Applied Mathematics 3B

Assignment #5 **Solution**

1.

(a) $\mathbf{x} \otimes I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) $I_2 \otimes \mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) $\mathbf{x} \otimes \mathbf{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (1 \quad 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{x}\mathbf{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \quad 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(d) $\mathbf{x}^* \otimes \mathbf{x} = (1 \quad 0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{x}^*\mathbf{x} = (1 \quad 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

2. We find

$$I_2 \otimes \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus $I_2 \otimes \sigma_x$ describes an observable. The characteristic equation is

$$\begin{aligned}
\det\left(\lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right) &= \det \begin{pmatrix} \lambda & -1 & 0 & 0 \\ -1 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & -1 & \lambda \end{pmatrix} \\
&= \lambda \det \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} + \det \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} \\
&= \lambda(\lambda^3 - \lambda) + (-\lambda^2 + 1) \\
&= \lambda^2(\lambda^2 - 1) - (\lambda^2 - 1) = (\lambda^2 - 1)^2 = 0.
\end{aligned}$$

The measurement outcomes are 1 (twice) and -1 (twice). The eigenstates corresponding to the eigenvalue 1 are given by

$$(I_2 \otimes \sigma_x) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{pmatrix}$$

i.e. $x_1 = x_2$ and $x_3 = x_4$:

$$\begin{pmatrix} x_1 \\ x_1 \\ x_3 \\ x_3 \end{pmatrix}.$$

Choosing $x_1 = 1$ and $x_3 = 0$ (respectively $x_1 = 0$ and $x_3 = 1$) and normalizing yields the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Other choices are also possible. The projection operator onto this eigenspace is given by

$$\Pi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^* + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}^* = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Try this with a different choice of orthonormal basis.

The eigenstates corresponding to the eigenvalue -1 are given by

$$(I_2 \otimes \sigma_x) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = - \begin{pmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{pmatrix}$$

i.e. $x_1 = -x_2$ and $x_3 = -x_4$:

$$\begin{pmatrix} x_1 \\ -x_1 \\ x_3 \\ -x_3 \end{pmatrix}.$$

Choosing $x_1 = 1$ and $x_3 = 0$ (respectively $x_1 = 0$ and $x_3 = 1$) and normalizing yields the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Other choices are also possible.
The projection operator onto this eigenspace is given by

$$\Pi_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}^* + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}^* = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Try this with a different choice of orthonormal basis.

Can these results be expressed in terms of the eigenvalues and eigenvectors of σ_x?

The probability that measurement of a system described by the state $(1, 0, 0, 0)^T$ yields the outcome 1 is given

$$\text{tr} \left(\Pi_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^* \right) = \frac{1}{2}.$$

The probability that measurement of a system described by the state $(1, 0, 0, 0)^T$ yields the outcome -1 is given

$$\text{tr} \left(\Pi_{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^* \right) = \frac{1}{2}.$$

The probability that measurement of a system described by the state $(1, 0, 0, 1)^T/\sqrt{2}$ yields the outcome 1 is given

$$\text{tr} \left(\Pi_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^* \right) = \frac{1}{2}.$$

The probability that measurement of a system described by the state $(1,0,0,1)^T/\sqrt{2}$ yields the outcome -1 is given

$$\mathrm{tr}\left(\Pi_{-1}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}^*\right)=\frac{1}{2}.$$

3. Note that since $\{|0\rangle,|1\rangle\}$ is an orthonormal basis on $\mathbf{C}^2$ we have $I_2=|0\rangle\langle 0|+|1\rangle\langle 1|$ (see assignment 3 nr. 5 2011). Suppose A is unitary then

$$\begin{aligned}
A^*A &= \left(\overline{\alpha}|b\rangle\langle a|+\overline{\beta}|d\rangle\langle c|\right)\left(\alpha|a\rangle\langle b|+\beta|c\rangle\langle d|\right)\\
&= |\alpha|^2|b\rangle\langle a|a\rangle\langle b|+\overline{\alpha}\beta|b\rangle\langle a|c\rangle\langle d|+\alpha\overline{\beta}|d\rangle\langle c|a\rangle\langle b|+|\beta|^2|d\rangle\langle c|c\rangle\langle d|=I_2\\
AA^* &= \left(\alpha|a\rangle\langle b|+\beta|c\rangle\langle d|\right)\left(\overline{\alpha}|b\rangle\langle a|+\overline{\beta}|d\rangle\langle c|\right)\\
&= |\alpha|^2|a\rangle\langle b|b\rangle\langle a|+\alpha\overline{\beta}|a\rangle\langle b|d\rangle\langle c|+\overline{\alpha}\beta|c\rangle\langle d|b\rangle\langle a|+|\beta|^2|c\rangle\langle d|d\rangle\langle c|=I_2
\end{aligned}$$

From $\langle 0|0\rangle=\langle 1|1\rangle=1$ we find $\langle a|a\rangle=\langle b|b\rangle=\langle c|c\rangle=\langle d|d\rangle=1$.

$$\begin{aligned}
A^*A &= \left(\overline{\alpha}|b\rangle\langle a|+\overline{\beta}|d\rangle\langle c|\right)\left(\alpha|a\rangle\langle b|+\beta|c\rangle\langle d|\right)\\
&= |\alpha|^2|b\rangle\langle b|+(\overline{\alpha}\langle a|c\rangle\beta)|b\rangle\langle d|+(\alpha\overline{\beta}\langle c|a\rangle)|d\rangle\langle b|+|\beta|^2|d\rangle\langle d|\\
&= |0\rangle\langle 0|+|1\rangle\langle 1|\\
AA^* &= \left(\alpha|a\rangle\langle b|+\beta|c\rangle\langle d|\right)\left(\overline{\alpha}|b\rangle\langle a|+\overline{\beta}|d\rangle\langle c|\right)\\
&= |\alpha|^2|a\rangle\langle a|+(\alpha\overline{\beta}\langle b|d\rangle)|a\rangle\langle c|+(\overline{\alpha}\beta\langle d|b\rangle)|c\rangle\langle a|+|\beta|^2|c\rangle\langle c|\\
&= |0\rangle\langle 0|+|1\rangle\langle 1|
\end{aligned}$$

Consider the first equation, if $b=d$ then we find

$$A^*A=(|\alpha|^2+\overline{\alpha}\langle a|c\rangle\beta+\alpha\overline{\beta}\langle c|a\rangle+|\beta|^2)|d\rangle\langle d|\neq|0\rangle\langle 0|+|1\rangle\langle 1|$$

since we either have $|0\rangle\langle 0|$ in the expression or $|1\rangle\langle 1|$ in the expression, but not both. Thus $b\neq d$. From $\langle 0|1\rangle=\langle 1|0\rangle=0$ we deduce $\langle b|d\rangle=\langle d|b\rangle=0$. Similarly from $AA^*=I_2$ we find $a\neq c$ and $\langle a|c\rangle=\langle c|a\rangle=0$. The two equations become

$$\begin{aligned}
A^*A &= |\alpha|^2|b\rangle\langle b|+|\beta|^2|d\rangle\langle d|=|0\rangle\langle 0|+|1\rangle\langle 1|,\\
AA^* &= |\alpha|^2|a\rangle\langle a|+|\beta|^2|c\rangle\langle c|=|0\rangle\langle 0|+|1\rangle\langle 1|.
\end{aligned}$$

Since $b\neq d$ and $a\neq c$ we only require that $|\alpha|^2=|\beta|^2=1$. Thus we find the solution ($a=0$, $b=0$, $c=1$ and $d=1$)

$$A=\alpha|0\rangle\langle 0|+\beta|1\rangle\langle 1|,\qquad |\alpha|=|\beta|=1$$

and the solution ($a=0$, $b=1$, $c=1$ and $d=0$)

$$A=\alpha|0\rangle\langle 1|+\beta|1\rangle\langle 0|,\qquad |\alpha|=|\beta|=1$$

The solution $a=1$, $b=1$, $c=0$ and $d=0$ is already provided by $a=0$, $b=0$, $c=1$ and $d=1$ and switching α and β. The solution $a=1$, $b=0$, $c=0$ and $d=1$ is already provided by $a=0$, $b=1$, $c=1$ and $d=0$ and switching α and β.

In general we need 4 terms:

$$U=\alpha|0\rangle\langle 0|+\beta|0\rangle\langle 1|+\gamma|1\rangle\langle 0|+\mu|0\rangle\langle 1|.$$

Note that because we restrict ourselves to $|0\rangle$ and $|1\rangle$ 4 terms are required, otherwise (from the spectral representation) only 2 terms would be required.

4. Let

$$\mathbf{a}:=\begin{pmatrix}a_1\\a_2\end{pmatrix},\qquad \mathbf{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

We have

$$\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} - \begin{pmatrix} b_1 a_1 \\ b_1 a_2 \\ b_2 a_1 \\ b_2 a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 b_2 - a_2 b_1 \\ -(a_1 b_2 - a_2 b_1) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \det \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \\ -\det \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \\ 0 \end{pmatrix}.$$

Setting $\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a} = \mathbf{0}$ yields $\det \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = 0$, i.e. $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. Consequently $\mathbf{a} = c\mathbf{b}$ for some $c \in \mathbb{C}$ and $\mathbf{b} \in \mathbb{C}^2$.