Applied Mathematics 3B

Assignment #8 **Solution**

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1. We find

$$\begin{aligned}
\sigma_x\sigma_y &= -i|0\rangle\langle 1|0\rangle\langle 1| + i|0\rangle\langle 1|1\rangle\langle 0| - i|1\rangle\langle 0|0\rangle\langle 1| + i|1\rangle\langle 0|1\rangle\langle 0| \\
&= i|0\rangle\langle 0| - i|1\rangle\langle 1| = i\sigma_z, \\
\sigma_y\sigma_x &= -i|0\rangle\langle 1|0\rangle\langle 1| - i|0\rangle\langle 1|1\rangle\langle 0| + i|1\rangle\langle 0|0\rangle\langle 1| + i|1\rangle\langle 0|1\rangle\langle 0| \\
&= -i|0\rangle\langle 0| + i|1\rangle\langle 1| = -i\sigma_z, \\
\sigma_x\sigma_z &= |0\rangle\langle 1|0\rangle\langle 0| - |0\rangle\langle 1|1\rangle\langle 1| + |1\rangle\langle 0|0\rangle\langle 0| - |1\rangle\langle 0|1\rangle\langle 1| \\
&= -|0\rangle\langle 1| + |1\rangle\langle 0| = -i\sigma_y, \\
\sigma_z\sigma_x &= |0\rangle\langle 0|0\rangle\langle 1| + |0\rangle\langle 0|1\rangle\langle 0| - |1\rangle\langle 1|0\rangle\langle 1| - |1\rangle\langle 1|1\rangle\langle 0| \\
&= |0\rangle\langle 1| - |1\rangle\langle 0| = i\sigma_y, \\
\sigma_y\sigma_z &= -i|0\rangle\langle 1|0\rangle\langle 0| + i|0\rangle\langle 1|1\rangle\langle 1| + i|1\rangle\langle 0|0\rangle\langle 0| - i|1\rangle\langle 0|1\rangle\langle 1| \\
&= i|0\rangle\langle 1| + i|1\rangle\langle 0| = i\sigma_x, \\
\sigma_z\sigma_y &= -i|0\rangle\langle 0|0\rangle\langle 1| + i|0\rangle\langle 0|1\rangle\langle 0| + i|1\rangle\langle 1|0\rangle\langle 1| - i|1\rangle\langle 1|1\rangle\langle 0| \\
&= -i|0\rangle\langle 1| - i|1\rangle\langle 0| = -i\sigma_x.
\end{aligned}$$

2. We begin with the state

$$|\psi\rangle := \frac{1}{2}(|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle + |0\rangle\otimes|0\rangle\otimes|1\rangle\otimes|1\rangle + |1\rangle\otimes|1\rangle\otimes|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\otimes|1\rangle\otimes|1\rangle).$$

(a) Expanding the second and third qubits in the Bell basis yields

$$\begin{aligned}
|\psi\rangle = {} & \frac{1}{2\sqrt{2}}|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle)\otimes|0\rangle + \frac{1}{2\sqrt{2}}|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle)\otimes|0\rangle \\
& + \frac{1}{2\sqrt{2}}|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle)\otimes|1\rangle + \frac{1}{2\sqrt{2}}|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle)\otimes|1\rangle \\
& + \frac{1}{2\sqrt{2}}|1\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle)\otimes|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle)\otimes|0\rangle \\
& + \frac{1}{2\sqrt{2}}|1\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle)\otimes|1\rangle - \frac{1}{2\sqrt{2}}|1\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle)\otimes|1\rangle.
\end{aligned}$$

(b) Note that

$$\begin{aligned}
(U_H\otimes I_2)U_{CNOT}\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle) &= |0\rangle\otimes|0\rangle, \\
(U_H\otimes I_2)U_{CNOT}\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle) &= |1\rangle\otimes|0\rangle, \\
(U_H\otimes I_2)U_{CNOT}\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle) &= |0\rangle\otimes|1\rangle, \\
(U_H\otimes I_2)U_{CNOT}\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle) &= |1\rangle\otimes|1\rangle.
\end{aligned}$$

Now

$$(I_2\otimes U_H\otimes I_2\otimes I_2)(I_2\otimes U_{CNOT}\otimes I_2)|\psi\rangle = \frac{1}{2\sqrt{2}}|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle + \frac{1}{2\sqrt{2}}|0\rangle\otimes|1\rangle\otimes|0\rangle\otimes|0\rangle$$

$$
\begin{aligned}
&+\frac{1}{2\sqrt{2}}|0\rangle\otimes|0\rangle\otimes|1\rangle\otimes|1\rangle+\frac{1}{2\sqrt{2}}|0\rangle\otimes|1\rangle\otimes|1\rangle\otimes|1\rangle\\
&+\frac{1}{2\sqrt{2}}|1\rangle\otimes|0\rangle\otimes|1\rangle\otimes|0\rangle-\frac{1}{2\sqrt{2}}|1\rangle\otimes|1\rangle\otimes|1\rangle\otimes|0\rangle\\
&+\frac{1}{2\sqrt{2}}|1\rangle\otimes|0\rangle\otimes|0\rangle\otimes|1\rangle-\frac{1}{2\sqrt{2}}|1\rangle\otimes|1\rangle\otimes|0\rangle\otimes|1\rangle.
\end{aligned}
$$

(c) Performing a measurement on the second and third qubit which can distinguish between $|0\rangle$ and $|1\rangle|$ (for example a measurement described by the observable $|0\rangle\langle 0|$) yields the states

State determined by measurement	Resulting state
$\|0\rangle\otimes\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|0\rangle\otimes\|0\rangle\otimes\|0\rangle+\|1\rangle\otimes\|0\rangle\otimes\|0\rangle\otimes\|1\rangle)$
$\|0\rangle\otimes\|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|0\rangle\otimes\|1\rangle\otimes\|1\rangle+\|1\rangle\otimes\|0\rangle\otimes\|1\rangle\otimes\|0\rangle)$
$\|1\rangle\otimes\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|1\rangle\otimes\|0\rangle\otimes\|0\rangle-\|1\rangle\otimes\|1\rangle\otimes\|0\rangle\otimes\|1\rangle)$
$\|1\rangle\otimes\|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|1\rangle\otimes\|1\rangle\otimes\|1\rangle-\|1\rangle\otimes\|1\rangle\otimes\|1\rangle\otimes\|0\rangle)$

(d) It is convenent to omit the second and third qubits at this stage:

State determined by measurement of second and third qubit	Resulting state of first and fourth qubit	
$\|0\rangle\otimes\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|0\rangle+\|1\rangle\otimes\|1\rangle)$	
$\|0\rangle\otimes\|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|1\rangle+\|1\rangle\otimes\|0\rangle)$	$I_2\otimes\sigma_x$
$\|1\rangle\otimes\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|0\rangle-\|1\rangle\otimes\|1\rangle)$	$I_2\otimes\sigma_z$
$\|1\rangle\otimes\|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle\otimes\|1\rangle-\|1\rangle\otimes\|0\rangle)$	$I_2\otimes(\sigma_z\sigma_x)$

The correction listed on the right hand side yields the original state

$$
\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle)
$$

which is still entangled.