# Linear Interpolation - Solutions

1. For the negative root choose $x_1 = -0.5773503$ and $x_2 = -1$. These give

$$\begin{aligned} y_1 &\equiv y(x_1) = 0.718233 \\ y_2 &\equiv y(x_2) = -1 \end{aligned}$$

where we use $y(x) = x^3 - x + 1 - 2x^2$. Implementing the Linear Interpolation method we have

$$\begin{aligned} x_3 &= \frac{x_1 y_2 - y_1 x_2}{y_2 - y_1} \\ &= -0.7540207 \\ y_3 &= 0.188229 \end{aligned}$$

The next iteration gives

$$\begin{aligned} x_4 &= \frac{x_2 y_3 - y_2 x_3}{y_3 - y_2} \\ &= -0.7929867 \\ y_4 &= 0.0366783 \end{aligned}$$

and the next gives

$$\begin{aligned} x_5 &= \frac{x_3 y_4 - y_3 x_4}{y_4 - y_3} \\ &= -0.8024173 \\ y_5 &= -0.0019851 \end{aligned}$$

Continuing like this eventually yields

$$\begin{aligned} x_7 &= \frac{x_5 y_6 - y_5 x_6}{y_6 - y_5} \\ &= -0.801937733 \\ y_7 &= 0.000000009 \end{aligned}$$

This value for $y_7$ is less than the tolerance of $10^{-6}$ that was required. Hence, the root, rounded to 6 decimal places, is $x_0 = -0.801938$.

For the positive root we choose $x_1 = 0$ and $x_2 = 0.577$, which give $y_1 = 1$ and $y_2 = -0.0515668$. The same procedure as above is implemented, eventually yielding

$$\begin{aligned} x_5 &= \frac{x_3 y_4 - y_3 x_4}{y_4 - y_3} \\ &= 0.554958 \\ y_5 &= -0.00000003 \end{aligned}$$

This value for $y_5$ is comfortably less than the tolerance of $10^{-6}$, and so the root is, rounded to 6 decimals, $x_0 = 0.554958$.

2. We seek the negative root of $y(x) = e^x - x - 2 = 0$. With $x_1 = -\sqrt{2}$ and $x_2 = -\sqrt{3} we$ have $y_1 = -0.342669$ and $y_2 = -0.091027$. Implementing Linear Interpolation gives

$$\begin{aligned} x_3 &= \frac{x_1 y_2 - y_1 x_2}{y_2 - y_1} \\ &= -1.847024 \\ y_3 &= 0.004729 \end{aligned}$$

Repeated iteration eventually yields

$$\begin{aligned} x_5 &= \frac{x_3 y_4 - y_3 x_4}{y_4 - y_3} \\ &= -1.841405 \\ y_5 &= -0.00000002 \end{aligned}$$

This satisfies the given tolerance of $10^{-5}$ and so the root is, rounded to 5 decimals, $x_0 = -1.84141$.

3. We implement Linear Interpolation to solve

$$y(c) = c \cosh \frac{45}{c} - 15 - c = 0$$

assuming $c_1 = 67.5$ and $c_2 = 70$. These give $y_1 = 0.56385$ and $y_2 = -0.03067$. The first iteration gives

$$\begin{aligned} c_3 &= \frac{c_1 y_2 - y_1 c_2}{y_2 - y_1} \\ &= 69.87103 \\ y_3 &= -0.0011 \end{aligned}$$

Further iteration yields

$$\begin{aligned} c_4 &= \frac{c_3 y_4 - y_3 c_4}{y_4 - y_3} \\ &= 69.86604 \\ y_4 &= -0.000002 \end{aligned}$$

This satisfies a tolerance of $10^{-3}$ and so the root is, rounded to 3 decimals, $c_0 = 69.866$.

4. Since $\sin \pi = 0$, we seek the root of $\sin x$ on $[2,4]$. Linear interpolation yields

| $x_1$ | $y_1 = \sin(x_1)$ | $x_2$ | $y_2 = \sin(x_2)$ | $x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$ | $y_3 = \sin(x_3)$ |
|---|---|---|---|---|---|
| 2 | 0.9092974 | 4 | $-0.7568025$ | 3.09152808 | 0.05004366 |
| 4 | $-0.7568025$ | 3.09152808 | 0.05004366 | 3.14787496 | $-0.006282262$ |
| 3.09152808 | 0.05004366 | 3.14787496 | $-0.006282262$ | **3.14159036** | $2.296 \times 10^{-6}$ |

Thus we find $\pi \approx 3.14159036$, since $|2.296 \times 10^{-6}| < 10^{-5}$.

5.

Applying Linear Interpolation to $y(x) = \cos x - \sin x$ gives

| $x_1$ | $y_1$ | $x_2$ | $y_2$ | $x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$ | $y_3$ |
|---|---|---|---|---|---|
| 0.7 | 0.12062450 | 0.8 | $-0.02064938$ | 0.78538344 | $-2.08 \times 10^{-5}$ |
| 0.8 | $-0.02064938$ | 0.78538344 | 0.00002082 | 0.78539816 | $-7.39 \times 10^{-10}$ |

Consequently, the approximation to the root is 0.78539816, since $|-7.391916 \times 10^{-10}| < 10^{-9}$.

6. We solve

$$y(x) = x^3 - a = 0.$$

Obviously, $\sqrt[3]{a}$ is a root of this equaion.

For Linear Intepolation, we have

$$\begin{aligned} x_{i+1} &= \frac{x_{i-1}y_i - x_i y_{i-1}}{y_i - y_{i-1}} \\ &= \frac{x_{i-1}\left(x_i^3 - a\right) - x_i\left(x_{i-1}^3 - a\right)}{\left(x_i^3 - a\right) - \left(x_{i-1}^3 - a\right)} \\ &= \frac{x_{i-1}x_i\left(x_i^2 - x_{i-1}^2\right) + a\left(x_i - x_{i-1}\right)}{x_i^3 - x_{i-1}^3} \\ &= \frac{x_{i-1}x_i\left(x_i - x_{i-1}\right)\left(x_i + x_{i-1}\right) + a\left(x_i - x_{i-1}\right)}{\left(x_i - x_{i-1}\right)\left(x_i^2 + x_{i-1}x_i + x_{i-1}^2\right)} \\ &= \frac{x_i x_{i-1}\left(x_i + x_{i-1}\right) + a}{\left(x_i^2 + x_i x_{i-1} + x_{i-1}^2\right)}. \end{aligned}$$

7. We consider

$$f(x) = \ln(x) - 1 = 0$$

which obviously has $x = e$ as a solution.

Linear Interpolation is

$$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

and so

| $i$ | $x_i$ | $\lvert f(x_i)\rvert$ |
|---|---|---|
| 1 | 2 | 0.306853 |
| 2 | 3 | 0.098612 |
| 3 | 2.756792 | 0.014068 |
| 4 | 2.716324 | 0.000721 |

We see that $\lvert f(x_4)\rvert < 10^{-3}$, and so $x_4 = 2.716324$ is our approximation to $e$.

8.

$$mx + c = 0 \Rightarrow x = -\frac{c}{m}.$$

Using Linear Interpolation t solve $mx + c = 0$ gives

$$\begin{aligned} x_{i+2} &= \frac{x_i (mx_{i+1} + c) - x_{i+1} (mx_i + c)}{mx_{i+1} + c - (mx_i + c)} \\ &= \frac{x_i c - x_{i+1} c}{mx_{i+1} - mx_i} \\ &= \frac{-c (x_{i+1} - x_i)}{m (x_{i+1} - x_i)} \\ &= -\frac{c}{m}. \end{aligned}$$

9. We use

$$x_{i+1} = \frac{x_{i-1} y_i - x_i y_{i-1}}{y_i - y_{i-1}}$$

with $x_1 = 3$ and $x_2 = 4$ to obtain

| $i$ | $x_i$ | $y_i$ |
|---|---|---|
| 1 | 3 | 0.141120 |
| 2 | 4 | $-0.756802$ |
| 3 | 3.157163 | $-0.015570$ |
| 4 | 3.139459 | 0.002134 |

Note that

$$|y_4| < \varepsilon = 10^{-2}$$

and so

$$\text{root } \approx 3.139459.$$

10. We consider

$$f(x) = x^2 - 0.8 = 0$$

which obviously has $x = \sqrt{0.8}$ as a solution. Linear Interpolation is

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

and so

$$\begin{aligned} x_{i+1} &= \frac{x_{i-1}\left(x_i^2 - 0.8\right) - x_i\left(x_{i-1}^2 - 0.8\right)}{\left(x_i^2 - 0.8\right) - \left(x_{i-1}^2 - 0.8\right)} \\ &= \frac{x_{i-1}x_i^2 - 0.8x_{i-1} - x_i x_{i-1}^2 + 0.8x_i}{x_i^2 - x_{i-1}^2} \\ &= \frac{x_{i-1}x_i\left(x_i - x_{i-1}\right) + 0.8\left(x_i - x_{i-1}\right)}{x_i^2 - x_{i-1}^2} \\ &= \frac{\left(x_{i-1}x_i + 0.8\right)\left(x_i - x_{i-1}\right)}{\left(x_i - x_{i-1}\right)\left(x_i + x_{i-1}\right)} \\ &= \frac{x_{i-1}x_i + 0.8}{x_i + x_{i-1}}. \end{aligned}$$

Applying this method gives

| $i$ | $x_i$ | $\lvert f(x_i)\rvert$ |
|---|---|---|
| 1 | 0 | 0.8 |
| 2 | 1 | 0.2 |
| 3 | 0.8 | 0.16 |
| 4 | 0.8889 | 0.0099 |
| 5 | 0.8947 | 0.0005 |

We see that $\lvert f(x_5)\rvert = 0.0005 < 10^{-3}$, and so $x_5 = 0.8947$ is our approximation to $\sqrt{0.8}$.

11.