Applied Mathematics 3B

Semester Test #1 **Solution**

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1. The solution is given by

$$\boldsymbol{\psi}(t) = \exp\left[i\omega t \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right]\boldsymbol{\psi}(0).$$

Let

$$A := i\omega t \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We have

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

where

$$A^0 = I_3, \qquad A^1 = A, \qquad A^2 = (i\omega t)^2 \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \qquad A^3 = (i\omega t)^3 \begin{pmatrix} 4 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 4 \end{pmatrix}.$$

We see a pattern emerging. Suppose, for $k \in \mathbb{N}$,

$$A^k = (i\omega t)^k \begin{pmatrix} 2^{k-1} & 0 & 2^{k-1} \\ 0 & 1 & 0 \\ 2^{k-1} & 0 & 2^{k-1} \end{pmatrix}$$

(which is clearly true for $k = 1$). It follows that

$$A^{k+1} = A \cdot A^k = i\omega t \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} (i\omega t)^k \begin{pmatrix} 2^{k-1} & 0 & 2^{k-1} \\ 0 & 1 & 0 \\ 2^{k-1} & 0 & 2^{k-1} \end{pmatrix} = (i\omega t)^{k+1} \begin{pmatrix} 2^k & 0 & 2^k \\ 0 & 1 & 0 \\ 2^k & 0 & 2^k \end{pmatrix}$$

Thus

$$A^k = (i\omega t)^k \begin{pmatrix} 2^{k-1} & 0 & 2^{k-1} \\ 0 & 1 & 0 \\ 2^{k-1} & 0 & 2^{k-1} \end{pmatrix} = \frac{1}{2}(2i\omega t)^k \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + (i\omega t)^k \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $k = 1, 2, \ldots$ by induction. It follows that

$$\begin{aligned} e^A &= \sum_{j=0}^{\infty} \frac{A^j}{j!} = I_3 + \sum_{j=1}^{\infty} \frac{A^j}{j!} \\ &= I_3 + \frac{1}{2}\sum_{j=1}^{\infty} \frac{(2i\omega t)^j}{j!} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \sum_{j=1}^{\infty} \frac{(i\omega t)^j}{j!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= I_3 + \frac{1}{2}\left(e^{2i\omega t} - 1\right) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \left(e^{i\omega t} - 1\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Consequently

$$\boldsymbol{\psi}(t) = \begin{pmatrix} \frac{1}{2}e^{2i\omega t} + \frac{1}{2} & 0 & \frac{1}{2}e^{2i\omega t} - \frac{1}{2} \\ 0 & e^{i\omega t} & \\ \frac{1}{2}e^{2i\omega t} - \frac{1}{2} & 0 & \frac{1}{2}e^{2i\omega t} + \frac{1}{2} \end{pmatrix} \boldsymbol{\psi}(0).$$

Alternatively we can use the eigenvalues and eigenvectors of A. The eigenvalues and corresponding representative orthonormal eigenvectors of A are

$$2i\omega t: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad i\omega t: \quad \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad 0: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Since A is normal we have

$$\begin{aligned} e^A &= e^{2i\omega t}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}^* + e^{i\omega t}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^* + e^0\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}^* \\ &= \begin{pmatrix}\frac{1}{2}e^{2i\omega t}+\frac{1}{2} & 0 & \frac{1}{2}e^{2i\omega t}-\frac{1}{2}\\ 0 & e^{i\omega t} & 0\\ \frac{1}{2}e^{2i\omega t}-\frac{1}{2} & 0 & \frac{1}{2}e^{2i\omega t}+\frac{1}{2}\end{pmatrix}. \end{aligned}$$

Thus

$$\boldsymbol{\psi}\left(t=\frac{\pi}{2\omega}\right) = \begin{pmatrix}0 & 0 & -1\\ 0 & i & 0\\ -1 & 0 & 0\end{pmatrix}\boldsymbol{\psi}(0).$$

For the initial value problems we find

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad\Rightarrow\quad \boldsymbol{\psi}\left(t=\frac{\pi}{2\omega}\right) = \begin{pmatrix}-1\\0\\-1\end{pmatrix} = -\boldsymbol{\psi}(0).$$

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad\Rightarrow\quad \boldsymbol{\psi}\left(t=\frac{\pi}{2\omega}\right) = \begin{pmatrix}0\\i\\0\end{pmatrix} = i\boldsymbol{\psi}(0).$$

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad\Rightarrow\quad \boldsymbol{\psi}\left(t=\frac{\pi}{2\omega}\right) = \begin{pmatrix}-1\\i\\-1\end{pmatrix}.$$

2. We have

$$\rho = \begin{pmatrix}\frac{1}{4}\cos^2\theta & 0 & \frac{1}{4}\cos^2\theta\\ 0 & \frac{1}{2}\sin^2\theta & 0\\ \frac{1}{4}\cos^2\theta & 0 & \frac{1}{4}\cos^2\theta+\frac{1}{2}\end{pmatrix}.$$

(a) The characteristic equation is

$$\begin{aligned} \det\left(\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix} - \lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right) &= (1-\lambda)^2(2-\lambda)-(2-\lambda) = ((1-\lambda)^2-1)(2-\lambda)\\ &= ((1-\lambda)-1)((1-\lambda)+1)(2-\lambda) = -\lambda(2-\lambda)^2 = 0 \end{aligned}$$

The measurement outcomes are 2 and 0.

(b) A set of orthonormal eigenvectors corresponding to these measurement outcomes (eigenvalues) are

$$2: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad 0: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

The corresponding projection operators are

$$\Pi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}^* + \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^* = \begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix}$$

and

$$\Pi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

The probability for the measurement outcome 2 is

$$p_2 = \operatorname{tr}(\rho\Pi_2) = \operatorname{tr} \begin{pmatrix} \frac{1}{4}\cos^2\theta & 0 & \frac{1}{4}\cos^2\theta \\ 0 & \frac{1}{2}\sin^2\theta & 0 \\ \frac{1}{4}\cos^2\theta + \frac{1}{4} & 0 & \frac{1}{4}\cos^2\theta + \frac{1}{4} \end{pmatrix} = \frac{3}{4}.$$

The probability for the measurement outcome 0 is

$$p_0 = \operatorname{tr}(\rho\Pi_0) = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} = \frac{1}{4}.$$

(c) The expectation value is

$$\operatorname{tr}(\rho \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}) = \operatorname{tr} \begin{pmatrix} \frac{1}{2}\cos^2\theta & 0 & \frac{1}{2}\cos^2\theta \\ 0 & \sin^2\theta & 0 \\ \frac{1}{2}\cos^2\theta + \frac{1}{2} & 0 & \frac{1}{2}\cos^2\theta + \frac{1}{2} \end{pmatrix} = \frac{3}{2} = 2\frac{3}{4} + 0\frac{1}{4}.$$

The probabilities are independent od θ since $(1,0,1)^T$ and $(0,1,0)$ both lie in the eigenspace corresponding to the eigenvalue 2 of the observable. Thus the observable does not enable us to distinguish between $(1,0,1)^T$ and $(0,1,0)$ and so the ratio of these states in the mixture is irrelevant for this measurement.

3. We have

$$\begin{aligned} AA^* &= (\mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*)(\mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*)^* = (\mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*)(\mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*) \\ &= \mathbf{x}(\mathbf{y}^*\mathbf{x})\mathbf{y}^* + \mathbf{x}(\mathbf{y}^*\mathbf{y})\mathbf{x}^* + \mathbf{y}(\mathbf{x}^*\mathbf{x})\mathbf{y}^* + \mathbf{y}(\mathbf{x}^*\mathbf{y})\mathbf{x}^* \\ &= \mathbf{x}\mathbf{x}^* + \mathbf{y}\mathbf{y}^* = I_2. \end{aligned}$$

To see this note that $\{\mathbf{x}, \mathbf{y}\}$ forms an orthonormal basis for $\mathbb{C}^2$. Any vector $\mathbf{z} \in \mathbb{C}^2$ can then be written as $\mathbf{z} = z_x\mathbf{x} + z_y\mathbf{y}$ (where $z_x, z_y \in \mathbb{C}$) so that

$$(\mathbf{x}\mathbf{x}^* + \mathbf{y}\mathbf{y}^*)\,\mathbf{z} = \mathbf{x}(\mathbf{x}^*\mathbf{z}) + \mathbf{y}(\mathbf{y}^*\mathbf{z}) = \mathbf{x}(z_x\mathbf{x}^*\mathbf{x} + z_y\mathbf{x}^*\mathbf{y}) + \mathbf{y}(z_x\mathbf{y}^*\mathbf{x} + z_y\mathbf{y}^*\mathbf{y}) = z_x\mathbf{x} + z_y\mathbf{y} = I_2\mathbf{z}.$$

It is easy to see that $A^*A = I_2$. Thus A is unitary.

We have

$$A\mathbf{x} = (\mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*)\mathbf{x} = \mathbf{x}(\mathbf{y}^{\mathbf{x}}) + \mathbf{y}(\mathbf{x}^*\mathbf{x}) = \mathbf{y}$$

and

$$A\mathbf{x} = (\mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*)\mathbf{y} = \mathbf{x}(\mathbf{y}^{\mathbf{y}}) + \mathbf{y}(\mathbf{x}^*\mathbf{y}) = \mathbf{x}.$$

Thus

$$A\frac{1}{\sqrt{2}}(\mathbf{x} + \mathbf{y}) = \frac{1}{\sqrt{2}}(A\mathbf{x} + A\mathbf{y}) = \frac{1}{\sqrt{2}}(\mathbf{y} + \mathbf{x}) = \frac{1}{\sqrt{2}}(\mathbf{x} + \mathbf{y}).$$

Since

$$\frac{1}{\sqrt{2}}(\mathbf{x} + \mathbf{y})^* \frac{1}{\sqrt{2}}(\mathbf{x} + \mathbf{y}) = \frac{1}{2}(\mathbf{x}^*\mathbf{x} + \mathbf{x}^*\mathbf{y} + \mathbf{y}^*\mathbf{x} + \mathbf{y}^*\mathbf{y}) = 1 \neq 0$$

we conclude that

$$\frac{1}{\sqrt{2}}(\mathbf{x} + \mathbf{y}) \neq \mathbf{0}$$

is an eigenvector of A corresponding to the eigenvalue 1.

We have

$$\begin{aligned} BB^* &= (\mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*)(\mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*)^* = (\mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*)(\mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*) \\ &= \mathbf{u}(\mathbf{v}^*\mathbf{u})\mathbf{v}^* + \mathbf{u}(\mathbf{v}^*\mathbf{v})\mathbf{u}^* + \mathbf{v}(\mathbf{u}^*\mathbf{u})\mathbf{v}^* + \mathbf{v}(\mathbf{u}^*\mathbf{v})\mathbf{u}^* \\ &= \mathbf{u}\mathbf{u}^* + \mathbf{v}\mathbf{v}^*. \end{aligned}$$

Since $n > 2$ there exists $\mathbf{w} \in \mathbb{C}^n$ with $\mathbf{w} \neq \mathbf{0}$ and $\mathbf{u}^*\mathbf{w} = \mathbf{v}^*\mathbf{w} = 0$. Thus

$$(BB^*)\mathbf{w} = (\mathbf{u}\mathbf{u}^* + \mathbf{v}\mathbf{v}^*)\mathbf{w} = \mathbf{u}(\mathbf{u}^*\mathbf{w}) + \mathbf{v}(\mathbf{v}^*\mathbf{w}) = 0.$$

Thus $BB^* \neq I_n$ and B is not unitary.

Bonus:
We have (as above for A) that

$$\frac{1}{\sqrt{2}}(\mathbf{u} + \mathbf{v})$$

is an eigenvector of B corresponding to the eigenvalue 1.
The remaining eigenvalues are -1 and 0 ($n - 2$ times).

Since we work in the Hilbert space $\mathbb{C}^n$ there exist $\mathbf{w}_3, \mathbf{w}_4, \ldots, \mathbf{w}_n \in \mathbb{C}^n$ such that $\mathbf{w}_j^*\mathbf{w}_k = \delta_{jk}$ and $\mathbf{u}^*\mathbf{w}_j = \mathbf{v}^*\mathbf{w}_j = 0$ for all $j = 3, 4, \ldots, n$. Now

$$B\mathbf{w}_j = (\mathbf{u}\mathbf{u}^* + \mathbf{v}\mathbf{v}^*)\mathbf{w}_j = \mathbf{u}(\mathbf{u}^*\mathbf{w}_j) + \mathbf{v}(\mathbf{v}^*\mathbf{w}_j) = \mathbf{0} = 0\mathbf{w_j}$$

for all $j = 3, 4, \ldots, n$. Thus we have $n - 2$ linearly independent eigenvectors corresponding to the eigenvalue 0.

From

$$B\frac{1}{\sqrt{2}}(\mathbf{u} - \mathbf{v}) = \frac{1}{\sqrt{2}}(B\mathbf{u} - B\mathbf{v}) = \frac{1}{\sqrt{2}}(\mathbf{v} - \mathbf{u}) = -\frac{1}{\sqrt{2}}(\mathbf{u} - \mathbf{v})$$

we find the eigenvector corresponding to the eigenvalue -1. It is important to verify that

$$\frac{1}{\sqrt{2}}(\mathbf{u} - \mathbf{v})^*\frac{1}{\sqrt{2}}(\mathbf{u} - \mathbf{v}) = \frac{1}{2}(\mathbf{u}^*\mathbf{u} - \mathbf{u}^*\mathbf{v} - \mathbf{v}^*\mathbf{u} + \mathbf{v}^*\mathbf{v}) = 1 \neq 0.$$

Thus we have found eigenvectors corresponding to each eigenvalue.