Numerical Differentiation - Solutions

1. We have

$$\begin{aligned}
y_{i-1} &= y_i - hy_i' + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{(4)} + O\left(h^5\right) \\
y_{i+1} &= y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{(4)} + O\left(h^5\right) \\
y_{i-2} &= y_i - 2hy_i' + 2h^2y_i'' - \frac{8h^3}{6}y_i''' + \frac{16h^4}{24}y_i^{(4)} + O\left(h^5\right) \\
y_{i+2} &= y_i + 2hy_i' + 2h^2y_i'' + \frac{8h^3}{6}y_i''' + \frac{16h^4}{24}y_i^{(4)} + O\left(h^5\right).
\end{aligned}$$

Hence

$$\begin{aligned}
y_{i+2} - 2y_{i+1} &= -y_i + h^2y_i'' + h^3y_i''' + \frac{14h^4}{24}y_i^{(4)} + O\left(h^5\right) \\
2y_{i-1} - y_{i-2} &= y_i - h^2y_i'' + h^3y_i''' - \frac{14h^4}{24}y_i^{(4)} + O\left(h^5\right)
\end{aligned}$$

and so

$$y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2} = 2h^3y_i''' + O\left(h^5\right)$$

$$\begin{aligned}
\Rightarrow \quad y_i''' &= \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3} + \frac{O\left(h^5\right)}{2h^3} \\
\Rightarrow \quad y_i''' &= \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3} + O\left(h^2\right).
\end{aligned}$$

Also,

$$\begin{aligned}
8y_{i+1} - y_{i+2} &= 7y_i + 6hy_i' + 2h^3y_i'' - \frac{8}{24}h^4y_i^{(4)} + O(h^5) \\
y_{i-2} - 8y_{i-1} &= -7y_i + 6hy_i' - 2h^3y_i'' + \frac{8}{24}h^4y_i^{(4)} + O(h^5)
\end{aligned}$$

which gives

$$8y_{i+1} - y_{i+2} + y_{i-2} - 8y_{i-1} = 12hy_i' + O(h^5)$$

$$\Rightarrow \quad y_i' = \frac{8y_{i+1} - y_{i+2} + y_{i-2} - 8y_{i-1}}{12h} + \frac{O\left(h^5\right)}{12h}$$
$$\Rightarrow \quad y_i' = \frac{8y_{i+1} - y_{i+2} + y_{i-2} - 8y_{i-1}}{12h} + O\left(h^4\right).$$

2. Using

$$\begin{aligned} y_{i+1} &= y_i + hy_i' + \frac{h^2}{2}y_i'' + O\left(h^3\right) \\ y_{i+2} &= y_i + 2hy_i' + 2h^2y_i'' + O\left(h^3\right) \end{aligned}$$

gives

$$y_{i+2} - 2y_{i+1} = -y_i + h^2 y_i'' + O(h^3)$$

$$\Rightarrow \quad y_i'' = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} + \frac{O\left(h^3\right)}{h^2}$$
$$\Rightarrow \quad y_i'' = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} + O\left(h\right).$$

Also,

$$4y_{i+1} - y_{i+2} = 3y_i + 2hy_i' + O(h^3)$$

$$\Rightarrow \quad y_i' = \frac{4y_{i+1} - y_{i+2} - 3y_i}{2h} + \frac{O\left(h^3\right)}{2h}$$
$$\Rightarrow \quad y_i' = \frac{4y_{i+1} - y_{i+2} - 3y_i}{2h} + O\left(h^2\right).$$

3. Using

$$y_{i-1} = y_i - hy_i' + O\left(h^2\right)$$

gives

$$\begin{aligned} y_i' &= \frac{-y_{i-1} + y_i}{h} + \frac{O\left(h^2\right)}{h} \\ &= \frac{y_i - y_{i-1}}{h} + O\left(h\right). \end{aligned}$$

Also,

$$\begin{aligned} y_{i-1} &= y_i - hy_i' + \frac{h^2}{2}y_i'' + O\left(h^3\right) \\ y_{i-2} &= y_i - 2hy_i' + 2h^2y_i'' + O\left(h^3\right) \end{aligned}$$

gives

$$y_{i-2} - 4y_{i-1} = -3y_i + 2hy_i' + O(h^3)$$

$$\Rightarrow \quad y_i' = \frac{y_{i-2} - 4y_{i-1} + 3y_i}{2h} + \frac{O(h^3)}{2h}$$
$$\Rightarrow \quad y_i' = \frac{y_{i-2} - 4y_{i-1} + 3y_i}{2h} + O(h^2).$$

4. **(a)** We use

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

with $h = \frac{1}{4}$ to get

$$y'\left(\frac{1}{2}\right) = \frac{2.117 - 1.284}{2(0.25)} = 1.666$$

(b) We use

$$y_i'' = \frac{-y_{i+3} + 4y_{i+2} - 5y_{i+1} + 2y_i}{h^2}$$

with $h = \frac{1}{4}$ to get

$$y''(0) = \frac{-2.117 + 4(1.6487) - 5(1.284) + 2(1)}{(0.25)^2} = 0.9248$$

(c) We use

$$y_i''' = \frac{-y_{i-3} + 3y_{i-2} - 3y_{i-1} + y_i}{h^3}$$

with $h = \frac{1}{4}$ to get

$$y'''(1) = \frac{-1.284 + 3(1.6487) - 3(2.117) + 2.7183}{(0.25)^3} = 1.8816$$

Compare this with the second-order Backward-difference formula

$$y_i''' = \frac{3y_{i-4} - 14y_{i-3} + 24y_{i-2} - 18y_{i-1} + 5y_i}{2h^3}$$

which yields the more accurate value

$$y'''(1) = 2.5056$$

(d) We use

$$y_i'' = \frac{-y_{i+2} + 16y_{i+1} - 30y_i + 16y_{i-1} - y_{i-2}}{12h^2}$$

with $h = \frac{1}{4}$ to get

$$y''\left(\frac{1}{2}\right) = \frac{-2.7183 + 16\,(2.117) - 30\,(1.6487) + 16\,(1.284) - 1}{12\,(0.25)^2} = 1.6489$$

(e) We use

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

with $h = \frac{1}{8}$ to get

$$y'\left(\frac{3}{8}\right) = \frac{1.6487 - 1.284}{2\,(0.125)} = 1.4588$$

5. **(a)** We use

$$y_i'' = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2}$$

to find

$$y''(0) = \frac{\sin\frac{\pi}{4} - 2\sin\frac{\pi}{8} + \sin 0}{\left(\frac{\pi}{8}\right)^2} = -0.3778$$

(b) We use

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

with

$$h = \frac{\pi}{16}$$

to find

$$y'\left(\frac{3\pi}{16}\right) = \frac{\sin\frac{\pi}{4} - \sin\frac{\pi}{8}}{2\left(\frac{\pi}{16}\right)} = 0.8261$$

Note that $\frac{3\pi}{16}$ is midway between $\frac{\pi}{8}$ and $\frac{\pi}{4}$.

6. **(a)** We use

$$y_i'' = \frac{y_{i-2} - 2y_{i-1} + y_i}{h^2}$$

to find

$$y''\left(\frac{\pi}{2}\right) = \frac{\cos\frac{\pi}{4} - 2\cos\frac{3\pi}{8} + \cos\frac{\pi}{2}}{\left(\frac{\pi}{8}\right)^2} = -0.3778$$

(b) We use

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h}$$

with

$$h = \frac{\pi}{16}$$

to find

$$y'\left(\frac{7\pi}{16}\right) = \frac{\cos\frac{\pi}{2} - \cos\frac{3\pi}{8}}{2\left(\frac{\pi}{16}\right)} = -0.9745$$

Note that $\frac{7\pi}{16}$ is midway between $\frac{3\pi}{8}$ and $\frac{\pi}{2}$.

7.