Lagrange Interpolation - Solutions

1. We have the discrete points

i	0	1	2	3
x_i	7	8	9	10
y_i	3	1	1	9

$$
\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)} \\
&= \frac{-x^3+27x^2-242x+720}{6} \\
y_0 &= 3 \\
L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)} \\
&= \frac{x^3-26x^2+223x-630}{2} \\
y_1 &= 1 \\
L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)} \\
&= \frac{-x^3+25x^2-206x+560}{2} \\
y_2 &= 1 \\
L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-7)(x-8)(x-9)}{(10-7)(10-8)(10-9)} \\
&= \frac{x^3-24x^2+191x-504}{6} \\
y_3 &= 9
\end{aligned}
$$

Hence,

$$
\begin{aligned}
P_3(x) &= \sum_{i=0}^{3} y_i L_i(x) \\
&= 3\left(\frac{-x^3+27x^2-242x+720}{6}\right)+1\left(\frac{x^3-26x^2+223x-630}{2}\right) \\
&\quad +1\left(\frac{-x^3+25x^2-206x+560}{2}\right)+9\left(\frac{x^3-24x^2+191x-504}{6}\right) \\
&= x^3-23x^2+174x-431.
\end{aligned}
$$

We may check this result by evaluating $P_3(x)$ at each of the discrete x_i - the answer should be the corresponding y_i since $P_3(x)$ passes through the discrete points. Finally, we have

$$P_3(9.5) = 3.625.$$

2. We note that in this problem $y(x) = e^x$. We have the discrete points

i	0	1	2
x_i	0	0.1	0.3
y_i	1	1.1052	1.3499

The Lagrange coefficient polynomials are

$$
\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.1)(x-0.3)}{(0-0.1)(0-0.3)} = \frac{x^2-0.4x+0.03}{0.03} \\
y_0 &= 1 \\
L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-0.3)}{(0.1-0)(0.1-0.3)} = \frac{-x^2+0.3x}{0.02} \\
y_1 &= 1.1052 \\
L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-0.1)}{(0.3-0)(0.3-0.1)} = \frac{x^2-0.1x}{0.06} \\
y_2 &= 1.3499
\end{aligned}
$$

and so the Lagrange interpolating polynomial is

$$\begin{aligned} P_2(x) &= \sum_{i=0}^{2} y_i L_i(x) \\ &= 1\left(\frac{x^2-0.4x+0.03}{0.03}\right)+1.1052\left(\frac{-x^2+0.3x}{0.02}\right)+1.3499\left(\frac{x^2-0.1x}{0.06}\right) \\ &= 0.57166x^2+0.99483x+1. \end{aligned}$$

We may check this result by evaluating $P_2(x)$ at each of the discrete x_i - the answer should be the corresponding y_i since $P_2(x)$ passes through the discrete points. Our approximation to $e^{0.2}$ is then

$$e^{0.2} \approx P_2(0.2) = 1.22183.$$

The actual value is 1.22140, so the error is 0.00043.

We may estimate the maximum error using

$$\Delta(x) \equiv \left| \frac{y^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{2} (x-x_i) \right|.$$

In this problem, $y(x)=e^x, n=2$ so that $y^{(n+1)}(x)=\frac{d^3e^x}{dx^3}=e^x$. We wish to determine the maximum error so we choose ξ such that $\frac{d^3e^x}{dx^3}=e^x$ is a maximum on the interval over which the discrete points were given. This interval is $[0,0.3]$, and the maximum value of e^x on this interval occurs at 0.3. Hence, $\xi=0.3$. We are trying to find the maximum error at $x=0.2$, so we have

$$\Delta(0.2) \leqslant \left| \frac{e^{0.3}}{3!}(0.2-0)(0.2-0.1)(0.2-0.3) \right| = 0.00045$$

which is larger than the actual error, as expected.

3. We are given

i	0	1	2	3	4
x_i	1	2	3	4	5
y_i	11.6	16.2	16.8	13.5	7.3

We construct the Lagrange coefficient polynomials:

$$
\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
&= \frac{(x-2)(x-3)(x-4)(x-5)}{(1-2)(1-3)(1-4)(1-5)} \\
&= \frac{x^4-14x^3+71x^2-154x+120}{24}
\end{aligned}
$$

$$
\begin{aligned}
L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
&= \frac{(x-1)(x-3)(x-4)(x-5)}{(2-1)(2-3)(2-4)(2-5)} \\
&= \frac{x^4-13x^3+59x^2-107x+60}{-6}
\end{aligned}
$$

$$
\begin{aligned}
L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
&= \frac{(x-1)(x-2)(x-4)(x-5)}{(3-1)(3-2)(3-4)(3-5)} \\
&= \frac{x^4-12x^3+49x^2-78x+40}{4}
\end{aligned}
$$

$$
\begin{aligned}
L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\
&= \frac{(x-1)(x-2)(x-3)(x-5)}{(4-1)(4-2)(4-3)(4-5)} \\
&= \frac{x^4-11x^3+41x^2-61x+30}{-6}
\end{aligned}
$$

$$
\begin{aligned}
L_4(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\
&= \frac{(x-1)(x-2)(x-3)(x-4)}{(5-1)(5-2)(5-3)(5-4)} \\
&= \frac{x^4-10x^3+35x^2-50x+24}{24}.
\end{aligned}
$$

The 4th order polynomial $p_4(x)$ that goes through the 5 given discrete points is now determined from

$$\begin{aligned} p_4(x) &= \sum_{i=0}^{4} y_i L_i(x) \\ &= 11.6L_0(x) + 16.2L_1(x) + 16.8L_2(x) + 13.5L_3(x) + 7.3L_4(x) \\ &= 0.0375x^4 - 0.3583x^3 - 0.7875x^2 + 8.9083x + 3.800. \end{aligned}$$

It is possible to check that this is the correct expression for $p_4(x)$: simply substitute the values of x_i into it; the results should be the corresponding y_i, because the interpolating polynomial passes through the discrete points.

To find a maximum put the first derivative of $p_4(x)$ equal to zero:

$$p_4'(x) = 0.15x^3 - 1.075x^2 - 1.575x + 8.9083 = 0.$$

This is a non-linear equation which we can solve using a numerical method such as Newton's Method. From the table of discrete data we see that the maximum is probably near $x = 3$, so we assume that 3 is a good initial estimate of the root. If we define

$$f(x) \equiv 0.15x^3 - 1.075x^2 - 1.575x + 8.9083$$

then

$$f'(x) = 0.45x^2 - 2.15x - 1.575.$$

The Newton procedure is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

with the initial estimate of $x_1 = 3$. After 3 iterations we obtain the root

$$x_0 = 2.64437$$

with $f(x_0) = 2 \times 10^{-13}$.

Thus, according to polynomial interpolation, the discrete data has a maximum at 2.64437 and $p_4(2.64437) = 17.05781$.

4. For each coefficient polynomial

$$L_j(x) = \frac{\prod_{\substack{k=0\\k\neq j}}^{n-1} x - x_k}{\prod_{\substack{k=0\\k\neq j}}^{n-1} x_j - x_k}$$

we have $n-1$ terms in each product, i.e. $n-2$ multiplications for the numerator and denominator and one division: $n-2+n-2+1 = 2n-3$ operations. Alternatively we can write

$$L_j(x) = \prod_{\substack{k=0\\k\neq j}}^{n-1} \frac{x - x_k}{x_j - x_k}$$

with 1 division for each of the $n-1$ terms in the product and $n-2$ multiplications: $(n-1)1+n-2 = 2n-3$. To calculate each of these n polynomials, $L_0, \ldots L_{n-1}$, takes $(2n-3)n$ operations. Now we calculate the sum

$$P_{n-1}(x) = \sum_{j=0}^{n-1} y_j L_j(x)$$

which entails another n multiplications for a total of $(2n-3)n+n = 2n^2 - 2n = 2n(n-1)$ operations.

(a) The Taylor polynomial is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!}.$$

(b) The data points

j	0	1	2	3	4
x_j	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos x_j$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0

Since we have $n+1 = 5$ data points we construct a quartic interpolating polynomial. The interpolating polynomial is

$$\begin{aligned} \sum_{j=0}^{4} y_j L_j &= 0 \cdot L_0 + \frac{1}{2} \cdot L_1 + 1 \cdot L_2 + \frac{1}{2} \cdot L_3 + 0 \cdot L_4 \\ &= \frac{L_1}{2} + L_2 + \frac{L_3}{2} \end{aligned}$$

where the L_j's are the Lagrange coefficient polynomials. To compute these polynomials easily, we introduce the variable

$$t \equiv \frac{6x}{\pi}$$

which ensures that each $t_j \left(\equiv \frac{6x_j}{\pi}\right)$ is an integer. We determine the Lagrange coefficient polynomials, and the Lagrange interpolating polynomial, in terms of t, and then substitute $t = \frac{6x}{\pi}$ to obtain the Lagrange interpolating polynomial in terms of x.

Hence, using

j	0	1	2	3	4
t_j	-3	-2	0	2	3

we find

$$\begin{aligned} L_1(t) &= \frac{(t+3)(t-0)(t-2)(t-3)}{(-2+3)(-2-0)(-2-2)(-2-3)} = \frac{t^4-2t^3-9t^2+18t}{-40} \\ L_2(t) &= \frac{(t+3)(t+2)(t-2)(t-3)}{(0+3)(0+2)(0-2)(0-3)} = \frac{t^4-13t^2+36}{36} \\ L_3(t) &= \frac{(t+3)(t+2)(t-0)(t-3)}{(2+3)(2+2)(2-0)(2-3)} = \frac{t^4+2t^3-9t^2-18t}{-40}. \end{aligned}$$

Consequently the interpolating polynomial is given by

$$\sum_{j=0}^{4} y_j L_j = \frac{t^4-49t^2+360}{360}.$$

In terms of x,

$$\begin{aligned} \sum_{j=0}^{3} y_j L_j &= \frac{\left(\frac{6}{\pi}x\right)^4 - 49\left(\frac{6}{\pi}x\right)^2 + 360}{360} = \frac{328x^4 - 441\pi^2x^2 + 90\pi^4}{90\pi^4} \\ &= 0.0374138x^4 - 0.49647378x^2 + 1. \end{aligned}$$

For comparison, the Taylor polynomial is

$$T_4(x) = 0.04166666x^4 - 0.5x^2 + 1.$$

(c) For the residual term we have $n = 4$ and $\frac{d^5}{dx^5}\cos x = -\sin x$ so that

$$\left|\frac{f^{(5)}(\xi)}{5!}x^5\right| = \left|-\frac{\sin\xi}{120}x^5\right| \leqslant \left|\frac{x^5}{120}\right|$$

where we used $|\sin\xi| \leq 1$. Since $\frac{x^5}{120}$ is monotonically increasing on $[-\frac{\pi}{2},\frac{\pi}{2}]$ we find the maximum at $\frac{\pi}{2}$ of $\frac{\pi^5}{3840} \approx 0.07969$. For the approximation error for the interpolating polynomial we find, with $t = \frac{6}{\pi}x$,

$$\begin{aligned}
\left|\frac{f^{(5)}(\xi)}{5!}\prod_{j=0}^{4}(x-x_i)\right| &= \left|\frac{\sin\xi}{120}\left(\frac{\pi}{6}\right)^5 (t+3)(t+2)(t-0)(t-2)(t-3)\right| \\
&\leqslant \left|\left(\frac{\pi}{6}\right)^5 \frac{t^5 - 13t^3 + 36t}{120}\right| \\
&= \left|\left(\frac{\pi}{6}\right)^5 \frac{\left(\frac{6}{\pi}x\right)^5 - 13\left(\frac{6}{\pi}x\right)^3 + 36\left(\frac{6}{\pi}x\right)}{120}\right| \\
&= \left|\frac{36x^5 - 13\pi^2x^3 + \pi^4 x}{4320}\right|.
\end{aligned}$$

We need to find the extrema of

$$\frac{36x^5 - 13\pi^2x^3 + \pi^4 x}{4320}$$

or, equivalently,

$$f(t) = t^5 - 13t^3 + 36t$$

which is easier to analyze. We differentiate and set equal to zero:

$$f'(t) = 5t^4 - 39t^2 + 36 = 0.$$

We find

$$t^2 = \frac{39 \pm \sqrt{801}}{10}$$

so that we have the 4 roots

$$t_1 = \frac{39 \pm \sqrt{801}}{10} = -\sqrt{\frac{39+\sqrt{801}}{10}} \approx -2.59426181$$

$$t_2 = \frac{39 \pm \sqrt{801}}{10} = -\sqrt{\frac{39-\sqrt{801}}{10}} \approx -1.0343141$$

$$t_3 = \frac{39 \pm \sqrt{801}}{10} = +\sqrt{\frac{39+\sqrt{801}}{10}} \approx 2.59426181$$

$$t_4 = \frac{39 \pm \sqrt{801}}{10} = +\sqrt{\frac{39-\sqrt{801}}{10}} \approx 1.0343141$$

Thus at t_1, t_2, t_3 and t_4 we have

$$\left| \left(\frac{\pi}{6}\right)^5 \frac{t_1^5 - 13t_1^3 + 36t_1}{120} \right| \approx 0.00528$$

$$\left| \left(\frac{\pi}{6}\right)^5 \frac{t_2^5 - 13t_2^3 + 36t_2}{120} \right| \approx 0.00789$$

$$\left| \left(\frac{\pi}{6}\right)^5 \frac{t_3^5 - 13t_3^3 + 36t_3}{120} \right| \approx 0.00528$$

$$\left| \left(\frac{\pi}{6}\right)^5 \frac{t_4^5 - 13t_4^3 + 36t_4}{120} \right| \approx 0.00789$$

and at the boundaries $t = -3$ and $t = 3$ we obviously have 0. Thus we have an upper bound of 0.00789 on the error for polynomial interpolation on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, which is some 10 times better (smaller) than the bound of $\frac{\pi^5}{3840} \approx 0.0797$ for the Taylor polynomial.

5. We are given

i	0	1	2	3	4
x_i	-2	$-\frac{5}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	1
$y_i = e^{x_i} \sin x_i$	-0.1231	-0.2719	-0.2908	0.3177	2.2874

We construct the Lagrange coefficient polynomials:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\ &= 0.1317x^4 + 0.0658x^3 - 0.1728x^2 - 0.0453x + 0.0206 \end{aligned}$$

$$\begin{aligned} L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\ &= -0.5267x^4 - 0.6584x^3 + 0.9877x^2 + 0.3292x - 0.1317 \end{aligned}$$

$$\begin{aligned} L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\ &= 0.7901x^4 + 1.5802x^3 - 1.0370x^2 - 1.8272x + 0.4938 \end{aligned}$$

$$\begin{aligned} L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\ &= -0.5267x^4 - 1.4486x^3 - 0.1975x^2 + 1.5144x + 0.6584 \end{aligned}$$

$$\begin{aligned} L_4(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\ &= 0.1317x^4 + 0.4609x^3 + 0.4198x^2 + 0.0288x - 0.0412. \end{aligned}$$

The 4th order polynomial $p_4(x)$ that goes through the 5 given discrete points is given by

$$\begin{aligned} p_4(x) &= \sum_{i=0}^{4} y_i L_i(x) \\ &= 0.0311x^4 + 0.3055x^3 + 0.9517x^2 + 0.9943x + 0.0047. \end{aligned}$$

The error $\Delta(x)$ is given by

$$\Delta(x) \equiv \frac{y^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{4} (x - x_i)$$

so that

$$|\Delta(x)| \leqslant \frac{\max_{[-2,1]} \left|y^{(5)}\right|}{5!} \left| \prod_{i=0}^{4} (x - x_i) \right|$$

serves as an expression for the upper bound on the error. Clearly, we need to determine $\max_{[-2,1]} \left|y^{(5)}\right|$. Since $y(x) = e^x \sin x$ we find

$$y^{(5)}(x) = -4e^x (\cos x + \sin x)$$

which yields

$$\begin{aligned}\left|y^{(5)}(-2)\right| &= 0.7175 \\ \left|y^{(5)}(1)\right| &= 15.0242\end{aligned}$$

at the endpoints of $[-2,1]$. Also,

$$y^{(6)} = -8e^{x}\cos x = 0 \Rightarrow x = -\frac{\pi}{2}.$$

This is the only stationary point on $[-2,1]$ and, since

$$\left|y^{(5)}\left(-\frac{\pi}{2}\right)\right| = 0.8315,$$

the maximum value of $\left|y^{(5)}\right|$ on $[-2,1]$ occurs at the endpoint $x=1$. Consequently,

$$\begin{aligned}|\Delta(x)| &\leqslant \frac{15.0242}{5!}\left|\prod_{i=0}^{4}(x-x_i)\right| \\ &= 0.1252\left|\prod_{i=0}^{4}(x-x_i)\right| \\ &= 0.1252\left|x^5+\frac{5}{2}x^4-\frac{5}{16}x^3-\frac{95}{32}x^2-\frac{17}{32}x+\frac{5}{16}\right|.\end{aligned}$$

6. We have three nodes, and so $n=2$. The error $\Delta(x)$ is given by

$$\Delta(x) \equiv \frac{y^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{2}(x-x_i)$$

so that

$$\begin{aligned}|\Delta(x)| &\leqslant \left(\frac{\max_{[-1,1]}\left|y^{(3)}\right|}{3!}\right)|(x-1)(x-0)(x+1)| \\ &= \left(\frac{\max_{[-1,1]}\left|y^{(3)}\right|}{6}\right)\left|x^3-x\right|\end{aligned}$$

serves as an expression for the upper bound on the error. Clearly, we need to determine $\max\limits_{[-1,1]} \left|y^{(3)}\right|$. Since $y(x) = \frac{1}{1+x^2}$ we find

$$y^{(3)}(x) = -\frac{48x^3}{A^4} + \frac{24x}{A^3}$$

where

$$A \equiv 1 + x^2$$

and hence

$$\begin{aligned} \left|y^{(3)}(-1)\right| &= 0 \\ \left|y^{(3)}(1)\right| &= 0 \end{aligned}$$

at the endpoints of $[-1,1]$. Also,

$$y^{(4)} = \frac{384x^4}{A^5} - \frac{288x^2}{A^4} + \frac{24}{A}$$

so that

$$\begin{aligned} y^{(4)} = 0 \Rightarrow \quad & 384x^4 - 288x^2 A + 24A^2 = 0 \\ \Rightarrow \quad & 384x^4 - 288x^2\left(1+x^2\right) + 24\left(1+x^2\right)^2 = 0 \\ \Rightarrow \quad & 120x^4 - 240x^2 + 24 = 0. \end{aligned}$$

Solving this gives

$$x^2 = 1.8944 \quad \text{or} \quad 0.1056$$

so that

$$x = \pm 1.3764 \quad \text{or} \quad \pm 0.3249.$$

Obviously, only the solutions $x = \pm 0.3249$ are on the interval $[-1,1]$, and so

$$\begin{aligned} \left|y^{(3)}(-0.3249)\right| &= 4.6686 \\ \left|y^{(3)}(0.3249)\right| &= 4.6686. \end{aligned}$$

This all gives

$$\max_{[-1,1]} \left|y^{(3)}\right| = 4.6686$$

and so

$$|\Delta(x)| \leqslant 0.7781 \left|x^3 - x\right|.$$

Hence,

$$\begin{aligned} |\Delta(-0.8)| &\leqslant 0.7781 \left|(-0.8)^3 - (-0.8)\right| = 0.2241 \\ |\Delta(0.5)| &\leqslant 0.7781 \left|(0.5)^3 - (0.5)\right| = 0.2918. \end{aligned}$$

7. We have $x_i = \left\{-1, -\frac{1}{3}, \frac{1}{3}, 1\right\}$ and $y(x) = e^{-x^2} - \frac{1}{e}$ so that

$$\begin{aligned} y_0 &= y(-1) = 0 \\ y_1 &= y\left(-\frac{1}{3}\right) = 0.5270 \\ y_2 &= y\left(-\frac{1}{3}\right) = 0.5270 \\ y_3 &= y(1) = 0. \end{aligned}$$

Now,

$$P(x) = \sum_{i=0}^{3} y_i L_i(x) = y_1 L_1(x) + y_2 L_2(x)$$

where

$$\begin{aligned} L_1(x) &= \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ &= \frac{(x - (-1))\left(x - \frac{1}{3}\right)(x - 1)}{\left(-\frac{1}{3} - (-1)\right)\left(-\frac{1}{3} - \frac{1}{3}\right)\left(-\frac{1}{3} - 1\right)} \\ &= \frac{27x^3 - 9x^2 - 27x + 9}{16} \end{aligned}$$

and

$$\begin{aligned} L_2(x) &= \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\ &= \frac{(x - (-1))\left(x - \left(-\frac{1}{3}\right)\right)(x - 1)}{\left(\frac{1}{3} - (-1)\right)\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right)\left(\frac{1}{3} - 1\right)} \\ &= \frac{-27x^3 - 9x^2 + 27x + 9}{16}. \end{aligned}$$

Hence,

$$P(x) = -0.5928x^2 + 0.5928.$$

$$P(0.6) = 0.3794$$

For the error $\Delta(x)$ we have

$$\begin{aligned}\Delta(0.6) &\leqslant \frac{\max_{[-1,1]} \left|y^{(4)}\right| \left|(0.6 - x_0)(0.6 - x_1)(0.6 - x_2)(0.6 - x_3)\right|}{4!} \\ &= \left(\frac{0.1593}{24}\right) \max_{[-1,1]} \left|y^{(4)}\right|.\end{aligned}$$

We have

$$\begin{aligned}y^{(5)} &= \frac{dy^{(4)}}{dx} = \frac{d}{dx}\left(e^{-x^2}\left(12 - 48x^2 + 16x^4\right)\right) \\ &= e^{-x^2}\left(-120x + 160x^3 - 32x^5\right)\end{aligned}$$

and, to find the stationary points of $y^{(4)}$,

$$\begin{aligned}y^{(5)} = 0 \;&\Rightarrow\; e^{-x^2}\left(-120x + 160x^3 - 32x^5\right) = 0 \\ &\Rightarrow\; -120x + 160x^3 - 32x^5 = 0 \\ &\Rightarrow\; 120x - 160x^3 + 32x^5 = 0 \\ &\Rightarrow\; x = \pm 0.9586 \text{ and } 0.\end{aligned}$$

So we have

$$\begin{aligned}\left|y^{(4)}(-1)\right| &= 7.3576 \\ \left|y^{(4)}(-0.9586)\right| &= 7.4195 \\ \left|y^{(4)}(0)\right| &= 12 \\ \left|y^{(4)}(0.9586)\right| &= 7.4195 \\ \left|y^{(4)}(1)\right| &= 7.3576\end{aligned}$$

The first and last of these are $\left|y^{(4)}\right|$ at the endpoints of the interval. Clearly, the maximum of $\left|y^{(4)}\right|$ is 12, and so

$$\Delta(0.6) \leqslant \left(\frac{0.1593}{24}\right)(12) = 0.0797.$$

The actual error is

$$|y(0.6) - P(0.6)| = 0.0496$$

which, as expected, is less than the upper bound.

8. We have $x_i = \{0, 1, 2, 3\}$ and $y(x) = (x-3)\sin x$ so that

$$\begin{aligned} y_0 &= y(0) = 0 \\ y_1 &= y(1) = -1.68294 \\ y_2 &= y(2) = -0.90930 \\ y_3 &= y(3) = 0. \end{aligned}$$

Now,

$$P(x) = \sum_{i=0}^{3} y_i L_i(x) = y_1 L_1(x) + y_2 L_2(x)$$

where

$$\begin{aligned} L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &= \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} \\ &= \frac{x^3-5x^2+6x}{2} \end{aligned}$$

and

$$\begin{aligned} L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\ &= \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} \\ &= \frac{-x^3+4x^2-3x}{2}. \end{aligned}$$

Hence,

$$P(x) = -0.3868x^3 + 2.3888x^2 - 3.6849x.$$

$$P\left(\frac{3}{2}\right) = -1.4581$$

For the error $\Delta(x)$ we have

$$\begin{aligned}
\Delta\left(\frac{3}{2}\right) &\leqslant \frac{\max\limits_{[0,3]}\left|y^{(4)}\right|\left|\left(\frac{3}{2}-x_0\right)\left(\frac{3}{2}-x_1\right)\left(\frac{3}{2}-x_2\right)\left(\frac{3}{2}-x_3\right)\right|}{4!} \\
&= \left(\frac{0.5625}{24}\right)\max\limits_{[0,3]}\left|y^{(4)}\right|.
\end{aligned}$$

We have

$$\begin{aligned}
y^{(5)} &= \frac{dy^{(4)}}{dx} = \frac{d}{dx}\left((x-3)\sin x - 4\cos x\right) \\
&= (x-3)\cos x + 5\sin x
\end{aligned}$$

and

$$\begin{aligned}
y^{(5)} = 0 \quad &\Rightarrow \quad (x-3)\cos x + 5\sin x = 0 \\
&\Rightarrow \quad x + 5\tan x = 3 \\
&\Rightarrow \quad x \approx 0.46865
\end{aligned}$$

So we have

$$\begin{aligned}
\left|y^{(4)}(0)\right| &= 4 \\
\left|y^{(4)}(0.46865)\right| &= 4.71208 \\
\left|y^{(4)}(3)\right| &= 3.96
\end{aligned}$$

which gives

$$\Delta\left(\frac{3}{2}\right) \leqslant \left(\frac{0.5625}{24}\right)(4.71208) = 0.1104$$

The actual error is

$$\left|y\left(\frac{3}{2}\right) - P\left(\frac{3}{2}\right)\right| = 0.0381$$

which, as expected, is less than the upper bound.

9.