# Jacobi Method - Solutions

1. From

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{b}}$$

we have

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and, with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

we find

$$\begin{aligned} \mathbf{x}^{(1)} &= \overbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix}}^{\mathbf{D}^{-1}} \left( \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{\mathbf{b}} - \left( \overbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{\mathbf{L}+\mathbf{U}} \right) \overbrace{\begin{bmatrix} 0.5 \\ 0 \end{bmatrix}}^{\mathbf{x}^{(0)}} \right) \\ &= \begin{bmatrix} 0.5 \\ 0.125 \end{bmatrix} \end{aligned}$$

After 5 iterations we have

$$\mathbf{x}^{(5)} = \begin{bmatrix} 0.4297 \\ 0.1426 \end{bmatrix}$$

The residual vector is given by

$$\begin{aligned} \mathbf{r}^{(5)} &\equiv \mathbf{A}\mathbf{x}^{(5)} - \mathbf{b} \\ &= \begin{bmatrix} 0.002 \\ 0 \end{bmatrix} \end{aligned}$$

which has magnitude $\left\|\mathbf{r}^{(5)}\right\| = 0.002$.

2. From

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 6 & \alpha \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

we have

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & \alpha \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and, with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we find

$$\begin{aligned} \mathbf{x}^{(1)} &= \overbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & \beta \equiv \frac{1}{\alpha} \end{bmatrix}}^{\mathbf{D}^{-1}} \left( \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\mathbf{b}} - \left( \overbrace{\begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{\mathbf{L}+\mathbf{U}} \right) \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\mathbf{x}^{(0)}} \right) \\ &= \begin{bmatrix} 0.5 \\ -6\beta \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}^{(2)} &= \overbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & \beta \end{bmatrix}}^{\mathbf{D}^{-1}} \left( \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\mathbf{b}} - \left( \overbrace{\begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{\mathbf{L}+\mathbf{U}} \right) \overbrace{\begin{bmatrix} 0.5 \\ -6\beta \end{bmatrix}}^{\mathbf{x}^{(0)}} \right) \\ &= \begin{bmatrix} 0.5 + 3\beta \\ -3\beta \end{bmatrix}. \end{aligned}$$

The residual is given by

$$\begin{aligned} \mathbf{r}^{(2)} &\equiv \mathbf{A}\mathbf{x}^{(2)} - \mathbf{b} \\ &= \begin{bmatrix} 3\beta \\ 18\beta \end{bmatrix} \end{aligned}$$

and has magnitude

$$\left\|\mathbf{r}^{(2)}\right\| = \sqrt{333}\beta.$$

Now, $\left\|\mathbf{r}^{(2)}\right\| = 1$ gives

$$\beta = \frac{1}{\sqrt{333}} \Rightarrow \alpha = \frac{1}{\beta} = \sqrt{333} \approx 18.248$$

3. (b) because $3 \not> 4$, (c) because $0 \not> |-1|$, and (d) because $6 \not> |-6|$.