# Euler's Method - Solutions

1. We have

$$\begin{aligned} y\left(x_{i+1}\right) &= y\left(x_i + h\right) \\ &= y\left(x_i\right) + hy'\left(x_i\right) + \frac{h^2}{2}y''\left(\xi\right) \end{aligned}$$

   where $\frac{h^2}{2}y''\left(\xi\right)$ is the residual term. Since $y' = f\left(x, y\right)$ we have

$$y\left(x_{i+1}\right) = y\left(x_i\right) + hf\left(x_i, y\left(x_i\right)\right) + \frac{h^2}{2}y''\left(\xi\right)$$

   and so

$$\left[y\left(x_i\right) + hf\left(x_i, y\left(x_i\right)\right)\right] - y\left(x_{i+1}\right) = -\frac{h^2}{2}y''\left(\xi\right) \equiv \varepsilon_{i+1}.$$

2. Euler's method is

$$y_{m+1} = y_m + hf(x_m, y_m)$$

   and so we obtain

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_m$ | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 |
| $y_m$ | 1 | 1.0100 | 1.0203 | 1.0309 | 1.0418 | 1.0531 |

| $m$ | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $x_m$ | 0.06 | 0.07 | 0.08 | 0.09 | 0.10 |
| $y_m$ | 1.0646 | 1.0765 | 1.0887 | 1.1013 | 1.1142 |

   We are now required to estimate the global error at $x_{10} = 0.1$. We have

$$\Delta_{10} = \varepsilon_{10} + \alpha_9\varepsilon_9 + \ldots + \alpha_9\alpha_8\cdots\alpha_1\varepsilon_1$$

   so that

$$\begin{aligned} \left|\Delta_{10}\right| &\leqslant \max_{[x_0,x_{10}]}\left|\varepsilon_m\right|\left(1 + \alpha + \alpha^2 + \ldots + \alpha^9\right) \\ &= \max_{[x_0,x_{10}]}\left|\varepsilon_m\right|\underbrace{\left(\frac{\alpha^{10} - 1}{\alpha - 1}\right)}_{\text{geometric sum}} \end{aligned}$$

where

$$\alpha \equiv \max_{[x_0,x_{10}]} |\alpha_m| = 1 + h \max_{[x_0,x_{10}]} |f_y|$$

and

$$\max_{[x_0,x_{10}]} |\varepsilon_m| = \max_{[x_0,x_{10}]} \left| \frac{h^2}{2} y'' \right| = \max_{[x_0,x_{10}]} \left| \frac{h^2}{2} (f_x + f f_y) \right|.$$

In this problem $f(x, y) = x + y + xy$ and so

$$\begin{aligned} f_x + f f_y &= 1 + y + (x + y + xy)(1 + x) \\ f_y &= 1 + x. \end{aligned}$$

We now substitute the values of $x_m$ and $y_m$ which we have obtained using Euler's method, for each $m \in [0, 10]$, to find

$$\begin{aligned} \alpha &\equiv 1 + h \max_{[x_0,x_{10}]} |f_y| = 1.0110 \\ \max_{[x_0,x_{10}]} |\varepsilon_m| &= \max_{[x_0,x_{10}]} \left| \frac{h^2}{2} (f_x + f f_y) \right| = 0.0001786 \end{aligned}$$

both of which happen to occur at $x = 0.1$. Thus, we have

$$\begin{aligned} |\Delta_{10}| &\leqslant (0.0001786) \left(1 + 1.011 + 1.011^2 + \ldots + 1.011^9\right) \\ &= 0.00187. \end{aligned}$$

This is an estimate of the maximum error in our result $y(0.1) = 1.1142$. The exact value is $y(0.1) = 1.1159$, so that the actual error is 0.0017.

3. We must solve

$$\frac{dy}{dx} = \frac{x}{y} \equiv f(x, y)$$

with the initial condition $y(0) = 1$, using Euler's Method with $h = 0.1$, up to $x = 0.9$. Euler's Method is

$$y_{m+1} = y_m + h f(x_m, y_m)$$

and so we obtain

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_m$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| $y_m$ | 1 | 1 | 1.01 | 1.0298 | 1.0589 | 1.0967 |

| $m$ | 6 | 7 | 8 | 9 |
|---|---|---|---|---|
| $x_m$ | 0.6 | 0.7 | 0.8 | 0.9 |
| $y_m$ | 1.1423 | 1.1948 | 1.2534 | 1.3172 |

We are now required to estimate the global error at $x_5 = 0.5$ and $x_9 = 0.9$. We have

$$\Delta_5 = \varepsilon_5 + \alpha_4\varepsilon_4 + \ldots + \alpha_4\alpha_3\alpha_2\alpha_1\varepsilon_1$$

so that

$$\begin{aligned} |\Delta_5| &\leqslant \max_{[x_0,x_5]} |\varepsilon_m| \left(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4\right) \\ &= \max_{[x_0,x_5]} |\varepsilon_m| \underbrace{\left(\frac{\alpha^5 - 1}{\alpha - 1}\right)}_{\text{geometric sum}} \end{aligned}$$

where

$$\alpha \equiv \max_{[x_0,x_5]} |\alpha_m| = 1 + h \max_{[x_0,x_5]} |f_y|$$

and

$$\max_{[x_0,x_5]} |\varepsilon_m| = \max_{[x_0,x_5]} \left|\frac{h^2}{2} y''\right| = \max_{[x_0,x_5]} \left|\frac{h^2}{2}\left(f_x + f f_y\right)\right|.$$

In this problem $f(x, y) = \frac{x}{y}$ and so

$$\begin{aligned} f_x + f f_y &= \frac{1}{y} - \frac{x^2}{y^3} \\ f_y &= -\frac{x}{y^2}. \end{aligned}$$

We now substitute the values of $x_m$ and $y_m$ which we have obtained using Euler's method, for each $m \in [0, 5]$, to find

$$\begin{aligned} \alpha &\equiv 1 + h \max_{[x_0,x_5]} |f_y| = 1.0416 \\ \max_{[x_0,x_5]} |\varepsilon_m| &= \max_{[x_0,x_5]} \left|\frac{h^2}{2}\left(f_x + f f_y\right)\right| = 0.005 \end{aligned}$$

The maximum in $|f_y|$ occurs at $x = 0.5$ and the maximum in $|f_x + f f_y|$ occurs at $x = 0$. Thus, we have

$$\begin{aligned} |\Delta_5| &\leqslant (0.005)\left(\frac{1.0416^5 - 1}{1.0416 - 1}\right) \\ &= 0.0272. \end{aligned}$$

The exact value is $y(0.5) = 1.1180$, so that the actual error is 0.0213.

For $y(x_9)$ we consider each $m \in [0,9]$, and we find

$$\begin{aligned} \alpha &\equiv 1 + h \max_{[x_0,x_9]} |f_y| = 1.0519 \\ \max_{[x_0,x_9]} |\varepsilon_m| &= \max_{[x_0,x_9]} \left| \frac{h^2}{2} (f_x + f f_y) \right| = 0.005 \end{aligned}$$

Here, the maximum in $|f_y|$ occurs at $x = 0.9$ and the maximum in $|f_x + f f_y|$ occurs at $x = 0$. So, we have

$$\begin{aligned} |\Delta_9| &\leqslant \max_{[x_0,x_9]} |\varepsilon_m| \left(1 + \alpha + \alpha^2 + \ldots + \alpha^8\right) \\ &= \max_{[x_0,x_9]} |\varepsilon_m| \left( \frac{\alpha^9 - 1}{\alpha - 1} \right) \end{aligned}$$

which gives

$$\begin{aligned} |\Delta_9| &\leqslant (0.005) \left( \frac{1.0519^9 - 1}{1.0519 - 1} \right) \\ &= 0.0556. \end{aligned}$$

The exact value is $y(0.5) = 1.3454$, so that the actual error is 0.0282.

Incidentally, the analytical solution is

$$y(x) = \sqrt{x^2 + 1}$$

which you may verify for yourself.

4. Euler's method is

$$y_{m+1} = y_m + h f(x_m, y_m)$$

and so we obtain

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_m$ | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 |
| $y_m$ | 2 | 2.0400 | 2.0811 | 2.1233 | 2.1667 | 2.2113 |

| $m$ | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $x_m$ | 0.06 | 0.07 | 0.08 | 0.09 | 0.10 |
| $y_m$ | 2.2572 | 2.3043 | 2.3527 | 2.4024 | 2.4535 |

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We are now required to estimate the global error at $x_8 = 0.8$. We have

$$\Delta_8 = \varepsilon_8 + \alpha_7\varepsilon_7 + \ldots + \alpha_7\alpha_6\cdots\alpha_1\varepsilon_1$$

so that

$$\begin{aligned} |\Delta_8| &\leqslant \max_{[x_0,x_8]} |\varepsilon_m| \left(1 + \alpha + \alpha^2 + \ldots + \alpha^7\right) \\ &= \max_{[x_0,x_8]} |\varepsilon_m| \underbrace{\left(\frac{\alpha^8 - 1}{\alpha - 1}\right)}_{\text{geometric sum}} \end{aligned}$$

where

$$\begin{aligned} \alpha &\equiv \max_{[x_0,x_8]} |\alpha_m| = 1 + h \max_{[x_0,x_8]} |f_y| \\ h &= 0.01 \end{aligned}$$

and

$$\max_{[x_0,x_8]} |\varepsilon_m| = \max_{[x_0,x_8]} \left|\frac{h^2}{2} y''\right| = \max_{[x_0,x_8]} \left|\frac{h^2}{2}\left(f_x + f f_y\right)\right|.$$

In this problem $f(x,y) = x + 2y + xy$ and so

$$\begin{aligned} f_x + f f_y &= 1 + y + (x + 2y + xy)(2 + x) \\ f_y &= 2 + x. \end{aligned}$$

We now substitute the values of $x_m$ and $y_m$ which we have obtained using Euler's method, for each $m \in [0, 8]$, to find

$$\begin{aligned} \alpha &= 1 + h \max_{[x_0,x_8]} |f_y| = 1.0208 \\ \max_{[x_0,x_8]} |\varepsilon_m| &= \max_{[x_0,x_8]} \left|\frac{h^2}{2}\left(f_x + f f_y\right)\right| = 0.0006849 \end{aligned}$$

both of which happen to occur at $x = 0.08$. Thus, we have

$$\begin{aligned} |\Delta_8| &\leqslant (0.0006849)\left(1 + 1.0208 + 1.0208^2 + \ldots + 1.0208^7\right) \\ &= 0.0059. \end{aligned}$$

This is an estimate of the maximum error in our result $y(0.08) = 2.3527$. The exact value is $y(0.08) = 2.3579$, so that the actual error is 0.0052.

5. Euler's method is

$$y_{m+1} = y_m + hf(x_m, y_m)$$

and so we obtain

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_m$ | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 |
| $y_m$ | 1 | 1.0100 | 1.0203 | 1.0309 | 1.0418 | 1.0531 |

| $m$ | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $x_m$ | 0.06 | 0.07 | 0.08 | 0.09 | 0.10 |
| $y_m$ | 1.0646 | 1.0765 | 1.0887 | 1.1013 | 1.1142 |

We are now required to estimate the global error at $x_{10} = 0.1$. We have

$$\Delta_{10} = \varepsilon_{10} + \alpha_9\varepsilon_9 + \ldots + \alpha_9\alpha_8 \cdots \alpha_1\varepsilon_1$$

so that

$$\begin{aligned} |\Delta_{10}| &\leqslant \max_{[x_0,x_{10}]} |\varepsilon_m| \left(1 + \alpha + \alpha^2 + \ldots + \alpha^9\right) \\ &= \max_{[x_0,x_{10}]} |\varepsilon_m| \underbrace{\left(\frac{\alpha^{10} - 1}{\alpha - 1}\right)}_{\text{geometric sum}} \end{aligned}$$

where

$$\alpha \doteqdot \max_{[x_0,x_{10}]} |\alpha_m| = 1 + h \max_{[x_0,x_{10}]} |f_y|$$

and

$$\max_{[x_0,x_{10}]} |\varepsilon_m| = \max_{[x_0,x_{10}]} \left|\frac{h^2}{2} y''\right| = \max_{[x_0,x_{10}]} \left|\frac{h^2}{2}\left(f_x + ff_y\right)\right|.$$

In this problem $f(x, y) = x + y + xy$ and so

$$\begin{aligned} f_x + ff_y &= 1 + y + (x + y + xy)(1 + x) \\ f_y &= 1 + x. \end{aligned}$$

We now substitute the values of $x_m$ and $y_m$ which we have obtained using Euler's method, for each $m \in [0, 10]$, to find

$$\begin{aligned} \alpha &\doteqdot 1 + h \max_{[x_0,x_{10}]} |f_y| = 1.0110 \\ \max_{[x_0,x_{10}]} |\varepsilon_m| &= \max_{[x_0,x_{10}]} \left|\frac{h^2}{2}\left(f_x + ff_y\right)\right| = 0.0001786 \end{aligned}$$

both of which happen to occur at $x = 0.1$. Thus, we have

$$\begin{aligned}|\Delta_{10}| &\leqslant (0.0001786)\left(1 + 1.011 + 1.011^2 + \ldots + 1.011^9\right) \\ &= 0.00187.\end{aligned}$$

This is an estimate of the maximum error in our result $y(0.1) = 1.1142$. The exact value is $y(0.1) = 1.1159$, so that the actual error is 0.0017.

6.