Applied Mathematics 3B

Assignment #6 **Solution**

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1. The table for f_1 is

x	$f_1(x)$
0	1
1	1

We see that from the right hand column we cannot reproduce the left hand column. The function f_1 is obviously not invertible, thus we need one qubit to store the input (domain) and one qubit for the output (range). A unitary operator which implements f_1 is

$$\begin{aligned} U_1 &= |0\rangle\langle 0| \otimes |f_1(0)\rangle\langle 0| + |1\rangle\langle 1| \otimes |f_1(1)\rangle\langle 0| \\ &+ |0\rangle\langle 0| \otimes |\overline{f_1(0)}\rangle\langle 1| + |1\rangle\langle 1| \otimes |\overline{f_1(1)}\rangle\langle 1| \\ &= |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 0| + |0\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 1| \end{aligned}$$

which can also be written as

$$U_{NOT} = |0\rangle\langle 0| \otimes U_{NOT} + |1\rangle\langle 1| \otimes I_2$$

where

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

Other implementations are, of course, possible.

The table for f_{AND} is

x	y	$f_1(x)$
0	0	0
0	1	0
1	0	0
1	1	1

We see that from the right hand column we cannot reproduce the two left columns. The function f_{AND} is obviously not invertible, thus we need two qubits to store the input (domain) and one qubit for the output (range). A unitary operator which implements f_{AND} is

$$\begin{aligned} U_{AND} &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |f_{AND}(0,0)\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |f_{AND}(0,1)\rangle\langle 0| \\ &+ |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |f_{AND}(1,0)\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |f_{AND}(1,1)\rangle\langle 0| \\ &+ |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |\overline{f_{AND}(0,0)}\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |\overline{f_{AND}(0,1)}\rangle\langle 1| \\ &+ |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |\overline{f_{AND}(1,0)}\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |\overline{f_{AND}(1,1)}\rangle\langle 1| \\ &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 0| \\ &+ |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 1| \end{aligned}$$

which can also be written as

$$U_{AND} = (|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|) \otimes I_2 + (|1\rangle\langle 1| \otimes |1\rangle\langle 1|) \otimes U_{NOT}.$$

Other implementations are, of course, possible.

2. First we find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 10 \end{pmatrix}.$$

Since the rank of this matrix is 2, one of the eigenvalues must be zero. The characteristic equation is

$$\det\begin{pmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-10 \end{pmatrix} = \lambda^3 - 12\lambda^2 + 18\lambda = 0$$

with solutions $\lambda = 0$, $\lambda = 6 \pm 3\sqrt{2}$. For $\lambda = 0$ we find the eigenvectors $(x, y, z)^T$ from

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e. $z = 0$ and $x = -y$. Thus a representative normalized eigenvector is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = 6 \pm 3\sqrt{2}$ we find the eigenvectors $(x, y, z)^T$ from

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (6 \pm 3\sqrt{2})\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e.

$$\begin{aligned} x + y + z &= (6 \pm 3\sqrt{2})x \\ x + y + z &= (6 \pm 3\sqrt{2})y \\ x + y + 10z &= (6 \pm 3\sqrt{2})z \end{aligned}$$

The first two equations give $x = y$. Thus

$$\begin{aligned} 2x + z &= (6 \pm 3\sqrt{2})x \\ 2x + 10z &= (6 \pm 3\sqrt{2})z \end{aligned}$$

$$9z = (6 \pm 3\sqrt{2})(z - x) \quad \Rightarrow \quad (3 \mp 3\sqrt{2})z = (-6 \mp 3\sqrt{2})x \quad \Rightarrow \quad -9z = (3 \pm 3\sqrt{2})(-6 \mp 3\sqrt{2})x = (-36 \mp 27\sqrt{2})$$

so that the respective representative normalized eigenvectors are

$$\frac{1}{\sqrt{36 \pm 24\sqrt{2}}}\begin{pmatrix} 1 \\ 1 \\ 4 \pm 3\sqrt{2} \end{pmatrix}.$$

The columns of U are given by

$$\frac{1}{\sqrt{6 \pm 3\sqrt{2}}}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}\frac{1}{\sqrt{36 \pm 24\sqrt{2}}}\begin{pmatrix} 1 \\ 1 \\ 4 \pm 3\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{6 \pm 3\sqrt{2}}\sqrt{36 \pm 24\sqrt{2}}}\begin{pmatrix} 6 \pm 3\sqrt{2} \\ 12 \pm 9\sqrt{2} \end{pmatrix} = \frac{\sqrt{6 \pm 3\sqrt{2}}}{\sqrt{36 \pm 24\sqrt{2}}}\begin{pmatrix} 1 \\ 1 \pm \sqrt{2} \end{pmatrix}.$$

The singular value decomposition is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} =$$

$$\underbrace{\begin{pmatrix} \frac{\sqrt{6+3\sqrt{2}}}{\sqrt{36+24\sqrt{2}}} & \frac{\sqrt{6-3\sqrt{2}}}{\sqrt{36-24\sqrt{2}}} \\ \frac{(1+\sqrt{2})\sqrt{6+3\sqrt{2}}}{\sqrt{36+24\sqrt{2}}} & \frac{(1-\sqrt{2})\sqrt{6-3\sqrt{2}}}{\sqrt{36-24\sqrt{2}}} \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} \sqrt{6+3\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{6-3\sqrt{2}} & 0 \end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix} \frac{1}{\sqrt{36+24\sqrt{2}}} & \frac{1}{\sqrt{36-24\sqrt{2}}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{36+24\sqrt{2}}} & \frac{1}{\sqrt{36-24\sqrt{2}}} & -\frac{1}{\sqrt{2}} \\ \frac{4+3\sqrt{2}}{\sqrt{36+24\sqrt{2}}} & \frac{4-3\sqrt{2}}{\sqrt{36-24\sqrt{2}}} & 0 \end{pmatrix}^*}_{V^*}.$$

By taking the complex conjugate transpose we find the singular value decomposition

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{\sqrt{36+24\sqrt{2}}} & \frac{1}{\sqrt{36-24\sqrt{2}}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{36+24\sqrt{2}}} & \frac{1}{\sqrt{36-24\sqrt{2}}} & -\frac{1}{\sqrt{2}} \\ \frac{4+3\sqrt{2}}{\sqrt{36+24\sqrt{2}}} & \frac{4-3\sqrt{2}}{\sqrt{36-24\sqrt{2}}} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{6+3\sqrt{2}} & 0 \\ 0 & \sqrt{6-3\sqrt{2}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{6+3\sqrt{2}}}{\sqrt{36+24\sqrt{2}}} & \frac{\sqrt{6-3\sqrt{2}}}{\sqrt{36-24\sqrt{2}}} \\ \frac{(1+\sqrt{2})\sqrt{6+3\sqrt{2}}}{\sqrt{36+24\sqrt{2}}} & \frac{(1-\sqrt{2})\sqrt{6-3\sqrt{2}}}{\sqrt{36-24\sqrt{2}}} \end{pmatrix}^*.$$

3. From the eigenvalue equation $U_{CNOT}|\psi\rangle = \lambda|\psi\rangle$ we find

$$\begin{aligned} U_{CNOT}|\psi\rangle &= aU_{CNOT}(|0\rangle \otimes |0\rangle) + bU_{CNOT}(|0\rangle \otimes |1\rangle) + cU_{CNOT}(|1\rangle \otimes |0\rangle) + dU_{CNOT}(|1\rangle \otimes |1\rangle) \\ &= a|0\rangle \otimes |0\rangle + b|0\rangle \otimes |1\rangle + c|1\rangle \otimes |1\rangle + d|1\rangle \otimes |0\rangle \\ &= \lambda a|0\rangle \otimes |0\rangle + \lambda b|0\rangle \otimes |1\rangle + \lambda c|1\rangle \otimes |0\rangle + \lambda d|1\rangle \otimes |1\rangle. \end{aligned}$$

Since $\{|0\rangle, |1\rangle\}$ is an orthonormal basis in $\mathbb{C}^2$ $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$ is an orthonormal basis in $\mathbb{C}^4$. Consequently we can compare coefficients in the above equation to obtain

$$a = \lambda a, \qquad b = \lambda b, \qquad c = \lambda d, \qquad d = \lambda c.$$

The last two equations provide

$$c = \lambda^2 c, \qquad d = \lambda^2 d.$$

Eigenvectors cannot be zero, i.e. we need to consider 4 possibilities:

1. $a \neq 0 \Rightarrow \lambda = 1$
2. $b \neq 0 \Rightarrow \lambda = 1$
3. $c \neq 0 \Rightarrow \lambda^2 = 1$
4. $d \neq 0 \Rightarrow \lambda^2 = 1$

Thus $\lambda = 1$ or $\lambda = -1$. For $\lambda = 1$ we obtain the eigenvectors

$$a|0\rangle \otimes |0\rangle + b|0\rangle \otimes |1\rangle + c|1\rangle \otimes |0\rangle + c|1\rangle \otimes |1\rangle = |0\rangle \otimes (a|0\rangle + b|1\rangle) + c|1\rangle \otimes (|0\rangle + |1\rangle)$$

where $(a, b, c) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$. For $\lambda = -1$ we obtain the eigenvectors

$$c|1\rangle \otimes |0\rangle - c|1\rangle \otimes |1\rangle = c|1\rangle \otimes (|0\rangle - |1\rangle)$$

where $c \in \mathbb{C} \setminus \{0\}$.