Fixed Point - Solutions

1. We have
$$f(x) = e^x - 3x^2 = 0$$
and we are told this equation has roots near $x = -\frac{1}{2}, 1$ and 4. There are three ways we can turn this equation into a fixed-point system:
$$\begin{aligned} 3x^2 &= e^x \\ \Rightarrow \quad x^2 &= \frac{e^x}{3} \\ \Rightarrow \quad x &= \pm\sqrt{\frac{e^x}{3}} \end{aligned}$$
$$\begin{aligned} 3x^2 &= e^x \\ \Rightarrow \quad x^2 &= \frac{e^x}{3} \\ \Rightarrow \quad x &= \frac{e^x}{3x} \end{aligned}$$
$$\begin{aligned} e^x &= 3x^2 \\ \Rightarrow \quad x &= \ln\left(3x^2\right) \end{aligned}$$
Define
$$\begin{aligned} g_1(x) &\equiv \pm\sqrt{\frac{e^x}{3}} \\ g_2(x) &\equiv \frac{e^x}{3x} \\ g_3(x) &\equiv \ln\left(3x^2\right) \end{aligned}$$
We then have
$$\begin{aligned} |g_1'(x)| &\equiv \left|\frac{e^x}{6}\sqrt{\frac{3}{e^x}}\right| \\ |g_2'(x)| &\equiv \left|\frac{3xe^x - 3e^x}{9x^2}\right| \\ |g_3'(x)| &\equiv \left|\frac{2}{x}\right| \end{aligned}$$

Evaluating these derivatives at the given values of x yields

x	$\lvert g_1'(x)\rvert$	$\lvert g_2'(x)\rvert$	$\lvert g_3'(x)\rvert$
$-\frac{1}{2}$	0.2	1.2	4.0
1	0.5	0	2.0
4	2.1	3.4	0.5

Only the systems for which $|g'(x)| < 1$ are attractive.

2. We have

$$f(x) = x^2 - 10\ln(x) = 0$$

and we are told this equation has roots near $x = 1.1384$ and 3.5656. There are three ways we can turn this equation into a fixed-point system:

$$\begin{aligned} x^2 &= 10\ln(x) \\ \Rightarrow \quad x &= \frac{10\ln(x)}{x} \end{aligned}$$

$$\begin{aligned} x^2 &= 10\ln(x) \\ \Rightarrow \quad x &= \pm\sqrt{10\ln(x)} \end{aligned}$$

$$\begin{aligned} 10\ln(x) &= x^2 \\ \Rightarrow \quad \ln(x) &= \frac{x^2}{10} \\ \Rightarrow \quad x &= e^{\frac{x^2}{10}} \end{aligned}$$

Define

$$\begin{aligned} g_1(x) &\equiv \frac{10\ln(x)}{x} \\ g_2(x) &\equiv \pm\sqrt{10\ln(x)} \\ g_3(x) &\equiv e^{\frac{x^2}{10}} \end{aligned}$$

We then have

$$
\begin{aligned}
|g_1'(x)| &\equiv \left|\frac{10 - 10\ln(x)}{x^2}\right| \\
|g_2'(x)| &\equiv \left|\frac{5}{x}\sqrt{\frac{1}{10\ln(x)}}\right| \\
|g_3'(x)| &\equiv \left|\frac{xe^{\frac{x^2}{10}}}{5}\right|
\end{aligned}
$$

Evaluating these derivatives at the given values of x yields

x	$\lvert g_1'(x)\rvert$	$\lvert g_2'(x)\rvert$	$\lvert g_3'(x)\rvert$
1.1384	6.7	3.9	0.3
3.5656	0.2	0.4	2.5

Only the systems for which $|g'(x)| < 1$ are attractive.

3. For

$$x_{j+1} = x_j \cot x_j$$

we have

$$\left.\frac{d}{dx} x\cot x\right|_{x=\frac{\pi}{4}} = \cot x - x\csc^2 x\big|_{x=\frac{\pi}{4}} = 1 - 2\frac{\pi}{4} = -0.57$$

which has absolute value < 1. Thus we have convergence near the root. On the other hand

$$\left.\frac{d}{dx} x\tan x\right|_{x=\frac{\pi}{4}} = \tan x + x\sec^2 x\big|_{x=\frac{\pi}{4}} = 1 + 2\frac{\pi}{4} > 1$$

and so this method diverges.

4. We divide by x^2 add x to both sides of the equation $\sin(x^2) = 0$ to obtain

$$\frac{\sin(x^2)}{x^2} + x = x, \qquad x \neq 0$$

Adding x on both sides is an identity, i.e. no roots are added or removed by this operation. Division by x^2 excludes the root $x = 0$, which is desirable since we seek non-zero roots.

Hence, the iteration scheme is

$$x_{i+1} = \frac{\sin\left(x_i^2\right)}{x_i^2} + x_i \equiv g_1\left(x_i\right)$$

For convergence we require $|g_1'(x)| < 1$ in the neighbourhood of the root. Thus

$$\begin{aligned} g_1'(x) &= \frac{2x\cos x^2}{x^2} - \frac{2\sin x^2}{x^3} + 1 \\ &= \frac{2\cos x^2}{x} - \frac{2\sin x^2}{x^3} + 1 \end{aligned}$$

and at the root $x = \sqrt{\pi}$

$$|g_1'(\sqrt{\pi})| \approx 0.128 < 1$$

So we expect convergence to the root $x = \sqrt{\pi}$.

Implementing the iteration scheme gives

i	x_i	$x_{i+1} = \frac{\sin\left(x_i^2\right)}{x_i^2} + x_i$	$\lvert \sin\left(x_{i+1}^2\right) \rvert$
1	2	1.810799376	0.1370
2	1.810799376	1.769027485	0.0121
3	1.769027485	1.77290487	0.0016
4	1.77290487	1.772396144	2×10^{-4}
5	1.772396144	1.772461262	$2.6 \times 10^{-5} < 10^{-4}$

Thus we find the approximation $x = 1.772461262$.

For

$$g_2(x) \equiv \frac{\sin\left(x^2\right)}{x} + x$$

We have

$$\begin{aligned} g_2'(x) &= \frac{2x\cos x^2}{x} - \frac{\sin x^2}{x^2} + 1 \\ &= 2\cos x^2 - \frac{\sin x^2}{x^2} + 1 \end{aligned}$$

and at the root $x = \sqrt{\pi}$

$$|g_2'(\sqrt{\pi})| = |-2 + 1| = 1 \not< 1.$$

Fixed Point iteration with g_2 will not converge to the root $\sqrt{\pi}$.