Taylor series - Questions

1. Determine Taylor series and intervals of convergence for $\sin x$ and $\cos x$.
2. Determine Taylor series and intervals of convergence for $\sinh x$ and $\cosh x$.
3. Determine Taylor series and interval of convergence for $\ln(1+x)$.
4. Use the first five terms of the Taylor series for $\sin x$ to determine $\sin\frac{\pi}{3}$.
5. Evaluate $(0.7)^{0.7}$ using the first five terms of the Taylor series for $\ln(1+x)$ and e^x.
6. Evaluate $\ln(1.3)$ using the first five terms of the Taylor series for $\ln(1+x)$.
7. Evaluate $\ln(0.0002)$ using the first five terms of the Taylor series for $\ln(1+x)$. Compare with a calculator value. Resolve the problem.
8. Determine for each of the series the minimum number of terms required in the series expansion for an accuracy of 10^{-3} in term magnitude:

$$\begin{aligned}(a)\ \ln(1+x), \quad x &= 1\\ (b)\ \ln(1-x), \quad x &= \frac{1}{2}\\ (c)\ \ln\left(\frac{1+x}{1-x}\right), \quad x &= \frac{1}{3}\end{aligned}$$

9. Find the series expansion for e^{-x^2} using the Taylor series for e^x around 0. Compare with the first three non-zero terms of the Taylor expansion of e^{-x^2} around 0.
10. Find the series expansion of $\mathrm{erf}(x)$. Use the first four terms of the series to approximate $\mathrm{erf}(1)$ and compare with $\mathrm{erf}(1) \approx 0.8427$, where

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

11. Find the Taylor expansion and interval of convergence of $\cos x$ about $x = \pi$.

12. Find the series expansion for $\sin(x^2)$ using the Taylor series for $\sin x$ around 0. Compare with the first 2 non-zero terms of the direct Taylor expansion of $\sin(x^2)$ around 0. Determine an interval of convergence.

13. Use the Taylor expansion for $\sin(x^2)$ up to 6th order in x to approximate 2 non-zero real roots of $\sin(x^2) = 0$. Find the approximation error. Repeat using the expansion up to 10th order.

14. Derive the Taylor series expansion of

$$\frac{\sin x}{x}.$$

Use this expansion to explain

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

15. We could find the Taylor expansion of $\sin^2 x$ about 0 by squaring the Taylor expansion of $\sin x$ about 0. This is not a simple solution. Use **two** other methods to derive the Taylor expansion of $\sin^2 x$ about 0. Both methods should, of course, yield the same expansion.

16. Use the Lagrange estimate of the residual term to find an **upper bound** on the residual term in the Taylor expansion of $\sin x$ around 0. Use this upper bound to estimate how many terms are necessary to calculate $\sin \frac{\pi}{4}$ accurate to 10^{-5} (empirically), and to estimate how many terms are necessary to calculate $\sin \pi$ accurate to 10^{-5} (again empirically).

17. Use a Taylor series expansion up to and including terms in x^2 to approximate a solution of

$$\cos x = \sin x.$$

18. Use the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

to show that

$$\sinh^2 x = \left(\frac{e^x - e^{-x}}{2} \right)^2 = \sum_{j=1}^{\infty} \frac{2^{2j-1} x^{2j}}{(2j)!}.$$

What is the interval of convergence for this series?

19. Use a suitable Taylor series for $\cos(x+\pi)$ to show that $\cos(x+\pi) = -\cos x$.

20. Use the first four terms of the Taylor series for $\ln(1+x)$ and e^x to estimate $\sqrt{0.8}$. Work to a precision of at least 6 decimal places throughout.

21.