UNIVERSITY OF JOHANNESBURG

APPLIED MATHEMATICS 3B

Semester Test: 23 August 2011

Duration: 80 minutes **Marks:** 30

Instructions: Answer all the questions
All calculations must be shown
Pocket calculators are permitted
All angles are measured in radians
The prescribed text book is allowed

Question 1

Solve the differential equation (initial value problem)

$$\frac{d\boldsymbol{\psi}}{dt} = i\omega \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \boldsymbol{\psi}$$

for $\boldsymbol{\psi}(t) \in \mathbb{C}^3$. Find $\boldsymbol{\psi}(t = \frac{\pi}{2\omega})$. Consider the cases

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{\psi}(t=0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{\psi}(t=0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(10)

Question 2

Consider the Hilbert space $\mathbb{C}^3$ and the observable described by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Determine

(a) the measurement outcomes

(b) associated probabilities

(c) expectation value

when observing a system described by the density matrix

$$\rho := \frac{1}{4}\cos^2\theta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}^* + \frac{1}{2}\sin^2\theta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^* + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}^*, \qquad \theta \in \mathbb{R}$$

Do the associated probabilities depend on θ ? Explain. **(10)**

Question 3 follows

Question 3

Let $\mathbf{x}, \mathbf{y} \in \mathbf{C}^2$ with $\mathbf{x}^*\mathbf{y} = 0$ and $\mathbf{x}^*\mathbf{x} = \mathbf{y}^*\mathbf{y} = 1$. Is the matrix

$$A := \mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*$$

unitary? Prove or disprove.

Calculate $A\mathbf{x}$ and $A\mathbf{y}$. Calculate

$$A\,\frac{1}{\sqrt{2}}(\mathbf{x} + \mathbf{y}).$$

What can you conclude?

Let $n \in \mathbb{N}$ with $n > 2$. Let $\mathbf{u}, \mathbf{v} \in \mathbf{C}^n$ with $\mathbf{u}^*\mathbf{v} = 0$ and $\mathbf{u}^*\mathbf{u} = \mathbf{v}^*\mathbf{v} = 1$. Is the matrix

$$B := \mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*$$

unitary? Prove or disprove.

Bonus 3 marks:

Find all the eigenvalues of B (and the corresponding multiplicities).
All answers must include a proof.

(10)

END OF QUESTION PAPER