RK2 Method - Solutions

1. We have

$$\begin{aligned} y(x_{i+1}) &= y(x_i + h) \\ &= y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(x_i) + \ldots \end{aligned}$$

Since $y' = f(x, y)$ we have

$$\begin{aligned} y'' &= f_x + f f_y \\ y''' &= f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2. \end{aligned}$$

Also,

$$\varepsilon_{i+1} = [y(x_i) + hF(x_i, y(x_i))] - y(x_{i+1}).$$

Now, consider

$$\begin{aligned} f(x_i + h, y_i + k) &= f(x_i, y_i + k) + h f_x(x_i, y_i + k) + \frac{h^2}{2} f_{xx}(x_i, y_i + k) + \ldots \\ &= f(x_i, y_i) + k f_y(x_i, y_i) + \frac{k^2}{2} f_{yy}(x_i, y_i) + \ldots \\ &\quad + h \{ f_x(x_i, y_i) + k f_{xy}(x_i, y_i) + \ldots \} \\ &\quad + \frac{h^2}{2} \{ f_{xx}(x_i, y_i) + \ldots \} \end{aligned}$$

For RK2

$$hF(x_i, y(x_i)) = \frac{h}{2} f(x_i, y(x_i)) + \frac{h}{2} f(x_i + h, y(x_i) + h f(x_i, y_i)),$$

i.e. $k = hf\left(x_i, y\left(x_i\right)\right)$ in the above, we find

$$\begin{aligned}
\varepsilon_{i+1} &= \left[y\left(x_i\right) + hF\left(x_i, y\left(x_i\right)\right)\right] - y\left(x_{i+1}\right) \\
&= y\left(x_i\right) + \frac{h}{2} f \\
&\quad + \frac{h}{2} f + \frac{h^2}{2} f f_y + \frac{h^3}{4} f^2 f_{yy} + \ldots \\
&\quad + \frac{h^2}{2} f_x + \frac{h^3}{2} f f_{xy} + \ldots \\
&\quad + \frac{h^3}{4} f_{xx} + \ldots \\
&\quad - y\left(x_i\right) - hf - \frac{h^2}{2}\left(f_x + f f_y\right) \\
&\quad - \frac{h^3}{6}\left(f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2\right) + \ldots
\end{aligned}$$

Here, it is understood that $f, f_x, f_y, f_{yy}, f_{xy}$ and f_{xx} are all evaluated at $\left(x_i, y\left(x_i\right)\right)$. Terms up to second order in h cancel, yielding

$$\begin{aligned}
\varepsilon_{i+1} &= \frac{h^3}{4} f^2 f_{yy} + \frac{h^3}{2} f f_{xy} + \frac{h^3}{4} f_{xx} \\
&\quad - \frac{h^3}{6}\left(f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2\right) + \ldots \\
&= h^3\left(\frac{f^2 f_{yy}}{12} + \frac{f f_{xy}}{6} + \frac{f_{xx}}{12} - \frac{f_x f_y}{6} - \frac{f f_y^2}{6}\right) + \ldots \\
&= O\left(h^3\right)
\end{aligned}$$

since the term in parentheses is, in general, not equal to zero.

2. If

$$\frac{dy}{dx} = x + y + xy \equiv f(x, y)$$

then, for RK2, we have

$$
\begin{aligned}
F(x,y) &= \frac{1}{2}f(x,y)+\frac{1}{2}f(x+h,y+hf(x,y)) \\
&= \frac{1}{2}(x+y+xy) \\
&\quad +\frac{1}{2}((x+h)+(y+hf)+(x+h)(y+hf)) \\
&= \frac{1}{2}(x+y+xy) \\
&\quad +\frac{1}{2}(x+h)+\frac{1}{2}(y+h(x+y+xy)) \\
&\quad +\frac{1}{2}(x+h)(y+h(x+y+xy))
\end{aligned}
$$

which gives

$$
\begin{aligned}
F_y &= \frac{\partial F}{\partial y}=\frac{h}{2}(1+x)+\frac{h}{2}(1+h+hx)+\frac{h}{2}(x+h)(1+h+hx) \\
&= \left(\frac{h}{2}\right)x^2+\left(1+\frac{h^2}{2}+h\right)x+\left(1+\frac{h^2}{2}+h\right).
\end{aligned}
$$

We also have

$$
\begin{aligned}
\lim_{h\to 0} F &= \frac{1}{2}(x+y+xy) \\
&\quad +\frac{1}{2}((x+0)+(y+0f)+(x+0)(y+0f)) \\
&= \frac{1}{2}(x+y+xy)+\frac{1}{2}(x+y+xy) \\
&= x+y+xy \\
&= f(x,y).
\end{aligned}
$$

This property is known as *consistency*.

3. The RK2 method is

$$
\begin{aligned}
y_{m+1} &= y_m+hF(x_m,y_m) \\
F(x_m,y_m) &= \frac{f(x_m,y_m)+f(x_m+h,y_m+hf(x_m,y_m))}{2}
\end{aligned}
$$

and so we obtain

m	0	1	2	3	4	5
x_m	0	0.01	0.02	0.03	0.04	0.05
y_m	1	1.01015	1.02061	1.03137	1.04245	1.05386

m	6	7	8	9	10
x_m	0.06	0.07	0.08	0.09	0.10
y_m	1.06559	1.07765	1.09005	1.10279	1.11589

We are now required to estimate the global error at $x_{10} = 0.1$. We have

$$\Delta_{10} = \varepsilon_{10} + \alpha_9\varepsilon_9 + \ldots + \alpha_9\alpha_8 \cdots \alpha_1\varepsilon_1$$

so that

$$\begin{aligned} |\Delta_{10}| &\leqslant \max_{[x_0,x_{10}]} |\varepsilon_m| \left(1 + \alpha + \alpha^2 + \ldots + \alpha^9\right) \\ &= \max_{[x_0,x_{10}]} |\varepsilon_m| \underbrace{\left(\frac{\alpha^{10} - 1}{\alpha - 1}\right)}_{\text{geometric sum}} \end{aligned}$$

where

$$\alpha \equiv \max_{[x_0,x_{10}]} |\alpha_m| = 1 + h \max_{[x_0,x_{10}]} |F_y|$$

and

$$\begin{aligned} \max_{[x_0,x_{10}]} |\varepsilon_m| &= \max_{[x_0,x_{10}]} \left|\frac{h^3}{6} y'''\right| \\ &= \max_{[x_0,x_{10}]} \left|\frac{h^3}{6}\left(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2\right)\right|. \end{aligned}$$

In this problem $f(x, y) = x + y + xy$ and so

$$\begin{aligned} f_{xx} &= 0 \\ f_{xy} &= 1 \\ f_{yy} &= 0 \\ f_x &= 1 + y \\ f_y &= 1 + x \end{aligned}$$

and

$$F_y = \left(\frac{h}{2}\right)x^2 + \left(1 + \frac{h^2}{2} + h\right)x + \left(1 + \frac{h^2}{2} + h\right).$$

We now substitute the values of x_m and y_m which we have obtained using RK2, for each $m \in [0, 10]$, to find

$$\begin{aligned} \alpha &\equiv 1 + h \max_{[x_0, x_{10}]} |F_y| = 1.01111 \\ \max_{[x_0, x_{10}]} |\varepsilon_m| &= 1.1 \times 10^{-6} \end{aligned}$$

both of which happen to occur at $x = 0.1$. Thus, we have

$$\begin{aligned} |\Delta_{10}| &\leqslant \left(1.1 \times 10^{-6}\right)\left(\frac{1.01111^{10} - 1}{1.01111 - 1}\right) \\ &= 1.16 \times 10^{-5}. \end{aligned}$$

This is an estimate of the maximum error in our result $y(0.1) = 1.11589$. The exact value is $y(0.1) = 1.115895$, so that the actual error is 5×10^{-6}.

If we use

$$\max_{[x_0, x_{10}]} |\varepsilon_m| = h^3 \max_{[x_0, x_{10}]} \left| \left(\frac{f^2 f_{yy}}{12} + \frac{f f_{xy}}{6} + \frac{f_{xx}}{12} - \frac{f_x f_y}{6} - \frac{f f_y^2}{6} \right) \right|$$

we find

$$\max_{[x_0, x_{10}]} |\varepsilon_m| = 4.3 \times 10^{-7}$$

(which also occurs at $x = 0.1$) and hence

$$\begin{aligned} |\Delta_{10}| &\leqslant \left(4.3 \times 10^{-7}\right)\left(\frac{1.01111^{10} - 1}{1.01111 - 1}\right) \\ &= 4.5 \times 10^{-6} \end{aligned}$$

which is much closer to the actual value of the error, albeit slightly lower.

4. RK2 is

$$y_{m+1} = y_m + \frac{1}{2} f(x_m, y_m) + \frac{1}{2} f(x_m + h, y_m + h f(x_m, y_m))$$

and so we obtain

m	0	1	2	3	4	5
x_m	1	1.1	1.2	1.3	1.4	1.5
y_m	1	1.11050	1.24554	1.41094	1.61426	1.86528

We are now required to estimate the global error at $x_5 = 1.5$. We have

$$\Delta_5 = \varepsilon_5 + \alpha_4\varepsilon_4 + \ldots + \alpha_4\alpha_3\alpha_2\alpha_1\varepsilon_1$$

so that

$$\begin{aligned} |\Delta_5| &\leqslant \max_{[x_0,x_5]} |\varepsilon_m| \left(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4\right) \\ &= \max_{[x_0,x_5]} |\varepsilon_m| \underbrace{\left(\frac{\alpha^5 - 1}{\alpha - 1}\right)}_{\text{geometric sum}} \end{aligned}$$

where

$$\begin{aligned} \alpha &\equiv \max_{[x_0,x_5]} |\alpha_m| = 1 + h \max_{[x_0,x_5]} |F_y| \\ h &= 0.1 \end{aligned}$$

and

$$\begin{aligned} \max_{[x_0,x_5]} |\varepsilon_m| &= \max_{[x_0,x_5]} \left| \frac{h^3}{6} y''' \right| \\ &= \max_{[x_0,x_5]} \left| \frac{h^3}{6} \left(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2\right) \right|. \end{aligned}$$

In this problem $f(x, y) = xy$ and so

$$\begin{aligned} f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2 &= xy\left(3 + x^2\right) \\ F(x, y) &= \frac{1}{2}\left(2xy + hx^2 y + hy + h^2 xy\right) \\ F_y &= \frac{1}{2}\left(2x + hx^2 + h + h^2 x\right). \end{aligned}$$

We now substitute the values of x_m and y_m which we have obtained using RK2, for each $m \in [0, 5]$, to find

$$\begin{aligned} \alpha &= 1 + h \max_{[x_0,x_5]} |F_y| = 1.167 \\ \max_{[x_0,x_5]} |\varepsilon_m| &= \max_{[x_0,x_5]} \left| \frac{h^3}{6} \left(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2\right) \right| = 0.00244818 \end{aligned}$$

both of which happen to occur at $x = 1.5$. Thus, we have

$$\begin{aligned} |\Delta_5| &\leqslant (0.00244818)\left(1 + 1.167 + 1.167^2 + 1.167^3 + 1.167^4\right) \\ &= 0.017. \end{aligned}$$

This is an estimate of the maximum error in our result $y(1.5) = 1.86528$. The exact value is $y(1.5) = 1.8682459..$, so that the actual error is ~ 0.003.

5.