UNIVERSITY OF JOHANNESBURG

Applied Mathematics 3B

Assignment #7 **7:30, 27 September 2011**

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1. Consider the Hilbert space $\mathbb{C}^4 \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$ and the density matrix

$$\rho := \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate the partial traces $\mathrm{tr}_1\rho$ and $\mathrm{tr}_2\rho$ where the first system is $\mathbb{C}^2$ and the second system is $\mathbb{C}^2$.

2. Find the von Neumann entropy for the density matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and for the matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Let $\mathcal{H}_A$ be a finite dimensional Hilbert space with orthonormal basis $\{|0\rangle_A, \ldots, |m-1\rangle_A\}$ and $\mathcal{H}_B$ be a finite dimensional Hilbert space with orthonormal basis $\{|0\rangle_B, \ldots, |n-1\rangle_B\}$. Here $m = \dim \mathcal{H}_A$ and $n = \dim \mathcal{H}_B$.

Any state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written in the form

$$|\psi\rangle = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} s_{j,k}|j\rangle_A \otimes |k\rangle_B$$

where $s_{j,k} \in \mathbb{C}$. Let S be the matrix

$$S := \begin{pmatrix} s_{0,0} & s_{0,2} & \cdots & s_{0,n} \\ s_{1,0} & s_{1,2} & \cdots & s_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m-1,0} & s_{m-1,2} & \cdots & s_{m-1,n} \end{pmatrix}.$$

Now let $S = U\Sigma V^*$ be the singular value decomposition of S. where U is an $m \times m$ unitary matrix, V is an $n \times n$ unitary matrix and Σ is an $m \times n$ "diagonal" matrix with the non-zero singular values $\sigma_1, \ldots, \sigma_r$ on the diagonal. Show that

$$|\psi\rangle = \sum_{l=1}^{r}\sigma_l\left(\sum_{j=1}^{m}(U)_{j,l}|j-1\rangle_A\right)\otimes\left(\sum_{k=1}^{n}\overline{(V)_{k,l}}|k-1\rangle_B\right).$$

Illustrate this result with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and

$$|\psi\rangle = \frac{1}{2}(|0\rangle\otimes|0\rangle + |0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle).$$
