UNIVERSITY OF JOHANNESBURG

# Applied Mathematics 3B

**Assignment #4** **7:30, 30 August 2011**

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**1.** Consider a mixture of 25% of the pure state $(1,0)^T$, 25% of the pure state $(0,1)^T$, and 50% of the pure state

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

described by the density operator

$$\rho = \frac{1}{4}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}^* + \frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}^* + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}^*$$

Find the spectral representation of $\rho$. Use the spectral representation of $\rho$ to find another mixture of pure states with the same (measurement) statistical properties as $\rho$.

What is the minimum number of pure states from $\mathbb{C}^2$ required to physically realize any mixed state on $\mathbb{C}^2$ ?

**2.** Consider the state

$$\boldsymbol{\psi} := \begin{pmatrix}\cos\theta\\ e^{i\phi}\sin\theta\end{pmatrix}$$

where $\theta, \phi \in \mathbb{R}$. We construct the density matrix $\rho(0) := \boldsymbol{\psi}\boldsymbol{\psi}^*$.

Given the Hamilton operator $\hat{H} = \hbar\omega\sigma_x$, solve the von Neumann equation for $\rho(t)$.

**3.** Show that the $2\times 2$ identity matrix $I_2$ together with the Pauli matrices

$$\left\{ \frac{I_2}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\}$$

forms a basis for the vector space of $2\times 2$ matrices over $\mathbb{C}$. Furthermore, show that the set is an orthonormal basis with respect to the Hilbert-Schmidt scalar product

$$\langle A, B\rangle := \mathrm{tr}(AB^*).$$

Expand the matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}$$

in terms of this basis.