UNIVERSITY OF JOHANNESBURG

APPLIED MATHEMATICS 3B

Semester Test: 21 October 2011

Duration: 80 minutes **Marks:** 25

Instructions: Answer all the questions
All calculations must be shown
Pocket calculators are permitted
All angles are measured in radians
The prescribed text book is allowed

Question 1

Consider the Hilbert space $\mathbb{C}^4 \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$. We prepare a classical mixture of $100\cos^2\theta\%$ of the Bell state $(1,0,0,1)^T/\sqrt{2}$ and $100\sin^2\theta\%$ of the Bell state $(0,1,1,0)^T/\sqrt{2}$ where $\theta \in \mathbb{R}$.

(a) Find the density matrix ρ that models this mixture. (2)

(b) Calculate the partial trace $\mathrm{tr}_1\rho$ where the first system is $\mathbb{C}^2$ and the second system is $\mathbb{C}^2$. (3)

(c) Use the von Neumann entropy measure to determine the amount of entanglement of ρ. (3)

(8)

Question 2

Let $f : \{0,1\}^2 \to \{0,1\}$. Let $\{\,|0\rangle, |1\rangle\,\}$ denote an orthonormal basis in $\mathbb{C}^2$. Consider the mapping

$$U(f) := \sum_{j=0}^{1}\sum_{k=0}^{1}(-1)^{f(j,k)}|j\rangle\langle j| \otimes |k\rangle\langle k| \equiv \sum_{j=0}^{1}\sum_{k=0}^{1}(-1)^{f(j,k)}(|j\rangle \otimes |k\rangle)(\langle j| \otimes \langle k|).$$

(a) Show that

$$U_H|a\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^a|1\rangle\right), \qquad \forall a \in \{0,1\}$$

where U_H is the Walsh-Hadamard transform (Hadamard gate). (2)

(b) Calculate

$$(U_H \otimes I_2 \otimes I_2)U_{C\text{-}f}(U_H \otimes I_2 \otimes I_2)(|0\rangle \otimes |a\rangle \otimes |b\rangle), \qquad a, b \in \{0,1\}$$

where I_2 is the identity operator on $\mathbb{C}^2$ and

$$U_{C\text{-}f} := |0\rangle\langle 0| \otimes I_2 \otimes I_2 + |1\rangle\langle 1| \otimes U(f).$$

Conclude how to calculate $f(a,b)$ using U_f. (6)

(8)

Question 3

Find the singular value decomposition of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

(9)

END OF QUESTION PAPER