Applied Mathematics 3B

Assignment #4 **Solution**

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1. We have

$$\rho = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

with characteristic equation

$$\det(\lambda I_2 - \rho) = \left(\lambda - \frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2 = \left(\lambda - \frac{1}{2} - \frac{1}{4}\right)\left(\lambda - \frac{1}{2} + \frac{1}{4}\right) = \left(\lambda - \frac{3}{4}\right)\left(\lambda - \frac{1}{4}\right) = 0.$$

Thus the eigenvalues are $\frac{3}{4}$ and $\frac{1}{4}$. Notice that the eigenvalues of a density matrix always sum to 1.

For the eigenvectors corresponding to the eigenvalue $\frac{3}{4}$ we solve $\rho(x,y)^T = \frac{3}{4}(x,y)^T$, i.e.

$$\frac{1}{4}(2x+y) = \frac{3}{4}x \qquad \frac{1}{4}(x+2y) = \frac{3}{4}y$$

with the solution $x = y$. Thus the eigenspace is given by

$$\left\{ \begin{pmatrix} t \\ t \end{pmatrix} : t \in \mathbb{C} \right\}.$$

A representative normalized eigenvector from this space is $(1,1)^T/\sqrt{2}$.

For the eigenvectors corresponding to the eigenvalue $\frac{1}{4}$ we solve $\rho(x,y)^T = \frac{1}{4}(x,y)^T$, i.e.

$$\frac{1}{4}(2x+y) = \frac{1}{4}x \qquad \frac{1}{4}(x+2y) = \frac{1}{4}y$$

so that $x = -y$. A representative normalized eigenvector from this space is $(1,-1)^T/\sqrt{2}$.

Thus we find

$$\rho = \frac{3}{4}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^* + \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}^*.$$

Consequently ρ can be realized by a mixture of 75% $(1,1)^T/\sqrt{2}$ and 25% $(1,-1)^T/\sqrt{2}$.

From the spectral decomposition of Hermitian matrices on $\mathbb{C}^2$ we see that two pure states (the eigenstates) are required to realize a mixed state on $\mathbb{C}^2$. The mixture is given by the eigenvalues. A mixture of only one pure state is obviously a pure state. Thus the minimum is 2.

2. We have

$$\rho(0) = \begin{pmatrix} \cos^2\theta & e^{-i\phi}\cos\theta\sin\theta \\ e^{i\phi}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

and

$$\rho(t) = e^{-i\hat{H}t/\hbar}\rho(0)e^{i\hat{H}t/\hbar}.$$

Since $-i\hat{H}t/\hbar = -i\omega t\sigma_x$ we find

$$(-i\hat{H}t/\hbar)^j = \begin{cases} (-i\omega t)^j\sigma_x & \text{if } j \text{ is odd, i.e. } j = 2k+1 \\ (-i\omega t)^j I_2 & \text{if } j \text{ is even, i.e. } j = 2k \end{cases}$$

where $k \in \mathbb{N}_0$. It follows that

$$\begin{aligned} e^{-i\hat{H}t/\hbar} &= \sum_{k=0}^{\infty} \frac{(-i\omega t)^{2k+1}}{(2k+1)!}\sigma_x + \sum_{k=0}^{\infty} \frac{(-i\omega t)^{2k}}{(2k)!} I_2 \\ &= -i\sum_{k=0}^{\infty} \frac{(-1)^k(\omega t)^{2k+1}}{(2k+1)!}\sigma_x + \sum_{k=0}^{\infty} \frac{(-1)^k(\omega t)^{2k}}{(2k)!} I_2 \\ &= -i\sin\omega t\,\sigma_x + \cos\omega t\, I_2 = \begin{pmatrix} \cos\omega t & -i\sin\omega t \\ -i\sin\omega t & \cos\omega t \end{pmatrix}. \end{aligned}$$

It follows that

$$e^{i\hat{H}t/\hbar} = \left(e^{-i\hat{H}t/\hbar}\right)^* = \begin{pmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{pmatrix}.$$

Finally

$$\begin{aligned} \rho(t) &= \begin{pmatrix} \cos\omega t & -i\sin\omega t \\ -i\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \cos^2\theta & e^{-i\phi}\cos\theta\sin\theta \\ e^{i\phi}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{pmatrix} \\ &= \begin{pmatrix} \cos\omega t\cos^2\theta - \frac{i}{2}e^{i\phi}\sin 2\theta\sin\omega t & \frac{1}{2}e^{-i\phi}\sin 2\theta\cos\omega t - i\sin\omega t\sin^2\theta \\ \frac{1}{2}e^{i\phi}\sin 2\theta\cos\omega t - i\sin\omega t\cos^2\theta & -\frac{i}{2}e^{-i\phi}\sin 2\theta\sin\omega t + \cos\omega t\sin^2\theta \end{pmatrix} \begin{pmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta\cos^2\omega t + \sin^2\theta\sin^2\omega t + \frac{\sin\phi}{2}\sin 2\theta\sin 2\omega t & \frac{\sin 2\theta}{2}(e^{i\phi}\sin^2\omega t + e^{-i\phi}\cos^2\omega t) + \frac{i}{2}\cos 2\theta\sin 2\omega t \\ \frac{\sin 2\theta}{2}(e^{-i\phi}\sin^2\omega t + e^{i\phi}\cos^2\omega t) - \frac{i}{2}\cos 2\theta\sin 2\omega t & \sin^2\theta\cos^2\omega t + \cos^2\theta\sin^2\omega t - \frac{\sin\phi}{2}\sin 2\theta\sin 2\omega t \end{pmatrix}. \end{aligned}$$

The initial state $\rho(0)$ describes any initial two state quantum system for appropriate choice of θ and ϕ. Thus we have found the the time evolution for any initial configuration of the system.

3. We must show that the set is linearly independent and that any 2×2 matrix over $\mathbb{C}$ can be represented as a linear combination of these matrices. To show linear independence we solve

$$a\frac{I_2}{\sqrt{2}} + b\frac{\sigma_x}{\sqrt{2}} + c\frac{\sigma_y}{\sqrt{2}} + d\frac{\sigma_z}{\sqrt{2}} = 0$$

for $a, b, c, d \in \mathbb{C}$. Multiplying the above equation by $\sqrt{2}$ we find

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$

i.e.

$$\begin{aligned} a + d &= 0 \\ b - ci &= 0 \\ b + ci &= 0 \\ a - d &= 0 \end{aligned}$$

with the only solution $a = b = c = d = 0$. Thus the set is linearly independent. Next we solve the matrix equation

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \alpha\frac{I_2}{\sqrt{2}} + \beta\frac{\sigma_x}{\sqrt{2}} + \gamma\frac{\sigma_y}{\sqrt{2}} + \epsilon\frac{\sigma_z}{\sqrt{2}}$$

for $\alpha, \beta, \gamma, \epsilon \in \mathbb{C}$ in terms of $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Thus we obtain the equations

$$\begin{aligned} a_1 &= \frac{\alpha + \epsilon}{\sqrt{2}} \\ a_2 &= \frac{\beta - i\gamma}{\sqrt{2}} \\ a_3 &= \frac{\beta + i\gamma}{\sqrt{2}} \\ a_4 &= \frac{\alpha - \epsilon}{\sqrt{2}} \end{aligned}$$

with the solution

$$\alpha = \frac{a_1 + a_4}{\sqrt{2}}, \qquad \beta = \frac{a_2 + a_3}{\sqrt{2}}, \qquad \gamma = i\frac{a_2 - a_3}{\sqrt{2}}, \qquad \epsilon = \frac{a_1 - a_4}{\sqrt{2}}.$$

To show that we have an orthonormal basis we first note that

$$\left(\frac{I_2}{\sqrt{2}}\right)^2 = \left(\frac{\sigma_x}{\sqrt{2}}\right)^2 = \left(\frac{\sigma_y}{\sqrt{2}}\right)^2 = \left(\frac{\sigma_z}{\sqrt{2}}\right)^2 = \frac{1}{2}I_2.$$

$$\left(\frac{I_2}{\sqrt{2}}\right)^* = \frac{I_2}{\sqrt{2}}, \qquad \left(\frac{\sigma_x}{\sqrt{2}}\right)^* = \frac{\sigma_x}{\sqrt{2}}, \qquad \left(\frac{\sigma_y}{\sqrt{2}}\right)^* = \frac{\sigma_y}{\sqrt{2}}, \qquad \left(\frac{\sigma_z}{\sqrt{2}}\right)^* = \frac{\sigma_z}{\sqrt{2}}.$$

The scalar products are

$$\left\langle \frac{I_2}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, I_2 = 1, \quad \left\langle \frac{I_2}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_x = 0, \quad \left\langle \frac{I_2}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_y = 0, \quad \left\langle \frac{I_2}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_z = 0$$

$$\left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_x = 0, \quad \left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, I_2 = 1, \quad \left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_z = 0, \quad \left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = -\frac{1}{2}\mathrm{tr}\, \sigma_y = 0$$

$$\left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_y = 0, \quad \left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = -\frac{1}{2}\mathrm{tr}\, \sigma_z = 0, \quad \left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, I_2 = 1, \quad \left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_x = 0$$

$$\left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_z = 0, \quad \left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, \sigma_y = 0, \quad \left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = -\frac{1}{2}\mathrm{tr}\, \sigma_x = 0, \quad \left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = \frac{1}{2}\mathrm{tr}\, I_2 = 1$$

Thus the basis is orthonormal. Expanding the matrix we find

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{I_2}{\sqrt{2}} \right\rangle \frac{I_2}{\sqrt{2}} + \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{\sigma_x}{\sqrt{2}} \right\rangle \frac{\sigma_x}{\sqrt{2}} + \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{\sigma_y}{\sqrt{2}} \right\rangle \frac{\sigma_y}{\sqrt{2}} + \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{\sigma_z}{\sqrt{2}} \right\rangle \frac{\sigma_z}{\sqrt{2}} \\ &= \left(\frac{a+d}{\sqrt{2}}\right)\frac{I_2}{\sqrt{2}} + \left(\frac{b+c}{\sqrt{2}}\right)\frac{\sigma_x}{\sqrt{2}} + \left(i\frac{b-c}{\sqrt{2}}\right)\frac{\sigma_y}{\sqrt{2}} + \left(\frac{a-d}{\sqrt{2}}\right)\frac{\sigma_z}{\sqrt{2}}. \end{aligned}$$

Notice that this result coincides exactly with the solution found when we showed that the set forms a basis.