Applied Mathematics 3B

Assignment #3 **Solution**

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1. The expectation value for a system described by the pure state $(1,0)^T$ is

$$\mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^*\right) = \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0.$$

The expectation value for a system described by the mixed state $1/2 I_2$ is

$$\mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = 0.$$

The mixed state can be considered as a mixture of 50% pure state $(1,0)^T$ and 50% pure state $(0,1)^T$ which are orthogonal and therefore distinguishable. The expectation value for $(1,0)^T$ and $(0,1)^T$ are both 0. Thus the expectation value for the mixed state is also 0. The expectation value does not allow us to distinguish between the given pure state and mixed state.

2. Let $\mathbf{x} = (a,b)^T$ where $a, b \in \mathbb{C}$ with $\mathbf{x}^*\mathbf{x} = |a|^2 + |b|^2 = 1$. We maximize the expectation value

$$\begin{aligned} f(a,b) &= \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}^*\right) \\ &= \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} \overline{a} & \overline{b} \end{pmatrix}\right) \\ &= \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |a|^2 & a\overline{b} \\ \overline{a}b & |b|^2 \end{pmatrix}\right) \\ &= \mathrm{tr}\begin{pmatrix} \overline{a}b & |b|^2 \\ |a|^2 & a\overline{b} \end{pmatrix} \\ &= \overline{a}b + a\overline{b} = 2\Re(a\overline{b}). \end{aligned}$$

Let $a = a_r + a_i i$ and $b = b_r + b_i i$ where $a_r, a_i, b_r, b_i \in \mathbb{R}$. Then

$$f(a,b) = a_r b_r + a_i b_i, \qquad a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1.$$

We use the Lagrange multiplier method:

$$g(a,b) = a_r b_r + a_i b_i + \lambda(a_r^2 + a_i^2 + b_r^2 + b_i^2 - 1)$$

where λ is the Lagrange multiplier. We obtain the equations

$$\frac{\partial g}{\partial a_r} = b_r + 2\lambda a_r = 0, \qquad \frac{\partial g}{\partial a_i} = b_i + 2\lambda a_i = 0, \qquad \frac{\partial g}{\partial b_r} = a_r + 2\lambda b_r = 0, \qquad \frac{\partial g}{\partial b_i} = a_i + 2\lambda b_i = 0$$

which yields

$$a_r = 4\lambda^2 a_r, \qquad a_i = 4\lambda^2 a_i, \qquad b_r = 4\lambda^2 b_r, \qquad b_i = 4\lambda^2 b_i.$$

Obviously $\lambda \neq 0$ (otherwise x is not normalized). Since at least one of a_r, a_i, b_r and b_i are non-zero we find $\lambda = \pm\frac{1}{2}$.

For $\lambda = -\frac{1}{2}$ we obtain $a_r = b_r$ and $a_i = b_i$ which yields $f(a,b) = 2|a|^2$. From the normalization constraint $|a|^2 + |b|^2 = 1$ we find $|a|^2 = \frac{1}{2}$ so that $f(a,b) = 1$. Thus we find a maximum at $\mathbf{x} = (a,a)^T$ where $|a|^2 = \frac{1}{2}$.

For $\lambda = \frac{1}{2}$ we find $a_r = -b_r$ and $a_i = -b_i$ which gives $f(a,b) = -2|a|^2$. From the normalization constraint $|a|^2 + |b|^2 = 1$ we find $|a|^2 = \frac{1}{2}$ so that $f(a,b) = -1$. This is a minimum.

The pure states for which the maximum is obtained can be written as

$$\mathbf{x} = \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For the expectation value for a system described by ρ we find

$$\begin{aligned} \operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(p\mathbf{x}\mathbf{x}^* + (1-p)\mathbf{y}\mathbf{y}^*)\right) &= p\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x}\mathbf{x}^*\right) + (1-p)\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y}\mathbf{y}^*\right) \\ &= pM + (1-p)\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y}\mathbf{y}^*\right) \\ &< pM + (1-p)M = M \end{aligned}$$

where we used that $p < 1$ and

$$\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y}\mathbf{y}^*\right) < M$$

since $\mathbf{y}$ is linearly independent of $\mathbf{x}$.

This result can be extended to an arbitrary number $n+1$ of pure states in the mixture. Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be pure states, all linearly independent of $\mathbf{x}$ and let $p_x, p_1, \ldots, p_n \in (0,1)$ with $p_x + p_1 + \cdots + p_n = 1$. Then

$$\begin{aligned} \operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(p_x\mathbf{x}\mathbf{x}^* + \sum_{j=1}^{n} p_j \mathbf{y}_j\mathbf{y}_j^*\right)\right) &= p_x\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x}\mathbf{x}^*\right) + \sum_{j=1}^{n} p_j\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y}_j\mathbf{y}_j^*\right) \\ &= p_x M + \sum_{j=1}^{n} p_j\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y}_j\mathbf{y}_j^*\right) \\ &< p_x M + \left(\sum_{j=1}^{n} p_j\right) M = M. \end{aligned}$$

3. Since

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

the matrix describes an observable. The eigenvalues and eigenvectors and corresponding representative normalized eigenvectors are

$$\begin{aligned} 2: &\quad \frac{1}{\sqrt{2}}(1,0,0,1)^T \\ 1: &\quad \frac{1}{\sqrt{2}}(0,1,0,0)^T, \quad \frac{1}{\sqrt{2}}(0,0,1,0)^T \\ 0: &\quad \frac{1}{\sqrt{2}}(1,0,0,-1)^T \end{aligned}$$

The projection operators onto the eigenspaces are

$$\begin{aligned} \Pi_2 &:= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \\ \Pi_1 &:= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \Pi_0 &:= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

The probability of obtaining the outcome 2 is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\Pi_2\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = 1.$$

The probability of obtaining the outcome 1 is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\Pi_1\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = 0.$$

The probability of obtaining the outcome 0 is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\Pi_0\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = 0.$$

Since the only possible outcome is 2, the state after measurement is

$$\Pi_2\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}.$$

Usually we would normalize this state, it is already normalized in this case.

4. Since A and B commute $[A, B] = AB - BA = 0$. Let $\mathbf{x}$ be an eigenvector of A corresponding to the eigenvalue λ. Then

$$A(B\mathbf{x}) = B(A\mathbf{x}) = \lambda(B\mathbf{x}).$$

Thus $B\mathbf{x}$ is either $\mathbf{0}$ or $B\mathbf{x}$ is also an eigenvector of A corresponding to λ.

Since the eigenspace of A corresponding to λ is one dimensional,

$$B\mathbf{x} = \mu\mathbf{x}$$

for some $\mu \in \mathbb{R}$. Thus $\mathbf{x}$ is an eigenvector of B corresponding to the eigenvalue μ.

It follows that A and B share one-dimensional eigenspaces and that measurement of A and B are compatible in the sense that performing the measurement described by B after the measurement described by A does not change the system after the measurement described by A (and vice-versa). If the system is in state described by an eigenstate of A (and consequently also B), then the results of measurement are independent of the order of measurement (A then B or B then A).

5. Since $\mathbf{x}$ describes a pure state we have $\|\mathbf{x}\|^2 = \mathbf{x}^*\mathbf{x} = 1$ and consequently $\mathbf{x} \neq \mathbf{0}$. Similarly $\mathbf{y} \neq \mathbf{0}$. Multiplying on the right by $\mathbf{x}$ we obtain

$$\left(\frac{1}{2}\mathbf{x}\mathbf{x}^* + \frac{1}{2}\mathbf{y}\mathbf{y}^*\right)\mathbf{x} = \frac{1}{2}\mathbf{x}(\mathbf{x}^*\mathbf{x}) + \frac{1}{2}\mathbf{y}(\mathbf{y}^*\mathbf{x}) = \frac{1}{2}\mathbf{x} + (\mathbf{y}^*\mathbf{x})\frac{1}{2}\mathbf{y} = \frac{1}{2}I_2\mathbf{x} = \frac{1}{2}\mathbf{x}.$$

Thus

$$(\mathbf{y}^*\mathbf{x})\frac{1}{2}\mathbf{y} = \mathbf{0}$$

and since $\mathbf{y} \neq \mathbf{0}$ we find $\mathbf{y}^*\mathbf{x} = 0$. Thus $\mathbf{x}$ and $\mathbf{y}$ must be orthogonal to each other. This condition is necessary but also sufficient. To see this note that $\{\mathbf{x}, \mathbf{y}\}$ forms an orthonormal basis for $\mathbb{C}^2$. Any vector $\mathbf{z} \in \mathbb{C}^2$ can then be written as $\mathbf{z} = z_x\mathbf{x} + z_y\mathbf{y}$ (where $z_x, z_y \in \mathbb{C}$) so that

$$\left(\frac{1}{2}\mathbf{x}\mathbf{x}^* + \frac{1}{2}\mathbf{y}\mathbf{y}^*\right)\mathbf{z} = \frac{1}{2}\mathbf{x}(\mathbf{x}^*\mathbf{z}) + \frac{1}{2}\mathbf{y}(\mathbf{y}^*\mathbf{z}) = \frac{1}{2}\mathbf{x}(z_x\mathbf{x}^*\mathbf{x} + z_y\mathbf{x}^*\mathbf{y}) + \frac{1}{2}\mathbf{y}(z_x\mathbf{y}^*\mathbf{x} + z_y\mathbf{y}^*\mathbf{y}) = \frac{1}{2}z_x\mathbf{x} + \frac{1}{2}z_y\mathbf{y} = \frac{1}{2}I_2\mathbf{z}.$$

Consequently from $\mathbf{y}^*\mathbf{x} = 0$ and $\mathbf{x}^*\mathbf{x} = \mathbf{y}^*\mathbf{y} = 1$ we obtain

$$\frac{1}{2}\mathbf{x}\mathbf{x}^* + \frac{1}{2}\mathbf{y}\mathbf{y}^* = \frac{1}{2}I_2.$$