Newton's Method - Solutions

1. We have $f(x) = e^x - 3x^2$, $f'(x) = e^x - 6x, x_1 = -0.464$ and tolerance $\varepsilon = 10^{-6}$. We see that $|f(x_1)| = 0.0171 > \varepsilon$, and so we must proceed with Newton's Method. The Newton iterations are as follows:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_{1)}}{f'(x_1)} = -0.464 - \frac{-0.0171}{3.4128} = -0.45898 \\ f(x_2) &= -0.000067 \Rightarrow |f(x_2)| > \varepsilon \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_{2)}}{f'(x_2)} = -0.459 - \frac{-0.000067}{3.3858} = -0.45896 \\ f(x_3) &= -1 \times 10^{-9} \Rightarrow |f(x_3)| < \varepsilon \end{aligned}$$

So the root is, to 5 decimal places, $x_0 = -0.45896$.

For the positive root we use $x_1 = 0.864$ ($f(x_1) = 0.133 > \varepsilon$) and we obtain

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_{1)}}{f'(x_1)} = 0.864 - \frac{0.13314}{-2.81137} = 0.9114 \\ f(x_2) &= -0.00402 \Rightarrow |f(x_2)| > \varepsilon \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_{2)}}{f'(x_2)} = 0.9114 - \frac{-0.00402}{-2.9804} = 0.9100 \\ f(x_3) &= -3.2 \times 10^{-6} \Rightarrow |f(x_3)| > \varepsilon \end{aligned}$$

$$\begin{aligned} x_4 &= x_3 - \frac{f(x_{3)}}{f'(x_3)} = 0.9101 - \frac{-3.2 \times 10^{-6}}{-2.97571} = 0.9100 \\ f(x_4) &= -2 \times 10^{-12} \Rightarrow |f(x_4)| < \varepsilon \end{aligned}$$

So the root is, to 5 decimal places, $x_0 = 0.91000$.

2. We have $f(x) = x^2 - 1 - \sin(x)$, $f'(x) = 2x - \cos(x)$, $x_1 = -0.6$ and tolerance $\varepsilon = 10^{-6}$. We see that $|f(x_1)| = 0.0753 > \varepsilon$, and so we must proceed with Newton's Method. The Newton iterations are as follows:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_{1})}{f'(x_1)} = -0.6 - \frac{-0.0753}{-2.02533} = -0.63721 \\ f(x_2) &= -0.00099 \Rightarrow |f(x_2)| > \varepsilon \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_{2})}{f'(x_2)} = -0.637 - \frac{-0.00099}{-2.07817} = -0.63673 \\ f(x_3) &= 1.6 \times 10^{-7} \Rightarrow |f(x_3)| < \varepsilon \end{aligned}$$

So the root is, to 5 decimal places, $x_0 = -0.63673$.

For the other root we have $x_1 = 1.6$ and tolerance $\varepsilon = 10^{-6}$. We see that $|f(x_1)| = 0.56 > \varepsilon$, and so we must proceed with Newton's Method:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_{1})}{f'(x_1)} = 1.6 - \frac{0.5604}{3.229199} = 1.42645 \\ f(x_2) &= 0.045 \Rightarrow |f(x_2)| > \varepsilon \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_{2})}{f'(x_2)} = 1.4264 - \frac{0.04516}{2.709055} = 1.40978 \\ f(x_3) &= 0.0004 \Rightarrow |f(x_3)| > \varepsilon \end{aligned}$$

$$\begin{aligned} x_4 &= x_3 - \frac{f(x_{3})}{f'(x_3)} = 1.409781 - \frac{0.000415}{2.659239} = 1.409624 \\ f(x_4) &= 3.6 \times 10^{-8} \Rightarrow |f(x_4)| < \varepsilon. \end{aligned}$$

So the root is, to 5 decimal places, $x_0 = 1.40962$.

3. To find the area enclosed by $y(x) = \cos(x)$ and $y(x) = e^{-x}$ we must first find the points where they intersect. It is clear that $\cos(0) = e^{-0}$ so that $(0, 1)$ is a point of intersection. To find the other point we must solve

$$f(x) \equiv \cos(x) - e^{-x} = 0$$

A low-order Taylor expansion helps us to estimate the root:

$$\left(1-\frac{x^2}{2!}\right)-\left(1-x+\frac{x^2}{2!}\right)=0$$

$$\Rightarrow x^2-x=0$$

$$\Rightarrow x=0 \text{ (as expected) or } x=1.$$

We have

$$f'(x)=e^{-x}-\sin(x)$$

and so, with $x_1=1$ and using Newton's Method with a tolerance of $\varepsilon=10^{-5}$, we obtain

$$x_0=1.29269$$

after 3 iterations.

The enclosed area is then given by

$$\begin{aligned}\text{Area} &= \left|\int_0^{1.29269}\left(\cos(x)-e^{-x}\right)dx\right| \\ &= \left|\left[\sin(x)+e^{-x}\right]_0^{1.29269}\right| \\ &= 0.23611.\end{aligned}$$

4. We have

$$\begin{aligned}y(x) &= 2\sin(x) \\ y(x) &= \ln(x)-c\end{aligned}$$

These curves **touch** each other near $x=8$. This means that their gradients must be equal at the point where they touch. Thus we have

$$f(x)\equiv 2\cos(x)-\frac{1}{x}=0.$$

This is the equation we must solve to find the value of x at which they touch. From this we have

$$f'(x)=-2\sin(x)+\frac{1}{x^2}.$$

Implementing Newton's Method with a tolerance of $\varepsilon = 10^{-5}$ and $x = 8$ as the initial estimate for the root yields

$$x_0 = 7.78975.$$

Now, since the curves touch at x_0, we must have

$$2\sin(x_0) = \ln(x_0) - c$$

which gives

$$c = 0.05693.$$

5.

$$x = \frac{1}{\sqrt{a}} \Rightarrow x^2 = \frac{1}{a} \Rightarrow a = \frac{1}{x^2} \Rightarrow a - \frac{1}{x^2} = 0$$

Of course, the root of $a - \frac{1}{x^2} = 0$ is $\frac{1}{\sqrt{a}}$. Hence, define

$$f(x) \equiv a - \frac{1}{x^2}$$

so that

$$f'(x) = \frac{2}{x^3}$$

which gives, for Newton's Method,

$$\begin{aligned} x_{i+1} = x_i - \frac{\left(a - \frac{1}{x_i^2}\right)}{\frac{2}{x_i^3}} &= x_i - \frac{1}{2}(ax_i^3 - x_i) \\ &= \frac{x_i(3 - ax_i^2)}{2}. \end{aligned}$$

6. Let the error in x after the ith iteration be denoted by η_i. In other words, $x_i = x_0 + \eta_i$. Hence, Newton's Method becomes

$$x_0 + \eta_{i+1} = x_0 + \eta_i - \frac{f(x_0 + \eta_i)}{f'(x_0 + \eta_i)}.$$

A Taylor expansion about $x = x_0$ for both $f(x_0 + \eta_i)$ and $f'(x_0 + \eta_i)$ gives

$$\eta_{i+1} = \eta_i - \frac{f(x_0) + \eta_i f'(x_0) + \frac{1}{2}\eta_i^2 f''(x_0) + \cdots}{f'(x_0) + \eta_i f''(x_0) + \cdots}.$$

Now, $f(x) = x^2$ gives $f(x_0 = 0) = 0$, $f'(0) = 0$ and $f''(0) = 2$. Hence

$$\begin{aligned}
\eta_{i+1} &\approx \eta_i - \frac{\frac{1}{2}\eta_i^2 2}{2\eta_i} \\
&= \frac{1}{2}\eta_i
\end{aligned}$$

which indicates linear convergence.

7.

$$\begin{aligned}
x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} \qquad \text{(Newton's Method)} \\
&\approx x_i - \frac{f(x_i)}{\left(\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}\right)} \\
&= x_i - \left(\frac{f(x_i)}{f(x_i) - f(x_{i-1})}\right)(x_i - x_{i-1}) \\
&= \frac{x_i\left(f(x_i) - f(x_{i-1})\right) - f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \\
&= \frac{-x_i f(x_{i-1}) + f(x_i) x_{i-1}}{f(x_i) - f(x_{i-1})} \\
&= \frac{f(x_i) x_{i-1} - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}. \qquad \text{(Linear Interpolation)}
\end{aligned}$$

8. The function $y(x)$ is defined by the relation

$$\sin(xy) = y - x \Rightarrow y = \sin(xy) + x \equiv g(x, y)$$

We seek the values of x and y where $y(x)$ has a maximum.

At a maximum, we have $\frac{dy}{dx} = 0$. Now,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial y}\frac{dy}{dx} = \frac{\partial g}{\partial x}(1) + \frac{\partial g}{\partial y}(0) = \frac{\partial g}{\partial x} \\
&= \frac{d}{dx}\left(\sin(xy) + x\right) \\
&= \frac{d\sin(xy)}{dx} + 1 \\
&= \frac{d\sin(xy)}{dxy}\frac{dxy}{dx} + 1 \\
&= y\cos(xy) + 1
\end{aligned}$$

So the two equations that must be solved simultaneously are

$$\begin{aligned} f_1(x,y) &\equiv y\cos(xy)+1=0 \\ f_2(x,y) &\equiv \sin(xy)-y+x=0 \end{aligned}$$

From these we have

$$\begin{aligned} \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix} &= \left(\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}^{-1}_{(x_1,y_1)} \right) \begin{bmatrix} -f_1(x_1,y_1) \\ -f_2(x_1,y_1) \end{bmatrix} \\ &= \left(\begin{bmatrix} -y^2\sin(xy) & \cos(xy)-xy\sin(xy) \\ y\cos(xy)+1 & x\cos(xy)-1 \end{bmatrix}^{-1}_{(x_1,y_1)} \right) \begin{bmatrix} -(y\cos(xy)+1) \\ -(\sin(xy)-y+x) \end{bmatrix}_{(x_1,y_1)} \end{aligned}$$

To estimate the roots we assume $x=1$ (the given value) which gives

$$\begin{aligned} & \sin(y)-y+1=0 \\ \Rightarrow\ & y-\frac{y^3}{3!}=y-1 \\ \Rightarrow\ & y^3=6 \\ \Rightarrow\ & y\approx 1.817 \end{aligned}$$

The first iteration of the Newton process is

$$\begin{aligned} \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix} &= \left(\begin{bmatrix} -3.2019 & -2.0059 \\ 0.5572 & -1.2437 \end{bmatrix}^{-1} \right) \begin{bmatrix} -0.5571 \\ -0.1528 \end{bmatrix} \\ &= \begin{bmatrix} 0.0758 \\ 0.1568 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.0758 \\ 1.9738 \end{bmatrix} \end{aligned}$$

Note that

$$\begin{aligned} |f_1(x,y)| &= 0.0361 \\ |f_2(x,y)| &= 0.0468 \end{aligned}$$

neither of which satisfy the tolerance of 10^{-3}. The second iteration is

$$\begin{aligned} \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix} &= \left(\begin{bmatrix} -3.3160 & -2.3323 \\ -0.0361 & -1.5647 \end{bmatrix}^{-1} \right) \begin{bmatrix} -0.0361 \\ -0.0468 \end{bmatrix} \\ &= \begin{bmatrix} 0.0103 \\ -0.0301 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.0861 \\ 1.9437 \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} |f_1(x,y)| &= 2.4 \times 10^{-4} \\ |f_2(x,y)| &= 2.4 \times 10^{-5}. \end{aligned}$$

Hence, we stop after the second iteration, since the tolerance condition is satisfied. Thus, the roots are

$$\begin{aligned} x_0 &= 1.0861 \\ y_0 &= 1.9437. \end{aligned}$$

9. Obviously, the root of

$$f(x) \doteqdot (2x-1)^3 = 0.$$

is

$$x_0 = \frac{1}{2}.$$

Note that

$$\begin{aligned} f'(x_0) &= 6(2x_0-1)^2 = 0 \\ f''(x_0) &= 24(2x_0-1) = 0 \\ f'''(x_0) &= 48. \end{aligned}$$

For Newton's Method we have

$$\begin{aligned} \eta_{i+1} &= \eta_i - \left(\frac{f(x_0) + \eta_i f'(x_0) + \frac{1}{2}\eta_i^2 f''(x_0) + \frac{1}{6}\eta_i^3 f'''(x_0) + \cdots}{f'(x_0) + \eta_i f''(x_0) + \frac{1}{2}\eta_i^2 f'''(x_0) + \cdots} \right) \\ &= \eta_i - \left(\frac{0+0+0+\frac{1}{6}\eta_i^3 48 + \cdots}{0+0+\frac{1}{2}\eta_i^2 48 + \cdots} \right) \\ &= \eta_i - \left(\frac{\frac{1}{6}\eta_i^3 48 + \cdots}{\frac{1}{2}\eta_i^2 48 + \cdots} \right) \\ &= \eta_i - \left(\frac{\frac{1}{6}\eta_i + \cdots}{\frac{1}{2} + \cdots} \right) \\ &\approx \eta_i - \frac{\eta_i}{3} \\ &= \frac{2}{3}\eta_i \end{aligned}$$

which indicates linear convergence.

Since we have

$$\eta_{i+1} \approx \frac{1}{2}\eta_i$$

for the Bisection Method, and $\frac{1}{2} < \frac{2}{3}$, we would expect the Bisection Method to converge faster for this particular problem.

10.

$$\begin{aligned} x &= \sqrt{0.8} \Rightarrow x^2 = 0.8 = \frac{4}{5} \\ &\Rightarrow \frac{5}{4} = \frac{1}{x^2} \Rightarrow \frac{5}{4} - \frac{1}{x^2} = 0 \end{aligned}$$

Of course, the root of $\frac{5}{4} - \frac{1}{x^2} = 0$ is $\sqrt{0.8}$. Hence, define

$$f(x) \equiv \frac{5}{4} - \frac{1}{x^2}$$

so that

$$f'(x) = \frac{2}{x^3}$$

which gives, for Newton's Method,

$$\begin{aligned} x_{i+1} = x_i - \frac{\left(\frac{5}{4} - \frac{1}{x_i^2}\right)}{\frac{2}{x_i^3}} &= x_i - \frac{1}{2}\left(\frac{5}{4}x_i^3 - x_i\right) \\ &= \frac{x_i(3 - \frac{5}{4}x_i^2)}{2} \\ &= \frac{12x_i - 5x_i^3}{8}. \end{aligned}$$

We have

$$x_{i+1} = \frac{12x_i - 5x_i^3}{8} \equiv g\left(x_i\right).$$

For convergence, we require that

$$\left|g'\left(x\right)\right| < 1$$

in a neighbourhood of the root. Now,

$$g'\left(x\right) = \frac{12 - 15x^2}{8}$$

and so

$$
\begin{aligned}
& \left|\frac{12-15x^2}{8}\right| < 1 \\
\Rightarrow\ & -1 < \frac{12-15x^2}{8} < 1 \\
\Rightarrow\ & -8-12 < -15x^2 < 8-12 \\
\Rightarrow\ & -20 < -15x^2 < -4 \\
\Rightarrow\ & 20 > 15x^2 > 4 \\
\Rightarrow\ & \frac{20}{15} > x^2 > \frac{4}{15} \\
\Rightarrow\ & \sqrt{\frac{4}{15}} < x < \sqrt{\frac{4}{3}}.
\end{aligned}
$$

Note that the root $x_0 \approx 0.89$ is in this interval, so that this interval is a neighbourhood of the root.

11.

$$
\begin{aligned}
x &= \sqrt{0.75} \Rightarrow x^2 = 0.75 = \frac{3}{4} \\
&\Rightarrow \frac{4}{3} = \frac{1}{x^2} \Rightarrow \frac{4}{3} - \frac{1}{x^2} = 0
\end{aligned}
$$

Of course, the root of $\frac{4}{3} - \frac{1}{x^2} = 0$ is $\sqrt{0.75}$. Hence, define

$$f(x) \equiv \frac{4}{3} - \frac{1}{x^2}$$

so that

$$f'(x) = \frac{2}{x^3}$$

which gives, for Newton's Method,

$$
\begin{aligned}
x_{i+1} = x_i - \frac{\left(\frac{4}{3} - \frac{1}{x_i^2}\right)}{\frac{2}{x_i^3}} &= x_i - \frac{1}{2}\left(\frac{4}{3}x_i^3 - x_i\right) \\
&= \frac{x_i(3 - \frac{4}{3}x_i^2)}{2} \\
&= \frac{9x_i - 4x_i^3}{6}.
\end{aligned}
$$

Application of this method gives

x	$\left\lvert \frac{4}{3} - \frac{1}{x^2} \right\rvert$
1	0.333
0.833	0.107
0.864	0.005
0.866019	1.7×10^{-5}

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12.