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Chapter 2

SERIES EXPANSIONS

2.1 Introduction

Sequences and series play an important role in numerical analysis. For example, Newton's method (see §3.4) generates a sequence of approximations to the root of a non-linear equation.

We review the theory of sequences and series and describe a useful power series that may be used to approximate functions about a given point.

2.2 Sequences

A *sequence of real numbers* is a function from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ onto the real numbers $\mathbb{R}$, i.e.

$$\begin{aligned} 1 &\mapsto x_1 \\ 2 &\mapsto x_2 \\ 3 &\mapsto x_3 \\ 4 &\mapsto x_4 \\ &\vdots \\ n &\mapsto x_n \\ &\vdots \end{aligned}$$

Thus $f(n) = x_n$. The numbers $x_1, x_2, x_3, \ldots, x_n, \ldots$, in the range of the function, are called the *elements* or *terms* of the sequence. A sequence is called *infinite* if it has an infinite amount of terms, otherwise it is called *finite*.

Example 2.1. Consider the following

$$\begin{aligned} 1 &\mapsto \sqrt{1} \\ 2 &\mapsto \sqrt{2} \\ 3 &\mapsto \sqrt{3} \\ 4 &\mapsto \sqrt{4} \\ &\vdots \end{aligned}$$

This is indeed a sequence, because the numbers on the left are in $\mathbb{N}$ and the numbers on the right are in some subset of $\mathbb{R}$. We note that the sequence is generated by the function $f(n) = \sqrt{n}$.

A sequence is denoted by its elements $x_1, x_2, x_3, \ldots$, or using the shorter notation $(x_n)_{n=1}^{\infty}$, where $n \in \mathbb{N}$. This latter notation is simplified to (x_n) if clear from context that we are dealing with an infinite or a finite sequence. Parentheses are deliberately used to emphasize the importance of ordering in a sequence. If we consider the set $\{1, 2, 3, 4\}$ and rewrite it as $\{4, 3, 2, 1\}$, then it is still considered to be the same set. However, when we consider the sequence $(1, 2, 3, 4)$ and rewrite it as $(4, 3, 2, 1)$, then the two sequences are considered to be two different. *Why are these two sequences considered to be different from each other?*

Most often a sequence is defined by giving a formula for its nth term x_n. For example, consider the sequence of reciprocals of the odd numbers

$$\left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right).$$

This sequence may be written as

$$\left(\frac{1}{2n+1} : n \in \mathbb{N}\right)$$

or more simply

$$x_n = \frac{1}{2n+1},$$

where it is understood that $n \in \mathbb{N}$. Another way of defining a sequence is to specify the value for x_1 and giving a formula for x_{n+1} in terms of x_n, the sequence is then said to be defined *recursively* or *inductively*.

Example 2.2. Consider the Fibonacci sequence $F = (f_n)$ given by

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

If we specify $f_1 = 1$ and $f_2 = 1$, then we may give the recursion formula for every other term in the sequence as $f_{n+1} = f_n + f_{n-1}$, where $n \geq 2$.

The limit of a sequence (x_n) is the real number ℓ such that

$$\lim_{n\to\infty} x_n = \ell \tag{2.1}$$

A sequence (x_n) is called *convergent* if its limit ℓ exists; if this limit does not exist, then the sequence is called *divergent*. It should be noted that the limit of a sequence is unique. Given two sequences (x_n) and (y_n) such that $\lim_{n\to\infty} x_n = \ell_1$ and $\lim_{n\to\infty} y_n = \ell_2$, then

$$\begin{aligned}
\lim_{n\to\infty} (x_n \pm y_n) &= \ell_1 \pm \ell_2 \\
\lim_{n\to\infty} (cx_n) &= c\ell_1, \quad c \in \mathbb{R} \\
\lim_{n\to\infty} (x_n y_n) &= \ell_1 \ell_2 \\
\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) &= \frac{\ell_1}{\ell_2}, \quad y_n \neq 0 \text{ and } \ell_2 \neq 0 \\
\lim_{n\to\infty} (x_n)^{(y_n)} &= x^y, \quad x > 0 \text{ and } x_n > 0
\end{aligned}$$

There are numerous other useful limit theorems and the reader is referred to consult an analysis text.

Example 2.3. Consider the sequence $\left(\frac{1}{n}\right)$. If $n \to \infty$ then $\frac{1}{n} \to 0$ and thus

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$$

Hence 0 is the limit of the sequence.

A sequence (x_n) is *bounded* if there exists numbers $a, b \in \mathbb{R}$ such that $a < x_n < b$ for all $n \in \mathbb{N}$. The number a is called the lower bound of the sequence and the number b is called the upper bound of the sequence.

Example 2.4. Consider the sequence of even, positive integers $(2, 4, 6, 8, \dots)$. It should not be difficult to see that the number 1 is a lower bound of this sequence. Next we consider the sequence of reciprocals of the even, positive integers $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right)$. In this case, the number 1 is the upper bound of the sequence.

A sequence (x_n) is called *increasing* if

$$x_1 \le x_2 \le x_3 \le \cdots \le x_n \le x_m \le \cdots$$

for all $n < m$ and $n, m \in \mathbb{N}$. Similarly, the sequence (x_n) is called *decreasing* if

$$x_1 \ge x_2 \ge x_3 \ge \cdots \ge x_n \ge x_m \ge \cdots$$

for all $n < m$ and $n, m \in \mathbb{N}$. A sequence (x_n) is called *monotone* if it is increasing or decreasing. It can be shown that a monotonic sequence is convergent if and only if it is bounded.

Example 2.5. A ball, with diameter ϵ, is dropped from a height h_0. Each time it drops h metres, it rebounds rh metres. Determine how many times the ball bounces before it stops moving.

Solution: We note that if the initial height was h_0, then after the first bounce the ball will reach a height of $h_1 = rh_0$, the height after the second bounce would be $h_2 = rh_1$, and after n bounces the height would be $h_n = rh_{n-1}$. But it follows that

$$h_n = rh_{n-1} = r\big(rh_{n-2}\big) = r^2 h_{n-2} = \cdots$$

and we conclude that $h_n = r^n h_0$. Clearly, the bounces of the ball form a recursive sequence.

The ball stops bouncing when $|h_n| \le \epsilon$. Since $h_n = r^n h_0$, it follows that

$$\begin{aligned} r^n h_0 &\le \epsilon \\ r^n &\le \frac{\epsilon}{h_0} \\ n\ln(r) &\le \ln\left(\frac{\epsilon}{h_0}\right) \\ \therefore \quad n &\ge \frac{\ln(\epsilon) - \ln(h_0)}{\ln(r)} \end{aligned}$$

If $h_0 = 8$m, $r = 0.7$ and $\epsilon = 0.07$m, then we find $n \ge 13.2858$ and we conclude that it would take 14 bounces before the balls stops.

2.3 Series

Informally, a series is the sum of the terms of a sequence. If we let (x_n) be a sequence, then sum of the first k terms of this sequence, i.e.

$$s_k = x_1 + x_2 + \cdots + x_k = \sum_{i=1}^{k} x_i, \quad k \leq n$$

is called the *kth partial sum* of the sequence. Note that the partial sums form a sequence by themselves, i.e.

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= s_1 + x_2 \quad (= x_1 + x_2) \\ s_3 &= s_2 + x_3 \quad (= x_1 + x_2 + x_3) \\ &\vdots \\ s_n &= s_{n-1} + x_n \quad (= x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n) \\ &\vdots \end{aligned}$$

This pair of sequences $\big((x_n), (s_n)\big)$ is called the *series* generated by the sequence (x_n). The numbers x_i, $i = 1, 2, \ldots, n$, in the partial sums are called the *terms* of the sequence. A series is called *infinite* if it has an infinite amount of terms, otherwise it is called *finite* if it has a finite amount of terms.

Instead of writing $\big((x_n), (s_n)\big)$ every time to denote a series generated by (x_n), it is convention to use the notation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$$

if dealing with a finite series; for an infinite series one would similarly have

$$\sum_{i=1}^{\infty} x_i = x_1 + x_2 + x_3 + \cdots .$$

The symbol Σ is the Greek letter sigma, and one often refers to the notation above as "sigma-notation". When clear from context whether we are dealing with a finite or infinite series, the sigma-notation is shortened to $\sum_i x_i$.

Let $\sum_{i=1}^{\infty} x_i$ be a series. If the sequence (s_n) of partial sums of this series converges to the limit s, then the series is called *convergent* and the limit s is called the *sum* of the series. The sum is denoted

$$s = \sum_{i=1}^{\infty} x_i$$

If this limit does not exist, then the series is said to be *divergent.*

Example 2.6. Consider the series $\sum_{i=1}^{\infty} c$, where $c \in \mathbb{R}$ is a constant. Clearly this infinite series does not have a sum and thus is divergent. However, if we were to consider the finite series $\sum_{i=1}^{n} c$, then it should be clear that this series does have a sum and is convergent. The sum is none other than $s = nc$.

Two very important questions arise when studying series.

1. Does the series converge or diverge?
2. What is the sum of the series if it is convergent?

Example 2.7. Given the infinite series

$$\sum_{i=0}^{\infty} ar^i = a + ar + ar^2 + ar^3 + \cdots + ar^i + \cdots , \tag{2.2}$$

where $a \in \mathbb{R}$ is a non-zero constant.

Consider the nth partial sum $s_n = a + ar + ar^2 + \cdots + ar^n$. If we multiply s_n by r and subtract the result from s_n, then we obtain

$$\begin{aligned} s_n - rs_n &= (a + ar + ar^2 + \cdots + ar^n) - r(a + ar + ar^2 + \cdots + ar^n) \\ &= a + ar + ar^2 + \cdots + ar^n - ar - ar^2 - \cdots - ar^{n+1} \\ &= a - ar^{n+1} \\ s_n(1-r) &= a(1 - r^{n+1}) \end{aligned}$$

and therefore

$$s_n = \frac{a(1-r^{n+1})}{1-r}.$$

If $0 < |r| < 1$, then the term $r^{n+1} \to 0$ as $n \to \infty$ and we obtain the sum of the geometric series (2.2) as

$$s = \frac{a}{1-r}. \tag{2.3}$$

The series (2.2) is called a *geometric series.*

A simpler way of determining whether a series converges or diverges is to make use of the following theorem:

Theorem 2.1 (Ratio Test)**.** The series $\sum_{n=0}^{\infty} a_n$ converges if

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| < 1$$

and diverges if

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| > 1.$$

If this limit is equal to 1, then we cannot conclude anything about the convergence or divergence of the series.

Example 2.8. Let us reconsider the geometric series (2.2). By the ratio test, we now have

$$\left| \frac{x_n}{x_{n-1}} \right| = \left| \frac{ar^n}{ar^{n-1}} \right| = |r|$$

and thus we require $|r| < 1$ for convergence. This concurs with our earlier analysis in example 2.7.

2.4 Taylor series expansions

From analysis it is known that a continuous function may be approximated by finite or infinite series, and these approximations are normally done by power series expansions. A *power series* is an infinite series of the form

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{2.4}$$

Example 2.7 is a special case of a power series. In the following sections we shall discuss a very useful power series expansion—the *Taylor series expansion* of a function—which is a powerful analytical and numerical tool in applied mathematics and is used extensively in later chapters.

2.4.1 Taylor's Theorem (1715)

We consider a function f that has continuous derivatives up to $(n+1)$th order on an interval $[a, b]$. From the fundamental theorem of calculus we have

$$f(b) = f(a) + \int_a^b f'(x)\,dx.$$

Repeated integration by parts yields

$$\begin{aligned} f(b) &= f(a) + (x-b)\big[f'(x)\big]_a^b + \int_a^b (x-b) f''(x)\,\mathrm{d}x \\ &= f(a) + (b-a)f'(a) + \left[\frac{(b-x)^2}{2} f''(x)\right]_a^b + \int_a^b \frac{(b-x)^2}{2} f'''(x)\,\mathrm{d}x \\ &= f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) \\ &\quad + \left[-\frac{(b-x)^3}{2\times 3} f'''(x)\right]_a^b + \int_a^b \frac{(b-x)^3}{2\times 3} f^{(4)}(x)\,\mathrm{d}x \\ &= f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) \\ &\quad + \frac{(b-a)^3}{3!} f'''(a) + \int_a^b \frac{(b-x)^3}{3!} f^{(4)}(x)\,\mathrm{d}x \end{aligned}$$

After n steps we have Taylor's Theorem:

$$\begin{aligned} f(b) &= f(a) + (b-a)f'(a) + \cdots + \frac{(b-a)^n}{n!} f^{(n)}(a) + R_n \\ R_n &= \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x)\,\mathrm{d}x \end{aligned}$$

With the substitutions $b \to x, a \to x_0, x \to t$ we obtain the more familiar form

$$\begin{aligned} f(x) &= f(x_0) + (x-x_0)f'(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_n \\ R_n &= \int_{x_0}^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t \end{aligned} \tag{2.5}$$

We refer to R_n as the *residual term*.

2.4.2 Taylor series

Consider a function f for which derivatives of all orders exist at x_0. In other words, we may write (2.5) for arbitrary n. The power series in $(x - x_0)$ on the right-hand side of (2.5) will converge to a finite value if $\lim_{n\to\infty} R_n = 0$. Hence, we may expand $f(x)$ then as an infinite power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) \tag{2.6}$$

The convergence, or lack thereof, of the series (2.6) may be investigated by means of estimates of R_n. Cauchy and Lagrange have given estimates for the residual term (see §2.4.3), but we could also make use of theorem 2.1.

Example 2.9. We want to write $f(x) = e^x$ as an infinite power series. Since e^x and all its derivatives exist at $x = 0$, we choose x_0 in (2.6). Then

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1 \end{aligned}$$

and so

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We investigate the convergence of this series using theorem 2.1:

$$e^x = \sum_{n=0}^{\infty} a_n \qquad a_n = \frac{x^n}{n!}$$

Hence

$$\left|\frac{a_n}{a_{n-1}}\right| = \left(\frac{x^n}{n!}\right) \Big/ \left(\frac{x^{n-1}}{(n-1)!}\right) = \frac{x}{n}$$

so that

$$\left|\frac{a_n}{a_{n-1}}\right| = \frac{|x|}{n} \to 0 \text{ as } n \to \infty$$

Thus, this series converges for all values of x.

Example 2.10. We obtain the so-called *binomial expansion* by determining the Taylor series of $f(x) = (1+x)^p$ for all real values of p. Let $x_0 = 0$. Then

$$\begin{aligned} f(x) &= (1+x)^p & f(0) &= 1 \\ f'(x) &= p(1+x)^{p-1} & f'(0) &= p \\ f''(x) &= p(p-1)(1+x)^{p-2} & f''(0) &= p(p-1) \\ &\vdots & &\vdots \\ f^{(n)} &= p(p-1)\cdots(p-n+1)(1+x)^{p-n} & f^{(n)}(0) &= p(p-1)\cdots(p-n+1) \end{aligned}$$

and so

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \tag{2.7}$$

where

$$\binom{p}{n} = \frac{p\,(p-1)\cdots(p-n+1)}{n!}$$

are called the *binomial coefficients.* We note that $\binom{p}{0} = 1$.

A special case is $p = \frac{1}{2}$, e.g.

$$\begin{aligned} (1+x)^{\frac{1}{2}} &= 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \end{aligned}$$

We use theorem 2.1 to test the convergence of this series. From (2.7) the series has the form

$$(1+x)^p = \sum_{n=0}^{\infty} a_n$$

with

$$a_n = \frac{p(p-1)\cdots(p-n+1)}{n!} x^n.$$

Hence

$$\frac{a_n}{a_{n-1}} = \frac{p(p-1)\cdots(p-n+2)(p-n+1)x^n}{(1)(2)\cdots(n-1)n} \times \frac{(1)(2)\cdots(n-1)}{p(p-1)\cdots(p-n+2)x^{n-1}}$$

and so

$$\left|\frac{a_n}{a_{n-1}}\right| = \left|\frac{(p-n+1)x}{n}\right| = \left|\frac{p+1}{n} - 1\right| |x|.$$

Clearly $\left|\frac{a_n}{a_{n-1}}\right| \to |x|$ as $n \to \infty$, and so this series converges only if $|x| < 1$.

Example 2.11. Find e^x to an accuracy of ϵ. We calculate here the series

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!},$$

where the Nth term is the first one for which $\left|\frac{x^N}{N!}\right| < \epsilon$. A numerical complication that arises is that $69! \approx 10^{100}$ and that $n!$ cannot be evaluated using a pocket calculator for $n \geq 70$. Instead we make use of the recursion relation (from example 2.9)

$$a_0 = 1 \qquad a_n = \left(\frac{x}{n}\right) a_{n-1} \qquad n > 0$$

We obtain $e^1 = 2.718282$ with 10 terms. Note the required accuracy: $\epsilon = 10^{-6}$, therefore *six* decimal places are shown.

2.4.3 Lagrange estimate of the error term

Lagrange obtained an estimate of the error term R_n in (2.5) which is very useful for analytical purposes. The generalized mean-value theorem for integral calculus allows us to write the residual term as

$$R_n = f^{(n+1)}(\xi_x) \int_{x_0}^{x} \frac{(x-t)^n}{n!}\, \mathrm{d}t$$

where $x_0 < \xi_x < x$. The integral in the residual term is now easily determined to be

$$R_n = f^{(n+1)}(\xi_x) \left[\frac{-1}{n!} \times \frac{(x-t)^{n+1}}{(n+1)}\right]_{x_0}^{x} = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x).$$

The series in (2.5) now becomes

$$\begin{aligned} f(x) &= f(x_0) + (x-x_0)f'(x_0) + \cdots \\ &\quad + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x). \end{aligned} \tag{2.8}$$

It is often useful to define $x = x_0 + h$. Then (2.8) becomes

$$\begin{aligned} f(x_0+h) &= f(x_0) + hf'(x_0) + \cdots \\ &\quad + \frac{h^n}{n!} f^{(n)}(x_0) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x) \end{aligned} \tag{2.9}$$

where $x_0 < \xi < x_0 + h$.

2.4.4 Expansion of multivariable functions

If a function is dependent on more than one variable, we use *consecutive* Taylor expansions with respect to each of the variables. Since all other variables are held constant while expanding with respect to a particular variable, all derivatives in the expansion are partial. Here, we expand a function of two variables. We consider $f(x_0 + h, y_0 + k)$ and use (2.9) to obtain an expansion with respect to x (holding y constant) and then we expand each term of this expansion with respect to y (holding x constant), i.e.

$$\begin{aligned}
f(x_0+h, y_0+k) &= f(x_0, y_0+k) + h\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0+k)} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0+k)} + \cdots \\
&= f(x_0,y_0) + k\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\bigg|_{(x_0,y_0)} + \cdots \\
&\quad + \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0+k)} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0+k)} + \cdots \right) \\
&= f(x_0,y_0) + k\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\bigg|_{(x_0,y_0)} + \cdots \\
&\quad + h\left(\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + k\frac{\partial^2 f}{\partial y \partial x}\bigg|_{(x_0,y_0)} + \cdots \right) \\
&\quad + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0)} + \frac{\partial^3 f}{\partial y \partial x^2}\bigg|_{(x_0,y_0)} \cdots \right) + \cdots \\
&= f(x_0,y_0) + \left(h\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + k\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} \right) \\
&\quad + \left(\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0)} + hk\frac{\partial^2 f}{\partial y \partial x}\bigg|_{(x_0,y_0)} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\bigg|_{(x_0,y_0)} \right) + \cdots
\end{aligned}$$

and in summary

$$f(x_0+h, y_0+k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{h^n k^m}{n! m!} \frac{\partial^{n+m} f}{\partial x^n \partial y^m}\bigg|_{(x_0,y_0)}.$$

The series for a function of more than two variables is analogous.