Applied Mathematics 3B

Assignment #7 **Solution**

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1. We have

$$\begin{aligned}
\mathrm{tr}_1\rho &= \begin{pmatrix}1 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\rho\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\begin{pmatrix}0 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\rho\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & -1 & 0\\0 & -1 & 1 & 0\\1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & -1 & 0\\0 & -1 & 1 & 0\\1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 0\\1 & 0\\0 & 1\end{pmatrix}\\
&= \frac{1}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}_2\rho &= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\end{pmatrix}\rho\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 1\end{pmatrix}\rho\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & -1 & 0\\0 & -1 & 1 & 0\\1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\\0 & 1\\0 & 0\end{pmatrix}+\begin{pmatrix}0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & -1 & 0\\0 & -1 & 1 & 0\\1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 0\\0 & 0\\0 & 1\end{pmatrix}\\
&= \frac{1}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.
\end{aligned}$$

2. The matrix

$$\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

has eigenvalues 1 and 0. The von Neumann entropy is

$$S\left(\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right)=-1\log_2 1-0\log_2 0=0.$$

The matrix

$$\frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

has eigenvalues $\frac{1}{2}$ and $\frac{1}{2}$. The von Neumann entropy is

$$S\left(\frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)=-\frac{1}{2}\log_2\frac{1}{2}-\frac{1}{2}\log_2\frac{1}{2}=1.$$

3. Straightforward matrix multiplication yields

$$(\Sigma V^*)_{j,k}=\sum_{l=1}^{n}(\Sigma)_{j,l}(V^*)_{l,k}=\sum_{l=1}^{n}(\Sigma)_{j,l}\overline{(V)_{k,l}}=\sigma_j\overline{(V)_{k,j}}.$$

and

$$s_{j+1,k+1} = (U\Sigma V^*)_{j,k} = \sum_{l=1}^{m} (U)_{j,l} (\Sigma V^*)_{l,k} = \sum_{l=1}^{m} (U)_{j,l} \sigma_l \overline{(V)_{k,l}}.$$

Now

$$\begin{aligned}
|\psi\rangle &= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} s_{j,k} |j\rangle_A \otimes |k\rangle_B \\
&= \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{l=1}^{m} (U)_{j,l} \sigma_l \overline{(V)_{k,l}} |j-1\rangle_A \otimes |k-1\rangle_B \\
&= \sum_{l=1}^{m} \sigma_l \left(\sum_{j=1}^{m} (U)_{j,l} |j-1\rangle_A \right) \otimes \left(\sum_{k=1}^{n} \overline{(V)_{k,l}} |k-1\rangle_B \right) \\
&= \sum_{l=1}^{r} \sigma_l \left(\sum_{j=1}^{m} (U)_{j,l} |j-1\rangle_A \right) \otimes \left(\sum_{k=1}^{n} \overline{(V)_{k,l}} |k-1\rangle_B \right).
\end{aligned}$$

For

$$|\psi\rangle = a|0\rangle \otimes |0\rangle + b|0\rangle \otimes |1\rangle + c|1\rangle \otimes |0\rangle + d|1\rangle \otimes |1\rangle)$$

we have

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

so for

$$|\psi\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

we find

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^*$$

We have $\sigma_1 = 1$ and $\sigma_2 = 0$. The singular value σ_1 corresponds to the first column of U and V respectively so that

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

The general case would proceed as follows

$$S = \underbrace{\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}^*}_{V}$$

with

$$|\psi\rangle = \sigma_1(u_{11}|0\rangle + u_{21}|1\rangle) \otimes (\overline{v_{11}}|0\rangle + \overline{v_{21}}|1\rangle) + \sigma_2(u_{12}|0\rangle + u_{22}|1\rangle) \otimes (\overline{v_{12}}|0\rangle + \overline{v_{22}}|1\rangle).$$