UNIVERSITY OF JOHANNESBURG

Applied Mathematics 3B

Assignment #5 **7:30, 13 September 2011**

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1. Let $\mathbf{x} = (1,0)^T$. Calculate

(a) $\mathbf{x} \otimes I_2$,

(b) $I_2 \otimes \mathbf{x}$,

(c) $\mathbf{x} \otimes \mathbf{x}^*$ and $\mathbf{x}\mathbf{x}^*$,

(d) $\mathbf{x}^* \otimes \mathbf{x}$ and $\mathbf{x}^*\mathbf{x}$.

2. Consider the Hilbert space $\mathbb{C}^4$. Show that the matrix

$$I_2 \otimes \sigma_x, \qquad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

describes an observable. Describe the measurement outcomes and associated probabilities when observing (performing the measurement on) a system described by the state

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and a system described by the state

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

3. Consider the Hilbert space $\mathbb{C}^2$ with an arbitrary orthonormal basis $\{\, |0\rangle, \, |1\rangle \,\}$. Let

$$A := \alpha |a\rangle\langle b| + \beta |c\rangle\langle d|$$

where $a, b, c, d \in \{0, 1\}$ and $\alpha, \beta \in \mathbb{C}$. Solve for $\alpha, \beta, a, b, c, d$ such that A is unitary.

We have used two terms in the sum describing A. Is it necessary to consider more terms to solve the problem in general?

4. Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$. Solve

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{b} \otimes \mathbf{a}$$

for $\mathbf{a}$ and $\mathbf{b}$. **Hint:** One approach is to show that

$$\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a} = \mathbf{0} \quad \Leftrightarrow \quad \det \left(\mathbf{a} \quad \mathbf{b} \right) = 0.$$
