin{aligned} \ln(1+x) &= \frac{(x-0)^0 \ln(1+0)}{0!} + \sum_{n=1}^{\infty} \frac{x^n(-1)^{n+1}(n-1)!}{n!} \\ &= 0 + \sum_{n=1}^{\infty} \frac{x^n(-1)^{n+1}}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \end{aligned}$$

Convergence:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| &= \lim_{n\to\infty} \left| \frac{x^n(-1)^{n+1}}{n} \frac{n-1}{x^{n-1}(-1)^n} \right| = \lim_{n\to\infty} \left| \frac{x(n-1)}{n} \right| \\ &= \lim_{n\to\infty} \left| x - \frac{x}{n} \right| = |x| \,. \end{aligned}$$

For convergence we require that $\lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| < 1$ and so we require $|x| < 1$. Interval of convergence is thus $-1 < x < 1$.

4.

$$\begin{aligned}
\sin\frac{\pi}{3} &= \frac{\pi}{3} - \left(\frac{\pi}{3}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{3}\right)^5 \frac{1}{5!} - \left(\frac{\pi}{3}\right)^7 \frac{1}{7!} + \left(\frac{\pi}{3}\right)^9 \frac{1}{9!} \\
&= 1.04720 - 0.19140 + 0.01049 - 0.00027 + 0.000006 \\
&= 0.86602 \quad (= 0.86603 \text{ using calculator directly})
\end{aligned}$$

5. We first evaluate $\ln(0.7)^{0.7}$ because $(0.7)^{0.7} = \exp\left(\ln(0.7)^{0.7}\right)$ and we know how to expand $\ln(1+x)$ and $e^x$

$$\begin{aligned}
\ln(0.7)^{0.7} &= 0.7\ln 0.7 = 0.7\ln(1-0.3) = 0.7\ln(1+(-0.3)) \\
&= 0.7\left\{-0.3 - \frac{(-0.3)^2}{2} + \frac{(-0.3)^3}{3} - \frac{(-0.3)^4}{4} + \frac{(-0.3)^5}{5}\right\} \\
&= 0.7\{-0.3 - 0.04500 - 0.00900 - 0.00203 - 0.00049\} \\
&= 0.7\{-0.35652\} \\
&= -0.24956.
\end{aligned}$$

Now

$$\begin{aligned}
(0.7)^{0.7} &= \exp\left(\ln(0.7)^{0.7}\right) = e^{-0.24956} \\
&= 1 + (-0.24956) + \frac{(-0.24956)^2}{2} + \frac{(-0.24956)^3}{6} + \frac{(-0.24956)^4}{4} \\
&= 1 - 0.24956 + 0.03114 - 0.00259 + 0.00016 \\
&= 0.77915 \quad (= 0.77906 \text{ using calculator directly}).
\end{aligned}$$

6.

$$\begin{aligned}
\ln(1.3) &= \ln(1+0.3) \\
&= 0.3 - \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3} - \frac{(0.3)^4}{4} + \frac{(0.3)^5}{5} \\
&= 0.3 - 0.04500 + 0.00900 - 0.00203 + 0.00049 \\
&= 0.26246 \quad (= 0.26236 \text{ using caclulator directly}).
\end{aligned}$$

7.

$$\begin{aligned}
\ln(0.0002) &= \ln(1-0.9998) = \ln\left(1+(-0.9998)\right) \\
&= -0.9998 - \frac{(-0.9998)^2}{2} + \frac{(-0.9998)^3}{3} - \frac{(-0.9998)^4}{4} + \frac{(-0.9998)^5}{5} \\
&= -0.9998 - 0.49980 - 0.33313 - 0.24980 - 0.19980 \\
&= -2.28233 \\
&= -8.51719 \text{ using calculator directly!!!}
\end{aligned}$$

We may resolve this problem by writing 0.0002 in a different manner:

$$\begin{aligned}
\ln(0.0002) &= \ln\left(\frac{2}{10000}\right) = \ln 2 - \ln 10^4 = \ln 2 - 4\ln 10 \\
&= \ln 2 - 4\ln\left(2^3 1.25\right) = \ln 2 - 4\left(3\ln 2 + \ln 1.25\right) \\
&= \ln 2 - 12\ln 2 - 4\ln 1.25 \\
&= -11\ln 2 - 4\ln 1.25
\end{aligned}$$

Now

$$\begin{aligned}
\ln 2 &= \ln\left(\frac{3}{2}\frac{4}{3}\right) = \ln(1.5) + \ln(1.33333) \\
&= \ln(1+0.5) + \ln(1+0.33333)
\end{aligned}$$

$$\begin{aligned}
\ln(1+0.5) &= 0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} + \frac{(0.5)^5}{5} \\
&= 0.5 - 0.12500 + 0.04167 - 0.01563 + 0.00625 \\
&= 0.40729 \quad (= 0.40547 \text{ using calculator directly})
\end{aligned}$$

$$\begin{aligned}
\ln(1+0.33333) &= 0.33333 - 0.05555 + 0.01235 - 0.00309 + 0.00082 \\
&= 0.28786 \quad (= 0.28768 \text{ using calculator directly}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\ln 2 &= 0.40729 + 0.28786 \\
&= 0.69515 \quad (= 0.69315 \text{ using calculator directly}).
\end{aligned}$$

Also

$$\begin{aligned}
\ln(1.25) &= \ln(1+0.25) = 0.25 - \frac{(0.25)^2}{2} + \frac{(0.25)^3}{3} - \frac{(0.25)^4}{4} + \frac{(0.25)^5}{5} \\
&= 0.25 - 0.03125 + 0.00521 - 0.00098 + 0.00020 \\
&= 0.22318 \quad (= 0.22314 \text{ using calculator directly}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\ln(0.0002) &= -11\ln 2 - 4\ln 1.25 \\
&= -11(0.69515) - 4(0.22318) \\
&= -8.53937 \quad (= -8.51719 \text{ using calculator directly}).
\end{aligned}$$

8. **(a)** For $x = 1$ we calculate

$$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} = \sum_{j=0}^{\infty} a_j.$$

We seek the smallest value of j for which $|a_j| < 10^{-3}$. Thus we require

$$\left|\frac{(-1)^j}{j+1}\right| < 10^{-3} \quad \Rightarrow \quad j > 999$$

i.e. we need the terms $j = 0, 1, \ldots, 1000$. So the answer is 1001 terms.

**(b)** For $x = \frac{1}{2}$ we calculate

$$\ln\left(1 - \frac{1}{2}\right) = -\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} - \ldots = -\sum_{j=1}^{\infty} \frac{\left(\frac{1}{2}\right)^j}{j}.$$

Thus we require

$$\left|-\frac{\left(\frac{1}{2}\right)^j}{j}\right| < 10^{-3} \quad \Rightarrow \quad j \geqslant 8.$$

We find the bound for $j$ empirically, i.e. we tried 1, 2, 3, … until we found that 8 satisfied the inequality. Thus the answer is 8 terms.

**(c)** For $x = \frac{1}{3}$ we calculate

$$\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \dots\right) = 2\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^{2j+1}}{2j+1}.$$

Thus we require

$$\left|2\frac{\left(\frac{1}{3}\right)^{2j+1}}{2j+1}\right| < 10^{-3} \quad \Rightarrow \quad j \geqslant 3.$$

We find the bound for $j$ empirically, i.e. we tried 1, 2 and 3. Thus the answer is 4 terms.

9. We know that

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

and consequently

$$e^{-x^2} = \sum_{j=0}^{\infty} \frac{(-x^2)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{j!} = 1 - x^2 + \frac{x^4}{2} - \dots.$$

Let $f(x) = e^{-x^2}$, thus we have

| $j$ | $f^{(j)}$ | $f^{(j)}(0)$ |
|---|---|---|
| 0 | $e^{-x^2}$ | 1 |
| 1 | $-2xe^{-x^2}$ | 0 |
| 2 | $-2e^{-x^2} + 4x^2e^{-x^2}$ | $-2$ |
| 3 | $12xe^{-x^2} - 8x^3e^{-x^2}$ | 0 |
| 4 | $12e^{-x^2} - 48x^2e^{-x^2} + 16x^4e^{-x^2}$ | 12 |
| $\vdots$ | $\vdots$ | $\vdots$ |

Thus we find

$$e^{-x^2} = 1 + 0x - 2\frac{x^2}{2!} + 0\frac{x^3}{3!} + 12\frac{x^4}{4!} + \dots = 1 - x^2 + \frac{x^4}{2} + \dots$$

and the two expansions are identical in the first three terms.

10. We have

$$\begin{aligned}
\operatorname{erf}(x) := & \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \\
= & \quad \frac{2}{\sqrt{\pi}} \int_0^x \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{j!} dt = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \int_0^x \frac{(-1)^j t^{2j}}{j!} dt \\
= & \quad \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \left[ \frac{(-1)^j t^{2j+1}}{(2j+1)(j!)} \right]_0^x \\
= & \quad \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)(j!)} \\
= & \quad \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3(1!)} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \dots \right).
\end{aligned}$$

Consequently

$$\operatorname{erf}(1) = \frac{2}{\sqrt{\pi}} \left( 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \dots \right).$$

Summing the first four terms yields 0.8382. Thus we have an approximate error of $|0.8427 - 0.8382| = 0.0045$.

11. Using $f(x) = \cos x$ we tabulate

| $j$ | $f^{(j)}$ | $f^{(j)}(\pi)$ |
|---|---|---|
| 0 | $\cos x$ | $-1$ |
| 1 | $-\sin x$ | 0 |
| 2 | $-\cos x$ | 1 |
| 3 | $\sin x$ | 0 |
| 4 | $\cos x$ | $-1$ |
| 5 | $-\sin x$ | 0 |
| 6 | $-\cos x$ | 1 |
| 7 | $\sin x$ | 0 |

where we see a repeating pattern. We have

$$f^{(j)}(\pi) = \begin{cases} 0, & j = 2k+1, k \in \mathbb{Z} \\ -(-1)^k, & j = 2k, k \in \mathbb{Z} \end{cases}.$$

Consequently the Taylor series is given by

$$f(x) = \sum_{j=0}^{\infty} \frac{(x-\pi)^j}{j!} f^{(j)}(\pi)$$

$$\cos x = -\sum_{k=0}^{\infty} \frac{(-1)^k (x-\pi)^{2k}}{(2k)!}.$$

Note the similarity with the series around 0:

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Shifting $x$ by $\pi$ causes a change in sign for $\cos x$ from

$$\cos x = -\cos(\pi - x) = -\cos(x-\pi).$$

For convergence we identify the term

$$a_k \equiv -\frac{(-1)^k (x-\pi)^{2k}}{(2k)!}$$

so that

$$\begin{aligned} \left|\frac{a_{k+1}}{a_k}\right| &= \left|\frac{(-1)^{k+1}(x-\pi)^{2(k+1)}}{(2(k+1))!} \frac{(2k)!}{(-1)^k (x-\pi)^{2k}}\right| \\ &= \left|-\frac{(x-\pi)^2}{(2k+2)(2k+1)}\right|. \end{aligned}$$

Thus

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 0 < 1 \quad \forall x \in (-\infty, \infty)$$

and so the interval of convergence is $(-\infty, \infty)$.

12. We know that

$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$$

and consequently

$$\sin\left(x^2\right) = \sum_{j=0}^{\infty} (-1)^j \frac{(x^2)^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} (-1)^j \frac{x^{4j+2}}{(2j+1)!} = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} + \ldots.$$

We identify in

$$\sin\left(x^2\right) = \sum_{j=0}^{\infty}(-1)^j \frac{x^{4j+2}}{(2j+1)!}$$

the term

$$a_j = (-1)^j \frac{x^{4j+2}}{(2j+1)!}$$

and applying the convergence criteria

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| < 1$$

where

$$\left|\frac{a_{j+1}}{a_j}\right| = \left|(-1)^{j+1}\frac{x^{4j+6}}{(2j+3)!}\cdot(-1)^j\frac{(2j+1)!}{x^{4j+2}}\right| = \left|\frac{x^4}{(2j+2)(2j+2)}\right|$$

yields

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = 0 < 1$$

for all $x \in (-\infty, \infty)$. Thus convergence holds for all real $x$.

13. Using $\sin x^2 \approx x^2 - \frac{x^6}{6}$ in the equation

$$x^2 - \frac{x^6}{6} = 0$$

yields

$$x^2(1 - \frac{x^4}{6}) = 0$$

with solutions $x = 0$ and $x^4 = 6$ (i.e $x = \pm\sqrt{\sqrt{6}} \approx 1.565$). Obviously $\sin\left(x^2\right) = 0$ implies $x^2 = k\pi$ for $k \in \mathbb{N}_0$. In other words $x = \pm\sqrt{k\pi}$. The approximation $\sqrt{\sqrt{6}}$ is near the root with $k = 1$ with $x = \sqrt{\pi} \approx 1.77$. Thus the approximation error

$$|\sqrt{\pi} - \sqrt{\sqrt{6}}| \approx 0.207$$

is not very small. Attempting to improve the approximation

$$\sin\left(x^2\right) \approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} = 0$$

yields the equation

$$x^2(1 - \frac{x^4}{6} + \frac{x^8}{120}) = 0$$

which has roots $x = 0$ and $(x^4)^2 - 20(x^4) + 120 = 0$. This last equation for the non-zero roots has no real roots for

$$x^4 = \frac{20 \pm \sqrt{400 - 480}}{2}.$$

Thus adding terms to the Taylor series approximation does not always yield better results (in this case no solution at all). Interestingly, though, expanding to an order 14 approximation yields good results.

14. We have

$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$$

so that

$$\frac{\sin x}{x} = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j+1)!}.$$

when $x \neq 0$. It follows that

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{1}{x} \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} \lim_{x\to 0} (-1)^j \frac{x^{2j+1}}{x(2j+1)!} \\
&= \sum_{j=0}^{\infty} \lim_{x\to 0} (-1)^j \frac{x^{2j}}{(2j+1)!} \\
&= \lim_{x\to 0} (-1)^0 \frac{(x^0)}{1!} + \sum_{j=1}^{\infty} \lim_{x\to 0} (-1)^j \frac{x^{2j}}{(2j+1)!} \\
&= 1 + \sum_{j=1}^{\infty} (-1)^j \times 0 = 1
\end{aligned}$$

where $\lim_{x\to 0} (x^0) = 1$.

15. Using the expansion

$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$$

yields

$$(\sin x)^2 = \left( \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \right) \left( \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right)$$

so that

$$(\sin x)^2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \frac{x^{2(j+k+1)}}{(2j+1)!(2k+1)!}.$$

To find the expansion we would have to determine the coefficient of $x^0$ (which is zero), $x^2$ (for $j = k = 0$), $x^4$ (for $j = 0, k = 1$ and $j = 1, k = 0$), $x^6$ (for $j = 0, k = 2$, $j = 1, k = 1$ and $j = 2, k = 0$) and so on. Obviously for higher degree in $x^{2n}$ we have more terms to sum for the different $j$ and $k$ such that $2(j + k + 1) = 2n$.

**(1)** Obviously we can determine the Taylor series expansion directly:

$$\begin{aligned}
\frac{d^0}{dx^0} \sin^2 x &= \sin^2 x \\
\frac{d^1}{dx^1} \sin^2 x &= 2 \sin x \cos x = \sin 2x \\
\frac{d^2}{dx^2} \sin^2 x &= 2 \cos 2x \\
\frac{d^3}{dx^3} \sin^2 x &= -4 \sin 2x \\
\frac{d^4}{dx^4} \sin^2 x &= -8 \cos 2x \\
&\vdots
\end{aligned}$$

Generally, we have

$$\frac{d^n}{dx^n} \sin^2 x = \begin{cases} \sin^2 x & n = 0 \\ (-1)^k \, 2^{2k-1} \sin 2x & n = 2k+1, k = 0, 1, 2, \ldots \\ (-1)^{k-1} \, 2^{2k-1} \cos 2x & n = 2k, k = 1, 2, 3, \ldots \end{cases}.$$

Consequently

$$\left.\frac{d^n}{dx^n}\sin^2 x\right|_{x=0} = \begin{cases} 0 & n=0 \\ 0 & n=2k+1, k=0,1,2,... \\ (-1)^{k-1}\,2^{2k-1} & n=2k, k=1,2,3,... \end{cases} .$$

Thus the Taylor expansion follows

$$\sin^2 x = \sum_{j=0}^{\infty} \frac{x^j}{j!}\left.\frac{d^j}{dx^j}\sin^2 x\right|_{x=0} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k-1}x^{2k}}{(2k)!} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}(2x)^{2k}}{2(2k)!}$$

where $(-1)^{k-1} = (-1)^{k+1}$.

**(2)** Since $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ we can use the Taylor expansion of $\cos x$

$$\cos x = \sum_{j=0}^{\infty}(-1)^j\frac{x^{2j}}{(2j)!}$$

to obtain

$$\begin{aligned}\sin^2 x &= \frac{1}{2}\left(1-\sum_{j=0}^{\infty}(-1)^j\frac{(2x)^{2j}}{(2j)!}\right) \\ &= \frac{1}{2}\left(1-1-\sum_{j=1}^{\infty}(-1)^j\frac{(2x)^{2j}}{(2j)!}\right) \\ &= \sum_{j=1}^{\infty}\frac{(-1)^{j+1}(2x)^{2j}}{2(2j)!}.\end{aligned}$$

16. We have

$$\frac{d^n}{dx^n}\sin x = \begin{cases} (-1)^k\sin x & n=2k, k\in\mathbb{Z} \\ (-1)^k\cos x & n=2k+1, k\in\mathbb{Z} \end{cases}$$

so that

$$\left|\frac{d^n}{dx^n}\sin x\right| \leqslant 1.$$

Suppose we seek accuracy $\varepsilon$, then we require $|R_n| < \epsilon$ where

$$|R_n| = \left|\frac{x^{n+1}}{(n+1)!}\left.\frac{d^{n+1}}{dx^{n+1}}\sin x\right|_{x=\xi}\right| = \left|\frac{x^{n+1}}{(n+1)!}\right|\left|\left.\frac{d^{n+1}}{dx^{n+1}}\sin x\right|_{x=\xi}\right| \leqslant \left|\frac{x^{n+1}}{(n+1)!}\right|.$$

where $\xi \in (0, x)$ or $\xi \in (x, 0)$ depending on whether $x$ is positive or negative. Thus it is sufficient (but not necessary) to require that

$$\left| \frac{x^{n+1}}{(n+1)!} \right| < \varepsilon$$

which is equivalent to

$$|x|^{n+1} < \varepsilon(n+1)!.$$

The value of $n$ satisfying the inequality depends on $x$.

For $x = \pi/4$ and $\varepsilon = 10^{-5}$ we find (empirically) the smallest $n$ satisfying the inequality is 7.

For $x = \pi$ and $\varepsilon = 10^{-5}$ we find (empirically) the smallest $n$ satisfying the inequality is 15.

17. Using the Taylor expansion of $\cos x$ and $\sin x$ we set

$$\cos x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = \sin x$$

and truncating after $x^2$ yields the approximation

$$1 - \frac{x^2}{2} \approx x$$

which results in a quadratic equation

$$\frac{x^2}{2} + x - 1 \approx 0$$

This equation has two solutions $x \approx -\sqrt{3} - 1$ and $x \approx \sqrt{3} - 1$. However, $x \approx -\sqrt{3} - 1$ yields

$$\cos(-\sqrt{3} - 1) = -0.91730335 \not\approx \sin(-\sqrt{3} - 1) = -0.3981891$$

which differ by more than 0.5. The second solution yields

$$\cos(\sqrt{3} - 1) = 0.7438052 \not\approx \sin(\sqrt{3} - 1) = 0.66839644$$

which differ by less than 0.076.

Consequently $\sqrt{3}-1$ gives a first approximation for a solution (there are infinitely many). To obtain a more accurate solution we could apply root finding techniques using $\sqrt{3}-1$ as a starting point.

Note that a solution to $\cos x = \sin x$ is $x = \pi/4$ and

$$\left|\frac{\pi}{4} - (\sqrt{3}-1)\right| \approx 0.05335$$

so that $\sqrt{3}-1$ approximate a root with accuracy better than 0.1.

($-\sqrt{3}-1$ approximates $-3\pi/4$ with accuracy $\epsilon \in (0.3, 0.4)$ and could also be used as a starting point for root finding, but obviously appears to be a worse candidate than $\sqrt{3}-1$.)

18.

$$\sinh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right).$$

Now,

$$\begin{aligned}
e^{2x} + e^{-2x} &= \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \frac{(-2x)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \frac{(-1)^n (2x)^n}{n!} \\
&= \begin{cases} 0 & n \text{ odd} \\ \frac{2(2x)^n}{n!} & n \text{ even} \end{cases} \\
&= \frac{2^{2j+1} x^{2j}}{(2j)!}
\end{aligned}$$

where $j = 0, 1, 2, \ldots$. We replace the index $n$ with $2j$ becasue $2j$ is always even, and the only nonzero term in the series are those with even index.

Hence, we have

$$\begin{aligned} e^{2x}+e^{-2x} &= \sum_{j=0}^{\infty}\frac{2^{2j+1}x^{2j}}{(2j)!} \\ &= 2+\sum_{j=1}^{\infty}\frac{2^{2j+1}x^{2j}}{(2j)!} \end{aligned}$$

and so

$$\begin{aligned} \frac{1}{4}\left(e^{2x}+e^{-2x}-2\right) &= \frac{1}{4}\left(2+\sum_{j=1}^{\infty}\frac{2^{2j+1}x^{2j}}{(2j)!}-2\right) \\ &= \frac{1}{4}\sum_{j=1}^{\infty}\frac{2^{2j+1}x^{2j}}{(2j)!} \\ &= \sum_{j=1}^{\infty}\frac{2^{2j+1}x^{2j}}{2^2\,(2j)!} \\ &= \sum_{j=1}^{\infty}\frac{2^{2j-1}x^{2j}}{(2j)!}. \end{aligned}$$

Interval of convergence:

$$\left|\frac{a_{j+1}}{a_j}\right| = \left|\left(\frac{2^{2j+1}x^{2j+2}}{(2j+2)!}\right)\left(\frac{(2j)!}{2^{2j-1}x^{2j}}\right)\right| = \left|\frac{2^2x^2}{(2j+2)\,(2j+1)}\right|$$

which tends to zero as $j\to\infty$ for all $x$. Hence, the interval of convergence is $(-\infty,\infty)$.

19.

$$\begin{aligned} f(x) &= \cos(x+\pi)=\cos(\pi)+x\,(-\sin(\pi))+\frac{x^2}{2!}\,(-\cos(\pi))+\frac{x^3}{3!}\sin(\pi)+\dots \\ &\quad +\frac{x^4}{4!}\,(\cos(\pi))+\frac{x^5}{5!}\,(-\sin(\pi))+\frac{x^6}{6!}\,(-\cos(\pi))+\dots \\ &= -1+0+\frac{x^2}{2!}+0-\frac{x^4}{4!}+0+\frac{x^6}{6!}+\dots \\ &= -1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\frac{x^{10}}{10!}+\dots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!} = -\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n}}{(2n)!} = -\cos(x). \end{aligned}$$

20. Of course, we have

$$\sqrt{0.8} = (0.8)^{0.5}.$$

We first evaluate $\ln(0.8)^{0.5}$ because $(0.8)^{0.5} = \exp\left(\ln(0.8)^{0.5}\right)$ and we know how to expand $\ln(1+x)$ and $e^x$

$$\begin{aligned}
\ln(0.8)^{0.5} &= 0.5\ln 0.8 = 0.5\ln(1-0.2) = 0.5\ln(1+(-0.2)) \\
&\approx 0.5\left\{-0.2 - \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{3} - \frac{(-0.2)^4}{4}\right\} \\
&= 0.5\{-0.2 - 0.02 - 0.002667 - 0.0004\} \\
&= 0.5\{-0.223067\} \\
&= -0.111533.
\end{aligned}$$

Now

$$\begin{aligned}
(0.8)^{0.5} &= \exp\left(\ln(0.8)^{0.5}\right) \\
&\approx e^{-0.111533} \\
&\approx 1 + (-0.111533) + \frac{(-0.111533)^2}{2} + \frac{(-0.111533)^3}{6} \\
&= 1 - 0.111533 + 0.006219 - 0.000231 \\
&= 0.894455 \quad (= 0.894427 \text{ using calculator directly}).
\end{aligned}$$