Applied Mathematics APM3B10

Exam: November 2011 **Solution**

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1. We have

$$\rho(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$\rho(t) = e^{-i\hat{H}t/\hbar}\rho(0)e^{i\hat{H}t/\hbar}.$$

Since $-i\hat{H}t/\hbar = -i\omega t\sigma_y$ we find

$$(-i\hat{H}t/\hbar)^j = \begin{cases} (-i\omega t)^j\sigma_y & \text{if } j \text{ is odd, i.e. } j = 2k+1 \\ (-i\omega t)^j I_2 & \text{if } j \text{ is even, i.e. } j = 2k \end{cases}$$

where $k \in \mathbb{N}_0$. It follows that

$$\begin{aligned} e^{-i\hat{H}t/\hbar} &= \sum_{k=0}^{\infty}\frac{(-i\omega t)^{2k+1}}{(2k+1)!}\sigma_y + \sum_{k=0}^{\infty}\frac{(-i\omega t)^{2k}}{(2k)!}I_2 \\ &= -i\sum_{k=0}^{\infty}\frac{(-1)^k(\omega t)^{2k+1}}{(2k+1)!}\sigma_y + \sum_{k=0}^{\infty}\frac{(-1)^k(\omega t)^{2k}}{(2k)!}I_2 \\ &= -i\sin\omega t\,\sigma_y + \cos\omega t\, I_2 = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}. \end{aligned}$$

It follows that

$$e^{i\hat{H}t/\hbar} = \left(e^{-i\hat{H}t/\hbar}\right)^* = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Finally

$$\begin{aligned} \rho(t) &= \frac{1}{2}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\omega t - \sin\omega t & \cos\omega t - \sin\omega t \\ \cos\omega t + \sin\omega t & \cos\omega t + \sin\omega t \end{pmatrix}\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 - \sin 2\omega t & \cos 2\omega t \\ \cos 2\omega t & 1 + \sin 2\omega t \end{pmatrix}. \end{aligned}$$

2. We begin with the state

$$|\psi\rangle := \frac{1}{2}(|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle - |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle).$$

Expanding the second and third qubits in the Bell basis yields

$$\begin{aligned} |\psi\rangle = & \frac{1}{2\sqrt{2}}|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \otimes |0\rangle - \frac{1}{2\sqrt{2}}|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \otimes |0\rangle \\ & + \frac{1}{2\sqrt{2}}|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \otimes |1\rangle - \frac{1}{2\sqrt{2}}|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle) \otimes |1\rangle \\ & - \frac{1}{2\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \otimes |0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle) \otimes |0\rangle \\ & - \frac{1}{2\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \otimes |1\rangle - \frac{1}{2\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \otimes |1\rangle. \end{aligned}$$

Now

$$
\begin{aligned}
(I_2 \otimes U_H \otimes I_2 \otimes I_2)(I_2 \otimes U_{CNOT} \otimes I_2)|\psi\rangle \;=\; & \frac{1}{2\sqrt{2}}|0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle - \frac{1}{2\sqrt{2}}|0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \\
& + \frac{1}{2\sqrt{2}}|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle - \frac{1}{2\sqrt{2}}|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \\
& - \frac{1}{2\sqrt{2}}|1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \\
& - \frac{1}{2\sqrt{2}}|1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle - \frac{1}{2\sqrt{2}}|1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle.
\end{aligned}
$$

Performing a measurement on the second and third qubit which can distinguish between $|0\rangle$ and $|1\rangle|$ (for example a measurement described by the observable $|0\rangle\langle 0|$) yields the states

State determined by measurement	Resulting state
$\|0\rangle \otimes \|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle \otimes \|0\rangle \otimes \|0\rangle \otimes \|1\rangle - \|1\rangle \otimes \|0\rangle \otimes \|0\rangle \otimes \|0\rangle)$
$\|0\rangle \otimes \|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle \otimes \|0\rangle \otimes \|1\rangle \otimes \|0\rangle - \|1\rangle \otimes \|0\rangle \otimes \|1\rangle \otimes \|1\rangle)$
$\|1\rangle \otimes \|0\rangle$	$-\frac{1}{\sqrt{2}}(\|0\rangle \otimes \|1\rangle \otimes \|0\rangle \otimes \|1\rangle\|1\rangle \otimes \|1\rangle \otimes \|0\rangle \otimes \|0\rangle)$
$\|1\rangle \otimes \|1\rangle$	$-\frac{1}{\sqrt{2}}(\|0\rangle \otimes \|1\rangle \otimes \|1\rangle \otimes \|0\rangle\|1\rangle \otimes \|1\rangle \otimes \|1\rangle \otimes \|1\rangle)$

It is convenent to omit the second and third qubits at this stage:

State determined by measurement of second and third qubit	Resulting state of first and fourth qubit	
$\|0\rangle \otimes \|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle \otimes \|1\rangle - \|1\rangle \otimes \|0\rangle)$	
$\|0\rangle \otimes \|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle \otimes \|0\rangle - \|1\rangle \otimes \|1\rangle)$	$I_2 \otimes \sigma_x$
$\|1\rangle \otimes \|0\rangle$	$\frac{1}{\sqrt{2}}(-\|0\rangle \otimes \|1\rangle - \|1\rangle \otimes \|0\rangle)$	$I_2 \otimes \sigma_z$
$\|1\rangle \otimes \|1\rangle$	$\frac{1}{\sqrt{2}}(-\|0\rangle \otimes \|0\rangle - \|1\rangle \otimes \|1\rangle)$	$I_2 \otimes (\sigma_z \sigma_x)$

The correction listed on the right hand side yields the original state

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

which is still entangled.

3. For

$$\frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

we have the coefficient matrix

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^*}_{V^*}.$$

We have two non-zero singular values, thus the state is entangled. Each term is obtained from the singular value and corresponding columns of U and V:

$$\frac{1}{2}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle + \frac{1}{2}\left(|0\rangle - |1\rangle\right) \otimes |1\rangle.$$

4. We have

$$
\begin{aligned}
(U_{QFT,4} \otimes I_4)|\psi_1\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle) \otimes |0\rangle + (U_{QFT,4}|1\rangle) \otimes |1\rangle + (U_{QFT,4}|2\rangle) \otimes |0\rangle + (U_{QFT,4}|3\rangle) \otimes |1\rangle) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right) \otimes |0\rangle + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right) \otimes |1\rangle\right. \\
&\quad \left. + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right) \otimes |0\rangle + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right) \otimes |1\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|1\rangle\right.\\
&\qquad\left.+\left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1+e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|0\rangle+e^{-i4\pi j/4}|0\rangle+e^{-i2\pi j/4}|1\rangle+e^{-i6\pi j/4}|1\rangle\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)+|2\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{0}\to\frac{4}{4}=1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2}=2$ which appears to be the correct underlying periodicity.