Applied Mathematics 3B

Semester Test #1 APS **Solution**

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1. The solution is given by

$$\boldsymbol{\psi}(t) = \exp\left[i\omega t \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right] \boldsymbol{\psi}(0).$$

Let

$$A := i\omega t \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We can use the eigenvalues and eigenvectors of A. The eigenvalues and corresponding representative orthonormal eigenvectors of A are

$$2i\omega t: \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \frac{1+\sqrt{5}}{2} i\omega t: \quad \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} \begin{pmatrix} 1 \\ 0 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}, \qquad \frac{1-\sqrt{5}}{2} i\omega t: \quad \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \begin{pmatrix} 1 \\ 0 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

Since A is normal we have

$$A = 2i\omega t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^* + \frac{1+\sqrt{5}}{2} i\omega t \frac{2}{5+\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}^* + \frac{1-\sqrt{5}}{2} i\omega t \frac{2}{5-\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}^*$$

and

$$\begin{aligned}
e^A &= e^{2i\omega t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^* + e^{\frac{1+\sqrt{5}}{2} i\omega t} \frac{2}{5+\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}^* + e^{\frac{1-\sqrt{5}}{2} i\omega t} \frac{2}{5-\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}^* \\
&= e^{\frac{i\omega t}{2}} \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{\frac{3}{i\omega t}2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + e^{\frac{i\omega t\sqrt{5}}{2}} \frac{2}{5+\sqrt{5}} \begin{pmatrix} 1 & 0 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 & 0 \\ \frac{1+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} \end{pmatrix} + e^{-\frac{i\omega t\sqrt{5}}{2}} \frac{2}{5-\sqrt{5}} \begin{pmatrix} 1 & 0 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 & 0 \\ \frac{1-\sqrt{5}}{2} & 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix} \right] \\
&= e^{\frac{i\omega t}{2}} \begin{pmatrix} \cos\frac{\sqrt{5}\omega t}{2} - \frac{i}{\sqrt{5}} \sin\frac{\sqrt{5}\omega t}{2} & 0 & \frac{2i}{\sqrt{5}} \sin\frac{\sqrt{5}\omega t}{2} \\ 0 & e^{\frac{3i\omega t}{2}} & 0 \\ \frac{2i}{\sqrt{5}} \sin\frac{\sqrt{5}\omega t}{2} & 0 & \cos\frac{\sqrt{5}\omega t}{2} + \frac{i}{\sqrt{5}} \sin\frac{\sqrt{5}\omega t}{2} \end{pmatrix}
\end{aligned}$$

where we used $5 \pm \sqrt{5} \equiv \sqrt{5}(\sqrt{5} \pm 1)$ and

$$\frac{e^{ix}}{5+\sqrt{5}} + \frac{e^{-ix}}{5-\sqrt{5}} = \frac{e^{ix}(5-\sqrt{5})}{20} + \frac{e^{-ix}(5+\sqrt{5})}{20} = \frac{1}{2}\cos x - \frac{i}{2\sqrt{5}} \sin x$$

and other similar identities. Thus

$$\boldsymbol{\psi}\left(t = \frac{\pi}{2\omega}\right) = \begin{pmatrix} e^{\frac{i\pi}{4}} \cos\frac{\sqrt{5}\pi}{4} - \frac{i}{\sqrt{5}} e^{\frac{i\pi}{4}} \sin\frac{\sqrt{5}\pi}{4} & 0 & \frac{2i}{\sqrt{5}} e^{\frac{i\pi}{4}} \sin\frac{\sqrt{5}\pi}{4} \\ 0 & -1 & 0 \\ \frac{2i}{\sqrt{5}} e^{\frac{i\pi}{4}} \sin\frac{\sqrt{5}\pi}{4} & 0 & e^{\frac{i\pi}{4}} \cos\frac{\sqrt{5}\pi}{4} + \frac{i}{\sqrt{5}} e^{\frac{i\pi}{4}} \sin\frac{\sqrt{5}\pi}{4} \end{pmatrix} \boldsymbol{\psi}(0).$$

For the initial value problems we find

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\psi}\left(t = \frac{\pi}{2\omega}\right) = \begin{pmatrix} e^{\frac{i\pi}{4}} \cos\frac{\sqrt{5}\pi}{4} + \frac{i}{\sqrt{5}} e^{\frac{i\pi}{4}} \sin\frac{\sqrt{5}\pi}{4} \\ 0 \\ e^{\frac{i\pi}{4}} \cos\frac{\sqrt{5}\pi}{4} + \frac{3i}{\sqrt{5}} e^{\frac{i\pi}{4}} \sin\frac{\sqrt{5}\pi}{4} \end{pmatrix}.$$

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\psi}\left(t = \frac{\pi}{2\omega}\right) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\boldsymbol{\psi}(t=0).$$

$$\psi(t=0)=\begin{pmatrix}1\\1\\1\end{pmatrix}\qquad\Rightarrow\qquad\psi\left(t=\frac{\pi}{2\omega}\right)=\begin{pmatrix}e^{\frac{i\pi}{4}}\cos\frac{\sqrt{5}\pi}{4}+\frac{i}{\sqrt{5}}e^{\frac{i\pi}{4}}\sin\frac{\sqrt{5}\pi}{4}\\-1\\e^{\frac{i\pi}{4}}\cos\frac{\sqrt{5}\pi}{4}+\frac{3i}{\sqrt{5}}e^{\frac{i\pi}{4}}\sin\frac{\sqrt{5}\pi}{4}\end{pmatrix}.$$

2.

(a) First note that

$$\langle A\rangle_\rho\overline{\mathrm{tr}(\rho A)}=\overline{\mathrm{tr}(\rho a)^T}=\overline{\mathrm{tr}A^T\rho^T}=\mathrm{tr}(\overline{A^T\rho^T})=\mathrm{tr}(\overline{A^T\rho^T})=\mathrm{tr}(A^*\rho^*)=\mathrm{tr}(A\rho)=\mathrm{tr}(\rho A)=\langle A\rangle_\rho.$$

Now

$$[A(A-\langle A\rangle_\rho I_n)]^*=(A-\langle A\rangle_\rho I_n)^*A^*=(A^*-\overline{\langle A\rangle_\rho}I_n)A^*=(A-\langle A\rangle_\rho I_n)A=A^2-\langle A\rangle_\rho A=A(A-\langle A\rangle_\rho I_n).$$

Thus $A(A-\langle A\rangle_\rho I_n)$ is an observable. The expectation value of this observable with respect to ρ is the variance of A with respect to ρ.

(b) We have

$$\begin{aligned}\langle A^2\rangle_\rho-\langle A\rangle_\rho^2&=\mathrm{tr}(\rho A^2)-\mathrm{tr}(\rho A)\mathrm{tr}(\rho A)=\mathrm{tr}(\rho A^2-\mathrm{tr}(\rho A)\rho A)=\mathrm{tr}(\rho[A^2-\mathrm{tr}(\rho A)A])\\&=\mathrm{tr}(\rho[A(A-\mathrm{tr}(\rho A)I_n)])=\mathrm{tr}(\rho[A(A-\langle A\rangle_\rho I_n)])=\langle A(A-\langle A\rangle_\rho I_n)\rangle_\rho.\end{aligned}$$

Thus the expectation value of the observable in (a) with respect to ρ is the variance of A with respect to ρ.

(c) We find

$$\rho=\begin{pmatrix}\frac14\cos^2\theta&0&\frac14\cos^2\theta\\0&\frac12\sin^2\theta&0\\\frac14\cos^2\theta&0&\frac14\cos^2\theta+\frac12\end{pmatrix}.$$

It follows that

$$\mathrm{tr}(\rho A)=\mathrm{tr}\left[\begin{pmatrix}\frac14\cos^2\theta&0&\frac14\cos^2\theta\\0&\frac12\sin^2\theta&0\\\frac14\cos^2\theta&0&\frac14\cos^2\theta+\frac12\end{pmatrix}\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}\right]=\mathrm{tr}\begin{pmatrix}\frac12\cos^2\theta&0&\frac12\cos^2\theta\\0&\sin^2\theta&0\\\frac12\cos^2\theta+\frac12&0&\frac12\cos^2\theta+\frac12\end{pmatrix}=\frac32$$

and

$$\mathrm{tr}(\rho A^2)=\mathrm{tr}\left[\begin{pmatrix}\frac14\cos^2\theta&0&\frac14\cos^2\theta\\0&\frac12\sin^2\theta&0\\\frac14\cos^2\theta&0&\frac14\cos^2\theta+\frac12\end{pmatrix}\begin{pmatrix}2&0&2\\0&4&0\\2&0&2\end{pmatrix}\right]=\mathrm{tr}\begin{pmatrix}\cos^2\theta&0&\cos^2\theta\\0&2\sin^2\theta&0\\\cos^2\theta+1&0&\cos^2\theta+1\end{pmatrix}=3.$$

Thus

$$\langle A^2\rangle_\rho-\langle A\rangle_\rho^2=3-\frac94=\frac34.$$

3. Let $A:=\mathbf{x}\mathbf{u}^*+\mathbf{y}\mathbf{v}^*$. We have

$$\begin{aligned}AA^*&=(\mathbf{x}\mathbf{u}^*+\mathbf{y}\mathbf{v}^*)(\mathbf{x}\mathbf{u}^*+\mathbf{y}\mathbf{v}^*)^*=(\mathbf{x}\mathbf{u}^*+\mathbf{y}\mathbf{v}^*)(\mathbf{u}\mathbf{x}^*+\mathbf{v}\mathbf{y}^*)\\&=\mathbf{x}(\mathbf{u}^*\mathbf{u})\mathbf{x}^*+\mathbf{x}(\mathbf{u}^*\mathbf{v})\mathbf{y}^*+\mathbf{y}(\mathbf{v}^*\mathbf{u})\mathbf{x}^*+\mathbf{y}(\mathbf{v}^*\mathbf{v})\mathbf{y}^*\\&=\mathbf{x}\mathbf{x}^*+\mathbf{y}\mathbf{y}^*=I_2.\end{aligned}$$

To see this note that $\{\mathbf{x},\mathbf{y}\}$ forms an orthonormal basis for $\mathbb{C}^2$. Any vector $\mathbf{z}\in\mathbb{C}^2$ can then be written as $\mathbf{z}=z_x\mathbf{x}+z_y\mathbf{y}$ (where $z_x,z_y\in\mathbb{C}$) so that

$$(\mathbf{x}\mathbf{x}^*+\mathbf{y}\mathbf{y}^*)\,\mathbf{z}=\mathbf{x}(\mathbf{x}^*\mathbf{z})+\mathbf{y}(\mathbf{y}^*\mathbf{z})=\mathbf{x}(z_x\mathbf{x}^*\mathbf{x}+z_y\mathbf{x}^*\mathbf{y})+\mathbf{y}(z_x\mathbf{y}^*\mathbf{x}+z_y\mathbf{y}^*\mathbf{y})=z_x\mathbf{x}+z_y\mathbf{y}=I_2\mathbf{z}.$$

It is easy to see that $A^*A=I_2$. Thus A is unitary.

Since we work in the Hilbert space $\mathbb{C}^n$ there exist $\mathbf{w}_2,\mathbf{w}_3,\ldots,\mathbf{w}_n\in\mathbb{C}^n$ such that $\mathbf{w}_j^*\mathbf{w}_k=\delta_{jk}$ and $\mathbf{b}^*\mathbf{w}_j=0$ for all $j=2,4,\ldots,n$. Now any eigenvector $\mathbf{z}$ of $\mathbf{a}\mathbf{b}^*$ can be written in the form

$$\mathbf{z}=\alpha_b\mathbf{b}+\sum_{j=2}^n\alpha_j\mathbf{w}_j$$

where $\alpha_b, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$. It follows that

$$(\mathbf{ab}^*)\mathbf{z} = \alpha_b \mathbf{ab}^*\mathbf{b} + \sum_{j=2}^{n} \alpha_j \mathbf{ab}^*\mathbf{w}_j = \alpha_b \mathbf{a}.$$

Suppose λ an eigenvalue of $\mathbf{ab}^*$ corresponding to $\mathbf{z}$. Then

$$(\mathbf{ab}^*)\mathbf{z} = \alpha_b \mathbf{a} = \lambda \mathbf{b}.$$

Multiplying on the left by $\mathbf{a}^*$ yields

$$\alpha_b = \alpha_b \mathbf{a}^*\mathbf{a} = \lambda \mathbf{a}^*\mathbf{b} = 0$$

and multiplying on the left by $\mathbf{b}^*$ yields

$$0 = \alpha_b \mathbf{b}^*\mathbf{a} = \lambda \mathbf{b}^*\mathbf{b} = \lambda.$$

The only eigenvalue is 0, with corresponding eigenvectors

$$\mathbf{z} = \sum_{j=2}^{n} \alpha_j \mathbf{w}_j$$

where $(\alpha_2, \ldots, \alpha_n) \in \mathbb{C}^{n-1} \setminus (0, 0, \ldots, 0)$.