Applied Mathematics 3B

Semester Test #2 **Solution**

1.

(a) The density matrix is

$$\rho = \frac{3}{4}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} + \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix} = \begin{pmatrix}\frac{3}{8} & 0 & 0 & \frac{3}{8}\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ \frac{3}{8} & 0 & 0 & \frac{3}{8}\end{pmatrix}.$$

(b) For the partial traces we have

$$\begin{aligned}
\mathrm{tr}_1\rho &= \begin{pmatrix}1 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\rho\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}0 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\rho\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{3}{8} & 0 & 0 & \frac{3}{8}\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ \frac{3}{8} & 0 & 0 & \frac{3}{8}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{3}{8} & 0 & 0 & \frac{3}{8}\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ \frac{3}{8} & 0 & 0 & \frac{3}{8}\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 0\\1 & 0\\0 & 1\end{pmatrix}\\
&= \begin{pmatrix}\frac{3}{8} & 0\\0 & \frac{1}{8}\end{pmatrix} + \begin{pmatrix}\frac{1}{8} & 0\\0 & \frac{3}{8}\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix},\\
\mathrm{tr}_2\rho &= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\end{pmatrix}\rho\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 1\end{pmatrix}\rho\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\frac{3}{8} & 0 & 0 & \frac{3}{8}\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ \frac{3}{8} & 0 & 0 & \frac{3}{8}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\\0 & 1\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{3}{8} & 0 & 0 & \frac{3}{8}\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & \frac{1}{8} & 0\\ \frac{3}{8} & 0 & 0 & \frac{3}{8}\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 0\\0 & 0\\0 & 1\end{pmatrix}\\
&= \begin{pmatrix}\frac{3}{8} & 0\\0 & \frac{1}{8}\end{pmatrix} + \begin{pmatrix}\frac{1}{8} & 0\\0 & \frac{3}{8}\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix}.
\end{aligned}$$

(c) We must calculate the von Neumann entropy $S(\mathrm{tr}_1\rho)$ or $S(\mathrm{tr}_2\rho)$ (which give the same result). Since $\mathrm{tr}_1\rho = \mathrm{tr}_2\rho$ we calculate $S(\mathrm{tr}_1\rho)$. The eigenvalues of $\mathrm{tr}_1\rho$ are $\frac{1}{2}$ and $\frac{1}{2}$, so that

$$S(\mathrm{tr}_1\rho) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1.$$

2.

(a) We have

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

Any eigenvector $|\psi\rangle$ in $\mathbb{C}^2$ can be written in the form

$$|\psi\rangle = a|0\rangle + b|1\rangle, \qquad (a,b)\in\mathbb{C}^2\setminus\{(0,0)\}$$

so that the eigenvalue equation becomes

$$\begin{aligned}
U_{NOT}|\psi\rangle &= \lambda|\psi\rangle\\
U_{NOT}(a|0\rangle + b|1\rangle) &= \lambda a|0\rangle + \lambda b|1\rangle\\
aU_{NOT}|0\rangle + bU_{NOT}|1\rangle &= \lambda a|0\rangle + \lambda b|1\rangle\\
a|1\rangle + b|0\rangle &= \lambda a|0\rangle + \lambda b|1\rangle.
\end{aligned}$$

Comparing coefficients of $|0\rangle$ and $|1\rangle$ yields

$$a = \lambda b, \qquad b = \lambda a, \qquad \Rightarrow \qquad a = \lambda^2 a, \qquad b = \lambda^2 b.$$

Since either $a \neq 0$ or $b \neq 0$ (or both) we have $\lambda^2 = 1$. We have two (potential) solutions $\lambda = 1$ ($a = b$), and $\lambda = -1$ ($a = -b$). Since solutions for the eigenvalue equation exists for both values of λ, the eigenvalues are 1 with corresponding eigenvectors

$$a|0\rangle + a|1\rangle, \qquad a \in \mathbb{C}, \quad a \neq 0$$

and -1 with corresponding eigenvectors

$$a|0\rangle - a|1\rangle, \qquad a \in \mathbb{C}, \quad a \neq 0.$$

(b) A braket representation for V_{NOT} is given by (others are possible)

$$\begin{aligned} V_{NOT} &= |0\rangle\langle 0| \otimes |\mathrm{NOT}(0)\rangle\langle 0| + |1\rangle\langle 1| \otimes |\mathrm{NOT}(1)\rangle\langle 0| + |0\rangle\langle 0| \otimes |\overline{\mathrm{NOT}(0)}\rangle\langle 1| + |1\rangle\langle 1| \otimes |\overline{\mathrm{NOT}(1)}\rangle\langle 1| \\ &= |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\ &= |0\rangle\langle 0| \otimes U_{NOT} + |1\rangle\langle 1| \otimes I_2. \end{aligned}$$

Note that

$$V_{NOT} = (I_2 \otimes U_{NOT})U_{CNOT} = U_{CNOT}(I_2 \otimes U_{NOT}).$$

Any eigenvector $|\phi\rangle$ in $\mathbb{C}^4$ can be written in the form

$$|\phi\rangle = a|0\rangle \otimes |0\rangle + b|0\rangle \otimes |1\rangle + c|1\rangle \otimes |0\rangle + d|1\rangle \otimes |1\rangle, \qquad (a,b,c,d) \in \mathbb{C}^4 \setminus \{(0,0,0,0)\}$$

so that the eigenvalue equation becomes

$$\begin{aligned} V_{NOT}|\phi\rangle &= \mu|\phi\rangle \\ &= V_{NOT}(a|0\rangle \otimes |0\rangle + b|0\rangle \otimes |1\rangle + c|1\rangle \otimes |0\rangle + d|1\rangle \otimes |1\rangle) \\ &= aV_{NOT}|0\rangle \otimes |0\rangle + bV_{NOT}|0\rangle \otimes |1\rangle + cV_{NOT}|1\rangle \otimes |0\rangle + dV_{NOT}|1\rangle \otimes |1\rangle \\ &= a|0\rangle \otimes |1\rangle + b|0\rangle \otimes |0\rangle + c|1\rangle \otimes |0\rangle + d|1\rangle \otimes |1\rangle \\ &= \mu a|0\rangle \otimes |0\rangle + \mu b|0\rangle \otimes |1\rangle + \mu c|1\rangle \otimes |0\rangle + \mu d|1\rangle \otimes |1\rangle \end{aligned}$$

Comparing coefficients yields

$$a = \mu b, \qquad b = \mu a, \qquad c = \mu c, \qquad d = \mu d.$$

Similar to (a) we find $\mu = 1$ or $\mu = -1$. For $\mu = 1$ we find $a = b$, and for $\mu = -1$ we find $a = -b$ and $c = d = 0$. Thus we have the eigenvalue 1 with corresponding eigenvectors

$$a|0\rangle \otimes |1\rangle + a|0\rangle \otimes |0\rangle + c|1\rangle \otimes |0\rangle + d|1\rangle \otimes |1\rangle = a|0\rangle \otimes (|0\rangle + |1\rangle) + |1\rangle \otimes (c|0\rangle + d|\rangle), \qquad (a,c,d) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$$

and -1 with corresponding eigenvectors

$$a|0\rangle \otimes |0\rangle - a|0\rangle \otimes |1\rangle = a|0\rangle \otimes (|0\rangle - |1\rangle), \qquad a \in \mathbb{C}, \quad a \neq 0.$$

3. We have

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{pmatrix}.$$

The characteristic equation is given by

$$\det \begin{pmatrix} \lambda - 2 & -4 & -2 \\ -4 & \lambda - 8 & -4 \\ -2 & -4 & \lambda - 2 \end{pmatrix} = (\lambda-2)^2(\lambda-8) - 32 - 32 - 16(\lambda-2) - 16(\lambda-2) - 4(\lambda-8) = \lambda^2(\lambda-12).$$

Thus the singular values are $\sqrt{12}$, 0 and 0. For the singular value $\sqrt{12}$ we find that the eigenvalue equation

$$\begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 12 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

yields

$$2x + 4y + 2z = 12x, \qquad 4x + 8y + 4z = 12y, \qquad 2x + 4y + 2z = 12z$$

from which follows $x = z = \frac{1}{2}y$, i.e. a representative eigenvector is

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

For the singular value 0 we find that the eigenvalue equation

$$\begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

yields

$$2x + 4y + 2z = 0 \qquad \Rightarrow \qquad x = -2y - z.$$

Here we need *two* orthonormal representative eigenvectors, it is easy to select values for y and z to obtain them, but Gram-Schmidt orthonormalization could also be used. Here we chose $y = 1$, $z = -1$ and $z = 1$, $y = 0$ respectively and then normalize to obtain

$$\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus we find

$$V = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Note that the columns are orthonormal. Other values for V are also possible, this is not the only acceptable answer.

The first column of U follows from the first column of V:

$$\frac{1}{\sqrt{12}}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since the remaining singular values are 0, we choose the remaining columns of U to be orthonormal to the first column. Once again, Gram-Schmidt orthonormalization could have been used and the choice of U is not unique:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}^*.$$

Check this by performing the matrix multiplication.

Another way to get this result is to find the singular value decomposition of the complex conjugate transpose of the matrix and then to take the complex conjugate transpose of the resulting singular value decomposition. Try this as an exercise.