Applied Mathematics 3B

Semester Test #2 APS **Solution**

1.

(a) The density matrix is

$$\rho = \cos^2\theta \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} + \sin^2\theta \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos^2\theta & 0 & 0 & \cos^2\theta \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ \cos^2\theta & 0 & 0 & \cos^2\theta \end{pmatrix}.$$

(b) For the partial traces we have

$$\begin{aligned}
\mathrm{tr}_1\rho &= \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} \cos^2\theta & 0 & 0 & \cos^2\theta \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ \cos^2\theta & 0 & 0 & \cos^2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} \cos^2\theta & 0 & 0 & \cos^2\theta \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ \cos^2\theta & 0 & 0 & \cos^2\theta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \sin^2\theta & 0 \\ 0 & \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.
\end{aligned}$$

(c) We must calculate the von Neumann entropy $S(\mathrm{tr}_1\rho)$ or $S(\mathrm{tr}_2\rho)$ (which give the same result). Since $\mathrm{tr}_1\rho = \mathrm{tr}_2\rho$ we calculate $S(\mathrm{tr}_1\rho)$. The eigenvalues of $\mathrm{tr}_1\rho$ are $\frac{1}{2}$ and $\frac{1}{2}$, so that

$$S(\mathrm{tr}_1\rho) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1.$$

2.

(a) We have

$$U_H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^0|1\rangle)$$

$$U_H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^1|1\rangle)$$

i.e.

$$U_H|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^a|1\rangle)$$

for $a \in \{0, 1\}$.

(b) The calculation proceeds as follows

$$\begin{aligned}
&(U_H \otimes I_2 \otimes I_2)U_{C\text{-}f}(U_H \otimes I_2 \otimes I_2)(|0\rangle \otimes |a\rangle \otimes |b\rangle) \\
&= (U_H \otimes I_2 \otimes I_2)U_{C\text{-}f}\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |a\rangle \otimes |b\rangle)\right] \\
&= (U_H \otimes I_2 \otimes I_2)\left[\frac{1}{\sqrt{2}}U_{C\text{-}f}|0\rangle \otimes |a\rangle \otimes |b\rangle) + \frac{1}{\sqrt{2}}U_{C\text{-}f}|1\rangle \otimes |a\rangle \otimes |b\rangle)\right]
\end{aligned}$$

$$
\begin{aligned}
&= (U_H \otimes I_2 \otimes I_2)\left[\frac{1}{\sqrt{2}}|0\rangle \otimes |a\rangle \otimes |b\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes (U(f)(|a\rangle \otimes |b\rangle))\right] \\
&= (U_H \otimes I_2 \otimes I_2)\left[\frac{1}{\sqrt{2}}|0\rangle \otimes |a\rangle \otimes |b\rangle + \frac{1}{\sqrt{2}}(-1)^{f(a,b)}|1\rangle \otimes |a\rangle \otimes |b\rangle\right] \\
&= (U_H \otimes I_2 \otimes I_2)\left[\frac{1}{\sqrt{2}}(|0\rangle + (-1)^{f(a,b)}|1\rangle) \otimes |a\rangle \otimes |b\rangle\right] \\
&= |f(a,b)\rangle \otimes |a\rangle \otimes |b\rangle.
\end{aligned}
$$

3. We have

$$
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
$$

The characteristic equation is given by

$$
\det \begin{pmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 5 & 0 \\ -1 & 0 & \lambda - 1 \end{pmatrix} = (\lambda-1)^2(\lambda-5) - (\lambda-5) = ((\lambda-1)^2-1)(\lambda-5) = (\lambda^2-2\lambda)(\lambda-5).
$$

Thus the singular values are 5, 2 and 0. For the singular value 5 we find the normalized corresponding eigenvector $(0,1,0)^T$.

For the singular value 2 we find the normalized corresponding eigenvector $(1,0,1)^T/\sqrt{2}$.

For the singular value 0 we find the normalized corresponding eigenvector $(1,0,-1)^T/\sqrt{2}$.

Thus we find

$$
V = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
$$

Note that the columns are orthonormal. Other values for V are also possible, this is not the only acceptable answer.

The first and second columns of U follow from the first column and second columns of V:

$$
\frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.
$$

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
$$

Since the remaining singular value is 0, we choose the remaining column of U to be orthonormal to the first two columns. Gram-Schmidt orthonormalization can be used and the choice of U is not unique:

$$
U = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}.
$$

It follows that

$$
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^*.
$$

Check this by performing the matrix multiplication.