# Chebyshev Polynomials - Solutions

1.

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 - 9x \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1
\end{aligned}
$$

2.

$$
\begin{aligned}
x^0 &= T_0 \\
x^1 &= T_1 \\
x^2 &= \frac{1}{2}(T_2 + T_0) \\
x^3 &= \frac{1}{4}(T_3 + 3T_1) \\
x^4 &= \frac{1}{8}(T_4 + 4T_2 + 3T_0) \\
x^5 &= \frac{1}{16}(T_5 + 5T_3 + 10T_1) \\
x^6 &= \frac{1}{32}(T_6 + 6T_4 + 15T_2 + 10T_0) \\
x^7 &= \frac{1}{64}(T_7 + 7T_5 + 21T_3 + 35T_1) \\
x^8 &= \frac{1}{128}(T_8 + 8T_6 + 28T_4 + 56T_2 + 35T_0)
\end{aligned}
$$

$$
\begin{aligned}
x^9 &= \frac{1}{256}\left(T_9 + 9T_7 + 36T_5 + 84T_3 + 126T_1\right) \\
x^{10} &= \frac{1}{512}\left(T_{10} + 10T_8 + 45T_6 + 120T_4 + 210T_2 + 126T_0\right)
\end{aligned}
$$

3. We seek the zero points and the extreme points of

$$T_5(x) = 16x^5 - 20x^3 + 5x.$$

The zero points occur when

$$x = \cos\left(\left(\frac{2r+1}{2n}\right)\pi\right) = \cos\left(\left(\frac{2r+1}{10}\right)\pi\right) \qquad r = 0, 1, \ldots, 4$$

and so

$$
\begin{aligned}
x_0 &= \cos\left(\frac{\pi}{10}\right) = 0.95105 \\
x_1 &= \cos\left(\frac{3\pi}{10}\right) = 0.58778 \\
x_2 &= \cos\left(\frac{5\pi}{10}\right) = 0 \\
x_3 &= \cos\left(\frac{7\pi}{10}\right) = -0.58778 \\
x_4 &= \cos\left(\frac{9\pi}{10}\right) = -0.95105
\end{aligned}
$$

The extreme points occur when

$$x = \cos\left(\frac{r\pi}{n}\right) = \cos\left(\frac{r\pi}{5}\right) \qquad r = 0, 1, \ldots, 5$$

and so

$$x_0 = \cos\left(\frac{0\pi}{5}\right) = \cos(0) = 1$$

$$x_1 = \cos\left(\frac{\pi}{5}\right) = 0.80901$$

$$x_2 = \cos\left(\frac{2\pi}{5}\right) = 0.30901$$

$$x_3 = \cos\left(\frac{3\pi}{5}\right) = -0.30901$$
$$x_4 = \cos\left(\frac{4\pi}{5}\right) = -0.80901$$
$$x_5 = \cos\left(\frac{5\pi}{5}\right)\cos(\pi) = -1.$$

4. We have, up to order 6,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!}.$$

In terms of Chebyshev polynomials this becomes

$$\begin{aligned} e^{-x} &= T_0 - T_1 + \frac{1}{2}\left(\frac{T_2 + T_0}{2}\right) - \frac{1}{6}\left(\frac{T_3 + 3T_1}{4}\right) + \frac{1}{24}\left(\frac{T_4 + 4T_2 + 3T_0}{8}\right) \\ &\quad -\frac{1}{120}\left(\frac{T_5 + 5T_3 + 10T_1}{16}\right) + \frac{1}{720}\left(\frac{T_6 + 6T_4 + 15T_2 + 10T_0}{32}\right) \end{aligned}$$

which gives, up to $T_5$,

$$\begin{aligned} e^{-x} &= \left(\frac{29170}{23040}\right)T_0 + \left(\frac{-2170}{1920}\right)T_1 + \left(\frac{6255}{23040}\right)T_2 \\ &\quad + \left(\frac{-85}{1920}\right)T_3 + \left(\frac{126}{23040}\right)T_4 + \left(\frac{-1}{1920}\right)T_5. \end{aligned}$$

We substitute the appropriate polynomials into this expression to obtain

$$\begin{aligned} e^{-x} &= \left(\frac{29170}{23040}\right)(1) + \left(\frac{-2170}{1920}\right)(x) + \left(\frac{6255}{23040}\right)(2x^2 - 1) + \left(\frac{-85}{1920}\right)(4x^3 - 3x) \\ &\quad + \left(\frac{126}{23040}\right)(8x^4 - 8x^2 + 1) + \left(\frac{-1}{1920}\right)(16x^5 - 20x^3 + 5x) \end{aligned}$$

which gives

$$e^{-x} = \left(\frac{23041}{23040}\right) + \left(\frac{-1920}{1920}\right)x + \left(\frac{11502}{23040}\right)x^2 + \left(\frac{-320}{1920}\right)x^3 + \left(\frac{1008}{23040}\right)x^4 + \left(\frac{-16}{1920}\right)x^5.$$

In terms of the same denominators the Taylor expansion is

$$e^{-x} = \left(\frac{23040}{23040}\right) + \left(\frac{-1920}{1920}\right)x + \left(\frac{11520}{23040}\right)x^2 + \left(\frac{-320}{1920}\right)x^3 + \left(\frac{960}{23040}\right)x^4 + \left(\frac{-16}{1920}\right)x^5.$$

5. We have
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$
Substituting the appropriate Chebyshev polynomials gives
$$\sin(x) = T_1 - \left(\frac{T_3 + 3T_1}{24}\right) + \left(\frac{T_5 + 5T_3 + 10T_1}{1920}\right) - \left(\frac{T_7 + 7T_5 + 21T_3 + 35T_1}{322560}\right)$$
which yields, up to $T_3$,
$$\sin(x) = \left(\frac{283885}{322560}\right) T_1 + \left(\frac{-12621}{322560}\right) T_3.$$
Substituting the appropriate polynomials (in terms of $x$) gives
$$\sin(x) = \left(\frac{283885}{322560}\right)(x) + \left(\frac{-12621}{322560}\right)(4x^3 - 3x)$$
which yields
$$\sin(x) = \left(\frac{321748}{322560}\right) x + \left(\frac{-50484}{322560}\right) x^3.$$
The Taylor series, in terms of the same denominators, is
$$\sin(x) = \left(\frac{322560}{322560}\right) x + \left(\frac{-53760}{322560}\right) x^3$$

6. We have
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$
Substituting the appropriate Chebyshev polynomials gives
$$\cos(x) = T_0 - \left(\frac{T_2 + T_0}{4}\right) + \left(\frac{T_4 + 4T_2 + 3T_0}{192}\right) - \left(\frac{T_6 + 6T_4 + 15T_2 + 10T_0}{23040}\right)$$
which yields, up to $T_2$,
$$\cos(x) = \left(\frac{17630}{23040}\right) T_0 + \left(\frac{-5295}{23040}\right) T_2.$$
Substituting the appropriate polynomials (in terms of $x$) gives
$$\cos(x) = \left(\frac{17630}{23040}\right) + \left(\frac{-5295}{23040}\right)(2x^2 - 1)$$

which yields

$$\cos(x) = \left(\frac{22925}{23040}\right) + \left(\frac{-10590}{23040}\right) x^2.$$

The Taylor series, in terms of the same denominators, is

$$\cos(x) = \left(\frac{23040}{23040}\right) + \left(\frac{-11520}{23040}\right) x^2.$$

7. We seek the Chebyshev expansion of

$$\arccos(x)$$

such that

$$\arccos(x) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k T_k(x).$$

With the definition $\cos\theta = x$ we have

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_0^{\pi} \cos(k\theta)\arccos(\cos\theta)\,d\theta \\ &= \frac{2}{\pi}\int_0^{\pi} \theta\cos(k\theta)\,d\theta. \end{aligned}$$

For the case $k = 0$ we have

$$c_0 = \frac{2}{\pi}\int_0^{\pi} \theta\cos(0)\,d\theta = \frac{2}{\pi}\int_0^{\pi}\theta d\theta = \frac{2}{\pi}\left[\frac{\theta^2}{2}\right]_0^{\pi} = \pi.$$

For the case $k \neq 0$ we have

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_0^{\pi} \cos(k\theta)\arccos(\cos\theta)\,d\theta \\ &= \frac{2}{\pi}\left[\theta\frac{\sin(k\theta)}{k}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\frac{\sin(k\theta)}{k}d\theta \\ &= 0 + \frac{2}{\pi}\left[\frac{\cos(k\theta)}{k^2}\right]_0^{\pi} \\ &= \begin{cases} 0 & k \text{ even} \\ -\frac{4}{\pi k^2} & k \text{ odd} \end{cases}. \end{aligned}$$

Hence,

$$\begin{aligned}
\arccos(x) &= \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k T_k(x) \\
&= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{T_{2k+1}(x)}{(2k+1)^2} \\
&= \frac{\pi}{2} - \arcsin(x).
\end{aligned}$$

This is the expected result, as shown below:

We know $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$ and we have the definition $\cos(\theta) = x$. Hence,

$$\arccos(x) = \theta \quad \text{and} \quad x = \sin\left(\frac{\pi}{2} - \arccos(x)\right).$$

This implies that

$$\arcsin(x) = \arcsin\left(\sin\left(\frac{\pi}{2} - \arccos(x)\right)\right)$$

which gives

$$\arcsin(x) = \frac{\pi}{2} - \arccos(x)$$

and so

$$\arccos(x) = \frac{\pi}{2} - \arcsin(x).$$

8.

$$\begin{aligned}
\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}dx &= \int_{0}^{1} \frac{\cos(n \arccos x)}{\sqrt{1-x^2}}dx, \qquad x = \cos\theta,\ dx = -\sin\theta\, d\theta \\
&= -\int_{\frac{\pi}{2}}^{0} \cos(n\theta)d\theta \\
&= \int_{0}^{\frac{\pi}{2}} \cos(n\theta)d\theta \\
&= \left[\frac{\sin(n\theta)}{n}\right]_0^{\frac{\pi}{2}}, \qquad n \neq 0 \\
&= \frac{\sin\left(n\frac{\pi}{2}\right)}{n}, \qquad n \neq 0 \\
&= \begin{cases} 0, & n = 2k \quad (even) \\ \frac{(-1)^k}{n}, & n = 2k+1 \quad (odd) \end{cases}, \qquad n \neq 0
\end{aligned}$$

and for $n = 0$

$$\begin{aligned} \int_{-1}^{1} \frac{f(x)T_0(x)}{\sqrt{1-x^2}} dx &= \int_0^{\frac{\pi}{2}} \cos(0 \cdot \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}. \end{aligned}$$

Thus $c_0 = 1$ and

$$c_n = \frac{2}{\pi} \begin{cases} 0, & n = 2k \quad (even) \\ \frac{(-1)^k}{n}, & n = 2k+1 \quad (odd) \end{cases}, \qquad n \neq 0$$

so that

$$f(x) = \frac{1}{2} T_0 + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} T_{2k+1}(x).$$

More explicitly,

$$f(x) = \frac{1}{2} T_0 + \frac{2}{\pi} \left( \frac{T_1}{1} - \frac{T_3}{3} + \frac{T_5}{5} - \frac{T_7}{7} + \dots \right).$$

9.

$$\begin{aligned} T_m(T_n(x)) &= \cos(m \arccos T_n(x)) \\ &= \cos(m \arccos(\cos(n \arccos x))) \\ &= \cos(m(n \arccos x)) \\ &= \cos((mn) \arccos x) = T_{mn}(x). \end{aligned}$$

10. We have

$$\begin{aligned} T_{m+n}(x) &= \cos([m+n] \arccos x) \\ &= \cos(m \arccos x + n \arccos x) \\ &= \cos(m \arccos x) \cos(n \arccos x) \\ &\quad - \sin(m \arccos x) \sin(n \arccos x) \end{aligned}$$

and

$$\begin{aligned} T_{m-n}(x) &= \cos([m-n] \arccos x) \\ &= \cos(m \arccos x - n \arccos x) \\ &= \cos(m \arccos x) \cos(n \arccos x) \\ &\quad + \sin(m \arccos x) \sin(n \arccos x) \end{aligned}$$

Hence,

$$\begin{aligned}
T_{m+n}(x) + T_{m-n}(x) &= \cos(m \arccos x)\cos(n \arccos x) \\
&\quad + \cos(m \arccos x)\cos(n \arccos x) \\
&\quad - \sin(m \arccos x)\sin(n \arccos x) \\
&\quad + \sin(m \arccos x)\sin(n \arccos x) \\
&= 2\cos(m \arccos x)\cos(n \arccos x) \\
&= 2T_m(x)\,T_n(x)\,.
\end{aligned}$$