Bisection Method - Solutions

1. We seek the root of $f(x) \equiv e^x - 3x = 0$ on the interval $I_1 = [0, 1]$. We find $f(0) = 1$ and $f(1) = -0.28$, so that there is a sign change in $f(x)$ on I_1. In the first iteration ($n = 1$) we approximate the root by the midpoint of I_1, so $x_1 = 0.5$. We find $f(0.5) = 0.14 > 0$, so that the root must lie on the interval $I_2 = [0.5, 1]$. The second iteration ($n = 2$) gives $x_2 = 0.75$ (the midpoint of $[0.5, 1]$) and $f(0.75) = -0.13 < 0$. Thus the root must lie on $I_3 = [0.5, 0.75]$. The first 9 iterations are shown in the table below.

n	x_n	$f(x_n)$	I_{n+1}
1	0.5	0.14872	$[0.5, 1]$
2	0.75	-0.13300	$[0.5, 0.75]$
3	0.625	-0.00675	$[0.5, 0.625]$
4	0.5625	0.06755	$[0.5625, 0.625]$
5	0.59375	0.02952	$[0.59375, 0.625]$
6	0.609375	0.01115	$[0.609375, 0.625]$
7	0.6171875	0.00214	$[0.6171875, 0.625]$
8	0.62109375	-0.00230	$[0.6171875, 0.62109375]$
9	0.619140625	-0.00009	$[0.6171875, 0.619140625]$

We assumed a tolerance of $\varepsilon = 10^{-4}$, and so the root after 9 iterations is taken as the solution (since $|f(x_9)| < \varepsilon$). Thus the root, to a precision of 5 decimal places, is $x_0 = 0.61914$.

2. We seek the root of $f(x) \equiv \ln x - x + 2 = 0$. It is necessary to find a starting interval on which $f(x)$ changes sign. To do this we make a crude estimate of the root of $f(x)$. From $\ln x - x + 2 = 0$ we get

$$\begin{aligned} \ln x &= x - 2 \Rightarrow x = e^x e^{-2} \Rightarrow e^2 x \approx 1 + x + \frac{x^2}{2} \\ &\Rightarrow x^2 + (2 - 2e^2)x + 2 = 0 \Rightarrow x = 12.61963 \quad \text{or} \quad x = 0.15848 \end{aligned}$$

In the third equation we have expanded e^x in a Taylor series up to the third term. We find $f(12.61963) = -8.08$ and $f(0.15848) = -0.0006$, so clearly 0.15848 is a much better estimate of the root than 12.61963. Indeed, it is a very good estimate because $f(0.15848) = -0.0006 \approx 0$, but this should be regarded as a stroke of luck - usually rough estimates made using low-order Taylor expansions *are* rough. Nevertheless, the root seems to be near 0.15848, so let us consider the interval $I_1 = [0.1, 0.2]$. We find that $f(0.1) = -0.4 < 0$ and $f(0.2) = 0.19 > 0$ and so there is a sign change in $f(x)$ on I_1. The first seven iterations using the bisection method are shown below.

n	x_n	$f(x_n)$	I_{n+1}
1	0.15	-0.04	$[0.15, 0.2]$
2	0.175	0.08	$[0.15, 0.175]$
3	0.1625	0.02	$[0.15, 0.1625]$
4	0.15625	-0.013	$[0.15625, 0.1625]$
5	0.159375	0.004	$[0.15625, 0.159375]$
6	0.1578125	-0.004	$[0.1578125, 0.159375]$
7	0.15859375	-0.000003	$[0.15859375, 0.159375]$

We stop after seven iterations because at that stage $|f(x_7)| < \varepsilon$, where $\varepsilon = 10^{-4}$. At this point the root is, to 5 decimal places precision, $x_0 = 0.15859$.

3. We seek the root of $f(x) \equiv e^x - x - 2 = 0$. To estimate the root we expand the exponential term in a Taylor series up to the third term, which gives

$$1 + x + \frac{x^2}{2} - x - 2 = 0 \Rightarrow x = \pm\sqrt{2}$$

This result suggests that there are two roots (as indeed there are) but for the purposes of this tutorial we will only attempt to find the positive one (the one near $\sqrt{2}$). We know $1 < \sqrt{2} < 2$, so that the interval $[1, 2]$ looks promising as the starting interval. We find $f(1) = -0.28 < 0$ and $f(2) = 3.39 > 0$ and so we have a change in the sign of $f(x)$ on $I_1 = [1, 2]$. The first seven iterations are shown below.

n	x_n	$f(x_n)$	I_{n+1}
1	1.5	0.98	$[1, 1.5]$
2	1.25	0.24	$[1, 1.25]$
3	1.125	-0.04	$[1.125, 1.25]$
4	1.1875	0.09	$[1.125, 1.1875]$
5	1.15625	0.02	$[1.125, 1.15625]$
6	1.140625	-0.01	$[1.140625, 1.15625]$
7	1.1484375	0.005	$[1.140625, 1.1484375]$

More iterations would, of course, give a more accurate result, but at this point the root, precise to 5 decimal places, is $x_0 = 1.14843$.

4. We begin with $x_1 = 1$ and $x_2 = 2$. The root is contained in the interval $(x_1, x_2) = (1, 2)$. Clearly the approximation error ε is bounded above by $\varepsilon < |x_2 - x_1| = |2 - 1| = 1$. At each step of the bisection method we calculate the average, i.e. $x_3 = \frac{x_1+x_2}{2}$. Thus

$$|x_3 - x_2| = \left|\frac{x_1 + x_2}{2} - x_2\right| = \frac{|x_2 - x_1|}{2} = |x_3 - x_1|.$$

Since the root is contained either in the interval $(x_1, x_3]$ or $[x_3, x_2)$ we find the upper bound on the error becomes $\varepsilon < |x_3 - x_1| = \frac{|x_2-x_1|}{2}$ or $\varepsilon < |x_3 - x_2| = \frac{|x_2-x_1|}{2}$. Thus the upper bound is halved. Similarly, after the second iteration

$$\varepsilon < \frac{|x_3 - x_2|}{2} = \frac{|x_2 - x_1|}{2^2}.$$

After n iterations

$$\varepsilon < \frac{|x_2 - x_1|}{2^n}.$$

Insisting on $\frac{|x_2-x_1|}{2^n} < 10^{-2}$ yields the required upper bound

$$\varepsilon < \frac{|x_2 - x_1|}{2^n} < 10^{-2}.$$

Since $|x_2 - x_1| = 1$ we solve

$$\frac{1}{2^n} < 10^{-2}$$

to find

$$n > \log_2 100 \approx 6.643856\ldots$$

so that we can ensure the accuracy required after $n = 7$ iterations. Note that the accuracy might be achieved sooner, but we are certain that no more than 7 iterations are required.

j	x_1	x_2	$x_3 = \frac{x_1+x_2}{2}$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\lvert x_{j+2} - x_{j+1} \rvert$
1	1	**2**	**1.5**	$+$	$(-)$	$(+)$	0.5
2	**2**	1.5	**1.75**	$(-)$	$+$	$(+)$	0.25
3	2	**1.75**	**1.875**	$-$	$(+)$	$(-)$	0.125
4	**1.75**	1.875	**1.8125**	$(+)$	$-$	$(-)$	0.0625
5	**1.75**	1.8125	**1.78125**	$(+)$	$-$	$(-)$	0.03125
6	1.75	**1.78125**	**1.765625**	$+$	$(-)$	$(+)$	0.015625
7	1.78125	**1.765625**	**1.7734375**	$-$	$(+)$	$(-)$	$0.0078125 < 10^{-2}$

We may use either 1.765625 or 1.7734375 as an approximation of the root to the desired accuracy. However, since $\sin(1.765625^2) \approx 0.024$ and $\sin(1.7734375^2) \approx -0.0035$ the root 1.7734375 is clearly a better choice since the function value is closer to zero.